"Über Probleme der Spannungskonzentration im Cosserat-Körper," Acta Mech. II (1966), 48-69.

On problems of stress concentration in Cosserat bodies

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With 5 diagrams

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(*Received on 22 July 1965*)

Summary. Using his integration of the linear-elastic Cosserat body [**14**] presented at the Eleventh International Congress of Applied Mathematics, the author solves the stress concentration problem for a circular cylindrical hole and a spherical cavity in a homogeneous stress field. A reduction of the stress concentration appears in comparison with the classical theory of elasticity.

I. Introduction

In a Cosserat continuum [**1**], along with the components of the stress tensor and the displacement vector, the components of the moment stresses and the rotation vector also enter in; consequently, the *rotation vector* must be introduced as a *kinematically independent quantity*, as in the papers of GÜNTHER [**5**] and SCHÄEFER [**9**]. This fact was also considered in a recently-appearing paper of Palmov [**15**] on the basic equations of linear COSSERAT bodies [**14**]. As the author [**14**] showed, all of these quantities may be represented with the help of displacement and rotation functions. For the case of anisotropy, six functions appear, which leads to a characteristic differential equation of twelfth order. For the isotropic case, the elasto-kinetic problem leads to five functions, of which three of them fulfill a common differential equation of fourth order and two of them fulfill a differential equation of second order. Only six functions are required, as a rule. The Ansatz admits the fulfillment of all six outer surface conditions; it includes the two-dimensional theory of SCHÄEFER [**9**] as a special case, as well as the abbreviated theories of GRIOLI [**6**], TRUESDELL and TOUPIN [**7**], [**10**], AERO and KUSHINSKI [**8**], MINDLIN and THIERSTEN [**11**], [**12**], as well as KOITER [**13**], in which the rotation vector – as in classical continuum mechanics – can be generated by the displacement field and can fulfill only five boundary conditions.

For elasto-kinetics, this yields an additional, very high, velocity of propagation [**14**], [**15**]. In elasto-statics, the examination of problems of stress concentration leads to smaller stress magnitudes than in classical elasticity theory, as the author proved in the example of the circular-cylindrical hole and the spherical cavity in a one-axis tension field [**14**]. The solutions of these two boundary-value problems will be presented completely in the current paper.

II. The initial equations.

With the use of the same tensor notation as in the aforementioned paper [**14**], the equilibrium conditions of the COSSERAT bodies in the absence of volume forces may be written as follows:

$$t^{\lambda\mu}|_{\lambda} = 0, \qquad m^{\lambda\mu}|_{\lambda} + \varepsilon^{\mu\lambda\eta}\, t_{\lambda\eta} = 0. \tag{1}$$

According to the law of linear elasticity, the stress components $t^{\lambda\mu}$ and moment-stress components $m^{\lambda\mu}$ that appear in them are coupled with the distortion components $V_{\mu}\,|_{\lambda} + \varepsilon_{\mu\lambda\eta}\, \omega^{\eta}$ (in this, V_{λ} is the displacement vector and ω_{λ} is the kinematically independent rotation vector) and the components of the change in curvature tensor (torsion tensor, resp.) $\omega_{\mu\,|\lambda}$ by the tensor equations:

$$\begin{aligned} t^{\lambda\mu} &= G\left[(1+a)V^{\mu|\lambda} + (1-a)V^{\lambda|\mu} + 2a\varepsilon^{\mu\lambda\eta}\omega_{\eta} + \frac{2\nu}{1-2\nu}\, g^{\lambda\mu}\, V^{\eta}\Big|_{\eta}\right], \\ m^{\lambda\mu} &= 4Gl^2[\omega^{\mu|\lambda} + b\,\omega^{\lambda|\mu} + c\, g^{\lambda\mu}\, \omega^{\eta}\Big|_{\eta}\,. \end{aligned} \tag{2}$$

In addition to the classical elasticity constants G (shear modulus) and ν (POISSON constant), these relations also include the additional (dimensionless) matter constants a, b, c, and l (dimension of length). By substituting (2) in (1), the two vectorial principal equations of linear elastostatics of the isotropic COSSERAT continuum:

$$\begin{aligned} (1+a)\,\Delta V^{\mu} + \left(\frac{1}{1-2\nu} - a\right) V^{\lambda}\Big|_{\lambda}^{\;\mu} + 2a\,\varepsilon^{\mu\lambda\eta}\,\omega_{\eta|\lambda} &= 0, \\ (a - l^2\Delta)\,\omega^{\mu} - (b+c)\,\omega^{\lambda}\Big|_{\lambda}^{\;\mu} + \frac{a}{2}\varepsilon^{\mu\sigma\eta}\, V_{\sigma|\eta} &= 0. \end{aligned} \tag{3}$$

As the author has shown [**14**], the general solution leads to eight displacement (rotations, resp.) functions that fulfill the partial differential equations of second order:

$$\begin{aligned} &\Delta\Phi_0 = 0, \quad \Delta\Phi_k = 0, \\ &\Delta\Psi_k - \frac{1}{l_1^2}\Psi_k = 0, \quad l_1^2 = \left(1 + \frac{1}{a}\right) l^2, \\ &\Delta N - \frac{1}{l_2^2} N = 0, \quad l_2^2 = \frac{1+b+c}{a}\, l^2. \end{aligned} \tag{4}$$

In this, the functions Φ_k and Ψ_k are understood to be the Cartesian vector components (Cartesian coordinates $\bar{x}_k$); only in this way do the associated differential equations include only one of these functions by a transformation to curvilinear coordinates. The Ansatz has the form:

$$
\begin{aligned}
2\,G\,V_\lambda &= -\,F|_\lambda + 4\,(1-n)\,\Phi_\lambda + \Psi_\lambda\,,\\
F &= \Phi_0 + \bar{x}_k\,\Phi^k + l_1^2\,\Psi^\eta\big|_\eta\,,\\
2\,G\,\omega_\lambda &= \frac{1}{2}\varepsilon_{\lambda\eta\sigma}\chi^{\sigma|\eta} + \frac{1}{a}N\big|_\lambda\,,\\
\chi^\lambda &= 4(1-\nu)\,\Phi^\lambda + \left(1+\frac{1}{a}\right)\Psi^\lambda\,,\\
t^{\lambda\mu} &= -\,F|^{\lambda\mu} + 2(1-\nu)\,(\Phi^{\lambda\,|\mu} + \Phi^{\mu\,|\lambda}) + 2\,\nu\,g^{\lambda\mu}\,\Phi^\eta|_\eta + \Psi^{\lambda\,|\mu},\\
m^{\lambda\mu} &= l^2\left[\varepsilon^{\mu\eta\sigma}\,\chi_\sigma\big|_\eta^{\ \lambda} + b\,\varepsilon^{\lambda\eta\sigma}\,\chi_\sigma\big|_\eta^{\ \mu} + \frac{2}{a}(1+b)N^{|\lambda\mu}\right] + \frac{2c}{1+b+c}\,g^{\lambda\mu}N\,.
\end{aligned}
\tag{5}
$$

For the problems treated here, the transformation to cylindrical and spherical coordinates is required. Thus, the Cartesian coordinates will be denoted by $\bar{x}^1 = x$, $\bar{x}^2 = y$, $\bar{x}^3 = z$, and the curvilinear ones by $x^1 = u$, $x^2 = v$, $x^3 = w$.

