"Die Liniengeometrie nach den Prinzipien der Grassmanschen Ausdehnungslehre," Monastshefte f. Mathematik u. Physik, **II** (1891), 267-290.

Line geometry, according to the principles of Grassmann's theory of extensions.

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Although *Grassmann* had already expressed the idea of employing the straight line as a spatial element in his *Ausdehnungslehre* (Theory of Extensions) of 1844, and to that end, exhibited systems of line coordinates, and also included many things that showed how simply line geometry could be formulated using his method in the second edition of his book in 1862, to the best of my knowledge, no one has made such an attempt up to now. The lines that follow contain a treatment of line geometry that is based upon the principles of *Grassmann's* line theory of extensions. I therefore hope to give an example of the simple applicability of *Grassmann's* great creation that has, unfortunately, still found much too little circulation.

In order for me to be as brief as possible, it will often be necessary to refer to the two editions of *Ausdehnunglehre* from 1844 (*) and 1862 in order to explain the concepts that are used; as usual, they shall be denoted by A_1 and A_2 .

The linear ray complex.

1. If one let A, B, …, X, … denote line segments (**) of unit magnitude – i.e., rays in which segments of unit length and definite direction are found – then the line segments $\mathfrak{a}\,A$, $\mathfrak{b}\,B$, …, $\mathfrak{x}\,X$, where $\mathfrak{a}$, $\mathfrak{b}$, …, $\mathfrak{x}$, … are real or complex numbers, will represent those rays, in which one must now think of the segments as having lengths $\mathfrak{a}$, $\mathfrak{b}$, …, $\mathfrak{x}$, …, however. Any algebraic sum C of line segments can be reduced to the sum of two line segments that have no point in common, and can thus be represented in the form (A_2, no. 285):

$$C = \mathfrak{a}\,A + \mathfrak{b}\,B . \tag{1}$$

This expression is itself capable of no further reduction, and has no immediate geometric meaning, although it plays the main role in line geometry.

(*) A new edition of this version that was edited by *Grassmann* appeared in 1878 that was printed by O. Wigand in Leipzig. The 1862 version is out of print.

(**) I prefer the expressions: line segment, surface segment, space segment of *H. Hankel* (*Theorie der complexen Zahlen*, Leipzig, 1867) to *Grassmann's* expressions: line part, surface part, body part (A_2) or line quantity, plane quantity, body space (A_1).

Namely, if we ask what all rays (i.e., line segments) X would be that would give zero when exterior multiplied by C, which would then fulfill the equation:

$$[C\,X] = 0, \tag{2}$$

then it would be easy to see that they would all define a linear complex.

Then, since any ray X that goes through a point p is representable in the form:

$$X = \mathfrak{x}\,[p\,x],$$

in which x is any point of X and $\mathfrak{x}$ means a well-defined number, one must have:

$$[C \cdot p\,x] = [C\,p \cdot x] = 0$$

for the rays that go through p, or since $[C\,p] = \mathfrak{a}\,[A\,P] + \mathfrak{b}\,[B\,p]$ is equal to a plane segment π that goes through p, one must have:

$$[\pi\,p] = 0,$$

which says that every point x, and thus, also every ray X, will lie in the plane π. The rays X that go through a point x and satisfy equation (2) will then define a pencil of rays.

One likewise shows that the rays X that lie in the plane π and satisfy equation (2) will also define a pencil of rays.

In fact, since every ray X that lies in a plane π can be represented in the form:

$$X = \mathfrak{x}\,[\pi\,\xi],$$

in which x means any plane (i.e., plane segment) through X and $\mathfrak{x}$ means a well-defined number, one must have:

$$[C \cdot \pi\,\xi] = [C\,\pi \cdot \xi] = 0$$

for the rays that lie in π, which will then satisfy equation (2), or since $[C\,\pi] = \mathfrak{a}\,[A\,\pi] + \mathfrak{b}\,[b\,\pi]$ is equal to a multiple point p that lies in π, one must have:

$$[p\,\xi] = 0,$$

which says that every plane x, and therefore, also every ray X, goes through the point p.

With that, we have proved:

Any quantity $C = \mathfrak{a}\,A + \mathfrak{b}\,B$ determines a unique linear ray complex by equation (2).

The fact that, conversely, any linear ray complex determines a quantity C will be derived later on (no. 2).

If C reduces to a line segment then all rays X that satisfy equation (2) will have the property that they cut C; i.e., C will then determine a special linear complex whose axis will be represented by C.

By the considerations above, one can likewise find a simple expression for the relationship in the null system that the linear complex implies. Namely, any point p will correspond to the plane $[C\,p]$ as the null plane. The fact that the points of a line will correspond to the planes of a pencil is deduced from that immediately. Then, if p_1, p_2 are any two points then any point p on the line $[p_1\,p_2]$ will be representable in the form $p = \mathfrak{p}_1\,p_1 + \mathfrak{p}_2\,p_2$; one obtains the null plane of p by exterior multiplying by C:

$$[C\,p] = \mathfrak{p}_1\,[C\,p_1] + \mathfrak{p}_2\,[C\,p_2],$$

which is then derived from the null planes of p_1 and p_2, and thus belongs to the pencil of these planes. The ray $[p_1\,p_2]$ is then said to be *associated* with the ray $[\pi_1\,\pi_2]$.

One infers the converse theorem analogously.

A and B are associated rays in the linear ray complexes that are defined by equations (1) and (2); since $[A\,a] = 0$, any point a of A will then correspond to the plane:

$$[C\,a] = \mathfrak{a}\,[A\,a] + \mathfrak{b}\,[B\,b] = \mathfrak{b}\,[B\,a],$$

which goes through B. Likewise any point b of B will correspond to a plane through A.

If one represents C in all possible ways as the sum of two line segments then one will obtain all pairs of associated rays.

The fact that two pairs of such lines A, B; A', B' will lie in a ruled family is deduced immediately. The equation:

$$\mathfrak{a}\,A + \mathfrak{b}\,B = \mathfrak{a}'\,A' + \mathfrak{b}'\,B'$$

will then follow from the facts that $C = \mathfrak{a}\,A + \mathfrak{b}\,B$ and $C = \mathfrak{a}'\,A' + \mathfrak{b}'\,B'$.

If one exterior multiplies these by a line X that cuts three of the lines – say, A, B, A' – then one will one will also have $[B'\,X] = 0$ in the resulting equation:

$$\mathfrak{a}\,[a\,X] + \mathfrak{b}\,[B\,X] = \mathfrak{a}'\,[A'\,X] + \mathfrak{b}'\,[B'\,X]$$
$$[A\,X] = [B\,X] = [A'\,X] = 0\,;$$

i.e., X can also cut B'.

2. One can also represent the quantity C as the sum of a line segment and a surface space (i.e., an extended quantity of rank two, a line in the plane at infinity), and in particular, as the sum of a line segment and a surface space that is perpendicular to it. If a denotes a point, c, a segment of definite length an direction (i.e., an infinitely-distant point), and $\mathfrak{k} \mid c$ is the extension of that segment – i.e., the surface space that is perpendicular to c whose area is equal to the length of c – and k is a number then *Grassmann* (A_2, no. 346, 347) has proved that C can always be put into the form:

$$C = [a\ c] + \mathfrak{k} \mid c \tag{3}$$

uniquely. $[a\ c]$ is a line segment that is identical with the axis of the linear complex, and $\mathfrak{k}$ is the number that *Plücker* called the *parameter* of the linear complex.

