# On the Lagrange dynamical equations 

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In a famous paper that was inserted in volume LV of Crelle's Journal (*), Clebsch proved that the differential equations that one obtains by annulling the first variation of a definite integral with fixed limits that contains as many unknown functions as one desires that are coupled with other and to the independent variables by first-order differential equations can be integrated when one knows a complete integral of a first-order partial differential equation, and thus, by the classical Hamilton-Jacobi procedure. In the final analysis, as the known Jacobi theory teaches, that is equivalent to putting the aforementioned differential equations into the form that Hamilton gave to the dynamical equations. The starting point for Clebsch's deductions is the method of multipliers that Lagrange had transferred (without proof) from mechanics to the calculus of variations.

The justification for that procedure was given by Mayer, first in the article "Begründung der Lagrange'schen Multiplicatorenmethode in der Variationsrechnung" [Ber. kön. Sächs. Ges. Wiss. zu Leipzig 37 (1885), pp. 1], and then in a later one that was entitled "Die Lagrange'sche Multiplicatorenmethode und das allgemeinste Problem der Variationsrechnung, etc.," [ibid., 47 (1895), pp. 129], in which he considered a more general problem for which the differential equations that Mayer gave only reduced to the Hamiltonian form.

Hamilton's principle would initially suggest that the equations of any dynamical problem in which a force function exists can always be reducible by the Clebsch-Mayer procedure to the canonical Hamiltonian form, but a more attentive analysis will show that that is not generally true, due to the different ways that the variations are presented in dynamics and in isoperimetric problems.

1.     - Let $p_{1}, p_{2}, \ldots, p_{N}$ denote the position parameters. In dynamics, the first variation of the Hamilton integral $\delta \int(T+U) d t$ is annulled while the $\delta p$ are coupled by a certain number of homogeneous linear equations:

$$
\begin{equation*}
\varphi_{\delta}^{(\nu)} \equiv \varphi_{1}^{(\nu)} \delta p_{1}+\varphi_{2}^{(\nu)} \delta p_{2}+\cdots+\varphi_{N}^{(\nu)} \delta p_{N}=0 \quad(n=1,2, \ldots, N-n), \tag{1}
\end{equation*}
$$

[^0]while the $p$ and their derivatives $d p / d t$ are coupled by differential equations of the form:
\[

$$
\begin{equation*}
\varphi^{(\nu)} \equiv \frac{\varphi_{d}^{(\nu)}}{d t} \equiv \varphi_{0}^{(\nu)}+\varphi_{1}^{(\nu)} \frac{d p_{1}}{d t}+\varphi_{2}^{(\nu)} \frac{d p_{2}}{d t}+\cdots+\varphi_{N}^{(\nu)} \frac{d p_{N}}{d t}=0 . \tag{2}
\end{equation*}
$$

\]

However, in the isoperimetric problems, one assumes that when one sets $p_{i}^{\prime}=d p_{i} / d t$, the $\delta$ are coupled by equations of the type:

$$
\sum_{i}\left(\frac{\partial \varphi^{(\nu)}}{\partial p_{i}} \delta p_{i}+\frac{\partial \varphi^{(\nu)}}{\partial p_{i}^{\prime}} \delta p_{i}^{\prime}\right)=0
$$

In that case of dynamics, that will become:

$$
\delta \varphi^{(\nu)} \equiv \frac{\partial \varphi^{(\nu)}}{\partial p_{1}} \delta p_{1}+\cdots+\frac{\partial \varphi^{(\nu)}}{\partial p_{N}} \delta p_{N}+\varphi_{1}^{(\nu)} \delta p_{1}^{\prime}+\cdots+\varphi_{N}^{(\nu)} \delta p_{N}^{\prime}=0
$$

When those equations are multiplied by $d t$, they can be written more concisely as:

$$
\begin{equation*}
\delta \varphi_{i}^{(\nu)}=0 \tag{2"}
\end{equation*}
$$

while assuming that $\delta t=0$. However, they are generally inconsistent with (1), which give:

$$
\begin{equation*}
d \varphi_{\delta}^{(\nu)}=0 \tag{1'}
\end{equation*}
$$

If the total differential equations:

$$
\varphi_{d}^{(\nu)} \equiv \varphi^{(\nu)} d p_{1}+\varphi_{1}^{(\nu)} d p_{1}+\cdots+\varphi_{N}^{(\nu)} d p_{N}=0
$$

are completely integrable then from a known theorem, the bilinear covariants:

$$
\delta \varphi_{d}^{(\nu)}-d \varphi_{\delta}^{(\nu)} \quad(\delta t=0)
$$

will prove to be identically zero, due to (1) and (2), and therefore (2") will also be true for the dynamical problem.

