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SUMMARY REPORTS

Self-stresses in elastic structures [1])

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1. Generalities. One can divide all stress states of elastic bodies into two main groups: The ones that are caused by actual external loads (surface or volume forces), and the ones that exist without external loads. The latter can have their origin in either the history of the body or in current physical influences – principally, temperature influences. Correspondingly, we distinguish the two main groups as load stresses and proper stresses – or self-stresses – and inside the latter group, we distinguish "manufacturing stresses" and "heat or temperature stresses."

While the load stresses have already found numerous comprehensive, systematic representations in the literature [2]), up to now, there exists no overview of the widely diverse examinations into the realm of proper stresses that is oriented towards the theoretical viewpoint. The current state of development of these investigations does not offer a unified picture. Moreover, it seems that many times the individual investigations were carried out with no knowledge of the other ones that were closely related to them, and a clarification of the connection between the various investigations has hardly been sought, at present.

For a long time, only simple special cases of manufacturing stresses and temperature stresses in elastic structures were considered in the literature of structural statics. The manufacturing stresses that come about as a result of internally statically-indeterminate trusses have their origins mostly in the imprecise lengthening of the individual rods. By means of so-called statically-indeterminate external structures, manufacturing stresses can also come about through the impressing of non-uniform reductions of the stock (Lagersenkungen), and these cases have been examined many times already. Of a completely different sort are any cases of manufacturing stresses that are known and have been examined empirically in the different branches of metal and glass technology. Geology and geophysics have also been concerned with difficult special problems of self-stresses. Likewise, heat stresses have been considered in various places in the technical literature.

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[1]) The foregoing report is an essentially extended version of my same-titled citation in the Handbuch der Physikalischen und Technischen Mechanik, edited by Prof. Dr. F. Auerbach and Prof. Dr. W. Hort. Printed by Joh. Ambr. Barth, Leipzig.

[2]) Cf., e.g., the article of Müller-Timpe, Tedone, Tedone-Timpe in the Enzyklopädie der Mathrematischen Wissenschaften 4/4; the citation of Trefftz in the Handbuch der Physik; the citation of Korn in Handbuch der Physikalischen und Technischen Mechanik (Bd. 3, pp. 1).

In the theoretical literature of elasticity, heat stresses had been considered since the earliest beginnings of that discipline, since otherwise it would still not be possible to present general physical Ansätze for the thermoelastic problem that are completely flawless [3]). On the other hand, manufacturing stresses were first introduced into the scientific representation of advanced elasticity theory by Love [4]), and we particularly emphasize A Föppl [5]). Föppl was indeed the first one to clearly recognize the suitability of summarizing the heat stresses and additional proper stresses from a unified viewpoint and to extend the sense of the word "proper stress" correspondingly; he was also the first to expressly emphasize the necessary restrictions on the uniqueness theorem of elasticity theory in the context of manufacturing stresses. Certain special types of pressure stresses were already treated theoretically in general by Weingarten and Volterra and a somewhat more general type was introduced into elasticity theory by Somigliana. The Volterra pressure stresses (the so-called distortion stresses) can, however, be regarded as only an adequate general representation of the aforementioned simple types of manufacturing stresses that were treated in the literature of structural statics.

The organizing principle of the present report is to proceed from the theoretically simplest types to the most general ones. In 2, the Volterra distortions will be treated. In 3, some examples will be discussed, along with the possibility of the experimental verification of the theory. In 4, the Weingarten and Somigliana distortions will be discussed. No. 5 will be directed towards the most general type of manufacturing stresses, with the single restriction to small displacements. In 6, the manufacturing stresses for large displacements will be treated, and in 7, the heat stresses. Finally, in 8 we seek to show that the connection between the pressures and the load stresses will be produced by means of the concept of higher load singularities.

2. Volterra distortions. Starting from a hydrodynamical analogy, and on the grounds of a fundamental preliminary examination of Weingarten [6]), and later by Klein-Wieghart [7]), Volterra [8]) examined the following question: Can there exist, in an elastic body that is free of surface and volume forces, a state of distortion that is continuous, along with its first and second derivatives, and having single-valued dilatations (ε_x , ε_y , ε_z) and shears (γ_{xy} , γ_{xz} , γ_{yz})? The answer to this question is that this is possible only for multiply connected elastic bodies.

In order to prove this, and likewise to establish the Nature of the aforementioned state of distortion, Volterra represented the components of the displacements *u*, *v*, *w* with the help of line integrals of the distortion quantities ε_x , etc., γ_{xy} , etc. Investigation of this line

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[3]) Cf., on this, A. Korn, loc. cit.

[4]) Love, Mathematical Theory of Elasticity, 2nd ed., Cambridge 1906. Amongst the proper stresses, Love also counted the ones that arose from volume forces, as well as those that did not originate outside of the body, but from the different parts of that same body.

[5]) Vorlesungen über Technische Mechanik, Bd. 5, pp. 293.

[6]) Weingarten, Atti della reale academia dei Lincei, Rendiconti Classe di scienze fisiche matematiche e naturali (in the following, cited as "Lincei") (5) 10_1, 1901.

[7]) F. Klein and K. Wieghardt, Über Spannungsflächen und reziproke Diagramme, Archiv d. Mathematik u. Physik 8, pp. 1, 1904.

[8]) Volterra, Annales de l'École normale supériure (3) 24, 400, 1907. This comprehensive treatise is an extended summary of earlier papers by Volterra. See Lincei (5), 14_1, pp. 127, 193, 351, 431, 641; 14_2, pp. 329; Nuov. Cim. (5) 10, pp. 361.

integral showed that it is independent of the path of integration when and only when the Saint-Venant compatibility conditions are satisfied and the body is simply connected. By comparison, for multiply connected bodies, although u, v, w satisfy the compatibility conditions they are not generally independent of the path; hence, they are polydromic functions of position.

Cesàro [9]) gave a simplified representation of these integral formulas, and on the same grounds he generalized the Volterra statement of the problem to non-Euclidian spaces. Giuganino [10]) again picked up the problem of representing the displacement field as characteristic quantities of the distortion field and solved it in a completely different way. He started with the fact that the three components ρ_{xy} , ρ_{xz} , ρ_{yz} of the rotation of the displacement field can also be polydromic when the distortion components ε_x , …, γ_{xy} , … are single-valued. The multi-valuedness of the displacement field u, v, w enters into the multi-valuedness of the rotor field in an essential way. Giuganino showed that u, v, w can be constructed from the three components of ρ and three harmonic functions P, Q, R that can be determined from the form of the body in a completely similar way to the construction of the displacement field of a rigid body from the six screw components.