If one bases C upon this form then it will follow from the defining equation (2) of the linear complex:

$$[C\ X] = [ac \cdot X] + \mathfrak{k}\,[|\ c \cdot X] = 0$$

or

$$\mathfrak{k} = -\frac{[ac \cdot X]}{[|\,c \cdot X]}.$$

Since the exterior product of two line segments of magnitude 1 will be equal to the product of the shortest distance between them with the sine of the angle between them, the numerator of this fraction will be $\mathfrak{i}\ l \sin c^{\wedge}X$, if $\mathfrak{i}$ denotes the length of the segment c and l, the distance from the line X to the axis. As one immediately recognizes, the denominator will have the value $\mathfrak{i} \cos\ c^{\wedge}X$; one will then have:

$$\mathfrak{k} = -\, l \tan c^{\wedge}X,$$

or, in words:

The product of the distance from a ray of a complex to the axis with the tangent of the angle that the two lines define is constant, and indeed, its absolute value is equal to the parameter of the complex when the axis and the ray of the complex are arranged in no particular sense.

Since the latter equation can also be employed as the defining equation of every linear complex, and one can deduce $[C\ X] = 0$ from it, where C has the form of equation (3), it is thus proved that any linear ray complex can be represented by the equation (2).

The theorem above is a special case of a more general one. Namely, if C is taken to have the general form:

$$C = \mathfrak{a}\,A + \mathfrak{b}\,B$$

then one will have:

$$[C\ X] = \mathfrak{a}\,[A\ X] + \mathfrak{b}\,[B\ X] = 0$$

for any complex ray X, so:

$$-\frac{\mathfrak{b}}{\mathfrak{a}} = \frac{[AX]}{[BX]}.$$

If, as usual, one calls the exterior product of two rays of unit magnitude – viz., the product of the shortest distance between them with the sine of the angle between them – their "moment" then this theorem will read:

The moments of the rays of a linear complex relative to any two associated lines in it have a constant ratio (*).

One likewise easily arrives at another well-known theorem, as well as its generalization. Let C assume the general form, as before, so one will obtain the null point p of the plane π, from no. 1, by exterior multiplication of C with π:

$$[C\ \pi] = \mathfrak{a}\,[A\ \pi] + \mathfrak{b}\,[B\ \pi] \equiv p.$$

If one calls the simple points of intersection of π with A and B, a and b, respectively, when π is assumed to have magnitude 1 then:

$$[A\ \pi] = a \sin A^\wedge\pi,$$
$$[B\ \pi] = b \sin B^\wedge\pi,$$

so

$$p \equiv \mathfrak{a}\, A^\wedge\pi \cdot a + \mathfrak{b}\, B^\wedge\pi \cdot b,$$

from which it will follow that:

$$\frac{[a\ p]}{[b\ p]} = -\frac{b \sin B^\wedge\pi}{a \sin A^\wedge\pi},$$

or

$$\frac{[a\ p] \sin A^\wedge\pi}{[b\ p] \sin B^\wedge\pi} = -\frac{\mathfrak{b}}{\mathfrak{a}}.$$

In words, this reads:

If the null plane of a point p meets two associated lines A and B of the complex at the points a and b, respectively, then the ratio of the products of the distances from the point p to the points a and b with the sine of the angle that the corresponding lines define with the null plane of the point will have the same value for all points of space.

If C has the special form (3) then the last equation will express the known theorem:

The product of the distance from a point to the axis of a linear ray complex with the tangent of the angle that the null plane of the points makes with the axis has the same value for all points in space, and is, in fact, equal to the parameter of the complex.

Remark. The definition of linear ray complex that was given here is only the expression of a known mechanical property of it. The quantity C, as the sum of line quantities, can, in fact, be considered to be a force system that acts upon a rigid body (A_1, § 122), and $[C\ X]$ is its static moment relative to the axis X: $[C\ X] = 0$ then says that the force system possesses a zero static moment relative to all of the rays that satisfy this equation. The remaining equations that appear above also have immediate mechanical interpretations.

(*) *Drach*, Math. Ann., Bd. II.

Numerical relationships between linear ray complexes. Exterior products of them.

3. We have seen that any quantity C determines a unique linear complex by the equation $[C\ X] = 0$, and that every linear ray complex can be determined by an equation of this form. The linear ray complex that is determined by C shall be denoted by $\mathfrak{C}$. For any ray X that does not satisfy the equation above, one will have $[C\ X] \neq 0$ and $[C\ X]$ shall be called the "moment of complex $\mathfrak{C}$ with the ray X" or "the moment of the ray X relative to the complex $\mathfrak{C}$."

We say of n complexes $\mathfrak{C}_i$ that a numerical relationship:

$$\sum_{i=1}^{n} \mathfrak{a}_i\, \mathfrak{C}_i = 0$$

arises if and only if the same relation exists between their moments with *any* ray X, so the equation:

$$\sum_{i=1}^{n} \mathfrak{a}_i\, [C_i\ X] = 0$$

will be fulfilled for every ray X. Due to the fact that:

$$\sum_{i=1}^{n} \mathfrak{a}_i\, [C_i\ X] = \left[\left(\sum_{i=1}^{n} \mathfrak{a}_i\, C_i\right) \cdot X\right],$$

the assumption above will be fulfilled only when one has:

$$\sum_{i=1}^{n} \mathfrak{a}_i\, C_i = 0,$$

so when the same numerical relationship exists between the quantities C_i . Conversely, it is easy to see that the numerical relationship $\sum_{i=1}^{n} \mathfrak{a}_i\, \mathfrak{C}_i = 0$ will also follow from the last equation.

One then has the theorem:

A numerical relationship of the form:

$$\sum_{i=1}^{n} \mathfrak{a}_i\, \mathfrak{C}_i = 0$$

arises between n linear ray complexes $\mathfrak{C}_i$ *if and only if the same relationship exists between the quantities* C_i .

4. If a numerical relationship:

$$\sum_{i=1}^{n} \mathfrak{a}_i A_i = 0$$

arises between any n quantities A then:

$$A_r = -\frac{\mathfrak{a}_1}{\mathfrak{a}_r} A_1 - \frac{\mathfrak{a}_2}{\mathfrak{a}_r} A_2 - \cdots - \frac{\mathfrak{a}_{r-1}}{\mathfrak{a}_r} A_{r-1} - \frac{\mathfrak{a}_{r+1}}{\mathfrak{a}_r} A_{r+1} - \cdots - \frac{\mathfrak{a}_n}{\mathfrak{a}_r} A_n$$

will be numerically derivable from any of the remaining ones.

If one imagines a tetrahedron as being chosen in such a way that each point of space can be derived from its vertices then any ray, and therefore, also any quantity C, will be numerically derivable from the edges of the that tetrahedron (A_2, no. 346, A_1, § 117). It follows immediately from this that any quantity C can be derived from any six quantities C_i that satisfy no numerical relationship (A_2, no. 24). However (no. 3), any linear complex is then numerically derivable from any six linear complexes that satisfy no numerical relationship, or – what amounts to the same thing:

The linear ray complexes define a domain of rank six.

All linear ray complexes that are derivable from n (n = 1, 2, 3, 4, 5) mutually-independent linear ray complexes define a *complex domain of rank n*. The domains of rank two, three, four, five shall also be called *pencils, sheaves, bushes, webs,* respectively.

5. From the discussion above, the concept of exterior product can be applied to linear ray complexes immediately.

The exterior product $[\mathfrak{C}_1\ \mathfrak{C}_2 \ldots \mathfrak{C}_n]$ (n = 2, …, 6) of n linear ray complexes represents the domain that they determine (A_2, no. 70, rem.), up to a numerical value that is equal to the determinant of the numbers by which $\mathfrak{C}_1 \ldots \mathfrak{C}_n$ can be derived from n complexes whose exterior product can be assumed to be unity. The exterior product of n complexes is zero if and only the n complexes define a domain of rank lower than n, or – what amounts to the same thing – as long as a numerical relationship exists between them.

Since the complexes of a pencil have a linear congruence in common, and any such congruence determines a pencil of complexes that goes through them, one can also say that the exterior product of two complexes represents their common congruence. Likewise, the exterior product of three linear ray complexes represents their common ruled family, and the exterior product of four linear ray complexes represents their common pair of rays. However, any other exterior product will likewise determine just one numerical value, and two exterior products are equal when and only when they represent the same geometric object, as well as also determine the same numerical value.