III. Transition to cylindrical coordinates.

Let the cylindrical coordinates be introduced by means of the relations (Fig. I):

$$x = u\cos v, \qquad y = u\sin v, \qquad z = w. \tag{6}$$

The non-zero tensor components and CHRISTOFFEL symbols of the second kind are:

$$g_{11} = 1, \qquad g_{22} = u^2, \qquad g_{11} = 1, \qquad \Gamma^1_{22} = -\,u, \qquad \Gamma^2_{22} = \frac{1}{u}. \tag{7}$$

In the equations that follow, all vector and tensor components with the indices u, v, w are understood to mean *physical components*. Indices after a comma mean *partial differentiation*, and no longer covariant differentiation. The equilibrium conditions are:

$$
\begin{aligned}
(\sigma_u\,u)_{,\,u} - \sigma_v + \tau_{vu,v} + u\,\tau_{wu,w} &= 0,\\
\sigma_{v,\,v} + (\sigma_{uv}\,u)_{,\,u} + \tau_{vu} + u\,\tau_{wu,w} &= 0,\\
u\,\sigma_{w,\,w} + (\sigma_{uv}\,u)_{,\,u} + \tau_{vw,v} &= 0,\\
(m_{uu}\,u)_{,\,u} - m_{vv} + m_{vu,v,} + u\,(m_{wu,w} + \tau_{vw} - \tau_{wv}) &= 0,\\
m_{vv,\,v} - (m_{uv}\,u)_{,\,u} + m_{vu,} + u\,(m_{wv,w} + \tau_{wu} - \tau_{uw}) &= 0,\\
(m_{uw}\,u)_{,\,u} + m_{vw,\,v,} + u\,(m_{ww,w} + \tau_{uv} - \tau_{vu}) &= 0.
\end{aligned}
\tag{8}
$$

For the Δ operator, the physical components of the functions Φ_k, Ψ_k of the displacement and rotation vectors, as well as the stress and moment-stress tensors, one has:

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{1}{u}\frac{\partial}{\partial u} + \frac{1}{u^2}\frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial w^2},$$

$$\Phi_u = \Phi_x \cos v + \Phi_y \sin v, \qquad \Phi_v = -\Phi_x \sin v + \Phi_x \cos v, \qquad \Phi_w = \Phi_z,$$
$$\Psi_u = \Psi_x \cos v + \Psi_y \sin v, \qquad \Psi_v = -\Psi_x \sin v + \Psi_x \cos v, \qquad \Psi_w = \Psi_z,$$

$$\chi_u = 4(1-\nu)\,\Phi_u + \left(1 + \frac{1}{a}\right)\Psi_u, \qquad \chi_v = \ldots, \qquad \chi_w = \ldots,$$

$$S = \Phi_0 + \frac{1}{u}(\Phi_u + \Phi_{v,v}) + \Phi_{w,w},$$

$$F = \Phi_0 + u\,\Phi_u + w\,\Phi_w + l_1^2\,[\Psi_{u,u} + \frac{1}{u}(\Psi_u + \Psi_{v,v}) + \Phi_{w,w}],$$

$$2\,G\,V_u = -F_{,u} + 4(1-\nu)\,\Phi_u + \Psi_u,$$
$$2\,G\,V_v = -\frac{1}{u}\,F_{,v} + 4(1-\nu)\,\Phi_v + \Psi_v,$$
$$2\,G\,V_w = -F_{,w} + 4(1-\nu)\,\Phi_w + \Psi_w,$$

$$2\,G\,\omega_u = \frac{1}{2}\left[\frac{1}{u}\chi_{w,v} - \chi_{v,w}\right] + \frac{1}{a}N_{,u},$$
$$2\,G\,\omega_v = \frac{1}{2}\left[\chi_{w,v} - \chi_{v,w}\right] + \frac{1}{a}N_{,u},$$
$$2\,G\,\omega_w = \frac{1}{2}\left[\chi_{v,w} + \frac{1}{u}(\chi_v - \chi_{u,v})\right] + \frac{1}{a}N_{,w},$$

$$\sigma_u = -F_{,uu} + 4(1-\nu)\,\Phi_{u,u} + \Psi_{u,u} + 2\nu S,$$
$$\sigma_v = -\frac{1}{u^2}F_{,vv} - \frac{1}{u}F_{u} + 4(1-\nu)\frac{1}{u}(\Phi_{v,v} + \Phi_u) + \frac{1}{u}(\Psi_{v,v} + \Psi_u) + 2\nu S,$$
$$\sigma_w = -F_{,ww} + 4(1-\nu)\,\Phi_{w,w} + \Psi_{w,w} + 2\nu S, \tag{9}$$

$$\tau_{uv} = -\left(\frac{1}{u}F_{,v}\right)_{,u} + 2(1-\nu)\left[\frac{1}{u}(\Phi_{u,v} - \Phi_v) + \Phi_{v,u}\right] + \frac{1}{u}[\Psi_{u,v} - \Psi_v] - N_{,w},$$
$$\tau_{vu} = \ldots + \Psi_{v,u} + N_{,w},$$
$$\tau_{vw} = -\frac{1}{u}F_{,vw} + 2(1-\nu)\left[\Phi_{v,w} + \frac{1}{u}\Phi_{w,v}\right] + \Psi_{v,w} - N_{,u},$$
$$\tau_{wv} = \ldots + \frac{1}{u}\Psi_{w,v} + N_{,u},$$
$$\tau_{wu} = -F_{,wv} + 2(1-\nu)\,[\Phi_{w,u} + \Phi_{u,w}] + \Psi_{w,u} - N_{,v},$$

$$\tau_{uw} = \ldots + \Psi_{u,w} + \frac{1}{u} N_{,v} ,$$

$$m_{uu} = 4G\, l^2 (1+b)\, \omega_{u,u} + \frac{2cN}{1+b+c},$$

$$m_{vv} = 4G\, l^2 (1+b)\, \frac{1}{u} (\omega_{v,v} + \omega_u) + \frac{2cN}{1+b+c},$$

$$m_{ww} = 4G\, l^2 (1+b)\, \omega_{w,w} + \frac{2cN}{1+b+c},$$

$$m_{uv} = 4Gl^2 \left[\omega_{v,u} + \frac{b}{u}(\omega_{u,v} - \omega_v) \right],$$

$$m_{vu} = 4Gl^2 \left[\frac{1}{u}(\omega_{u,v} - \omega_v) + b\omega_{v,u} \right],$$

$$m_{vw} = 4Gl^2 \left[\frac{1}{u}\omega_{w,v} + b\omega_{v,w} \right], \qquad m_{wv} = 4Gl^2 \left[\omega_{v,w} + \frac{b}{u}\omega_{w,v} \right],$$

$$m_{wu} = 4Gl^2 \left[\omega_{u,w} + b\omega_{w,u} \right], \qquad m_{uw} = 4Gl^2 \left[\omega_{w,u} + b\omega_{u,w} \right].$$

IV. Transition to spherical coordinates

Let spherical coordinates (Fig. 4) be established by the relations:

$$x = u\cos v, \quad y = u\sin v\cos w, \quad z = u\sin v\sin w. \tag{10}$$

The non-zero metric tensor components and CHRISTOFFEL symbols of the second type are:

$$\begin{aligned} &g_{11} = 1, && g_{22} = u^2, && g_{33} = u^2\sin^2 v, \\ &\Gamma^1_{22} = -u, && \Gamma^2_{12} = \frac{1}{u}, && \Gamma^1_{33} = -\sin^2 v, \\ &\Gamma^3_{13} = \frac{1}{u}, && \Gamma^2_{33} = -\sin v\cos v, && \Gamma^3_{23} = \cot v. \end{aligned} \tag{11}$$