From the fundamental theorem of Volterra-Cesàro above, one can infer that an elastic body of simply connected form, when it is free of loads, must be found in the natural state in the event that the distortions are regular, while in a multiply connected body, a load-free state of stress is also possible for regular distortions.

Volterra has further shown that the polydromic displacement field thus defined may, in any case, be represented by a discontinuity that is linear in nature when expressed in spatial coordinates. The displacement field may also be physically produced in such a way that the body will be converted into a simply connected one by a corresponding number of cuts, the cuts being subject to arbitrary screw motions between them, and then the multiple connectivity will again be produced by filling in the intermediate space with foreign material and removing the superfluous material and welding together again. Volterra called these manipulations "distortions" and the stresses that resulted from them "distortion stresses." The manufacturing stresses can also be easily interpreted as the stresses. Namely, if we think of the body as having been produced by a simply connected body whose connecting surfaces agree with each other precisely, although its position exhibits indeterminacy, then the desired situation can be arrived at only through pressures that produce a state of stress that is identical with the Volterra distortion stresses. (Otherwise, the body would not generally exhibit the characteristic discontinuity properties that appear for Volterra for this way of creating proper stresses.)

Volterra showed that for these pressures there exists a reciprocity relation that is analogous to Maxwell's theorem. Namely, if we consider two arbitrary cuts through the multiply connected body and impose the two screw motions that bring these cuts together then the body can be assigned distortions with six components. Now, if one thinks of the resulting dynamics that arises from stresses in the two cut surfaces in question as being likewise decomposed into components and denotes the mutually corresponding screw (stress, resp.) dynamical components by the same notation then one can assert that the

[9]) Cesàro, Sulle formole del Volterra fondamentale nella teoria delle distortioni elastiche. Atti della R. Academia della scienze fis. e mat. di Napoli (3) 12, 1906, pp. 311.

[10]) Giuganino, Alcune formole analoghe a quelle del Volterra nella teoria delle distortioni elastiche. Lincei (5), 20_1.

screw component r that is appropriate to the cross-section α gives rise to a stress dynamic component S at the cross-section β that is just as large as the stress dynamic component R that the screw component s appropriate to the cross-section β gives rise to at the cross-section α.

Colonetti [11]) has likewise found another general reciprocity theorem on the basis of work considerations that exists between load stress states and self-stress states: The product sum of the six components of the dynamic of the cross-sectional stresses that results from a group of external loads with the corresponding components of a given distortion screw is equal and opposite to the work done by external forces along a path of displacement that is caused by this distortion. Various special cases of this noteworthy theorem are already known in structural statics [12]); Colonetti's contribution is only the general formulation and flawless proof of it.

Another important result of the energetic structural statics of distortions is the extension of Castigliano's theorem on the minimum work done in deformation due to the distortions, which was likewise carried out by Colonetti [13]).

To the Castigliano work expression, one must add yet another work quantity that Colonetti symbolically referred to as the "deformation work of the impressed distortions"; we will learn about the character of this theorem more closely with the help of a specialization.

Starting from the Volterra reciprocity theorem one may derive the connection between the stress field and the displacement field for the distortions directly. Volterra and Colonetti [14]) have solved this question for arbitrary plane distortions that relate to a plane-symmetric body. If one considers such distortions then it is obvious that the distortion can be characterized by a pure rotation. The associated stress state in each cross-section can be characterized by its resulting forces. In this way, one can associate each point of a doubly-connected plane system with a unique line, and reciprocally. An n-fold connected body establishes $n - 1$ such reciprocal associations. The study of these reciprocities now yields the remarkable result that for any reciprocity there are two mutually perpendicular directions that are distinguished by the fact that a pure displacement in the one direction belongs to a force in the other, and conversely.

3. Special cases of Volterra distortions. The simplest special case of a Volterra distortion was given by Volterra himself [8]), namely, he solved the distortion problem for an annulus between two concentric circular cylinders and bounded by planes perpendicular to them. He considered the six elementary distortions when he chose the reference axis system to be through the center of the body. He carried out the solution in two parts: The first part consisted in the search for such a system of stresses that is regular everywhere and in the radial cut surface that corresponds to the elementary distortions that the discontinuity in the displacements produces while leaving the outer surface of the cylinder stress-free. As a second step, he then ascertained the forces on the base surface of the cylinder that this system yielded, and overlaid the first stress field with a second one that delivered equal and opposite forces on the base surface. The

[11]) Colonetti, Su di una reciprocita fra deformazioni e distortioni, Lincei 24_1 (1915).
[12]) Cf., e.g., Ritter, Anwendungen der graphischen Statik, 3 Teil, Zürich 1900.
[13]) Colonetti, Sul problema delle coazioni elastiche, Torino Atti 54 (1918-19), pp. 864.
[14]) Colonetti, Sulle distortioni dei sistemi elastici piani più volte connessi, Lincei 24_1, 1915.

solution to the first is obtained in closed form with the usual functional notation, while the solution to the second part for single elementary distortions was obtained only approximately. The exact part of this problem is independent, and indeed was solved by Timpe [15]) before the appearance of the general theory of Volterra. All of the results were verified by Volterra using caoutchouc models, which also allow one to test the stress field. To that end, Rolla, and later Corbino and Trabacchi [16]) constructed transparent models from gelatine and fish glue (Fischleim), and by means of them, expressed the six elementary Volterra pressures as stresses. The examination of the cylinder by means of polarized light using the process of W. König yields dark lines that can also be computed theoretically. The experiments have admirably confirmed the theory.

The difficult special case of regions between two confocal ellipses and bounded by planes was solved by Timpe with the help of trigonometric and hyperbolic functions [17]).

A very general case of Volterra distortions was treated by Almansi [18]). He considered multiply connected cylindrical bodies with completely arbitrary cross-sections and reduced the solution of the problem in full generality to the determination of biharmonic stress-functions (for any elementary distortion 3) in two variables x, y (coordinates in the cross-section). The solution is analogous to the treatment of plane deformation problems with the help of Airy stress functions, although the problem to be solved here does not reduce to plane distortions. The investigations of Almansi were applied by E. Freda [19]) to a cylindrical body with an eccentric circular cross-section. He reduced the eccentric circle to a centered one with the help of a suitable reflection in the unit circle.