The concept of the regressive product (A_1, Chap. 3, A_2, § 5) can be applied here immediately.

The inner product of linear ray complexes.

6. If a linear ray complex $\mathfrak{C}$ is given then the positive square root of $[C \cdot C]$ shall be called its *numerical value*. (The basis for this term will be given later.)

A linear ray complex is *special* whenever C represents a line segment. This happens if and only if $[C\ C] = 0$ (A_2, no. 286, A_1, § 124). A special complex is then one with a numerical value of zero. Since the linear complex $\mathfrak{C}$ does not change when one multiplies its quantity C by an arbitrary number, one can always put into a form that has the numerical value of 1; to that end, one needs only to multiply C by $1 / \sqrt{[C\,C]}$.

Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ be any two linear ray complexes, so each complex $\mathfrak{C}$ of the pencil $[\mathfrak{C}_1\ \mathfrak{C}_2]$ can be represented in the form:

$$\mathfrak{C} = \mathfrak{a}_1\ \mathfrak{C}_1 + \mathfrak{a}_2\ \mathfrak{C}_2\ ;$$

since one then has:

$$C = \mathfrak{a}_1\ C_1 + \mathfrak{a}_2\ C_2\ ,$$

the condition for $\mathfrak{C}$ to be a special complex will be:

$$\begin{aligned} 0 = [C\ C] &= [(\mathfrak{a}_1\ C_1 + \mathfrak{a}_2\ C_2) \cdot (\mathfrak{a}_1\ C_1 + \mathfrak{a}_2\ C_2)] \\ &= \mathfrak{a}_1^2\,[C_1\ C_1] + 2\ \mathfrak{a}_1\ \mathfrak{a}_2\ [C_1\ C_2] + \mathfrak{a}_2^2\,[C_2\ C_2], \end{aligned}$$

which is an equation that will yield two values for $\mathfrak{a}_1 / \mathfrak{a}_2$. There are then two special complexes in any pencil of complexes. If $\mathfrak{A}$, $\mathfrak{B}$ are the special complexes of a pencil, and $\mathfrak{C}_1$, $\mathfrak{C}_2$ are any two complexes in it then one will have:

$$\begin{aligned} \mathfrak{C}_1 &= \mathfrak{u}_1\ \mathfrak{A} + \mathfrak{v}_1\ \mathfrak{B}\ , \\ \mathfrak{C}_2 &= \mathfrak{u}_2\ \mathfrak{A} + \mathfrak{v}_2\ \mathfrak{B}\ . \end{aligned}$$

The double ratio of these four complexes is then (A_1, § 165):

$$\frac{[\mathfrak{A}\,\mathfrak{C}_1]}{[\mathfrak{B}\,\mathfrak{C}_1]} : \frac{[\mathfrak{A}\,\mathfrak{C}_2]}{[\mathfrak{B}\,\mathfrak{C}_2]} = \left(-\frac{\mathfrak{v}_1}{\mathfrak{u}_1}\right) : \left(-\frac{\mathfrak{v}_2}{\mathfrak{u}_2}\right) = \frac{\mathfrak{v}_1}{\mathfrak{u}_1} : \frac{\mathfrak{v}_2}{\mathfrak{u}_2}\ .$$

From the corresponding equations for the quantities C:

$$C_1 = \mathfrak{u}_1 U + \mathfrak{v}_1\ V\ ,$$

$$C_2 = \mathfrak{u}_2 U + \mathfrak{v}_2 V,$$

it follows by multiplying by a plane π that:

$$[C_1\,\pi] = \mathfrak{u}_1\,[U\,\pi] + \mathfrak{v}_1\,[V\,\pi],$$
$$[C_2\,\pi] = \mathfrak{u}_2\,[U\,\pi] + \mathfrak{v}_2\,[V\,\pi],$$

or, when one has:

$$[C_1\,\pi] = p_1, \quad [C_2\,\pi] = p_2, \quad [U\,\pi] = u, \qquad [V\,\pi] = v,$$

that:

$$p_1 = \mathfrak{u}_1\,u + \mathfrak{v}_1\,v,$$
$$p_2 = \mathfrak{u}_2\,u + \mathfrak{v}_2\,v.$$

The double ratio of these four points:

$$\frac{[u\,p_1]}{[v\,p_1]} : \frac{[u\,p_2]}{[v\,p_2]} = \frac{\mathfrak{v}_1}{\mathfrak{u}_1} : \frac{\mathfrak{v}_2}{\mathfrak{u}_2}$$

then has the same value as the double ratio of the corresponding complex. In particular, if this double ratio has the value – 1 then, according to *F. Klein's* terminology, the two complexes $\mathfrak{C}_1$ and $\mathfrak{C}_2$ *will lie in involution.* The condition for that is then:

$$\mathfrak{u}_1\,\mathfrak{v}_2 + \mathfrak{u}_2\,\mathfrak{v}_1 = 0.$$

However, from the above, it will follow that:

$$[C_1\,C_2] = [(\mathfrak{u}_1\,U + \mathfrak{v}_1\,V) \cdot (\mathfrak{u}_2\,U + \mathfrak{v}_2\,V)] = (\mathfrak{u}_1\,\mathfrak{v}_2 + \mathfrak{u}_2\,\mathfrak{v}_1)\,[U\,V],$$

since one has $[U\,U] = [V\,V] = 0$ (by assumption). Therefore, if the two complexes $\mathfrak{C}_1$, $\mathfrak{C}_2$ lie in involution then one must have:

$$[C_1\,C_2] = 0.$$

Conversely, as long as this equation is valid, if $[U\,V] \neq 0$ then one must have $\mathfrak{u}_1\,\mathfrak{v}_2 + \mathfrak{u}_2\,\mathfrak{v}_1 = 0$, so the two complexes must lie in involution. $[U\,V] \neq 0$ corresponds to just the requirement that the axes of the two special complexes of the pencil should not intersect, so the pencil of complexes will not be a pencil of special complexes, and $\mathfrak{C}_1$, $\mathfrak{C}_2$ will not be special complexes. If we assume that for the case in which $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are special complexes they will be said to lie in involution if and only if their axes intersect then one can express the following theorem in full generality:

The necessary and sufficient condition for two complexes $\mathfrak{C}_1$, $\mathfrak{C}_2$ *to lie in involution is that* $[C_1\,C_2] = 0$.

7. All complexes $\mathfrak{Y}$ that lie in involution with a complex $\mathfrak{C}$ will be determined by the equation:

$$[C\,Y] = 0, \tag{4}$$

if Y refers to the sum of line segments that belongs to $\mathfrak{Y}$ (*). If one derives $\mathfrak{Y}$ from any six mutually-independent complexes $\mathfrak{Y}_i$ – say, $\mathfrak{Y} = \sum_{i=1}^{6} \mathfrak{y}_i\, \mathfrak{Y}_i$ – then one will also have (no. 3) $Y = \sum_{i=1}^{6} \mathfrak{y}_i\, Y_i$, and the equation above will then read:

$$\sum_{i=1}^{6} \mathfrak{y}_i\, [C\,Y_i] = 0,$$

which then expresses a linear relation between the deriving numbers $\mathfrak{y}_i$. However, as is easy to see, each of the complexes $\mathfrak{Y}$ that satisfy equation (4) can be derived from five mutually-independent complexes, or belong to a domain of rank five. Furthermore, any complex $\mathfrak{Y}$ that belongs to this domain of rank five will lie in involution with $\mathfrak{C}$. Thus, if $\mathfrak{Y}_1, \mathfrak{Y}_2, \ldots, \mathfrak{Y}_5$ are any five mutually-independent complexes that lie in involution with $\mathfrak{C}$ then $\mathfrak{Y}$ will be numerically derivable from them; perhaps:

$$\mathfrak{Y} = \sum_{i=1}^{5} \mathfrak{y}_i\, \mathfrak{Y}_i ,$$

and therefore:

$$Y = \sum_{i=1}^{5} \mathfrak{y}_i\, Y_i .$$

Upon multiplying by C, one will obtain:

$$[Y\,C] = \sum_{i=1}^{5} \mathfrak{y}_i\, [Y_i\; C] = 0,$$

since all products $[Y_i\; C]$ are zero, by assumption. Therefore, any complex $\mathfrak{Y}$ lies in involution with $\mathfrak{C}$, and one can state the theorem:

All linear ray complexes that lie in involution with a linear ray complex define a domain of rank five.