The equilibrium conditions are:

$$\begin{aligned} &\frac{1}{u}(\sigma_u u^2)_{,u} + \frac{1}{\sin v}[(\tau_{vu}\sin v)_{,v} + \tau_{wu,w}] - \sigma_v - \sigma_w = 0, \\ &\frac{1}{\sin v}[(\sigma_v \sin v)_{,v} + \tau_{wv,w}] + \frac{1}{u}(\tau_{uv}u^2)_{,u} + \tau_{vu} - \sigma_w \cot v = 0, \\ &\frac{1}{\sin v}[\sigma_{w,w} + (\tau_{vw}\sin v)_{,v}] + \frac{1}{u}(\tau_{uw}u^2)_{,u} + \tau_{vu} - \sigma_w \cot v = 0, \end{aligned} \tag{12}$$

$$\frac{1}{u}(m_{uu}u^2)_{,u} + \frac{1}{\sin v}[(m_{vu}\sin v)_{,v} + m_{wu,w}] - m_{vv} - m_{ww} + u\,(\tau_{vw} - \tau_{wv}) = 0, \qquad .$$

$$\frac{1}{\sin v}[(m_{vv}\sin v)_{,v} + m_{wv,w}] + \frac{1}{u}(m_{uv}u^2)_{,u} + m_{vu} - m_{ww}\cot v + u\,(\tau_{wu} - \tau_{uw}) = 0, \qquad .$$

$$\frac{1}{\sin v}[m_{ww,w} + (m_{vw}\sin v)_{,v}] + \frac{1}{u}(m_{uw}u^2)_{,u} + m_{wu} - m_{wv}\cot v + u\,(\tau_{uv} - \tau_{vu}) = 0. \qquad .$$

It further follows that:

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{2}{u}\frac{\partial}{\partial u} + \frac{1}{u^2}\frac{\partial^2}{\partial v^2} + \frac{\cot v}{u^2}\frac{\partial}{\partial v} + \frac{1}{u^2\sin^2 v}\frac{\partial^2}{\partial w^2},$$

$$\Phi_u = \Phi_x\cos v + \Phi_y\sin v\cos w + \Phi_z\sin v\sin w,$$
$$\Phi_v = -\Phi_x\sin v + \Phi_y\cos v\cos w + \Phi_z\cos v\sin w,$$
$$\Phi_w = -\Phi_x\sin w + \Phi_z\cos w,$$

$$\Psi_u = \Psi_x\cos v + \ldots,\ \Psi_v = \ldots,\ \Psi_w = \ldots,$$

$$\chi_u = 4(1-\nu)\,\Phi_u + \left(1+\frac{1}{a}\right)\Psi_u, \qquad \chi_v = \ldots, \qquad \chi_w = \ldots,$$

$$S = \Phi_{u,u} + \frac{1}{u}\left(2\Phi_u + \Phi_{v,u} + \Phi_v\cot v + \frac{1}{\sin v}\Phi_{w,w}\right),$$

$$F = \Phi_0 + u\,\Phi_u + l_1^2\left[\Psi_{u,u} + \frac{1}{u}\left(2\Psi_u + \Psi_{v,v} + \Psi_v\cot v + \frac{1}{\sin v}\Psi_{w,w}\right)\right],$$

$$2G\,V_u = -F_{,u} + 4(1-\nu)\,\Phi_u + \Psi_u,$$
$$2G\,V_v = -\frac{1}{u}F_{,v} + 4(1-\nu)\,\Phi_v + \Psi_v,$$
$$2G\,V_w = -\frac{1}{u\sin v}F_{,w} + 4(1-\nu)\,\Phi_w + \Psi_w,$$

$$2\,G\,\omega_u = \frac{1}{2u}\left(\chi_{w,v} - \frac{1}{\sin v}\chi_{v,w} + \cot v\,\chi_w\right) + \frac{1}{a}N_{,u},$$
$$2\,G\,\omega_v = \frac{1}{2u}\left(\frac{1}{\sin v}\chi_{u,w} - u\chi_{w,u} - \chi_w\right) + \frac{1}{au}N_{,v},$$
$$2\,G\,\omega_w = \frac{1}{2u}\left(u\chi_{w,v} + \chi_v - \chi_{u,v}\right) + \frac{1}{a\,u\,\sin v}N_{,w},$$

(13)

$$
\begin{aligned}
\sigma_u &= -F_{,uu} + 4(1-\nu)\,\Phi_{u,u} + \Psi_{u,u} + 2\nu S,\\
\sigma_v &= -\frac{1}{u^2}F_{,uu} - \frac{1}{u}F_{,u} + 4(1-\nu)\,\frac{1}{u}(\Phi_{v,v} + \Phi_u) + \frac{1}{u}(\Psi_{v,v} + \Psi_u) + 2\nu S,\\
\sigma_w &= -\frac{1}{u^2\sin^2 v}F_{,ww} - \frac{1}{u}F_{,u} - \frac{\cot v}{u^2}F_{,v} + 4(1-\nu)\frac{1}{u}\left(\Phi_u + \Phi_v\cot v + \frac{1}{\sin v}\Phi_{w,w}\right)\\
&\quad + \frac{1}{u}\left(\Psi_u + \Psi_v\cot v + \frac{1}{\sin v}\Psi_{w,w}\right) + 2\nu S,
\end{aligned}
$$

$$
\tau_{uv} = -\left(\frac{1}{u}F_{,v}\right)_{,u} + 2(1-\nu)\frac{1}{u}(\Phi_{u,v} + u\Phi_{v,u} - \Phi_v) + \frac{1}{u}(\Psi_{u,v} - \Psi_v) - \frac{1}{u\sin v}N_{,w},
$$

$$
\tau_{vu} = \ldots + \Psi_{v,u} + \frac{1}{u\sin v}N_{,w},
$$

$$
\tau_{vw} =
$$

$$
-\frac{1}{u}\left(\frac{1}{\sin v}F_{,w}\right)_{,v} + \frac{2(1-\nu)}{u}\left(\frac{1}{\sin v}\Phi_{v,w} + \Phi_{w,v} - \Phi_w\cot v\right) + \frac{1}{u}\left(\frac{1}{\sin v}\Psi_{u,v} - \Psi_w\cot v\right) - N_{,u},
$$

$$
\begin{aligned}
\tau_{wv} &= \ldots + \frac{1}{u}\Psi_{w,v} + N_{,u},\\
\tau_{wu} &= -\frac{1}{\sin v}\left(\frac{1}{u}F_{,w}\right)_{,u} + \frac{2(1-\nu)}{u}\left(u\Phi_{w,u} + \frac{1}{\sin v}\Phi_{u,w} - \Phi_w\right) + \Psi_{w,u} - \frac{1}{u}N_{,v},\\
\tau_{uw} &= \ldots + \frac{1}{u}\left(\frac{1}{\sin v}\Psi_{u,w} - \Psi_w\right) + \frac{1}{u}N_{,v},
\end{aligned}
$$

$$
\begin{aligned}
m_{uw} &= 4\,G\,l^2\,(1+b)\;\omega_{u,u} + \frac{2cN}{1+b+c},\\
m_{vv} &= 4\,G\,l^2\,\frac{1+b}{u}(\omega_{v,v} + \omega_u) + \frac{2cN}{1+b+c},\\
m_{ww} &= 4\,G\,l^2\,\frac{1+b}{u}\left(\frac{1}{\sin v}\omega_{w,w} + \omega_u + \omega_v\cot v\right) + \frac{2cN}{1+b+c},\\
m_{uv} &= 4\,G\,l^2\left[\omega_{v,u} + \frac{b}{u}(\omega_{u,v} - \omega_v)\right],\\
m_{vu} &= 4\,G\,l^2\left[\frac{1}{u}(\omega_{v,u} - \omega_v) + b\,\omega_{v,u}\right],\\
m_{vw} &= \frac{4Gl^2}{u}\left[\omega_{w,v} + b\left(\frac{1}{\sin v}\omega_{v,w} - \omega_w\cot v\right)\right],\\
m_{wv} &= \frac{4Gl^2}{u}\left[\frac{1}{\sin v}\omega_{v,w} - \omega_w\cot v + b\,\omega_{w,v}\right],
\end{aligned}
$$

$$m_{wu} = 4Gl^2\left[\frac{1}{u}\left(\frac{1}{\sin v}\omega_{u,w}-\omega_w\right)+b\,\omega_{w,u}\right]$$
$$m_{uw} = 4Gl^2\left[\omega_{w,u}+\frac{b}{u}\left(\frac{1}{\sin v}\omega_{u,w}-\omega_w\right)\right].$$