According to the Clebsch-Mayer procedure, one then sets:

$$
\Omega=T+U+\lambda_{1} \varphi^{(1)}+\lambda_{2} \varphi^{(2)}+\cdots+\lambda_{N-n} \varphi^{(N-n)}, \quad \frac{\partial \Omega}{\partial p_{i}^{\prime}}=q_{i} \quad(i=1,2, \ldots, N) .
$$

One combines those equations with (2), i.e.:

$$
\varphi^{(v)}\left(p_{1}, \ldots, p_{N} ; p_{1}^{\prime}, \ldots, p_{N}^{\prime} ; t\right)=0
$$

and solves the $2 N-n$ equations thus-defined for the $N-n$ multipliers $\lambda$ and the $p_{1}^{\prime}, \ldots, p_{N}^{\prime}$, which will be expressed as functions of $t, p_{i}, q_{i}$.

Transform the function:

$$
H=\sum_{i=1}^{N} p_{i}^{\prime} q_{i}-\Omega=\sum_{i=1}^{N} p_{i}^{\prime} q_{i}-(T+U)
$$

into those variables, whose transform will be denoted by $[H]$. If one calculates $\delta[H]$ in two different ways then one will easily find upon comparing the two expressions that ("):

$$
p_{i}^{\prime}=\frac{\partial[H]}{\partial q_{i}}, \quad \frac{\partial \Omega}{\partial p_{i}}=-\frac{\partial[H]}{\partial p_{i}},
$$

so the differential equations of motion will take on the appearance:

$$
\frac{d p_{i}}{d t}=\frac{\partial[H]}{\partial q_{i}}, \quad \frac{d q_{i}}{d t}=-\frac{\partial[H]}{\partial p_{i}} .
$$

If equations (2) can be integrated then their $N-n$ integrals will be just integrals of the preceding Hamiltonian system.

As a rule, in dynamics, one profits from the arbitrariness in the multipliers in order to reduce the differential equations of motion to the minimum possible number and not just to put them in the canonical Hamiltonian form by the Clebsch-Mayer procedure. Nonetheless, it is known that one increases the number of unknowns by $N-n$, but that is compensated by the advantage that Lie's theory of groups of integrals will permit one to derive the maximum benefit from the integrals that are known already in the integration of the canonical system. [Cf., Mayer, "Ueber die allgemeinen Integrale der dynamischen Diff. gleichungen, etc.," Math. Ann. 17, pp. 332, et seq., § 3]
2. - The Lagrangian form of the dynamical equations can be obtained by directly transforming the equation of virtual work, as Lagrange did in Mécanique analytique (Nouv. éd., t. I, pp. 304, et seq.), or as one better does nowadays with a stroke of the pen, so to speak, one appeals to the invariant theory of quadratic forms.
(*) Mayer, Leipziger Berichte (1895), pp. 141. The Lagrange differential equations are:

$$
\frac{d}{d t} \frac{\partial \Omega}{\partial p_{i}^{\prime}}=\frac{\partial \Omega}{\partial p_{i}}
$$

If one uses the notations of Mécanique analytique and considers the $d x, d y, d z$ to be functions of time that are continuous, differentiable, and satisfy the equations of constraint at any instant then the equations of virtual work, as Lagrange himself observed, can be written:

$$
\frac{d}{d t} \sum m\left(x^{\prime} \delta x+y^{\prime} \delta y+z^{\prime} \delta z\right)=\delta T+\delta \mathcal{L}
$$

in which:

$$
x^{\prime}=\frac{d x}{d t}, \quad \ldots, \quad T=\frac{1}{2} \sum m\left(x^{\prime 2}+y^{\prime 2}+z^{\prime 2}\right), \quad \delta \mathcal{L}=\sum(X \delta x+Y \delta y+Z \delta z) .
$$