It follows from this immediately (A_1, § 126) that:

(*) It follows from this equation that the axes of all special complexes that lie in involution with $\mathfrak{C}$ will define the rays of the complex $\mathfrak{C}$.

The linear complexes that lie in involution with n linear ray complexes ($n = 1, 2, \ldots, 5$) *define a domain of rank* $6 - n$.

One can then also choose six linear ray complexes, any two of which lie in involution.

8. If we choose six linear ray complexes $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_6$ with numerical values of unity, any two of which lie in involution, to be the original units and set their exterior product equal to one, so:

$$[\mathfrak{C}_1\, \mathfrak{C}_2\, \mathfrak{C}_3\, \mathfrak{C}_4\, \mathfrak{C}_5\, \mathfrak{C}_6] = 1,$$

then the necessary and sufficient conditions for the definition of the inner product are met, and one can all immediately apply all of the theorems that *Grassmann* derived in A_2, Chap. 4 here. However, one must establish the meanings that concepts like "normal, numerical value, value" that were presented there for arbitrary linear manifolds in general might have for the geometry of linear ray complexes.

We would first like to prove that two complexes that lie in involution are identical with two normal complexes.

We will have that proof when we have shown that:

Any six complexes with numerical values of unity, any two of which lie in involution, can be derived from the complexes that were chosen to be the original units by repeated circular alteration.

Then, since the original units are six quantities that are normal to each other (A_2, no. 162) and, in turn, only mutually-normal quantities will emerge by circular alterations (A_2, no. 155), any six complexes that are reciprocally in involution must also be six mutually-normal quantities.

In order to prove the theorem we need the lemma that:

Two complexes with numerical values of unity that lie in involution to each other will go to two such complexes under circular alteration.

If $\mathfrak{C}_1, \mathfrak{C}_2$ go to $\mathfrak{C}'_1$, $\mathfrak{C}'_2$ under circular alteration then, if the complexes are assumed to have numerical values of unity and $\mathfrak{a}_1^2 + \mathfrak{a}_2^2 = 1$, one will have (from A_2, no. 154):

$$\mathfrak{m}\,\mathfrak{C}'_1 = \mathfrak{a}_1\, \mathfrak{C}_1 + \mathfrak{a}_2\, \mathfrak{C}_2 ,$$
$$\mathfrak{n}\;\; \mathfrak{C}'_2 = \mathfrak{a}_1\, \mathfrak{C}_2 - \mathfrak{a}_2\, \mathfrak{C}_1 ,$$

so

$$\mathfrak{m}\, C'_1 = \mathfrak{a}_1\, C_1 + \mathfrak{a}_2\, C_2 , \qquad (\mu)$$
$$\mathfrak{n}\;\; C'_2 = \mathfrak{a}_1\, C_2 - \mathfrak{a}_2\, C_1 . \qquad (\nu)$$

In order to find the value of $\mathfrak{m}$, one exterior multiplies each side of equation (μ) by itself. One will then get:

$$\mathfrak{m}^2\,[C_1' C_1'] = \mathfrak{a}_1^2\,[C_1\,C_1] + 2\,\mathfrak{a}_1\mathfrak{a}_2\,[C_1\,C_2] + \mathfrak{a}_2^2\,[C_2\,C_2],$$

and since, by assumption, one will have:

$$[C_1' C_1'] = [C_1\,C_1] = [C_2\,C_2] = 1, \qquad [C_1\,C_2] = 0,$$

one will then have:

$$\mathfrak{m}^2 = \mathfrak{a}_1^2 + \mathfrak{a}_2^2 = 1.$$

One will similarly obtain from equation (ν) the fact that:

$$\mathfrak{n}^2 = 1.$$

The complexes that emerge from $\mathfrak{C}_1$ and $\mathfrak{C}_2$ by circular alteration will then possess the numerical value of unity, and equations (μ) and (ν) will then read:

$$C_1' = \mathfrak{a}_1\,C_1 + \mathfrak{a}_2\,C_2\,,$$
$$C_2' = \mathfrak{a}_1\,C_2 - \mathfrak{a}_2\,C_1\,.$$

If one exterior multiplies them with each other then it will follow that:

$$[C_1' C_2'] = (\mathfrak{a}_1^2 - \mathfrak{a}_2^2)\,[C_1\,C_2] - \mathfrak{a}_1\,\mathfrak{a}_2\,([C_1\,C_1] - [C_2\,C_2]),$$

so, since $[C_1\,C_2] = 0$ and $[C_1\,C_1] = ([C_2\,C_2] = 1$, one will have:

$$[C_1' C_2'] = 0;$$

i.e., the complexes $\mathfrak{C}_1'$, $\mathfrak{C}_2'$ lie in involution.

The main theorem – viz., that any six complexes $\mathfrak{C}_1'$, $\mathfrak{C}_2'$, …, $\mathfrak{C}_6'$ with numerical values of unity, any two of which lie in involution, can be derived from six complexes $\mathfrak{C}_1$, $\mathfrak{C}_2$, …, $\mathfrak{C}_6$ that are chosen to be the original units by repeated circular alteration – can now be proved in the following way: From A_2, no. 160, the system of original units can be changed circularly in such a way that one of them – say, $\mathfrak{C}_1$ – coincides with an arbitrary quantity of rank one – say, $\mathfrak{C}_1'$. Since, from the lemma that was just proved, any two complexes with numerical values of unity that are found to be in involution will again go to other such complexes under any circular alteration, the original units will go to six complexes that mutually lie in involution, so the original five will lie in involution with $\mathfrak{C}_1$. From no. 7, these five complexes then belong to the domain $[\mathfrak{C}_2'\,\mathfrak{C}_3' \ldots \mathfrak{C}_6']$, and, from A_2, no. 160, they can therefore be once more circularly altered in such a way that one of them will coincide with – say $-\mathfrak{C}_2'$, whereby the four remaining ones will lie in the domain $[\mathfrak{C}_3' \mathfrak{C}_4' \mathfrak{C}_5' \mathfrak{C}_6']$. If one repeats this argument then one will ultimately arrive at the realization that the original units $\mathfrak{C}_1, \ldots, \mathfrak{C}_6$ can actually go to $\mathfrak{C}_1', \ldots, \mathfrak{C}_6'$ under circular alteration. Since the numerical values of the complexes remain unchanged by that – so they will remain equal to one – we can say:

Any six ray complexes with numerical values of unity that mutually lie in involution define a complete, simple, normal system (in Grassmann's terminology).

Since any two complexes with numerical values of unity that lie in involution can be regarded as a part of a complete, simple, normal system, they will define two mutually-normal quantities of rank one, and conversely, any two normal quantities in the domain that we spoke of will define two complexes that lie in involution with each other, since (A_2, no. 161) they can be derived from the original units by circular alteration.

The concept of normal, linear ray complexes thus overlaps with that of complexes that lie in involution.

In the sequel, we would ordinarily like to speak of *normal*, linear complexes.
It follows from the foregoing that:

The inner product of two normal complexes is zero, and conversely.