V. Deformation energy and equivalent stress

Knowledge of the deformation energy is required for the assessment of material stress in a COSSERAT body. In the cited paper of the author [**14**], the total energy of the linearly deformed COSSERAT body was computed as a function of the deformation magnitudes. If the deformation quantities are expressed in terms of the stresses then one obtains from (2) in tensor notation:

$$V^{\mu\,|\lambda} + \varepsilon^{\mu\lambda\eta}\, w_\eta = \frac{1}{4G}\left[\left(1+\frac{1}{a}\right)t^{\lambda\mu}+\left(1-\frac{1}{a}\right)t^{\mu\lambda}-\frac{2\nu}{1+\nu}g^{\lambda\mu}t_\eta^{\ \eta}\right], \tag{14}$$

$$\omega^{\mu\,|\lambda} = \frac{1}{4Gl^2(1-b^2)}\left[m^{\lambda\mu}-b\,m^{\mu\lambda}-\frac{c(1-b)}{1+b+3c}g^{\lambda\mu}m_\eta^{\ \eta}\right]. \tag{15}$$

This yields the total deformation energy, expressed in terms of the stresses:

$$\begin{aligned} W &= \tfrac{1}{2}[t^{\lambda\mu}\,(V_{\mu|\lambda} + \varepsilon_{\mu\lambda\eta}\, w^\eta) + m^{\lambda\mu}\, w_{\mu|\lambda}] \\ &= \frac{1}{8G}\left\{\left(1+\frac{1}{a}\right)t^{\lambda\mu}t_{\lambda\mu}+\left(1-\frac{1}{a}\right)t^{\lambda\mu}t_{\mu\lambda}-\frac{2\nu}{1+\nu}t_\lambda^{\ \lambda}t_\mu^{\ \mu}\right. \\ &\qquad \left.+\frac{1}{l^2(1-b^2)}\left[m^{\lambda\mu}m_{\lambda\mu}-b\,m^{\lambda\mu}m_{\mu\lambda}-\frac{c(1-b)}{1+b+3c}m_\lambda^{\ \lambda}m_\mu^{\ \mu}\right]\right\}. \end{aligned} \tag{16}$$

After subtracting the pure volume change energy $\frac{1-2\nu}{12G(1+\nu)}t_\lambda^{\ \lambda}t_\mu^{\ \mu}$, this expression goes to the *form change energy of the* COSSERAT *body;* if it were identified with the corresponding amount of energy for a one-axis tensile load (*equivalent stress* σ^*) then it would follow that:

$$\begin{aligned} \sigma^{*2} = \frac{3}{4}&\left\{\left(1+\frac{1}{a}\right)t^{\lambda\mu}t_{\lambda\mu}+\left(1-\frac{1}{a}\right)t^{\lambda\mu}t_{\mu\lambda}-\frac{2}{3}t_\lambda^{\ \lambda}t_\mu^{\ \mu}+\right. \\ &\left.\frac{1}{(1-b^2)l^2}\left[m^{\lambda\mu}m_{\lambda\mu}-bm^{\lambda\mu}m_{\mu\lambda}-\frac{c(1-b)}{1+b+3c}m_\lambda^{\ \lambda}m_\mu^{\ \mu}\right]\right\}. \end{aligned} \tag{17}$$

For the problems that are treated here, only the physical components with the indices v and w will enter into the highly stressed places, such that equivalent stress has the following form:

$$\sigma^{*2} = \sqrt{\sigma_v^2 - \sigma_v\sigma_w + \sigma_w^2 + \frac{3}{4}(\tau_{vw} + \tau_{wv}) + \frac{3}{4a}(\tau_{vw} - \tau_{wv})}$$

$$\overline{+\frac{1}{2(1+b)l^2}[m_{vv}^2 - m_{vv}m_{ww} + m_{ww}^2 + \frac{3}{4}(m_{vw} + m_{ww})^2} \tag{18}$$

$$\overline{+\frac{3(1+b)}{4(1-b)}(m_{vw} - m_{wv})^2 + \frac{1+b}{2(1+b+3c)}(m_{vw} + m_{wv})^2]} \;.$$

VI. The cylindrical hole in the tension field of a planar state of distortion

The solution for this problem may be constructed when one adds other functions to the functions that appear within classical elasticity theory [**4**] that correspond to the schema that was presented in the third section and are suitable to fulfill the conditions of the free hole boundary and vanish for large values of *u*. The conditions for the free outer surface of the hole are (Fig. 1):

For $u = R$:

$$\begin{Bmatrix} \sigma_u = 0, & \tau_{uv} = 0, & \tau_{uw} = 0, \\ m_{uu} = 0, & m_{uv} = 0, & m_{uw} = 0. \end{Bmatrix} \tag{19}$$

The process of computation will be lightened essentially if one represents each additional function as the product of a function of *u* and a function of *v*, and indeed with the same dependency on *v* as for the corresponding functions of classical elasticity theory. Next, let us examine the state of distortion; it is obtained from the following Ansatz:

$$\begin{aligned} \Phi_0 &= \frac{p}{4}\left\{2A\ln u + \left[-2vu^2 + \frac{B}{u^2}\right]\cos(2v)\right\}, \\ \Phi_x &= \frac{p}{4}\left\{u + \frac{C}{u}\right\}\cos v\,, \quad \Phi_y = \Phi_z = 0, \\ \Psi_x &= p\,D\,K_1\frac{u}{l_1}\cos v\,, \qquad \Psi_y = \Psi_z = 0. \end{aligned} \tag{20}$$

In this, *A*, *B*, *C*, and *D* are constants; K_1 and the function K_0 that is employed in the later calculations are cylinder functions with imaginary arguments; they fulfill the differential equations:

$$\frac{d^2K_0}{du^2} + \frac{1}{u}\frac{dK_0}{du} - \frac{K_0}{l_1^2} = 0, \qquad \frac{d^2K_1}{du^2} + \frac{1}{u}\frac{dK_1}{du} - \left(\frac{1}{u^2} + \frac{1}{l_1^2}\right)K_1 = 0,$$

$$K_1 = -\, l_1 \, \frac{dK_0}{du}, \qquad \frac{dK_1}{du} = -\,\frac{K_0}{l_1} - \frac{K_1}{u}. \tag{21}$$

With the help of the schema (9), it follows that:

$$\Phi_u = \frac{p}{4}\left\{u + \frac{C}{u}\right\}[1+\cos(2v)], \qquad \Phi_u = \frac{p}{4}\left\{-u - \frac{C}{u}\right\}\sin(2v), \qquad \Phi_u = 0,$$

$$\Psi_u = \frac{p}{2} DK_1[1+\cos(2v)], \qquad \Psi_v = \frac{p}{2} DK_1 \sin(2v), \quad \Psi_w = 0,$$

$$\chi_u = \frac{p}{2}\left\{2(1-\nu)\left(u + \frac{C}{u}\right) + D\left(1 + \frac{1}{a}\right)K_1\right\}[1+\cos(2v)],$$

$$\chi_v = \frac{p}{2}\left\{-2(1-\nu)\left(u + \frac{C}{u}\right) - D\left(1 + \frac{1}{a}\right)K_1\right\}\sin(2v), \quad \chi_v = 0,$$

$$F = \frac{p}{4}\{u^2 + 2A\ln u + C - 2Dl_1K_0$$
$$+ \left[(1-2\nu)u^2 + \frac{B}{u^2} + C - 2Dl_1\left(K_0 + 2K_1\frac{l_1}{u}\right)\right]\cos(2v)\Big\},$$