The same things that Almansi derived for a cylindrical body were carried out by Laura [20]) for a body of revolution. This intrinsically three-dimensional problem was likewise reduced by the author to a system of simultaneous differential equations in only two r, z (coordinates in the meridian plane). If one now cuts out a piece of the body of revolution with the help of two meridian pieces then this theory allows one to deduce the stress state of this curve rod for arbitrary forces and moments acting on the end cross-section; The Laura theory then leads to a generalization of the Saint-Venant theory of rods to circularly curved rods constructed with arbitrary cross-sections and center angle (Zentriwinkel).

Volterra further communicated an important special case (limiting case) of his theory, namely, the pressures that are defined in frame structures constructed from thin rods with stiff corner links (Eckanschlüssen). He called this system a cyclic structure of flexible

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[15]) Timpe, Probleme der Spannungverteilung in ebenen Systemen, einfach gelöst mit Hilfe der Airyschen Funktion, Diss. Göttingen 1905.

[16]) O. M. Corbino, Le tensioni in un corpo elastico distortioni di Volterra e la conseguente doppia rifrazione accidentale. Lincei (5) 18, 437, 1909. – G. C. Trabacchi, I. fenomeni di doppia rifrazione accidentale prodotti dalle tensioni create in un corpo elastico dalle distortioni di Volterra. Lincei (5) 18, 444, 1909. – Cf., also Volterra, 3 Vorlesungen über neuere Fortschritte der mathematischen Physik. German by C. Lamia, Leipzig and Berlin 1914, pp. 142, et seq., where a brief synopsis of the Volterra theory of distortions is given.

[17]) Timpe, Die Airysche Funktion für den Ellipsenring, Mathematische Zeitschrift 17, pp. 189, 1923.

[18]) Almansi, Sopra una classe particolare di deformazione a spostamenti polidromi dei solidi cilindrica. Lincei (5) 161, pp. 26. – Almansi, Sulla deformazione a spostamenti polidromi de solidi cilindrici. Lombardo Instituto rendi conti (2) 40, pp. 937.

[19]) Freda, Sulle distortioni di un cilindrico elastico due volte connesso. Lincei (5) 25_1, pp. 582 and 679, 1916.

[20]) Laura, Sopra le deformazione per distortioni elastico di revoluzione. Nuov. Cim. (6) 7, 1914.

elements. They represent a theoretical generalization of the frameworks treated in structural statics, for which pressure stresses are occasionally treated. Volterra's solution is carried out in the form of a system of linear equations, just like the aforementioned special problems in structural statics. However, it is extraordinarily remarkable that the way of representing distortions that was given by Volterra exhibits a close analogy with the Kirchhoff equations for the distribution of electrical currents in a system of wires that define a network. Any equation of Kirchhoff's system of equations corresponds to six equations in the Volterra system of equations. The electrical current strength in a cross-section corresponds to the resulting flux (Dyname) of stresses in a cross-section of a rod (the six components of this flux, resp.) and the potential at a junction of the Kirchhoff network corresponds to the screw flux at a Volterra junction (the six components of it, resp.), and finally the electromotive force corresponds to the characteristics of the distortion; the reciprocal value of the Ohmian resistance corresponds to the elastic constants of the rod.

The ideal truss (network) may be regarded as a counterpart to the Volterra network. For this simple case the pressure stresses, et al., were treated by Zschetsche [21]), and for this case he had verified the legitimacy of the invertibility of the stresses independently of Volterra. For this case, the aforementioned general theorem of the minimum deformation work takes on a particularly simple form: It must be determined from the statically indeterminate quantities under the condition that the work quantity:

$$\sum\left[\frac{S^2 l}{E\,F} + S\Delta l\right]$$

is a minimum. In this, the second term is the "work done by the imprinted distortions" [22]). Colonetti applied this connection to the study of assembly stresses in the struts of biplanes [23]). He verified the exceptionally great meaning of the self-stresses and showed the ways in which one could make good use of it in economic systems.

Klein and Pfeiffer [24]) have examined the self-stresses in ideal trusses from the standpoint of geometry with the help of the Maxwellian polyhedron and deduced interesting topological links.

At this point, we may clarify why ideal trusses with uncountable rods behave the same way in relation to self-stresses as multiply-connected bodies. Namely, whereas trusses with stiff corners may be informally represented as special (limiting, resp.) cases of multiply-connected bodies, it is not at all geometrically intuitive why a statically determinate ideal truss or configuration of plates (Scheibenwerk) should behave like a simply-connected one and a statically indeterminate one, like a multiply-connected one. A simple geometrical-kinematical consideration clears up this question:

We think of our truss or configuration of plates, with the help of supporting members in the soil (Erdscheibe), as being united into a single configuration of plates; we can thus

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[21]) Zschetsche, Handbuch der Baustatik, Bd. 1, Düsseldorf 1912, pp. 490.

[22]) Colonetti, Aplicazione a Problemi tecnici di un nuovo teorema sulle coazioni elastiche. Atti di Torino 54, pp. 69, 1918/19.

[23]) Colonetti, Sforzi di montaggio nell armature dell, ala di un biplane. 54, pp. 426, 1918/19.

[24]) F. Klein, Selbstspannungen elastischer Diagramme. Mathematische Annalen 67, pp. 433, 1909. – Pfeiffer, Zeitschrift für Mathematik und Physik 58, 1909.

restrict our considerations to free (unsupported) bodies. We refer to a "cut" as a material separation along a planar piece of a surface whose boundary belongs completely to the outer surface of the body. A simply-connected body will be divided into two separate pieces by such a cut while this is not the case for multiply-connected ones. Kinematically speaking: If we think of our body as rigid then if it is simply-connected the edges of the cut have six degrees of freedom with respect to each other, while if it is multiply-connected they have none. If one now employs the same considerations to an ideal truss or configuration of plates then one sees that in the statically-indeterminate case a cut (except for the missing part) contributes no relative degree of freedom, while in the statically-determinate case one degree of motion exists across the cut.

In a remarkably fruitful way, Zimmermann [25]) added certain fictitious pressures, long before the general theory of Volterra, by means of which the treatment of the elastically (gebetteten) rods served as a lemma for the process of letting rods of finite length go to rods of infinite length.