Therefore, the equation of work can be put into the form:

$$
\begin{equation*}
\frac{d}{d t} \sum\left(\frac{\partial T}{\partial x^{\prime}} \delta x+\frac{\partial T}{\partial y^{\prime}} \delta y+\frac{\partial T}{\partial z^{\prime}} \delta z\right)=\delta T+\delta \mathcal{L} \tag{3}
\end{equation*}
$$

The proof of the following theorem of algebra is obvious:

The polar of a homogeneous quadratic form is a covariant, no matter what the number of new variables that one introduces in place of the original ones into the homogeneous linear cogredient transformations.

Now, if all of the $x, y, z$ are compatible with the constraints, and one expresses them as functions of as many new variables or position parameters as one pleases:

$$
p_{1}, p_{2}, \ldots, p_{N}, \text { and time } t
$$

then one will have:

$$
\begin{aligned}
& x^{\prime}=\sum_{i} \frac{\partial x}{\partial p_{i}} p_{i}^{\prime}+\frac{\partial x}{\partial t} t^{\prime}, \quad \ldots, \\
& \delta x=\sum_{i} \frac{\partial x}{\partial p_{i}} \delta p_{i}+\frac{\partial x}{\partial t} \delta t, \ldots
\end{aligned}
$$

and one will have to set:

$$
t^{\prime}=1, \quad \delta t=0
$$

after completing the transformation.
Hence, express the vis viva $T$ as a function of the $p_{i}, p_{i}^{\prime}$, and $t$, so the transform of the polar form:

$$
\sum\left(\frac{\partial T}{\partial x^{\prime}} \delta x+\frac{\partial T}{\partial y^{\prime}} \delta y+\frac{\partial T}{\partial z^{\prime}} \delta z\right)
$$

will certainly be:

$$
\sum_{i} \frac{\partial T}{\partial p_{i}^{\prime}} \delta p_{i}
$$

and consequently, the transform of (3) will be:

$$
\begin{equation*}
\frac{d}{d t} \sum_{i=1}^{N} \frac{\partial T}{\partial p_{i}^{\prime}} \delta p_{i}=\delta T+\delta \mathcal{L} \tag{III}
\end{equation*}
$$

in which $T$ must mean the vis viva, when expressed in the manner that was just defined, and $\delta \mathcal{L}$ must mean the expression for the virtual work done by the applied force, and therefore an expression of the form:

$$
\delta \mathcal{L} \equiv \sum_{i=1}^{N} \mathcal{P}_{i} \delta p_{i}
$$

Equation (III) gives the most general analytical expression for the principle of virtual work, and we shall call it Lagrange's principle. It is equivalent to Hamilton's principle.

In general, the equations of constraint are given in part in finite form and in part in the form of total differential equations between the coordinates and time. If one chooses the parameters $p$ in such a way that the finite equations are all satisfied identically and perform the transformation on the $p$ according to Lagrange's principle then (III) must be verified by all of the $\delta p$ that satisfy a certain number $N-n$ of total differential equations (1), while the differentials of the parameter and time are coupled by $N-n$ total differential equations of the type ( $2^{\prime}$ ).
3. - Let $\varphi$ be a linear function of the $p^{\prime}$ whose coefficients and terms are known functions of just the $p$ and $t$, i.e.:

$$
\varphi \equiv \varphi_{0}+\varphi_{1} p_{1}^{\prime}+\varphi_{2} p_{2}^{\prime}+\cdots+\varphi_{N} p_{N}^{\prime} .
$$

If one sets:

$$
\begin{aligned}
& \varphi_{d} \equiv \varphi_{0} d t+\varphi_{1} d p_{1}+\varphi_{2} d p_{2}+\cdots+\varphi_{N} d p_{N} \\
& \varphi_{\delta} \equiv \varphi_{0} \delta t+\varphi_{1} \delta p_{1}+\varphi_{2} \delta p_{2}+\cdots+\varphi_{N} \delta p_{N}
\end{aligned}
$$

then (III) can be written:

$$
\begin{equation*}
\frac{d}{d t} \sum_{i} \frac{\partial(T+\varphi)}{\partial p_{i}^{\prime}} \delta p_{i}=\delta(T+\varphi)+\delta \mathcal{L}-\frac{\delta \varphi_{d}-d \varphi_{\delta}}{d t} \tag{IV}
\end{equation*}
$$

in which the bilinear covariant is:

$$
\begin{equation*}
\delta \varphi_{d}-d \varphi_{\delta}=\frac{1}{2} \sum_{i} \sum_{k}\left(\frac{\partial \varphi_{i}}{\partial p_{k}}-\frac{\partial \varphi_{k}}{\partial p_{i}}\right)\left(d p_{i} \delta p_{k}-d p_{k} \delta p_{i}\right)+\sum_{k}\left(\frac{\partial \varphi_{0}}{\partial p_{k}}-\frac{\partial \varphi_{k}}{\partial t}\right)\left(d t \delta p_{k}-d p_{k} \delta t\right) \tag{5}
\end{equation*}
$$

and one sets:

$$
\delta t=0 .
$$

If $\varphi$ is the total derivative with respect to time of an arbitrary function $U$ of the $p_{i}$ and $t$ such that:

$$
\varphi_{d}=d U
$$

then the bilinear covariant of $\varphi_{d}$ will be identically zero, and (IV) will then become:

$$
\frac{d}{d t} \sum_{i} \frac{\partial(T+\varphi)}{\partial p_{i}^{\prime}} \delta p_{i}=\delta(T+\varphi)+\delta \mathcal{L}
$$

and consequently:
The Lagrange dynamical equations will remain unaltered when one adds the total derivatives of an arbitrary function of the position parameters and time to the vis viva.

That theorem is already found implicitly in Mécanique analytique (t. I, pp. 311, 8).
4. - If some of the differential equations of constraint (2) or some of their linear combinations are completely integrable then one sets:

$$
\varphi=\lambda \frac{d f}{d t}
$$

in which $f$ is an arbitrary integral and $\lambda$ is an arbitrary multiplier.
Therefore $\varphi_{d}=\lambda d f$, and since $d f$ and $\delta f$ are annulled due to equations (1) and (2), (IV) will give:

$$
\begin{aligned}
\frac{d}{d t} \sum_{i} \frac{\partial(T+\varphi)}{\partial p_{i}^{\prime}} \delta p_{i} & =\delta(T+\varphi)+\delta \mathcal{L}-\lambda \frac{\delta d f-d \delta f}{d t} \\
& =\delta(T+\varphi)+\delta \mathcal{L}
\end{aligned}
$$

That equation will still be true if one assumes that $\lambda$ is an arbitrary function of not only the $p_{i}$ and $t$, but also the $p_{i}^{\prime}$.

We therefore conclude the following theorem:
The Lagrange dynamical equations will remain unaltered if the product of an arbitrary multiplier with the total derivative of an integral of the equations of constraint is added to the vis viva.

The last theorem and the preceding one imply the following general theorem:

If $f_{1}, f_{2}, f_{3}, \ldots$ are integrals of the differential equations of constraint or also of the equations of motion, but ones that do not contain the $p_{i}^{\prime}$ then set:

$$
\varphi=\frac{d U}{d t}+\lambda_{1} \frac{d f_{1}}{d t}+\lambda_{2} \frac{d f_{2}}{d t}+\lambda_{3} \frac{d f_{3}}{d t}+\cdots
$$

in which $U$ is an arbitrary function $p_{i}$ and $t$ and $\lambda_{1}, \lambda_{2}, \lambda_{3}, \ldots$ are arbitrary functions of the $p_{i}, p_{i}^{\prime}$, and $t$. The Lagrange dynamical equations will then remain unaltered when one adds $\varphi$ to the vis viva.