$$2GV_u = \frac{p}{4}\Big\{(2-4\nu)u - \frac{2A}{u} + 4(1-\nu)\frac{C}{u}$$
$$+ \left[2u + \frac{2B}{u^3} + 4(1-\nu)\frac{C}{u} + D\left(-4K_0\frac{l_1}{u} - 8K_1\frac{l_1^2}{u^2}\right)\right]\cos(2v)\Big\},$$

$$2GV_v = \frac{p}{4}\left\{-2u + \frac{2B}{u^3} + (-2+4\nu)\frac{C}{u} + D\left(-4K_0\frac{l_1}{u} - 8K_1\frac{l_1^2}{u^2} - 2K_1\right)\right\}\sin(2v),$$

$$V_w = 0, \quad \omega_u = 0, \qquad \omega_v = 0,$$

$$2G\omega_w = \frac{p}{4}\left\{4(1-\nu)\frac{C}{u^2} + D\left(1+\frac{1}{a}\right)\left(\frac{K_0}{l_1} + 2\frac{K_1}{u}\right)\right\}\sin(2v), \tag{22}$$

$$\sigma_u = \frac{p}{2}\Big\{1 + \frac{A}{u^2} - 2(1-\nu)\frac{C}{u^2}$$
$$+ \left[1 - 3\frac{B}{u^4} - 2\frac{C}{u^2} + D\left(6K_0\frac{l_1}{u^2} + 12K_1\frac{l_1^2}{u^3} + 2\frac{K_1}{u}\right)\right]\cos(2v)\Big\},$$

$$\sigma_v = \frac{p}{2}\Big\{1 - \frac{A}{u^2} + 2(1-\nu)\frac{C}{u^2}$$

$$+\left[-1+3\frac{B}{u^4}+D\left(-6K_0\frac{l_1}{u^2}-12K_1\frac{l_1^2}{u^3}-2\frac{K_1}{u}\right)\right]\cos(2v)\Bigg\},$$

$$\sigma_w = p\left\{v-v\frac{C}{u^2}\cos(2v)\right\},$$

$$\tau_{uv} = \frac{p}{2}\left\{-1-\frac{3B}{u^4}-\frac{C}{u^2}+D\left(6K_0\frac{l_1}{u^2}+12K_1\frac{l_1^2}{u^3}+\frac{K_1}{u}\right)\right\}\sin(2v),$$

$$\tau_{vu} = \frac{p}{2}\left\{-1-\frac{3B}{u^4}-\frac{C}{u^2}+D\left(6K_0\frac{l_1}{u^2}+12K_1\frac{l_1^2}{u^3}+\frac{3K_1}{u}+\frac{K_0}{l_1}\right)\right\}\sin(2v),$$

$$\tau_{vu} = \tau_{wv} = \tau_{wu} = \tau_{uw} = 0,$$

$$m_{uw} = \frac{p}{2}l^2\left\{-8(1-v)\frac{C}{u^3}+D\left(1+\frac{1}{a}\right)\left(-\frac{2K_0}{l_1u}-\frac{4K_1}{u^2}-\frac{K_1}{l_1^2}\right)\right\}\sin(2v),$$

$$m_{vw} = \frac{p}{2}l^2\left\{8(1-v)\frac{C}{u^3}+D\left(1+\frac{1}{a}\right)\left(\frac{2K_0}{l_1u}+\frac{4K_1}{u^2}\right)\right\}\cos(2v),$$

$$m_{wu} = b\, m_{uw}, \quad m_{wv} = b\, m_{vw},$$

$$m_{uu} = m_{uv} = m_{vu} = m_{vv} = m_{ww} = 0.$$

With the abbreviation $R / l_1 = b$, the boundary conditions (19) lead to the following four linear equations:

$$\begin{aligned}
&\frac{A}{R^2}-2(1-v)\frac{C}{R^2} = -1,\\
&-\frac{3B}{R^4}-2\frac{C}{R^2}+\frac{D}{R}\left(\frac{6K_0}{\beta}+\frac{12K_1}{\beta^2}+2K_1\right) = -1,\\
&-\frac{3B}{R^4}-\frac{C}{R^2}+\frac{D}{R}\left(\frac{6K_0}{\beta}+\frac{12K_1}{\beta^2}+K_1\right) = 1,\\
&-8(1-v)\frac{C}{R^2}+\frac{D}{R}\left(1+\frac{1}{a}\right)\left(-2K_0\beta-4K_1-K_1\beta^2\right) = 0.
\end{aligned} \tag{23}$$

With the further abbreviations:

$$K_0(\beta) / K_1(\beta) = \kappa; \qquad \frac{8(1-v)}{8(1-v)+(1+1/a)(\beta^2+2\kappa\beta+4)} = \lambda, \tag{24}$$

it follows that:

$$\frac{A}{R^2} = 3 - 4\nu - 4(1-\nu)\,\lambda, \qquad \frac{B}{R^4} = -1 - 4\lambda\left(\frac{2}{\beta^2} + \frac{\kappa}{\beta}\right), \tag{25}$$
$$\frac{C}{R^2} = 2 - 2\lambda, \qquad \frac{D}{R^4} K_1(\beta) = -2\lambda\,.$$

The following stresses act at the location $u = R$, $v = 0$:

$$\sigma_v = p(-1 + 2\lambda), \qquad \sigma_w = \nu\sigma_v, \qquad m_{vw} = p\,R\lambda, \quad m_{vw} = b\,m_{vw}\,. \tag{26}$$

The *maximal stress* is found at the location $u = R$, $v = p/2$ on (Fig. 1); the following stresses act there:

$$\begin{aligned} \sigma_v &= p(3-2\lambda) = \sigma_{\max}, \quad \sigma_w = \nu\sigma_v, \\ m_{vw} &= -p\,R\,\lambda \quad = -m_{\max}, \; m_{wv} = b\,m_{vw}. \end{aligned} \tag{27}$$

The passage to the limit $a \to \infty$ leads to the abridged theory (the rotation vector will depend upon the displacement; cf., sec. I), and confirms the results of MINDLIN and THIERSTEN [**11**], [**12**].

For the *equivalent stress* at the location $u = R$, $v = \pi/2$, with the use of the relation (18), one obtains:

$$\sigma^* = p\,\sqrt{(1-\nu-\nu^2)(3-2\lambda)^2 + 3\lambda^2 R^2/(4l^2)}\,. \tag{28}$$

In the *limiting case* $l \ll R$ (the structural length l is small compared to the hole radius) will be:

$$\lambda \approx 8\,(1-\nu)\,l^2/R^2, \qquad \sigma^* \approx 3p\,\sqrt{1-\nu+\nu^2}\left[1 - \frac{8(1-\nu)(1-\nu+2\nu^2)l^2}{3(1-\nu-\nu^2)R^2}\right]. \tag{29}$$

In the *limiting case* $R \ll l$ (the radius of the hole is small compared to the length l), one has:

$$\lambda \approx \frac{2(1-\nu)}{3-2\nu+1/a}, \qquad \sigma^* \approx p\,\frac{5-2\nu+3/a}{3-2\nu+1/a}\sqrt{1-\nu+\nu^2}\,. \tag{30}$$

Formula (29) shows a *decrease in the equivalent stress* for increasing values of the parameter l/R. The structural length l is obviously connected with an *additional support reaction* for the COSSERAT body that leads to a reduction in the stress concentration.

VIII. The circular cylindrical hole in the stress field of a planar stress state

If one is dealing with a pure shear stress state in the distant neighborhood of a circular-cylindrical hole then there are two cases to distinguish according to whether the indices of the plane defined by the shear stress (*shear plane*) are oriented perpendicular or parallel to the axis of the hole. If the shear plane is perpendicular to the hole then the desired shear state can be obtained directly from the sections VI and VII when one overlays a second stress state with $v + \pi/2$, instead of v and $p_2 = -p$, instead of p (cf., Fig. 2). Constant symmetric shear stresses $\tau = p$ then act in the distant neighborhood of the hole at 45° to the x and y axes (corresponding to an elementary rule that arises from, e.g., the MOHR stress circle).