4. Weingarten and Somigliana distortions. The question that Weingarten [6]) posed, which defined the starting point for all of distortion theory, was somewhat more general than the one that Volterra examined so thoroughly. Namely, Weingarten asked: Under which conditions in an elastic body that is free of loads can there exist a system of stresses and distortions (ε_x , …, γ_{xy} , …) that it is continuous everywhere; continuous first and second derivatives of the distortion quantities were thus not required. The answer to this question is the following: In a simply-connected body, such a system of distortions can come about only when the displacement field *u*, *v*, *w* possesses a discontinuity surface *S* with the following property: The discontinuity must be such that it is either everywhere perpendicular to *S* or such that it takes *S* into an infinitely close surface S' that is developable from *S*, where the boundaries of *S* and S' define a common asymptotic line for both surfaces. If the body is multiply-connected then the conditions posed for this type of discontinuity surface obviously correspond to the Volterra type that was discussed in 2. The detailed examination of Weingarten distortions was carried out by Somigliana [26]), who also gave an example of a Weingarten distortion that did not correspond to a Volterra type.

However, in his examination of 1914 Somigliana took another step further: He subjected the discontinuity layer SS' to no other restriction than that the discontinuities should be very small (infinitely small). Then, as Somigliana showed, the stress quantities σ_x , etc., and the distortion quantities ε_x , etc., will nonetheless remain regular in the entire body, except that on the discontinuity layer the distortion field experiences a jump variation. If u_S, v_S, w_S mean the discontinuities in the displacement field at a point of *S* relative to an axis intersection whose *x* and *y* axes lie in the tangent plane to *S*, while the *z* axis is perpendicular to it, then the jumps in the six distortion components ε_x , …, γ_{xy} , … relative to the same axis intersection are expressed in the following form:

[25]) Zimmermann, Die Berechnung des Eisenbahnoberbaues. Berlin 1888, - As a result of the elastic (Bettung), it behaved in the case of a simply-connected rod similarly to the case of a doubly-connected one.

[26]) Somigliana, Mathematiker-Congress von Rom, Bd. III – Somigliana, Sulla teoria delle distortioni elastichi I and II. Lincei (5) 231, pp. 463, 1914 and 241, pp. 655, 1915.

$$[\varepsilon_x]_S = \frac{\partial u_S}{\partial x}, \quad [\varepsilon_y]_S = \frac{\partial v_S}{\partial y}, \quad [\varepsilon_z]_S = -\frac{\lambda}{\lambda + 2\mu}\left(\frac{\partial u_S}{\partial x} + \frac{\partial v_S}{\partial y}\right);$$

$$[\gamma_{xy}]_S = \frac{\partial u_S}{\partial y} + \frac{\partial v_S}{\partial x}, \qquad [\gamma_{xy}]_S = 0, \qquad [\gamma_{xy}]_S = 0.$$

From these formulas, from which it first drops out that w_S appears nowhere, one may easily read off the properties communicated above that u_S , v_S , w_S must satisfy if the Somigliana distortion is to likewise be a Weingarten one [27]).

This examination of the discontinuities, which is also noteworthy from the geometric standpoint, was extended by Maggi [28]) to the derivatives of ε_x , etc., and he likewise succeeded in freeing himself from the natural coordinate system so the result could relate to an arbitrary fixed *x*, *y*, *z* axis intersection.

Somigliana [29]) succeeded in carrying out the determination of the stress and distortion states of a body subjected to a completely arbitrary distortion in the following way: The desired displacement field *u*, *v*, *w* will be composed of two parts. The first part *U*, *V*, *W* shall vanish at infinity, and be regular everywhere except for *S*, where it shall exhibit the prescribed discontinuities. The thus uniquely defined displacement state *U*, *V*, *W* will then, on the grounds of the biharmonic generalization of potential theory, include double layers. Obviously, the remaining part u_1 , v_1 , w_1 must then satisfy the condition that it is again regular everywhere but *S* and on the outer surface of the body the surface stresses associated with *U*, *V*, *W* are annihilated; this second partial solution then demands the solution of a second boundary-value problem in elasticity theory.

It is worth mentioning the hydrodynamic analogy that Somigliana gave for his distortions: In a closed space *R* with rigid walls one finds a stationary double source with a direction that is perpendicular to a fixed surface *S*; one seeks the resulting stationary vortex-free motion; naturally, the analogy is only a purely qualitative one.

5. General type of manufacturing stresses. While the self-stresses that were treated in 2, 3, and 4 will lead back locally to indeterminacies that are restricted to certain cut surfaces, or at least to such indeterminacies that can be thought to exist, the proper stresses to be treated here are of a more general sort that are not superficially, but spatially, distributed throughout the entire body, or in a part of it. By the casting process that was already mentioned in 1, the indeterminacy comes about as a result of, for example, the non-uniform cooling of the individual parts that formerly went over into the elastic solid state and the rest of them hinder the change of shape that is linked with setting. Moreover, proper stresses appear in almost all other technological manufacturing processes; e.g., in forging. In concrete buildings, this phenomenon corresponds to shrinkage under non-uniform setting (abbinden). The follow-up treatment can also give rise to completely similar self-stresses, and likewise so can the piecewise plasticizing of

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[27]) With this consequence, Somigliana has corrected a mistake in the purely geometric examination of Weingarten.

[28]) Maggi, Calcolo delle discontinuità delle derivate di ordine superiore dello spostamento d'equilibrio elastico. Lincei (5) 30_2, pp. 71, 1921.

[29]) Somigliana, loc. cit., Treatise II.

the body that comes about from over-stretching it. In the latter case, one mostly calls the proper stresses “residual stresses.”

Reissner [30]) showed, in the context of a fundamental general examination of the self-stress problem, that the distortion stress field can also be regarded as generated by continuously-distributed indeterminacies. Namely, the distribution of indeterminacies – or, as Reissner expressed it, the distribution of proper stress sources – is, for a completely determined state of self-stress, infinitely multi-valued. The distortions thus distinguish, among all other self-stress states, that the first kind of manifold of possible systems of proper stress sources that one should concentrate on is isolated surfaces. Reissner, et al, have also given simple instructions for how one can establish a continuous distribution of proper stress sources of an especially simple type from a given self-stress state.

Mathematically speaking, the difference between the general self-stress state and the pressures treated above also lies in the fact that, whereas the Saint-Venant compatibility conditions between the deformation quantities ε_x , …, γ_{xy} , … are satisfied everywhere in the body with the exception of a surface, these conditions are invalid for the general state of self-stress in the entire body – or a spatially extended part of it – that are treated here [31]).