9. *The general definition of numerical value that was given by Grassmann overlaps with the one in no. 6 for linear ray complexes.*

If one lets $[C\,C] = 1$, first of all, then (from no. 8) the complex C will emerge from the original units by circular alteration, so, from Grassmann (A_2, no. 155), it will likewise have the numerical value of unity; i.e., one will then have $\mathfrak{C}^2 = 1$ (*). If, moreover, $[C\,C] \neq 1$ then one can set:

$$C = \sqrt{[C\,C]}\; C',$$

where one now has $[C'\,C'] = 1$, and then, from no. 3, the complexes $\mathfrak{C}$ and $\mathfrak{C}'$ that correspond to the quantities C and C' will then have the relationship:

$$\mathfrak{C} = \sqrt{[C\,C]}\; \mathfrak{C}',$$

so

$$\mathfrak{C}^2 = [C\,C]\,\mathfrak{C}'^2 = [C\,C], \qquad \text{since} \quad \mathfrak{C}'^2 = 1,$$

or

(*) We apply the simple square sign without a prime as the sign of the inner square, since no confusion is possible here.

$$\sqrt{\mathfrak{C}^2} = \sqrt{[C\,C]};$$

i.e., *the numerical value of* $\mathfrak{C}$, *according to Grassmann's definition, agrees with the definition in no. 6.*

Since a special complex has a numerical value of zero, one can also call it a *null complex.*

Since it follows from A_2, no. 168 that all theorems that are true for the original units will also remain valid for the quantities of a complete, simple, normal system, the numerical value of an exterior product of n (n = 2, 3, …, 6) complexes of numerical value unity that are normal to each other will by unity, in any case.

10. Another important concept is that of "extension." If $\mathfrak{C}_1, \ldots, \mathfrak{C}_6$ are the complexes of a simple, normal system – so $[\mathfrak{C}_1 \ldots \mathfrak{C}_6] = 1$ – then the *extension* of a product A of α of these complexes is equal to the product B of the remaining $(6 - \alpha)$ complexes, endowed with a sign that will make $[A\ B] = +1$ (A_2, no. 167). In order to obtain the extension of any other quantity A' of rank α, one assumes that one has a complete, simple, normal system such that α of its units come to lie in the domain of A'. If A is their exterior product, and a' is the numerical value of A' then one will have:

$$A' = a'\,A,$$

so

$$|\,A' = a'\,|\,A \qquad\qquad (A_2, \text{ no. } 91),$$

in which $|\,A$ denotes the extension of A, so, from the above, it will be the product of $(6 - \alpha)$ complexes that are normal to each other and to all complexes of A, and therefore also A'. If one calls the domain of rank $(6 - \alpha)$ that is defined by all complexes that are normal to all complexes of a domain A' of rank α *the extended domain of* A' then one can say:

The extension of a quantity A' *is a quantity that belongs to the extended domain of* A' *whose numerical value is equal to that of* A'.

Since any quantity in one of two extended domains is normal to any quantity of the other one, (from no. 7, remark) all complexes of two extended complex domains must go through the axes of the special complexes in the other one.

The extension of an exterior product of two linear ray complexes, which from no. 5, will be a linear congruence, represents the complex domain of rank four that is normal to it, so, from no. 5, it will be a line pair. Since, from what we just said, all complexes in this domain of rank four will go through the two directrices of that congruence, the pair of directrices will then be represented by the exterior product of any four complexes of the domain, and one can say:

The extension of a linear ray congruence is its pair of directrices.

One likewise finds that:

The extension of a ruled family is its guiding line.

These two theorems can also be expressed in the following form:

The axes of the special complexes of a complex bush define a ray congruence,
and
The axes of the special complex of a sheaf of complexes define a ruled family. (*)

11. We ask what the number would be that the inner product of two complexes $\mathfrak{C}'_1$, $\mathfrak{C}'_2$ determines.

If $\mathfrak{C}_1$, $\mathfrak{C}_2$ are two complexes with numerical values of unity that belong to the pencil $[C'_1 C'_2]$ then one will have the relations:

$$\mathfrak{C}'_1 = \mathfrak{a}_{11}\,\mathfrak{C}_1 + \mathfrak{a}_{12}\,\mathfrak{C}_2\,,$$
$$\mathfrak{C}'_2 = \mathfrak{a}_{21}\,\mathfrak{C}_1 + \mathfrak{a}_{22}\,\mathfrak{C}_2\,,$$

so if one interior multiplies (A_2, no. 143) the two equations then one will get:

$$[\mathfrak{C}'_1 \mid \mathfrak{C}'_2] = \mathfrak{a}_{11}\,\mathfrak{a}_{21} + \mathfrak{a}_{12}\,\mathfrak{a}_{22}\,,$$

since $\mathfrak{C}_1^2 = \mathfrak{C}_2^2 = 1$ and $[C_1 \mid C_2] = 0$.

However, one finds the equations:

$$C'_1 = \mathfrak{a}_{11}\,C_1 + \mathfrak{a}_{12}\,C_2\,,$$
$$C'_2 = \mathfrak{a}_{21}\,C_1 + \mathfrak{a}_{22}\,C_2$$

between the quantities C, in any case. If one exterior multiplies them then one will obtain

$$[C'_1 \mid C'_2] = \mathfrak{a}_{11}\,\mathfrak{a}_{21} + \mathfrak{a}_{12}\,\mathfrak{a}_{22}\,,$$

since $[C_1\,C_1] = [C_2\,C_2] = 1$ and $[C_1\,C_2] = 0$. Therefore:

$$[\mathfrak{C}'_1 \mid \mathfrak{C}'_2] = [C'_1 \mid C'_2],$$

or

(*) Five complexes determine a web of complexes and a complex that is normal to them that is defined by the axes of the special complexes of that web.

The inner product of two linear ray complexes is equal to the exterior product of their quantities C.

If $\mathfrak{C}_1$, $\mathfrak{C}_2$ are any two linear ray complexes then $[\mathfrak{C}_1\ \mathfrak{C}_2]$ shall be called the *moment of the two complexes relative to each other* (**). One can then also say:

The inner product of two linear ray complexes is equal to their moment relative to each other.

Two complexes are therefore normal as long as their moment relative to each other is zero.

12. *The angle between two linear ray complexes.* *Grassmann* (A_2, no. 195) understood the *angle* $\angle\ AB$ between two quantities A and B of equal rank, whose numerical values were a and b, to mean the angle between 0 and π whose cosine was equal to the inner product of those quantities, divided by the numerical values; i.e., he set:

$$\cos \angle AB = \frac{[A \mid B]}{ab}.$$

Therefore, for two linear ray complexes $\mathfrak{C}_1$, $\mathfrak{C}_2$ one will have:

$$\cos \mathfrak{C}_1{}^{\wedge}\, \mathfrak{C}_2 = \frac{[\mathfrak{C}_1 \mid \mathfrak{C}_2]}{\sqrt{\mathfrak{C}_1^2\, \mathfrak{C}_2^2}} = \frac{[C_1 \mid C_2]}{\sqrt{[C_1 \mid C_1][C_2 \mid C_2]}}.$$

This expression for the angle between two linear complexes agrees with the other definitions that are given for it. In particular, one can prove that arc cos $\mathfrak{C}_1{}^{\wedge}\, \mathfrak{C}_2$ is equal to the logarithm of the double ratio, multiplied by $i / 2$, that the two complexes define with the special complexes in their pencil. The proof of this is achieved in the following way: If $\mathfrak{C}_1$, $\mathfrak{C}_2$ are two complexes with unity numerical values that define the angle γ with each other then $\mathfrak{a}_1\mathfrak{C}_1 + \mathfrak{a}_2\mathfrak{C}_2$ will represent a special complex as long as one has $(\mathfrak{a}_1\mathfrak{C}_1 + \mathfrak{a}_2\mathfrak{C}_2)^2 = 0$. This gives the equation:

$$\mathfrak{a}_1^2\, \mathfrak{C}_1^2 + 2\mathfrak{a}_1\mathfrak{a}_2[\mathfrak{C}_1 \mid \mathfrak{C}_2] + \mathfrak{a}_2^2\, \mathfrak{C}_2^2 = 0,$$

or, since:

$$\mathfrak{C}_1^2 = \mathfrak{C}_2^2 = 1 \quad \text{and} \quad [\mathfrak{C}_1 \mid \mathfrak{C}_2] = \cos \mathfrak{C}_1{}^{\wedge}\, \mathfrak{C}_2 = \cos \gamma$$

$$\mathfrak{a}_1^2 + 2\mathfrak{a}_1\mathfrak{a}_2 \cos\gamma + \mathfrak{a}_2^2 = 0.$$

(**) See *F. Klein*, Math. Ann., Bd. II, page 368.