Observe that (IV) can be written:

$$
\sum_{i}\left\{\frac{d}{d t} \frac{\partial(T+\varphi)}{\partial p_{i}^{\prime}}-\frac{\partial(T+\varphi)}{\partial p_{i}}-\mathcal{P}_{i}\right\} \delta p_{i}+\frac{\delta \varphi_{d}-d \varphi_{\delta}}{d t}=0
$$

Suppose that the differential equations of constraint are completely integrable: The $f$ will then be $N-n$ in number, and then imagine that one has determined the $N-n$ multipliers $\lambda$ in the Lagrange manner, in such a way that the coefficients of just as many of the $\delta p_{i}$ in (4'), which are functions of the remaining ones, due to (1), are annulled. One will then arrive at the system of differential equations:

$$
\frac{d}{d t} \frac{\partial \Omega}{\partial p_{i}^{\prime}}-\frac{\partial \Omega}{\partial p_{i}}=\mathcal{P}_{i} \quad(i=1,2, \ldots, N)
$$

in which:

$$
\Omega \equiv T+\frac{d U}{d t}+\lambda_{1} \frac{\partial f_{1}}{\partial t}+\cdots+\lambda_{N-n} \frac{\partial f_{N-n}}{\partial t} .
$$

5.     - Regard $t$ as a constant parameter in (1) and suppose that those equations admit several integrable linear combinations. Let $P, Q, R$ be three arbitrary integrals. Set:

$$
\varphi_{d}=P d Q+R d t
$$

so one will have:

$$
\delta \varphi_{d}-d \varphi \delta=\delta P d Q-d P \delta Q+\delta R d t
$$

and since equations (1) imply that $\delta P=\delta Q=\delta R=0$, one can conclude that the bilinear covariant of $\varphi_{d}$ is zero identically. Therefore:

The Lagrange differential equations will remain unaltered when one adds $\varphi \equiv P(d Q / d t)+R$ to them, in which $P, Q, R$ are any three integrals of the equations of constraint when their state is fixed at time $t$.
6. - Set:

$$
\varphi \equiv \frac{\varphi_{d}}{d t}, \quad \varphi_{d} \equiv d U+\sum_{v=1}^{N-n} \lambda_{\nu} \varphi_{d}^{(\nu)}
$$

in (IV), in which $\lambda_{1}, \lambda_{2}, \ldots$ denote arbitrary functions of the $p_{i}, p_{i}^{\prime}$, and $t$. From what we just saw, it will obviously result that:

$$
\begin{equation*}
d \sum_{i=1}^{N} \frac{\partial(T+\varphi)}{\partial p_{i}^{\prime}} \delta p_{i}=d t[\delta(T+\varphi)+\delta \mathcal{L}]-\sum_{\nu=1}^{N-n} \lambda_{v}\left(\delta \varphi_{d}^{(\nu)}-d \varphi_{\delta}^{(\nu)}\right) \tag{6}
\end{equation*}
$$

In general, one supposes that (2) do not admit integrable linear combinations.
The system (1) of homogeneous linear equations in the $\delta p$ always admits $n$ linearlyindependent solutions ( ${ }^{*}$ ):

$$
\left.\begin{array}{lllll}
\text { 1.a) } & \xi_{11}, & \xi_{12}, & \cdots & \xi_{1 N} \\
2 . a) & \xi_{21}, & \xi_{22}, & \cdots & \xi_{2 N} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\text { n.a) } & \xi_{n 1}, & \xi_{n 2}, & \cdots & \xi_{n N}
\end{array}\right\},
$$

since the most-general solution of those equations will be:

$$
\begin{equation*}
\delta p_{k}=\sum_{v=1}^{n} \xi_{r k} \delta \pi_{r} \quad(k=1,2, \ldots, N) \tag{7}
\end{equation*}
$$

in which $\delta \pi_{r}$ are arbitrary infinitesimal multipliers.
Now let:

$$
\xi_{01}, \quad \xi_{02}, \quad \ldots, \quad \xi_{0 N}
$$

be any particular solution to the system of the inhomogeneous linear equations (2). Therefore:

$$
\xi_{01} \varphi_{1}^{(\nu)}+\xi_{02} \varphi_{2}^{(\nu)}+\cdots+\xi_{0 N} \varphi_{N}^{(\nu)}=-\varphi_{1}^{(\nu)} \quad(\nu=1,2, \ldots, N-n) .
$$

The most-general solution of ( $2^{\prime}$ ) will be:

$$
\begin{equation*}
d p_{k}=\sum_{s=1}^{n} \xi_{s k} d \pi_{s}+\xi_{0 k} d t \quad(k=1,2, \ldots, N) \tag{8}
\end{equation*}
$$

(*) It cannot happen that the $\varphi_{d}^{(v)}$ are not linearly independent, since if that were the case then one could eliminate the $d p$ from ( $2^{\prime}$ ), which must be independent, and at least one finite equations between the $p_{i}$ and $t$ would result from that.