Föppl [32]), in his classical presentation of this set of circumstances, showed that the theorem of the minimum deformation work under the variation of the state of deformation also remains valid for the proper stresses. By comparison, the theorem of the minimum deformation work under variation of the stress state does lose its validity. Colonetti [33]) has, however, also defined the extra term of “deformation work of the impressed deformations” for the presently most general case, which, when added to the deformation expression, secures the minimum property of the latter.

Föppl has showed that in elasticity theory, outside of the equilibrium condition for the internal stresses and the equations:

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\begin{aligned}
2\frac{\partial}{\partial x}\left(\varepsilon_x + \frac{e}{m-2}\right) + \frac{\partial \gamma_{xy}}{\partial y} + \frac{\partial \gamma_{xz}}{\partial z} &= 0 \\
2\frac{\partial}{\partial y}\left(\varepsilon_y + \frac{e}{m-2}\right) + \frac{\partial \gamma_{xy}}{\partial x} + \frac{\partial \gamma_{yz}}{\partial z} &= 0 \\
2\frac{\partial}{\partial z}\left(\varepsilon_x + \frac{e}{m-2}\right) + \frac{\partial \gamma_{xz}}{\partial x} + \frac{\partial \gamma_{yz}}{\partial y} &= 0
\end{aligned}
$$

which follow immediately from the elasticity law, and outside of the boundary conditions, there are no other independent equations for the solution of the boundary-value problem to add, such that the problem is undetermined, and cannot be solved at all without special physical assumptions on the manufacturing process of the self-stress state, when absolutely nothing else can usually be expected from the definition of the stress state.

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[30]) Reissner, Eigenspannungen und Eigenspannungsquellen, This Zeitschrift, this volume, pp. 1.

[31]) Colonetti, Su certi di coazione elastic ache non dipendono da azioni esterne. Lincei (5) 26_2, pp. 43, 1917.

[32]) Föppl, loc. cit.

[33]) Colonetti, Per una teoria generale delle coazioni elastiche. Torino atti 56, pp. 188, 1921.

Nonetheless, they allow one to show certain general properties of the deformation and stress fields. Colonetti [34]) used the Betti reciprocity theorem to derive such properties. Since a system of deformations is the one to be investigated, as a second system of deformations he first chose a homogeneous deformation. He proved for it that the total volume of the body due to proper stresses is precisely as large as it is in the natural state. Föppl has already presumably expressed this theorem for the special case of casting stresses in a ball. He further chose a Volterra distortion as a second system of deformations and came to the following result: The resulting force and moment that general proper stresses carry over into the cut can be computed when the displacements of the points of the cut surface that come about from the cutting and associated relaxation can be determined experimentally, and when, on the other hand, the stress state that the Volterra distortion gives rise is known [35]).

If, as it happened above, in the absence of knowledge of the manufacturing process for the elastic body in question, in general only the cutting of it can give satisfying information about its state of self-stress [36]) then the basic knowledge of the indeterminacy would be of great significance for the manufacturing process. The foundations of this knowledge are scattered throughout the technological literature, and we also find isolated conclusive investigations of proper stresses in technological papers.

We find generalities on the physical conditions under which proper stresses exist (on the way to reduce them, resp.), in, e.g., Martens and Hein: Materialkunde [37]). An incisive investigation of casting stress issues from the physical, as well as the mathematical, standpoint, has been given by Steiger [38]). The case that was already treated briefly by Föppl of casting stresses in a ball, as well as the casting stresses in cylindrical bodies, were examined by Honegger [39]) in close connection with questions of temperature stress, and he advanced numerical and graphic evaluations of his approximate theory.

In another paper [40]), Honegger examined the self-stress state that exists as a result of the piecewise application of a rapid rotation to the elasticity boundary, and has verified the practical applicability of such a temporary overstrain in increasing the use strength (Gebrauchfestigkeit). Also technically important are the self-stresses that arise in steel plates during welding (so-called Schrumpf stresses) [41]).

In conclusion, we come back to the general posing of the question, which shall lead us into the questions of the next section. August Föppl [42]) has shown that when one

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[34]) Colonetti, Una proprieta caracteristica delle coazioni elastiche nei solidi elasticamente omogenei. Lincei (5) 27_2, pp. 155, 1918; Sul problema delle coazioni elastiche. Lincei (5) 27_2, pp. 267, 331, 1918.

[35]) One finds further general theorems (considerations, resp.) in Masing "Zur Theorie der Wärmespannungen" Zeitschr. für Technische Physik 3, pp. 167 1922 (in which, by "heat stresses" he means stresses that come about during the deformation as a result of heat treatment and not temperature stresses), and M. v. Laue "Über die Eigenspannungen in planparallen Glasplatten und ihre Änderung beim Zerschneiden," Zeitschr. für Technische Physik 11. pp. 385 ff. 1930.

[36]) In the excluded case ("elastically indeterminate structures"), a loading experiment can give information about the self stresses.

[37]) Martens and Hein, Materialkunde. Bd. IIa Berlin 1912.

[38]) R. v. Steiger, Über Gussspannungen. Dissertation. Zürich 1913.

[39]) C. Honegger, Über Eigenspannungen. Festschrift. Prof. Dr. A. Stodola on his 70th birthday.

[40]) Honegger, Ausgleich der Beanspruchungen einer rasch rotierenden Radscheibe durch passenden Vorspannungszustand, B. B. C. – Mitteilungen, November 1919.

[41]) Lottmann, Schrumpfspannungen bei Lichtbogenschweissung. V. D. I. – Zeitschrift 1930, pp. 1340.

[42]) Föppl, loc. cit. Cf., also A. and L. Föppl, Zwang und Drang, Bd. II, 2nd ed.

considers only self-stresses with small (infinitely small) displacements, the load stresses under likewise infinitely small displacements will generally be simply superposed, or conversely: the stress and displacement field of an external load for small displacements is generally independent of the self-stresses that govern it. The latter can thus never be established through a load experiment – this is also true in the case where the system can be controlled theoretically. L. Prandtl [43]) has, however, shown that there are exceptional structures for which Föppl's theorem is invalid and has shown the conditions that such a structure must obey as a result of the deformation work. Prandtl calls such structures "elastically indeterminate" since their stress state that results from load cannot be calculated without any knowledge of the manufacturing of the body. The simplest example of such structures is the so-called "exceptional truss," like, say, the rod-triangle (Stab-dreieck) of vanishing height. Prandtl has shown that the thin planar plate likewise belongs to the elastically indeterminate structures. He has thought up a truss-like substitute structure for the thin, planar right-angle plate (Rechteckplatte) that allows one to understand the essentials of the characteristic exceptional properties of the plate.