This equation gives the two values $-e^{i\gamma}$, $-e^{-i\gamma}$ for $\mathfrak{a}_2 / \mathfrak{a}_1$; the two null complexes $\mathfrak{A}$, $\mathfrak{B}$ of the pencil can thus be represented in the form:

$$\mathfrak{A} = \mathfrak{C}_1 - e^{-i\gamma}\,\mathfrak{C}_2\,,$$
$$\mathfrak{B} = \mathfrak{C}_1 - e^{i\gamma}\,\mathfrak{C}_2\,,$$

from which, it will follow that:

$$\frac{[\mathfrak{C}_1\,\mathfrak{A}]}{[\mathfrak{C}_2\,\mathfrak{A}]} = e^{i\gamma}, \quad \frac{[\mathfrak{C}_1\,\mathfrak{B}]}{[\mathfrak{C}_2\,\mathfrak{B}]} = e^{-i\gamma}, \qquad \text{so} \qquad \frac{[\mathfrak{C}_1\,\mathfrak{A}]}{[\mathfrak{C}_2\,\mathfrak{A}]} : \frac{[\mathfrak{C}_1\,\mathfrak{B}]}{[\mathfrak{C}_2\,\mathfrak{B}]} = e^{2i\gamma},$$

If δ is the value of the double ratio, so $\delta = e^{2i\gamma}$, then (*):

$$\gamma = \frac{1}{2i}\log\,\delta.$$

One also arrives very easily at the expression for $\cos \mathfrak{C}_1 \wedge \mathfrak{C}_2$ that *F. Klein* found.

If we assume that the two complexes $\mathfrak{C}_1$, $\mathfrak{C}_2$ are given by:

$$C_1 = \mathfrak{a}_1\,A_1 + \mathfrak{a}_1' A_1',$$
$$C_2 = \mathfrak{a}_2\,A_2 + \mathfrak{a}_2' A_2',$$

in which (see no. 2) A_1, A_2 should denote the axes of the two complexes with magnitude unity and A_1', A_2' should denote the surface spaces that are normal to them, which likewise have magnitude 1, so $\mathfrak{a}_1' / \mathfrak{a}_1 = \mathfrak{k}_1$, $\mathfrak{a}_2' / \mathfrak{a}_2 = \mathfrak{k}_2$ will be the parameters of the two complexes, then, due to the fact that $[A_1' A_2'] = 0$, one will obtain:

$$[\mathfrak{C}_1 \mid \mathfrak{C}_2] = [C_1\,C_2] = \mathfrak{a}_1\,\mathfrak{a}_2\,[A_1\,A_2] + \mathfrak{a}_1\,\mathfrak{a}_2'\,[A_1\,A_2'] + \mathfrak{a}_2\,\mathfrak{a}_1'\,[A_2\,A_1']\,.$$

If the angle between the two axes is denoted by φ and the shortest distance between them by Δ then one will have:

$$[A_1\,A_2] = \Delta \sin \varphi, \qquad [A_1\,A_2'] = [A_2\,A_1'] = \cos \varphi,$$

so

$$[\mathfrak{C}_1 \mid \mathfrak{C}_2] = \mathfrak{a}_1\,\mathfrak{a}_2\,\Delta \sin \varphi + (\mathfrak{a}_1\,\mathfrak{a}_2' + \mathfrak{a}_2\,\mathfrak{a}_1')\cos \varphi\,.$$

Since one further has:

$$\mathfrak{C}_1^2 = [C_1\,C_1] = 2\,\mathfrak{a}_1\mathfrak{a}_1',$$
$$\mathfrak{C}_2^2 = [C_2\,C_2] = 2\,\mathfrak{a}_2\mathfrak{a}_2',$$

one will have (**):

(*) Cf., *Lindemann*, Math. Ann., VII, pp. 66.

$$\cos \mathfrak{C}_1 \wedge \mathfrak{C}_2 = \frac{[\mathfrak{C}_1 \mid \mathfrak{C}_2]}{\sqrt{\mathfrak{C}_1^2 \mathfrak{C}_2^2}} = \frac{\mathfrak{a}_1\mathfrak{a}_2 \Delta \sin\varphi + (\mathfrak{a}_1\mathfrak{a}_2' + \mathfrak{a}_2\mathfrak{a}_1')\cos\varphi}{2\sqrt{\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_1'\mathfrak{a}_2'}}$$

$$= \frac{\Delta \sin\varphi + \left(\dfrac{\mathfrak{a}_2'}{\mathfrak{a}_2} + \dfrac{\mathfrak{a}_1'}{\mathfrak{a}_1}\right)\cos\varphi}{2\sqrt{\dfrac{\mathfrak{a}_1'}{\mathfrak{a}_1}\dfrac{\mathfrak{a}_2'}{\mathfrak{a}_2}}} = \frac{\Delta \sin\varphi + (\mathfrak{k}_1 + \mathfrak{k}_2)\cos\varphi}{2\sqrt{\mathfrak{k}_1\mathfrak{k}_2}}.$$

13. If one chooses six mutually-normal complexes $\mathfrak{C}_1, \ldots, \mathfrak{C}_6$ with unity numerical values to be the original units then any complex $\mathfrak{C}$ can be derived from them numerically. Let $\mathfrak{x}_1, \ldots, \mathfrak{x}_6$ be the deriving numbers, so one has:

$$\mathfrak{C} = \mathfrak{x}_1 \mathfrak{C}_1 + \ldots + \mathfrak{x}_6 \mathfrak{C}_6 = \sum_{i=1}^{6} \mathfrak{x}_i \mathfrak{C}_i .$$

The numbers $\mathfrak{x}_i$ are nothing but *F. Klein's* complex coordinates. If one then inner multiplies the equation above by $\mathfrak{C}_i$ then, since $[\mathfrak{C}_i \mid \mathfrak{C}_k] = 0$ $(i \leq k)$ and $\mathfrak{C}_i^2 = 1$, one will have:

$$[\mathfrak{C} \mid \mathfrak{C}_i] = \mathfrak{x}_i ;$$

i.e., the numbers $\mathfrak{x}_i$ will be equal to the moments of the complex C relative to the six fundamental complexes.

By squaring the equation above, it follows that:

$$\mathfrak{C}^2 = \mathfrak{x}_1^2 + \cdots + \mathfrak{x}_6^2 .$$

$\mathfrak{C}^2$ is therefore identical with the invariant of a linear ray complex.

By inner multiplying the two complexes:

$$\mathfrak{C} = \sum_{i=1}^{6} \mathfrak{x}_i \mathfrak{C}_i ,$$

$$\mathfrak{C}' = \sum_{i=1}^{6} \mathfrak{x}_i' \mathfrak{C}_i ,$$

it follows that:

$$[\mathfrak{C} \mid \mathfrak{C}_i'] = \sum_{i=1}^{6} \mathfrak{x}_i \mathfrak{x}_i' .$$

(**) Cf., *Segre*, Borch. J., Bd. IC.