If one recalls (5), in which one sets $\delta t=0$, then one will obviously have:

$$
\sum_{v=1}^{N-n} \lambda_{v}\left(\delta \varphi_{d}^{(\nu)}-d \varphi_{\delta}^{(\nu)}\right)=\sum_{v=1}^{N-n} \lambda_{v}\left[\sum_{r=1}^{n} \sum_{s=1}^{n} P_{r s}^{(\nu)}\left(d \pi_{r} \delta \pi_{s}-d \pi_{s} \delta \pi_{r}\right)+\sum_{r=1}^{n} P_{0 r}^{(\nu)} d t \delta \pi_{r}\right],
$$

in which one sets:

$$
\begin{aligned}
& P_{r s}^{(\nu)}=\frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{N} \xi_{r i} \xi_{s k}\left(\frac{\partial \varphi_{i}^{(\nu)}}{\partial p_{k}}-\frac{\partial \varphi_{k}^{(\nu)}}{\partial p_{i}}\right)=-P_{s r}^{(v)}, \\
& P_{0 r}^{(\nu)}=\frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{N}\left(\xi_{0 i} \xi_{r k}-\xi_{0 k} \xi_{r s}\right)\left(\frac{\partial \varphi_{i}^{(\nu)}}{\partial p_{k}}-\frac{\partial \varphi_{k}^{(\nu)}}{\partial p_{i}}\right)+\sum_{i=1}^{N}\left(\frac{\partial \varphi_{0}^{(\nu)}}{\partial p_{i}}-\frac{\partial \varphi_{i}^{(\nu)}}{\partial p_{0}}\right) \xi_{r i},
\end{aligned}
$$

for brevity.
It is known that the necessary and sufficient conditions for the system of total differential equations ( $2^{\prime}$ ) to be completely integrable are:

$$
P_{r s}^{(v)}=0, \quad P_{0 r}^{(v)}=0 .
$$

The last term on the right-hand side of (6) can be made to vanish identically, due to (1) and ( $2^{\prime}$ ), when it is possible to determine the $\lambda_{\nu}$ in such a way that the $n(n+1) / 2$ equations:

$$
\sum_{v=1}^{N-n} \lambda_{v} P_{r s}^{(\nu)}=0, \quad \sum_{v=1}^{N-n} \lambda_{v} P_{0 r}^{(\nu)}=0
$$

are still satisfied. In the first of them, $r$ and $s$ are taken from all of the binary combinations of the indices $1,2, \ldots, n$. If the constraints are independent of time then one will have:

$$
\varphi_{0}^{(\nu)}=0, \quad \xi_{01}=\xi_{02}=\ldots=\xi_{0 N}=0, \quad \frac{\partial \varphi_{k}^{(\nu)}}{\partial t}=0 .
$$

Therefore: $P_{0 r}^{(\nu)}=0$, since the preceding equations will then reduce to the first $n(n-1) / 2$. That can always be satisfied, and it will determine the ratios of the $\lambda$ as functions of the $p_{i}$ and $t$ when $N>n(n+1) / 2$. Among the linear combinations of the $\varphi^{(\nu)}$, there will then exist ones that can be multiplied by arbitrary multipliers and added to the vis viva without altering the Lagrange dynamical equations, as Hadamard brought to light in his interesting paper: "Sur les mouvements de roulement," Mém. de la Société des Sciences phys. et nat. de Bordeaux, t. V, s. IV, pp. 397, which was reproduced in no. 4 of the collection Scientia.

However, if the constraints depend upon time, but in such a way that their state is fixed at an arbitrary instant $t$, so their differential equations will become completely integrable, then the first of the preceding equations will become an identity, while the last $n$ can always be satisfied and will determine the ratios of the $\lambda$ as functions of the $p$ and $t$ when:

$$
N>2 n
$$

Consequently, there will then be at least $N-2 n$ linear combinations of the $\varphi^{(\nu)}$ that can be added to the vis viva after one multiplies them by arbitrary multipliers without altering the dynamical equations.