6. Self-stresses under large displacements. In all of the investigations up to now, we have always assumed small displacements and have seen that under these restrictions – except for the exceptional structures (elastically indeterminate systems) – the law of superposition always remains valid. If one does without the consideration of small displacements then law of superposition generally fails completely, such that the differential equations of the problem are no longer linear, and the difficulties in their examination are enlarged considerably.

A general theory of such self-stresses has not been sought up to now. One suspects that all of the previously considered types of self-stresses for small displacements correspond to analogous self-stresses for large displacements. For example, one can come to new types of self-stresses for the Volterra and Somigliana distortions when one simply assumes that the dislocations are large. For large dislocations in the tangential direction of a thin, planar circular ring disc (Kreisringscheibe) there arises a thin, truncated conical shell that is marked with self-stresses. Also, for continuously-distributed dislocations self-stresses can come about that are similar to the ones linked with instability phenomena: For example, in foundry work the appearance of kinks (Ausknicken) in thin rods that connect massive components is known through casting stresses [44]). It would certainly be worthwhile to pursue a general theory of self-stresses with large displacements.

A narrowly bounded special case of a distortion that is analogous to the Volterra case with larger displacements was recently solved by Sadowsky [45]). He considered the self-stress state of an infinitely slender, planar, thin strip of plate shaped into a Möbius band, and applied the theory of one-dimensional elastic continua [46]) to this problem. The theory led to eight nonlinear differential equations in eight unknowns. The solution

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[43]) L. Prandtl, Elastisch bestimmte und elastisch unbestimmte Systeme. Beiträge zur technischen Mechanik und technischen Physik, pp. 52, 1924.

[44]) See R. v. Steiger, loc. cit., pp. 7.

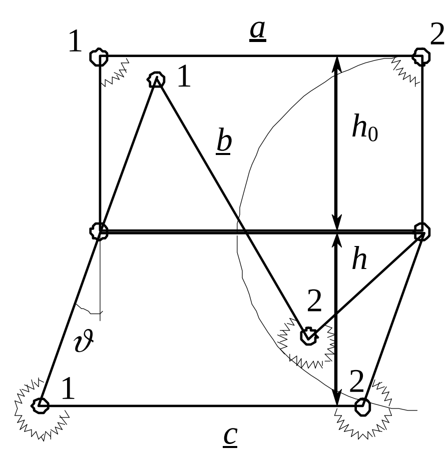
[45]) Sadowsky, Jahresberichte der Deutschen Mathematiker-Vereinigung 39, pp. 49, et seq., 1930.

[46]) Hamel, Handbuch der Physik, Bd. 5, Die Axiome der Mechanik, pp. 18 et seq.; Hamel, Sitzungsberichte der Berliner Mathematischen Ges. 25, 1925-26.

comes about by combining the theory with the experiment when certain characteristic unknowns are ascertained by observation (experimenting on a model, resp.). The main result of the investigation of Sadowsky is that the Möbius surface can not only be represented by an analytic function, but by a planar right-angled triangular surface and another analytic surface that is first of all continuous and has a continuous first derivative, but has a discontinuity connected with the curvature.

Whereas all of the self-stress states considered up to now are connected with certain inelastic operations (manufacturing processes, residual pressures, temporary overstrains, retightening by metalworkers, etc.), if one ignores large, finite displacements then there is also another distinguished group of self-stresses that will be generated by purely elastic operations, hence, by temporarily subjecting the body to external loads that are nowhere plastic. Displacements and stresses are everywhere continuous single-valued functions of position; the compatibility conditions are satisfied everywhere. This essential remark goes back to Armanni [47]). Such self-stresses arise any time a thin, flexible open shell with ellipsoidal curvature ratios – say, for instance, a domed or ringlike shell that is everted in such a way that the inner and outer surfaces have switched roles. The eversion is produced by external loads that can, however, be later removed so that the body does not revert to its original shape.

Almansi [48]) has occupied himself with the Armanni self-stresses in a discussion with Somigliana. He is of the opinion that connectivity phenomena due to the creation of such self-stresses are of no importance whatsoever. This hypothetically-posed assertion is, on closer examination, much more compelling. For the planar case, one immediately sees that the assertion is not correct. For the sake of example, if we think of a flexible rod that is clamped at only one end then it obviously has only an isolated stable equilibrium position: the unstressed one. By comparison, if it were anchored at both ends with two links such that it, along with the ground beneath it, represented a doubly-connected structure then it would have, in addition to the unstressed equilibrium position, at least one other equilibrium shape that is marked with self-stresses, and indeed, with ones of precisely the same nature as in the case of spatial self-stresses that was examined by Armanni; correspondingly, self-stresses of this sort must also be referred as being of the Armanni type.



In this way, we can best delve into the nature of such stress states by means of a further simplification of the example in question that admits the solution in its simplest and clearest form. We thus replace the two-linked arc with a right-angled two-link frame and think of its rods and members as being rigid, but the corners are linked in such a way that any change in angle is opposed by a reaction that is proportional to it (torsional spring links); the elasticity of the entire

[47]) Armanni, Sulle deformazioni finite dei solidi elastici isotropi. Nuovo Cimento (6) 10, pp. 427, 1915.

[48]) Almansi, La teoria delle distortioni e le deformazioni finite dei solidi elastici. Lincei 25_2, pp. 1919, 1916. Cf., also Almansi: L'ordinaria teoria dell'elastica e la teoria delle deformazioni finite (5), 26_2, pp. 3, 1917.

framework is thus concentrated into the two corners. Let the elastic compliance (change of angle per unit moment) of the one spring link be ε_1 and the other be ε_2 .

With the help of temporary external forces, we can now bring the structure from the position *b* to the position *c*, in which the outer loop of the frame is pulled towards the inside. We seek the criterion for this position to be an equilibrium position, and in the affirmative case we ask what sort of self-stress it is endowed with. By the use of the auxiliary angle ϑ the equilibrium condition is obviously:

$$M\,\varepsilon_1 = \pi + \vartheta$$
$$M\,\varepsilon_2 = \pi - \vartheta;$$

thus, one has:

$$M = \frac{2\pi}{\varepsilon_1 + \varepsilon_2}$$

$$\vartheta = \pi\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

Horizontal shear: $$H = \frac{2\pi \cdot h_0}{(\varepsilon_1 + \varepsilon_2)} \cdot \cos\left(\pi\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right),$$

from which all remaining internal force magnitudes follow with nothing further.