The quantity $[\mathfrak{C} \mid \mathfrak{C}']$ is then identical with the simultaneous invariant of the two linear ray complexes.

Metric relations between linear ray complexes (*):

Every complex in the pencil that is determined by the two complexes $\mathfrak{C}_1$, $\mathfrak{C}_2$ can be represented in the form $\mathfrak{a}_1 \mathfrak{C}_1 + \mathfrak{a}_2 \mathfrak{C}_2$. The two special complexes of the pencil will then be determined by the equation:

$$(\mathfrak{a}_1 \mathfrak{C}_1 + \mathfrak{a}_2 \mathfrak{C}_2)^2 = \mathfrak{a}_1^2 \mathfrak{C}_1^2 + 2\mathfrak{a}_1 \mathfrak{a}_2 [\mathfrak{C}_1 \mid \mathfrak{C}_2] + \mathfrak{a}_2^2 \mathfrak{C}_2^2 = 0 .$$

They will coincide whenthe equation above for $\mathfrak{a}_1 / \mathfrak{a}_2$ gives two equal roots, so when one has:

$$[\mathfrak{C}_1 \mid \mathfrak{C}_2]^2 - \mathfrak{C}_1^2 \mathfrak{C}_2^2 = 0 .$$

Since, from A_2, no. 177, one has:

$$[\mathfrak{C}_1\, \mathfrak{C}_2]^2 = \mathfrak{C}_1^2 \mathfrak{C}_2^2 - [\mathfrak{C}_1 \mid \mathfrak{C}_2]^2,$$

one can also write the condition equation for the pencil $[\mathfrak{C}_1\, \mathfrak{C}_2]$ to possess two coincident special complexes as:

$$[\mathfrak{C}_1\, \mathfrak{C}_2]^2 = 0,$$

which, from no. 9, says that the numerical value of the product $[\mathfrak{C}_1\, \mathfrak{C}_2]$, or the linear congruence $[\mathfrak{C}_1\, \mathfrak{C}_2]$, is zero. Since the congruence possesses two coincident directrices in this case – so it is a special congruence – one can say:

A linear congruence is a special one if and only if its numerical value is equal to zero.

If $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are themselves special then, due to the fact that $\mathfrak{C}_2^2 = \mathfrak{C}_1^2 = 0$, one will have:

$$[\mathfrak{C}_1\, \mathfrak{C}_2]^2 = -[\mathfrak{C}_1 \mid \mathfrak{C}_2]^2.$$

In this case, the congruence will be special for $[\mathfrak{C}_1 \mid \mathfrak{C}_2] = 0$; i.e., when the axes of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ cut each other. All complexes of the pencil will be special then, and their axes will define a pencil of rays; the common congruence will be a decomposable one (**).

(*) See: *F. Klein*, "Über gewisse in der Liniengeometrie auftretende Differentialgleichungen," Math. Ann., Bd. V.

(**) See *loc. cit.*

Every complex in the sheaf of complexes that is determined by the three complexes $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ can be represented in the form $\mathfrak{a}_1\mathfrak{C}_1 + \mathfrak{a}_2\,\mathfrak{C}_2 + \mathfrak{a}_3\,\mathfrak{C}_3$. The special complexes of the sheaf, whose axes define a ruled family, from no. 10, are determined by the equation:

$$(\mathfrak{a}_1\mathfrak{C}_1 + \mathfrak{a}_2\,\mathfrak{C}_2 + \mathfrak{a}_3\,\mathfrak{C}_3)^2$$
$$= \mathfrak{a}_1^2\mathfrak{C}_1^2 + \mathfrak{a}_2^2\mathfrak{C}_2^2 + \mathfrak{a}_3^2\mathfrak{C}_3^2 + 2\mathfrak{a}_1\mathfrak{a}_2\,[\mathfrak{C}_1 \mid \mathfrak{C}_2] + 2\mathfrak{a}_1\mathfrak{a}_3\,[\mathfrak{C}_1 \mid \mathfrak{C}_3] + 2\mathfrak{a}_2\mathfrak{a}_3\,[\mathfrak{C}_2 \mid \mathfrak{C}_3] = 0.$$

The polynomial in this equation can be represented as a product of two linear factors as long as the determinant satisfies:

$$\begin{vmatrix} \mathfrak{C}_1^2 & [\mathfrak{C}_1 \mid \mathfrak{C}_2] & [\mathfrak{C}_1 \mid \mathfrak{C}_3] \\ [\mathfrak{C}_2 \mid \mathfrak{C}_1] & \mathfrak{C}_2^2 & [\mathfrak{C}_2 \mid \mathfrak{C}_3] \\ [\mathfrak{C}_3 \mid \mathfrak{C}_1] & [\mathfrak{C}_3 \mid \mathfrak{C}_2] & \mathfrak{C}_3^2 \end{vmatrix} = 0$$

Since, from A_2, no. 175, this determinant is identical with $[\mathfrak{C}_1\mathfrak{C}_2\mathfrak{C}_3]^2$, and thus represents the square of the numerical value of $[\mathfrak{C}_1\mathfrak{C}_2\mathfrak{C}_3]$, one can also say that the polynomial in the equation above can be decomposed into two linear factors as long as the numerical value of $[\mathfrak{C}_1\mathfrak{C}_2\mathfrak{C}_3]$ is zero. The equation can then be written in the form:

$$(\mathfrak{a}_1\mathfrak{m}_1 + \mathfrak{a}_2\mathfrak{m}_2 + \mathfrak{a}_3\mathfrak{m}_3)\,(\mathfrak{a}_1\mathfrak{n}_1 + \mathfrak{a}_2\mathfrak{n}_2 + \mathfrak{a}_3\mathfrak{n}_3) = 0,$$

and the values of $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$ will the satisfy one of the two equations:

$$\mathfrak{a}_1\mathfrak{m}_1 + \mathfrak{a}_2\mathfrak{m}_2 + \mathfrak{a}_3\mathfrak{m}_3 = 0, \qquad (\mu)$$
$$\mathfrak{a}_1\,\mathfrak{n}_1 + \mathfrak{a}_2\,\mathfrak{n}_2 + \mathfrak{a}_3\,\mathfrak{n}_3 = 0. \qquad (\nu)$$

All deriving numbers $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$ that satisfy (μ) determine special complexes $\mathfrak{a}_1\mathfrak{C}_1 + \mathfrak{a}_2\mathfrak{C}_2 + \mathfrak{a}_3\mathfrak{C}_3$ with the property that any of them is numerically derivable from two of them, and is then a pencil of complexes that consists of only special complexes whose axes therefore define a pencil of rays.

Equation (ν) will also determine such a pencil of rays. Since equations (μ) and (ν) have a pair of solutions $\mathfrak{a}_1 / \mathfrak{a}_3$, $\mathfrak{a}_2 / \mathfrak{a}_3$ in common, the two pencils of rays must possess a common ray. The axes of the special complexes of the pencil $[\mathfrak{C}_1\mathfrak{C}_2\mathfrak{C}_3]$ then define a ruled family that decomposes into two pencils of rays. Since its guiding line then decomposes into two pencils of rays, in any case, one can say:

The necessary and sufficient condition for three complexes $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ *that do not belong to the same pencil to have a ruled family in common that decomposes into two pencils of rays is* $[\mathfrak{C}_1\mathfrak{C}_2\mathfrak{C}_3]^2 = 0$.

F. Klein called such a ruled family *special*.

If one is given four complexes $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$, $\mathfrak{C}_4$ whose exterior product is not equal to zero, but whose numerical values are equal to zero, then the equation:

$$[\mathfrak{C}_1\ \mathfrak{C}_2\ \mathfrak{C}_3\ \mathfrak{C}_4]^2 = 0$$

will be true for them, so it will follow immediately from the concept of inner product that:

$$[\ |\ \mathfrak{C}_1\ \mathfrak{C}_2\ \mathfrak{C}_3\ \mathfrak{C}_4]^2 = 0.$$

The extended congruence of the pair of rays $[\mathfrak{C}_1\ \mathfrak{C}_2\ \mathfrak{C}_3\ \mathfrak{C}_4]$ is therefore a special one, so, from no. 10, the ray-pair that is represented by $[\mathfrak{C}_1\ \mathfrak{C}_2\ \mathfrak{C}_3\ \mathfrak{C}_4]$ will consist of two coincident rays, or a *special* ray-pair.