The maximal stress appears at the locations $u = R$, $v = \pi/2$ and $u = R$, $v = 0$:

$$\sigma_v = \pm 4(1-\lambda)\, p, \qquad \sigma_v = \nu\sigma_v \text{ (planar distortion)} \quad 0, \text{ resp. (planar stress)},$$
$$m_{vw} = \pm 2\, p\, R\, \lambda = \pm m_{\max}, \qquad m_{wv} = b\, m_{vw}. \tag{34}$$

The *equivalent stress* will be:

$$\sigma^* = p\,\sqrt{16(1-\nu-\nu^2)(1-\lambda^2)+3\lambda^2R^2/l^2} \qquad \text{(planar distortion)},$$

resp.:

$$\sigma^* = p\,\sqrt{16(1-\bar{\lambda}^2)+3\bar{\lambda}^2R^2/[l^2(1-b^2)]} \qquad \text{(planar stress)}. \tag{35}$$

In the *limiting case* $l \ll R$ one has:

$$\sigma^* = p\,\sqrt{1-\nu-\nu^2}\left[4-\frac{8(1-\nu)(1-\nu+4\nu^2)}{1-\nu+\nu^2}\left(\frac{l}{R}\right)^2\right] \qquad \text{(planar distortion)},$$

resp.:

$$\sigma^* = p\left[4-\frac{8}{1+\nu^2}\left(1+4\nu-\frac{3b^2}{1-b^2}\right)\left(\frac{l}{R}\right)^2\right] \qquad \text{(planar stress)}. \tag{36}$$

In the limiting case $R \ll l$ one has:

$$\sigma^* = p\,\frac{4\left(1+\frac{1}{a}\right)}{3-2\nu+\frac{1}{a}}\sqrt{1-\nu+\nu^2} \qquad \text{(planar distortion)},$$

resp.:

$$\sigma^* = p\,\frac{4(1+\nu)\left(1+\frac{1}{a}\right)}{3+\nu+(1+\nu)/a} \qquad \text{(planar stress)}. \tag{37}$$

IX. The circular-cylindrical hole in a stress field parallel to the axis of the hole

If the shear plane is parallel to the axis of the hole and if the stress state that arises in the distant neighborhood of the hole is represented by, e.g., the shear stress $\tau_{xz} = \tau_{zx} = p$

(Fig. 3) then the associated solution must be presented anew. The immediate vicinity leads to the following Ansatz:

$$\Phi_z = -\frac{p}{2(1-\nu)}\left[u+\frac{A}{u}\right]\cos v, \qquad \Phi_0 = -z\,\Phi_z, \qquad \Phi_x = \Phi_y = 0,$$

$$\Psi_x = \Psi_y = 0, \quad \Psi_z = 2\,pBK_1\left(\frac{u}{l_1}\right)\cos v, \tag{38}$$

$$N = p\,C\,K_1\left(\frac{u}{l_2}\right)\sin v.$$

In this, A, B, and C are constants; K_1 and K_0 are cylinder functions with imaginary arguments that satisfy the differential equations (21). To abbreviate, let us set:

$$K_1\left(\frac{u}{l_1}\right) = K_{1(1)}, \qquad K_1\left(\frac{u}{l_2}\right) = K_{1(2)}, \qquad K_0\left(\frac{u}{l_1}\right) = K_{0(1)}, \qquad K_0\left(\frac{u}{l_2}\right) = K_{0(2)}. \tag{39}$$

With the help of the schema (9), it then follows that:

$$\Phi_u = \Phi_v = 0, \quad \Phi_w = \Phi_z, \qquad \Psi_u = \Psi_v = 0, \qquad \Psi_w = \Psi_z,$$

$$\chi_u = \chi_v = 0, \quad \chi_w = 2p\left\{u+\frac{A}{u}+B\left(1+\frac{1}{a}\right)K_{1(1)}\right\}\cos v,$$

$$F = 0, \qquad V_u = V_v = 0, \quad \omega_w = 0,$$

$$2\,GV_w = p\left\{2u+\frac{2A}{u}+2BK_{1(1)}\right\}\cos v,$$

$$2\,G\omega_u = p\left\{-1-\frac{A}{u^2}-B\left(1+\frac{1}{a}\right)\frac{1}{u}K_{1(1)}+\frac{C}{au}\left[-K_{1(2)}-\frac{u}{l_2}K_{0(2)}\right]\right\}\sin v,$$

$$2\,G\omega_v = p\left\{-1+\frac{A}{u^2}+B\left(1+\frac{1}{a}\right)\frac{1}{u}\left[K_{1(1)}+\frac{u}{l_2}K_{0(1)}\right]+\frac{C}{au}K_{1(2)}\right\}\cos v,$$

$$\sigma_u = \tau_{uv} = \tau_{vu} = \sigma_v = \sigma_w = 0,$$

$$\tau_{vw} = p\left\{-1-\frac{A}{u^2}+\frac{C}{u}\left[K_{1(2)}+\frac{u}{l_2}K_{0(2)}\right]\right\}\sin v,$$

$$
\begin{aligned}
\tau_{wv} &= p\left\{-1-\frac{A}{u^2}-\frac{2B}{u}K_{1(1)}+\frac{C}{u}\left[-K_{1(2)}-\frac{u}{l_2}K_{0(2)}\right]\right\}\sin v,\\
\tau_{wu} &= p\left\{1-\frac{A}{u^2}+\frac{2B}{u}\left[-K_{1(1)}-\frac{u}{l_1}K_{0(1)}\right]-\frac{C}{u}K_{1(2)}\right\}\cos v,\\
\tau_{uw} &= p\left\{1-\frac{A}{u^2}+\frac{C}{u}K_{1(2)}\right\}\cos v,\\
m_{uu} &= 2pl^2(1+b)\left\{\frac{2A}{u^3}+B\left(1+\frac{1}{a}\right)\left[\frac{2}{u^3}K_{1(1)}+\frac{1}{ul_1}K_{0(1)}\right]\right.\\
&\qquad\left.+\frac{C}{a}\left[\frac{2}{u^2}K_{1(2)}+\frac{1}{ul_2}K_{0(2)}+\frac{1+b+c}{(1+b)\,l_2^2}K_{1(2)}\right]\right\}\sin v,\\
m_{vv} &= 2pl^2(1+b)\left\{-\frac{2A}{u^3}+B\left(1+\frac{1}{a}\right)\left[-\frac{2}{u^3}K_{1(1)}-\frac{1}{ul_1}K_{0(1)}\right]\right.\\
&\qquad\left.+\frac{C}{a}\left[-\frac{2}{u^2}K_{1(2)}-\frac{1}{ul_2}K_{0(2)}+\frac{c}{(1+b)\,l_2^2}K_{1(2)}\right]\right\}\sin v,\\
m_{ww} &= 2pl^2\,\frac{Cc}{al_2^2}K_{1(2)}\sin v,\\
m_{uv} &= 2pl^2(1+b)\left\{-\frac{2A}{u^3}+B\left(1+\frac{1}{a}\right)\left[-\frac{2}{u^2}K_{1(1)}-\frac{1}{ul_1}K_{0(1)}-\frac{1}{l_1^2(1+b)}K_{1(1)}\right]\right.\\
&\qquad\left.+\frac{C}{a}\left[-\frac{2}{u^2}K_{1(2)}-\frac{1}{ul_2}K_{0(2)}\right]\right\}\cos v,\\
m_{vu} &= 2pl^2(1+b)\left\{-\frac{2A}{u^3}+B\left(1+\frac{1}{a}\right)\left[-\frac{2}{u^2}K_{1(1)}-\frac{1}{ul_1}K_{0(1)}-\frac{b}{l_1^2(1+b)}K_{1(1)}\right]\right.\\
&\qquad\left.+\frac{C}{a}\left[-\frac{2}{u^2}K_{1(2)}-\frac{1}{ul_2}K_{0(2)}\right]\right\}\cos v,\\
m_{wu} &= m_{uw}=m_{wv}=m_{vw}=0.
\end{aligned}
\tag{40}
$$