On the basis of this and other simple examples, such as the case examined by Armanni of a spherical dome, it can be tentatively asserted that the necessary condition for the existence of Armanni self-stresses is that the body shall represent a statically indeterminate, but elastically determinate structure.

Armanni then examined a peculiar case of self-stress that is indeed convertible to the aforementioned cases of eversion, yet proves to be especially remarkable: Namely, he considered a full spherical shell of uniform, small thickness, cut through it along a meridian (semi-circle), pulled the interior to the outside, and, with the help of a small pressure, brought the resulting open shell into a suitable form, and again converted it into a closed spherical shell; he also obtained the solution in closed form for this case.

Heretofore, self-stresses for large displacements seem to be technical, but not useful, so possibilities for technical applications might also present themselves.

7. Heat stresses. The stresses that are caused by actual temperature differences, which are then called heat stresses or temperature stresses, define a particular group of self-stresses whose connection with other types of self-stress states has still not be clarified from a mathematical standpoint [49]). In certain simple special cases the question of heat stresses and that of manufacturing stresses are precisely identical: e.g., for an ideal truss, a length discrepancy Δl causes precisely the same stress state in a rod as heating the rod in such a way that it would lead to the unchecked extension of the rod through a change in length Δl. On the other hand, it is, however, not difficult to give self-stress states that can in no way be thought to arise from a temperature field. It then is possible that one would like to allow "temperature singularities" in the temperature field of simple and higher order, in the manner of the sources, double sources, etc. of

[49]) Although the paper of Reissner cited above also contains important suggestions in this direction.

hydrodynamics, the poles, dipoles, etc., of electrostatics, as well as the "load singularities" [50]) of elasticity theory; in these cases, however, the immediately evident basic equations of thermo-elasticity lose their validity, and must be generalized accordingly.

In the most general known formulation of the thermo-elastokinetic boundary-value problem, as it would be presented on the basis of the early work of Duhamel, Neumann, and Voigt [51]), is the following one:

$$(\lambda+\mu)\frac{\partial e}{\partial x}+\mu\Delta u+q\frac{\partial \tau}{\partial x}+\rho(X-\ddot{u})=0$$
$$(\lambda+\mu)\frac{\partial e}{\partial y}+\mu\Delta v+q\frac{\partial \tau}{\partial y}+\rho(Y-\ddot{v})=0$$
$$(\lambda+\mu)\frac{\partial e}{\partial z}+\mu\Delta w+q\frac{\partial \tau}{\partial z}+\rho(Z-\ddot{w})=0$$
$$l\Delta\tau-\gamma_1\frac{\partial \tau}{\partial t}+qt^0\frac{\partial e}{\partial t}=0,$$

where t is the difference between the absolute temperature and the normal temperature t^0, ρ is the density, γ_1 is the specific heat of a unit volume, l is the constant of internal heat conductivity, and λ and μ are the Lamé constants. As boundary values, the components of the displacement are prescribed on the outer surface, and τ satisfies the condition $l\,\partial\tau/\partial\nu=-\bar{l}\,[\tau-\tau_0]$ there, where $\bar{l}$ is the coefficient of external heat conductivity and τ_0 means the difference between the outer surface temperature and t^0; ν is the surface normal. In addition, the initial values for $u, v, w, \dot{u}, \dot{v}, \dot{w}$, and τ are given at t (= time) = 0.

If the examination of this boundary-value problem is restricted to merely the equilibrium problem – hence, the fluctuation terms will drop away – then the problem will, in principle, lead back to the solution of elasto-static boundary-value problem and the heat conduction equation. By contrast, the acceleration terms bring a special difficulty into this problem that was examined by Rosenblatt [52]). He succeeded in solving the general problem by means of an examination of the eigenvalues and eigenfunctions of an integral equation with asymmetric kernel.

The formulation above of the boundary-value problems is not physically flawless, insofar as it does not take into account the temperature dependence of the material constants. It is further considerably restricted by the fact that it is valid only for temperature fluctuations τ that are everywhere differentiable and small (infinitely small) of the same order of magnitude as u, v, w. This restriction has the result that the temperature fluctuations behave just like volume forces whose potential is $q\,\tau$, plus outer surface forces of magnitude $q\,\tau$. The elasto-thermal problem may the be treated in the static case by determining the temperature distribution as an elasto-static problem with

[50]) See, these reports, pp. 70.

[51]) Voigt, Kompendium der theoretischer Physik. Leipzig 1895.

[52]) Rosenblatt, Über das allgemeine thermo-elastische Problem, Rendiconti di Palermo 29, a paper that would also remove the formulation of the Voigt Ansatz above.

volume forces [53]). For the elasto-thermal problem, when regarded as restricted by the Ansatz above, the compatibility condition, when extended to the case of volume forces, has validity.

For cylindrical bodies of arbitrary cross-section with a given temperature distribution, and indeed, under the assumption of constant heat flux (hence, the temperature τ is a harmonic function of x and y), Muschelišvili [54]) succeeded in determining heat stresses, on the one hand, by converting the problem into a logarithmic potential problem, and, on the other hand, a Volterra distortion. His results were valid for arbitrary connected cylindrical bodies of infinite length. Briefly, his result is the following:

$$u = \frac{\beta P(x, y)}{2(\lambda + \mu)} + u_1(x, y) \qquad\qquad v = \frac{\beta Q(x, y)}{2(\lambda + \mu)} + v_1(x, y),$$

where β is a material constant, while P and Q are the real (imaginary, resp.) part of the complex integral of any complex function whose real part is the harmonic function τ. However, u_1, v_1 represent the displacement field of a Volterra distortion whose characteristics can be determined by the condition that u and v must be single-valued, continuous functions. For simply-connected regions, the additional terms u_1, v_1 drop out since in this case P, Q are single-valued functions. The solution obtained may be extended to a cylindrical body of finite length with nothing further, and one obviously needs to solve a Saint-Venant rod problem, so the solution above must be augmented.