In general, there will always be such combinations when $N>n(n+3) / 2$, and their total number will be at least $N-n(n+3) / 2$.
7. - When the system of homogeneous linear partial equations is adjoint or reciprocal to (2') then we will have ("):

$$
\begin{equation*}
X_{r} f \equiv \xi_{r 0} \frac{\partial f}{\partial p_{0}}+\xi_{r 1} \frac{\partial f}{\partial p_{1}}+\cdots+\xi_{r N} \frac{\partial f}{\partial p_{N}}=0 \quad(r=0,1,2, \ldots, n), \tag{9}
\end{equation*}
$$

in which we have written $p_{0}$ instead of $t, \xi_{00}$ instead of 1 , and $\xi_{10}, \xi_{20}, \ldots, \xi_{n 0}$ instead of 0 , for the sake of symmetry. It is known that the conditions for the complete integrability of (2') are that the linear partial differential equations (9) must constitute a complete system or that the ( $X_{r}, X_{s}$ ) $f$ are linear combinations of those same $X f$.

Combine (9) with:

$$
\begin{equation*}
\left(X_{r}, X_{s}\right) f=0, \tag{10}
\end{equation*}
$$

which are not all a consequence of (9), in general. One will generally have an incomplete system then that consists of $m$ distinct equations, where:

$$
n<m<n+1+\frac{1}{2} n(n+1)=\frac{1}{2}(n+1)(n+2) .
$$

Suppose that $N>m$, so the reciprocal of that will consist of:

$$
H=N+1-m \geq N-\frac{1}{2} n(n+3)
$$

equations, and therefore, at least $N-\frac{1}{2} n(n+3)$ total differential equations.
Those equations are independent homogeneous linear combinations of the original ones (2'). Indeed, from (9), they are satisfied in the most general way by taking:

$$
\frac{\partial f}{\partial p_{i}}=\varphi_{i}=\lambda_{1} \varphi_{i}^{(1)}+\lambda_{2} \varphi_{i}^{(2)}+\cdots+\lambda_{N-n} \varphi_{i}^{(N-n)} \quad(i=0,1,2, \ldots, N) .
$$

[^1]If one substitutes that in (10) then one will have $m-n-1$ homogeneous linear equations in the multipliers $\lambda$ that will admit $H=N-m+1$ distinct solutions, and any one of them will correspond to the reciprocal system to an equation $\varphi_{d}=0$.

It is easy to prove the following theorem, which is an obvious generalization of another one that was published by Hadamard in the article "Sur certains systèmes d'équations aux diff. tot.," Procés-verbaux de la Société des Sciences phys. et nat. de Bordeaux, Année 1894-95, pp. 17.

The bilinear covariant of $\varphi_{d}$ is identically zero, due to (1) and (2).

Indeed, from the identity:

$$
\sum_{i=0}^{N} \xi_{r i} \varphi_{i}=0 \quad(r=0,1, \ldots, n)
$$

one will also have:

$$
\sum_{i=0}^{N} \xi_{r i} \frac{\partial \varphi_{i}}{\partial p_{k}}=-\sum_{i=0}^{N} \varphi_{r i} \frac{\partial \xi_{r i}}{\partial p_{k}} \quad(k=0,1,2, \ldots, N)
$$

and therefore:

$$
2 P_{r s}=\sum_{i} \sum_{k} \xi_{r i} \xi_{s k}\left(\frac{\partial \varphi_{i}}{\partial p_{k}}-\frac{\partial \varphi_{k}}{\partial p_{i}}\right)=\sum_{i} \varphi_{i} \sum_{k}\left(\xi_{r k} \frac{\partial \xi_{r i}}{\partial p_{k}}-\xi_{s k} \frac{\partial \xi_{r i}}{\partial p_{i}}\right),
$$

which is the expression that one will obtain from $\left(X_{r}, X_{s}\right) f$ by substituting the $\varphi_{i}$ for the $\partial f$ / $\partial p_{i}$, respectively, and that will therefore be zero identically.