With the abbreviations:

$$
R/l_1=\beta_1,\qquad R/l_1=\beta_1,\qquad K_0(\beta_1)/K_1(\beta_1)=\kappa_1,\quad K_0(\beta_2)/K_1(\beta_2)=\kappa_2, \tag{41}
$$

the boundary conditions lead to the following three linear equations for the constant A, B, C:

$$
-\frac{A}{R^2}+\frac{C}{R}K_1(\beta_2)=-1,
$$

$$\frac{2A}{R^2}+\frac{B}{R}\left(1+\frac{1}{a}\right)K_1(\beta_2)[2+\kappa_1\beta_1]+\frac{C}{aR}K_1(\beta_2)\left[2+\kappa_2\beta_2+\frac{1+b+c}{1+b}\beta_2^2\right]=0, \quad (42)$$
$$-\frac{2A}{R^2}+\frac{B}{R}\left(1+\frac{1}{a}\right)K_1(\beta_1)\left[-2-\kappa_1\beta_1-\frac{1}{1+b}\beta_2^2\right]+\frac{C}{aR}K_1(\beta_2)[-2-\kappa_2\beta_2]=0.$$

After the introduction of the further auxiliary quantity:

$$\eta=\frac{1}{(1+1/a)[4+\kappa_1\beta_1+\beta_1^2/(1+b)]+\kappa_2\beta_2/a}, \tag{43}$$

it follows that:

$$\frac{A}{R^2}=1-2\eta, \quad \frac{B}{R}K_1(\beta_1)=\frac{C}{R}K_1(\beta_2)=-2\eta. \tag{44}$$

The *maximal stress* occur at the locations $u = R$, $v = \pm\,\pi/2$ (Fig. 3); the stresses that act there are:

$$\begin{aligned}
\tau_{\max} &= \tau_{vw} = \mp p\,[2+2\,\kappa_2\,\beta_2\,\eta],\\
\tau_{vw} &= \mp p\,[2-2(4+\kappa_2\,\beta_2)\eta],\\
m_{\max} &= m_{vw} = \mp 4pR\frac{1+b+2c}{1+b+c}\eta = \mp\, m_{\max},\\
m_{ww} &= \mp 4pR\frac{c\eta}{1+b+c}.
\end{aligned} \tag{45}$$

Referring to (18), one obtains the *equivalent stress:*

$$\sigma^*=2p\sqrt{3\left[(1-2\eta)^2+\frac{1}{a}(2+\kappa_2\beta_2)^2\eta^2+\frac{(1+b+2c)\eta^2R^2}{(1+b)(1+b+c)l^2}\right]}; \tag{46}$$

In the *limiting case* $l \ll R$ one has:

$$\eta\approx(1+b)\,l^2/R^2, \qquad \kappa_1\approx\kappa_2\approx 1, \qquad \sigma^*\approx 2p\sqrt{3}\,[1-(1+b)\,l^2/R^2], \tag{47}$$

In the *limiting case* $R \ll l$ one has:

$$\eta\approx\frac{a}{4(a+1)}, \qquad \sigma^*\approx p\frac{2+a}{1+a}\sqrt{3}. \tag{48}$$

X. The spherical cavity in a stress field

One also arrives at the solution to this problem when one extends the associated functions that appear in classical elasticity theory [**4**] in a certain way; for a stress in the $\bar{x}_1$-direction one then has (Fig. 4):

$$\begin{aligned}
\Phi_0 &= p\left\{\left[\frac{\nu}{2(1+\nu)}u^2+\frac{B}{u^3}\right](3\sin^2 v-2)+\frac{A}{u}\right\},\\
\Phi_x &= p\left[\frac{u}{2(1+\nu)}+\frac{C}{u^3}\right]\cos v,\\
\Phi_y &= \Phi_z = 0,\\
\Psi_x &= p\,D\,e^{-u/l_1}\left(\frac{l}{u}+\frac{l_1}{u^2}\right)\cos v,\\
\Psi_y &= \Psi_z = 0.
\end{aligned} \tag{49}$$

In this, A, B, C, and D are constants. With the help of the equation schema (13), it follows that:

$$\Phi_u = p\left\{\frac{u}{2(1+\nu)}+\frac{C}{u^2}\right\}\cos^2 v,$$

$$\Phi_u = p\left\{-\frac{u}{2(1+\nu)}-\frac{C}{u^2}\right\}\sin v\cos v, \qquad \Phi_w = 0,$$

$$\Psi_u = p\,D\,e^{-u/l_1}\left\{\frac{1}{u}+\frac{l_1}{u^2}\right\}\cos^2 v,$$

$$\Psi_v = p\,D\,e^{-u/l_1}\left\{-\frac{1}{u}-\frac{l_1}{u^2}\right\}\sin v\cos v, \qquad \Psi_w = 0,$$

$$\chi_u = p\left\{4(1-\nu)\left[\frac{u}{2(1+\nu)}+\frac{C}{u^2}\right]+D\left(1+\frac{1}{a}\right)e^{-u/l_1}\left(\frac{1}{u}+\frac{l_1}{u^2}\right)\right\}\cos^2 v,$$

$$\chi_v = p\left\{4(1-\nu)\left[-\frac{u}{2(1+\nu)}-\frac{C}{u^2}\right]+D\left(1+\frac{1}{a}\right)e^{-u/l_1}\left(-\frac{1}{u}-\frac{l_1}{u^2}\right)\right\}\sin v\cos v, \qquad \chi_w = 0,$$

$$\begin{aligned}
F = p\Bigg\{&\frac{(1-2\nu)u^2}{2(1+\nu)}+\frac{A+C}{u}-\frac{2B}{u^3}+D\,e^{-u/l_1}\left(-\frac{l_1}{u}-\frac{2l_1^2}{u^2}-\frac{2l_1^3}{u^3}\right)\\
&+\left[\frac{(-1+3\nu)u^2}{2(1+\nu)}+\frac{3B}{u^3}-\frac{C}{u}+D\,e^{-u/l_1}\left(\frac{l_1}{u}+\frac{3l_1^2}{u^2}+\frac{3l_1^3}{u^3}\right)\right]\sin^2 v\Bigg\},
\end{aligned}$$

(50)

$$2\,G\,V_u = p\left\{\frac{u}{1+v}+\frac{A}{u^2}-\frac{6B}{u^2}+(5-4v)\frac{C}{u^2}+D\,e^{-u/l_1}\left(-\frac{2l_1}{u^2}-\frac{6l_1^2}{u^3}-\frac{2l_1^3}{u^4}\right)\right.$$

$$\left.+\left[-u+\frac{9B}{u^4}+(-5+4v)\frac{c}{u^2}+D\,e^{-u/l_1}\left(\frac{3l_1}{u^2}+\frac{9l_1^2}{u^3}+\frac{9l_1^3}{u^4}\right)\right]\sin^2 v\right\},$$

$$2\,G\,V_v = p\left\{-u-\frac{6B}{u^4}+(-2+4v)\frac{C}{u^2}+D\,e^{-u/l_1}\left(-\frac{1}{u}-\frac{3l_1}{u^2}-\frac{6l_1^2}{u^3}-\frac{6l_1^3}{u^4}\right)\sin v\cos v\right\}$$