If $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$, $\mathfrak{C}_4$ are special complexes then:

$$[\mathfrak{C}_1\ \mathfrak{C}_2\ \mathfrak{C}_3\ \mathfrak{C}_4]^2 = 0$$

will be the condition for there to be just one line that cuts its axes.

For five complexes $\mathfrak{C}_1$, …, $\mathfrak{C}_5$ whose exterior product is not equal to zero, the equation:

$$[\mathfrak{C}_1\ \mathfrak{C}_2\ \mathfrak{C}_3\ \mathfrak{C}_4\ \mathfrak{C}_5]^2 = 0$$

likewise represents the condition for one to have $[\ |\ \mathfrak{C}_1\ ,\ \mathfrak{C}_2\ ,\ \ldots\ \mathfrak{C}_5]^2 = 0$; i.e., for the complex that is normal to all five complexes to be a special one, or, since the five complexes must go through its axis, for the five complexes to have a common ray.

All complexes of the web of complexes that they determine then likewise contain this ray; such a web shall be called *special*.

If the five complexes are all special then $[\mathfrak{C}_1\ \ldots\ \mathfrak{C}_5]^2 = 0$ will represent the condition for its axis to be cut by that ray. If one writes this expression, from A_2, no. 175, as the determinant:

$$\begin{vmatrix} \mathfrak{C}_1^2 & [\mathfrak{C}_1\,|\,\mathfrak{C}_2] & [\mathfrak{C}_1\,|\,\mathfrak{C}_3] & \cdots & [\mathfrak{C}_1\,|\,\mathfrak{C}_3] \\ [\mathfrak{C}_2\,|\,\mathfrak{C}_1] & \mathfrak{C}_2^2 & [\mathfrak{C}_2\,|\,\mathfrak{C}_3] & \cdots & [\mathfrak{C}_2\,|\,\mathfrak{C}_5] \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ [\mathfrak{C}_5\,|\,\mathfrak{C}_1] & [\mathfrak{C}_3\,|\,\mathfrak{C}_2] & [\mathfrak{C}_5\,|\,\mathfrak{C}_3] & \cdots & \mathfrak{C}_5^2 \end{vmatrix} = 0$$

and remarks that $\mathfrak{C}_i^2 = 0$ and $[\mathfrak{C}_i\ |\ \mathfrak{C}_k] = [\mathfrak{C}_k\ |\ \mathfrak{C}_i] = m_{ik}$ is the moment of the axes of the special complexes relative to each other then one will obtain the condition equation for the five rays to determine a special complex in the form:

$$\begin{vmatrix} 0 & m_{12} & m_{13} & \cdots & m_{15} \\ m_{21} & 0 & m_{23} & \cdots & m_{25} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{51} & m_{52} & m_{53} & \cdots & 0 \end{vmatrix} = 0.$$

The exterior product of six arbitrary complexes $\mathfrak{C}_1, \ldots, \mathfrak{C}_6$ represents a number $\mathfrak{x}$, while the equation:

$$[\mathfrak{C}_1 \ldots \mathfrak{C}_6]^2 = 0$$

then means nothing but $\mathfrak{x}^2 = 0$, or:

$$\mathfrak{x} = [\mathfrak{C}_1 \ldots \mathfrak{C}_6] = 0;$$

i.e., the six complexes belong to the same bush.

If the complexes are all special then $[\mathfrak{C}_1 \ldots \mathfrak{C}_6] = 0$ will give the condition for their axes to belong to those linear complexes; if one writes this equation, as above, in the form:

$$\begin{vmatrix} 0 & m_{12} & m_{13} & \cdots & m_{16} \\ m_{21} & 0 & m_{23} & \cdots & m_{26} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{61} & m_{62} & m_{63} & \cdots & 0 \end{vmatrix} = 0$$

then it will represent the condition for the six rays to belong to a linear complex.

Since the exterior product of seven complexes $\mathfrak{C}_1, \ldots, \mathfrak{C}_7$ is always zero, one will also always have the equation:

$$[\mathfrak{C}_1 \ldots \mathfrak{C}_7]^2 = 0.$$

If the complexes are special then one will obtain the equation that exists between the moments of any seven rays in space in the form:

$$\begin{vmatrix} 0 & m_{12} & \cdots & m_{17} \\ m_{21} & 0 & \cdots & m_{27} \\ \cdots & \cdots & \cdots & \cdots \\ m_{71} & m_{72} & \cdots & 0 \end{vmatrix} = 0.$$

Only one of the other easily-obtained equations shall be derived in order to represent the method. From no. 10, any five linear ray complexes determine a complex that is normal to them.

Now, let two groups of any five linear ray complexes $\mathfrak{C}_1, \ldots, \mathfrak{C}_5$, $\mathfrak{C}'_1, \ldots, \mathfrak{C}'_5$ be given, so one seeks the condition for the normal complexes that are determined by the two groups to be normal to each other.

Since the two normal complexes are given by $\mid [\mathfrak{C}_1 \ldots \mathfrak{C}_5]$ and $\mid [\mathfrak{C}'_1 \ldots \mathfrak{C}'_5]$, from no. 8, the desired condition equation will read:

$$\mid [\mathfrak{C}_1\ \mathfrak{C}_2\ \mathfrak{C}_3\ \mathfrak{C}_4\ \mathfrak{C}_5] \mid [\mathfrak{C}'_1 \mathfrak{C}'_2 \mathfrak{C}'_3 \mathfrak{C}'_4 \mathfrak{C}'_5] = 0 = [\mathfrak{C}_1\ \mathfrak{C}_2 \ldots \mathfrak{C}_5 \mid \mathfrak{C}'_1 \ldots \mathfrak{C}'_5],$$

or in determinant form:

$$\begin{vmatrix} [\mathfrak{C}_1 \mid \mathfrak{C}'_1] & [\mathfrak{C}_1 \mid \mathfrak{C}'_2] & \cdots & [\mathfrak{C}_1 \mid \mathfrak{C}'_5] \\ [\mathfrak{C}_2 \mid \mathfrak{C}'_1] & [\mathfrak{C}_2 \mid \mathfrak{C}'_2] & \cdots & [\mathfrak{C}_2 \mid \mathfrak{C}'_5] \\ \cdots & \cdots & \cdots & \cdots \\ [\mathfrak{C}_5 \mid \mathfrak{C}'_1] & [\mathfrak{C}_5 \mid \mathfrak{C}'_2] & \cdots & [\mathfrak{C}_5 \mid \mathfrak{C}'_5] \end{vmatrix} = 0.$$

If one assumes that all complexes are special and sets $[\mathfrak{C}_i \mid \mathfrak{C}'_k] = m_{ik}$ here then the equation:

$$\begin{vmatrix} m_{11} & m_{12} & \cdots & m_{15} \\ m_{21} & m_{22} & \cdots & m_{25} \\ \cdots & \cdots & \cdots & \cdots \\ m_{51} & m_{52} & \cdots & m_{55} \end{vmatrix} = 0$$

will give the condition for the linear complexes that are determined by those two groups of five rays to be normal to each other (i.e., to lie in involution).

These, and similar, equations are analogous to the known equations between spheres and points that were found by *Darboux*, for the most part.

This analogy is a complete one, in that it allows one to treat sphere geometry by the same principles that were applied here to ray geometry, and every formula in the theory of extensions can then be interpreted immediately in one or the other realm.