From this solution, with nothing further, one has the solution for hollow cylindrical tube (chimney), which was already treated by Föppl [55]). Honegger [40]) has communicated good approximate solutions for the solid cylinder. Westergaard [56]) examined the stress distribution in a flat, elastic plate that covers the half-plane, under certain arbitrary assumptions about the temperature distribution – a problem that is meaningful for the dimensioning of iron reinforced pavement (Eisenbetonstrassendecken).

While these special solutions assume, from the outset, a completely determined, temporally unchanging – and thus, a more or less arbitrary temperature distribution – in an elastic body, the Swedish researchers, in particular, Berwald and Hellström [57]), posed the problem for certain special cases that are extremely important technically (in particular, plates and drums [Tonnengewölbe]) and solve the corresponding elasto-thermal problem for actual natural phenomena in a physically flawless way. Starting with the actual temperature variations in the atmosphere and the oceans, which they knew about from their work with the national meteorological institute in Sweden, they established the actual temperature distribution that one should expect in the body from the heat condition equation, and then based the approximate examination of heat stresses

[53]) Encyklopädie der Mathematischen Wissenschaften. Article of Tedone: Allgemeine Theoreme der Mathemischen Elastizitätslehre. Bd. 4/4. pp. 68.

[54]) Muschelivšili, Bulletin de l'Université de Tiflis 3, 1923.

[55]) Föppl, Vorlesungen über technische Mechanik, Bd. V, pp. 240 et seq.

[56]) Westergaard, Om bereguing af plader paa elastik underlag med saerlig henblik paa spörsmaalet om spändinger i Betonveje. Ingenieuren 32, 513, 1923. For other temperature stress issues in plates, see, e.g., Nádai, Elastische Platten, pp. 264 et seq., Berlin 1925.

[57]) F. R. Berwald and B. Hellström, Om temperaturvariationer och temperaturspänningar i betong konstruktioner, Stockholm 1921.

that was carried out on it, with the help of the work equations. The results are developed in such a way that they can easily be applied to arched bridges and iron-reinforced dams (Eisenbeton-Talsperren). The comparison with the temperature stresses arrived at on the basis of the usual assumptions of constant temperature shows not inconsiderable differences such that it would be worthwhile to make the most important results of the Berwald-Hellström work accessible to structural engineering circles.

Another work that likewise produces the temperature phenomena that are to be physically expected corresponding to temperature stresses is the examination of Grünberg [58]) of the temperature stresses in a ball that is suddenly brought into a hot medium. The examination is likewise supported by the basic equations for small temperature differences and based the further assumptions on the heat transition number (Wärmeübergangszahl) being infinitely large, such that in the first moment the outer surface takes on the new outside temperature. Under this assumption, the stresses are independent of the diameter and the heat conduction number. The tangential stress has its maximum value in the initial moment, and indeed, on the outer surface of the body; the greatest radial stress has the opposite sign and occurs in the center of the ball in the moment at which the temperature evolution has its inflection point at this place. The examination is carried out with regard to a physical way of posing the problem that has great significance. Namely, Joffé has shown the great influence that the outer surface condition of crystals exerts on the resistivity. Thus, in order to obtain information about the strength (Festigkeit) when outer surface influences are excluded, Joffé brought in the temperature stresses and thus stimulated the theoretical examination of temperature stresses. The Grünberg solution related to only the isotropic case, such that it offered no reliable foundation for the evaluation of the Joffé investigation of rock salt.

8. Connection between the self-stresses with the higher stresses produced by load singularities and volume forces. Starting with the unit loads, Michell and Love [59]) introduced certain higher load singularities under the name of "typical deformation kernels."

Neményi [60]) has treated the load singularity question from a completely general standpoint and showed that with the help of the influence field concept any internal force magnitude (stress magnitude) can be associated with a dual load singularity that is defined as the limiting form of an equilibrium system of external forces that acts on a narrowly bounded part of the body.

If we subject a simply-connected rod-like body to such a load singularity then it is obvious that except for the purely local stress conditions no other internal forces can arise in the body. On the other hand, if the rod-like body is multiply-connected then there would be a stress state produced in it that is – except for the point of application – identical with the Volterra pressure stresses. The new reciprocity theorem [60]) that is indicated for notion of load extended by load singularity thus concludes, like the Maxwell

[58]) Grünberg, Über die in einer isotropen Kugel durch ungleichförmige Erwärmung erregten Spannungzustände, Zeitschr. f. Physik 35, pp. 548, 1925.

[59]) Love-Timpe, Lehrbuch der Elastizität, pp. 220 and 247. Leipzig and Berlin 1907.

[60]) Neményi, Eine neue Singularitätenmethode für die Elastizitätstheorie. Zeitschr. f. angewandte Mathematik und Mechanik, 9, pp. 488, 1929.

theorem, with the reciprocity of displacements, just as the Volterra theorem concludes with the reciprocity of pressures as special cases of it.

Moreover, the reciprocity property above is also exhibited for the bending of plates and the higher singularities defined for discs. Naturally, the analogy with Volterra distortions applies here. By contrast, it is, however, easy to see that all stress fields can be thought of as being created by Somigliana distortions by the insertion of a line (in the most general case, a surface, resp.) with such load singularities.

From the discussion, it thus emerges that when one extends the notion of load with the help of the higher load singularities all distortion stress states can be thought of as being created by load stress states.

Meanwhile, Reissner [30]), in his cited paper, has, independently of the author, suggested another connection between proper stresses and load stresses, which first relates to the case of continuously distributed dislocations (proper stress sources), from which our theorem above might follow by passing to a limit. The Reissner theorem states that if a self-stress field σ, τ arises from a proper stress source field ${}^0\varepsilon$, ${}^0\gamma$ then σ, τ can be written in the form $\sigma = {}^0\sigma - {}^+\sigma$, $\tau = {}^0\tau - {}^+\tau$, where the stress system with the pre-index 0 is the system that is formally associated with the source field by means of Hooke's law, and the one denoted with the pre-index +, however, can be calculated as a load stress field that results from the volume force distribution:

$$X = -\frac{E\,m}{m+1}\left[\frac{1}{m-1}\frac{\partial\,{}^0\Theta}{\partial x} + \frac{\partial\,{}^0\varepsilon_x}{\partial x} + \frac{1}{2}\frac{\partial\,{}^0\gamma_{xy}}{\partial y} + \frac{1}{2}\frac{\partial\,{}^0\gamma_{xz}}{\partial z}\right]$$

(etc., with cyclic permutation of x, y, z) for unloaded outer surfaces.