Conversely, that will suggest that if $\varphi_{d}$ belongs to the array $\sum_{v} \lambda_{v} \varphi_{d}^{(\nu)}$ then in order for its bilinear covariant to be identically zero, it is necessary that $\varphi_{d}$ must belong to the adjoint or reciprocal system to (9) and (10). Therefore, form the system of total differential equations that is adjoint (9) and (10), which is denoted by:

$$
\begin{equation*}
\varphi_{d}^{I}=0, \quad \varphi_{d}^{I I}=0, \quad \ldots, \quad \varphi_{d}^{H}=0 \tag{11}
\end{equation*}
$$

and set:

$$
\Omega \equiv T+\mu_{I} \varphi^{I}+\mu_{I I} \varphi^{I I}+\cdots+\mu_{H} \varphi^{H},
$$

in which the multipliers $\mu$ are arbitrary functions of the $p_{i}, p_{i}^{\prime}$, and $t$. Equations (4') will become:

$$
\begin{equation*}
\sum_{i=1}^{N}\left(\frac{d}{d t} \frac{\partial \Omega}{\partial p_{i}^{\prime}}-\frac{\partial \Omega}{\partial p_{i}}-\mathcal{P}_{i}\right) \delta p_{i}=0 \tag{12}
\end{equation*}
$$

If one still assumes that (11) is soluble for the $p_{1}^{\prime}, p_{2}^{\prime}, \ldots, p_{H}^{\prime}$, and the multipliers $\mu$ then one can obviously determine them in such a way that $\Omega$ does not contain $p_{1}^{\prime}, \ldots, p_{H}^{\prime}$, from which, it will result that the $\mu$ are linear functions of the $p_{i}^{\prime}$. One must then have:

$$
\frac{\partial \Omega}{\partial p_{1}^{\prime}}=\frac{\partial \Omega}{\partial p_{2}^{\prime}}=\ldots=\frac{\partial \Omega}{\partial p_{H}^{\prime}}=0
$$

so from (12), one can then eliminate the $p_{1}^{\prime}, \ldots, p_{H}^{\prime}$.

Now express the $\delta p_{i}$ as functions of the independent infinitesimal parameters $\delta \pi_{r}$ by means of (7). (12) will split into $n$ second-order differential equations that no longer include the derivatives with respect to time of the $p_{1}, \ldots, p_{H}$.

The same thing will be true for those $N-n-H$ of the differential equations of constraint (2) that were not utilized for eliminating the $p_{1}^{\prime}, \ldots, p_{H}^{\prime}$.

In the particular case where (11) are completely integrable by means of the integrals of (12) and the aforementioned $N-n-H$ equations of constraint, one can also eliminate $p_{1}, p_{2}, \ldots, p_{H}$.

However, one can profit from the arbitrariness of the multipliers $\mu$ in order to annul $H$ terms in (12), or to satisfy the $H$ equations:

$$
\frac{d}{d t} \frac{\partial \Omega}{\partial p_{i}^{\prime}}-\frac{\partial \Omega}{\partial p_{i}}=\mathcal{P}_{i} \quad(i=1,2, \ldots, H)
$$

The equation of work will then become:

$$
\begin{equation*}
\sum_{i=H+1}^{N}\left(\frac{d}{d t} \frac{\partial \Omega}{\partial p_{i}^{\prime}}-\frac{\partial \Omega}{\partial p_{i}}-\mathcal{P}_{i}\right) \delta p_{i}=0 \tag{12'}
\end{equation*}
$$

The first $H$ equations can be associated with the $H$ equations that one obtains by solving (11) for the $p_{1}^{\prime}, \ldots, p_{H}^{\prime}$. The equation of virtual work is associated with the other $N-n-H$ equations of constraint, which are independent of (11), and one intends to eliminate $p_{1}^{\prime}, \ldots, p_{H}^{\prime}$ from them by means of the preceding ones. The missing $n$ dynamical equations are then deduced from (12') by the usual procedures, by keeping the $p$, which are coupled to each other and time $t$ by those $N$ $-n-H$ total differential equations that are obtained by the multiplying the aforementioned equations by $d t$.


[^0]:    (*) "Ueber diejenigen Probleme der Variationsrechnung welche nur eine unabhängige Variable enthalten," loc. cit., pp. 335.

[^1]:    (*) Cf., Pascal, I gruppi continui di trasformazioni, Chap. V, §§ 2, 3, Manuali Hoepli, Milano, 1903.