$$V_w = 0, \qquad \omega_u = 0, \qquad \omega_v = 0,$$

$$2\,G\,V_v = p\left\{6(1-v)\frac{C}{u^3}+D\left(1+\frac{1}{a}\right)e^{-u/l_1}\left(\frac{1}{2l_1u}+\frac{3}{2u^2}+\frac{3l_1}{2u^3}\right)\right\}\sin v\cos v\,,$$

$$\sigma_u = p\left\{1-\frac{2A}{u^3}+\frac{24B}{u^5}+(-10+4v)\frac{C}{u^3}+D\,e^{-u/l_1}\left(\frac{2}{u^2}+\frac{10l_1}{u^3}+\frac{24l_1^2}{u^4}+\frac{24l_1^3}{u^5}\right)\right.$$

$$\left.+\left[-1-\frac{36B}{u^5}+(10-2v)\frac{C}{u^3}+D\,e^{-u/l_1}\left(-\frac{3}{u^2}-\frac{15l_1}{u^3}-\frac{36l_1^2}{u^4}-\frac{36l_1^3}{u^5}\right)\right]\sin^2 v\right\}$$

$$\sigma_v = p\left\{\frac{A}{u^3}-\frac{12B}{u^5}+(3-4v)\frac{C}{u^3}+D\,e^{-u/l_1}\left(-\frac{1}{u^2}-\frac{5l_1}{u^3}-\frac{12l_1^2}{u^4}-\frac{12l_1^3}{u^5}\right)\right.$$

$$\left.+\left[1+\frac{21B}{u^5}+(-1+2v)\frac{C}{u^3}+D\,e^{-u/l_1}\left(\frac{2}{u^2}+\frac{9l_1}{u^3}+\frac{21l_1^2}{u^4}+\frac{21l_1^3}{u^5}\right)\right]\sin^2 v\right\},$$

$$\sigma_w = p\left\{\frac{A}{u^3}-\frac{12B}{u^5}+(3-4v)\frac{C}{u^3}+D\,e^{-u/l_1}\left(-\frac{1}{u^2}-\frac{5l_1}{u^3}-\frac{12l_1^2}{u^4}-\frac{12l_1^3}{u^5}\right)\right.$$

$$\left.+\left[\frac{15B}{u^5}+(-3+6v)\frac{C}{u^3}+D\,e^{-u/l_1}\left(\frac{1}{u^2}+\frac{6l_1}{u^3}+\frac{15l_1^2}{u^4}+\frac{15l_1^3}{u^5}\right)\right]\sin^2 v\right\},$$

$$\tau_{uv} = p\left\{-1-\frac{24B}{u^5}+(-2-2v)\frac{C}{u^3}+D\,e^{-u/l_1}\left(\frac{1}{u^2}+\frac{9l_1}{u^3}+\frac{24l_1^2}{u^4}+\frac{24l_1^3}{u^5}\right)\sin v\cos v\,,\right.$$

$$\tau_{vu} = p\left\{-1+\frac{24B}{u^5}+(-2-2v)\frac{C}{u^3}+D\,e^{-u/l_1}\left(\frac{1}{l_1u}+\frac{4}{u^2}+\frac{12l_1}{u^3}+\frac{24l_1^2}{u^4}+\frac{24l_1^3}{u^5}\right)\right\}\sin v\cos v\,,$$

$$\tau_{vw} = \tau_{wv} = \tau_{wu} = \tau_{uw} = 0,$$

$$m_{uw} = p\left\{-12(1-v)(3+b)l^2\,\frac{C}{u^4}\right.$$

$$+ D e^{-u/l_1}\left[-\frac{1}{u}-(4+b)\frac{l_1}{u^2}-3(3+b)\frac{l_1^2}{u^3}\left(1+\frac{l_1}{u}\right)\right]\Bigg\}\sin v\cos v,$$

$$m_{wu} = p\Bigg\{-12(1-\nu)(3+b)l^2\frac{C}{u^4}$$

$$+ D e^{-u/l_1}\left[-\frac{b}{u}-(1+4b)\frac{l_1}{u^2}-3(1+3b)\frac{l_1^2}{u^3}\left(1+\frac{l_1}{u}\right)\right]\Bigg\}\sin v\cos v,$$

$$m_{vw} = p\left\{12(1-\nu)\frac{Cl^2}{u^4}+D e^{-u/l_1}\left(\frac{l_1}{u^2}+\frac{l_1^2}{u^3}+\frac{3l_1^3}{u^4}\right)\right\}\left[1-b+(-2+b)\sin^2 v\right],$$

$$m_{wv} = p\left\{12(1-\nu)\frac{Cl^2}{u^4}+D e^{-u/l_1}\left(\frac{l_1}{u^2}+\frac{3l_1^2}{u^3}+\frac{3l_1^3}{u^4}\right)\right\}\left[b-1+(1-2b)\sin^2 v\right],$$

$$m_{uu} = m_{vv} = m_{ww} = m_{uv} = m_{vu} = 0.$$

The boundary conditions (19) lead to the following four linear equations (with the abbreviation $R / l_1 = \beta$):

$$-\frac{2A}{R^3}+24\frac{B}{R^5}+(-10+4\nu)\frac{C}{R^3}+\frac{D}{R^2}e^{-\beta}\left[2+\frac{10}{\beta}+\frac{24}{\beta^2}+\frac{24}{\beta^3}\right]=-1,$$

$$-36\frac{B}{R^5}+(10-2\nu)\frac{C}{R^3}+\frac{D}{R^2}e^{-\beta}\left[-3-\frac{15}{\beta}-\frac{36}{\beta^2}-\frac{36}{\beta^3}\right]=1,$$

$$24\frac{B}{R^5}+(-2-2\nu)\frac{C}{R^3}+\frac{D}{R^2}e^{-\beta}\left[1+\frac{9}{\beta}+\frac{24}{\beta^2}+\frac{24}{\beta^3}\right]=1, \tag{51}$$

$$12(1-\nu)(3+b)\frac{C}{R^3}+\frac{D}{R^2}e^{-\beta}\left(1+\frac{1}{a}\right)\left[\beta^2+(4+b)\beta+3(3+b)\left(1+\frac{1}{\beta}\right)\right]=0.$$

With the further auxiliary quantity:

$$\mu = \frac{18(1-\nu)(3+b)(\beta+1)}{\left(1+\frac{1}{a}\right)[\beta^3+(4+b)\beta^2+3(3+b)(\beta+1)]}, \tag{52}$$

it follows from (33) that:

$$\frac{A}{R^3} = \frac{1}{2(7-5\nu+\mu)}\left[\frac{\mu}{3}-1+5\nu\right],$$

$$\frac{B}{R^5} = \frac{1}{2(7-5\nu+\mu)}\left[\mu\left(\frac{2}{9}+\frac{10}{9(\beta+1)}+\frac{10}{\beta^2}\right)+1\right], \tag{53}$$
$$\frac{C}{R^3} = \frac{5}{2(7-5\nu+\mu)}, \qquad \frac{D}{R^2}e^{-\beta} = \frac{5\beta\mu}{3(\beta+1)(7-5\nu+\mu)}.$$

The passage to the limit $a \to \infty$ leads to the abbreviated theory and confirms the results of MINDLIN and TIERSTEN [**11**], [**12**].

For the evaluation of the stress, as well as making preparations for further reasoning, let the following stress values be computed for $u = R$ (cf., Fig. 4, sec. 5, with $p_1 = p$, $p_2 = p_3 = 0$):

$$\begin{aligned}
s_1 &= \frac{1}{p}(\sigma_v)_{v=\pi/2} = \frac{27-15\nu+\mu}{2(7-5\nu+\mu)} = \sigma_{\max}/p = \sigma_{\mathrm{I}}/p = \sigma_{\mathrm{II}}/p,\\
s_2 &= \frac{1}{p}(\sigma_w)_{v=\pi/2} = \frac{-3+15\nu+\mu}{2(7-5\nu+\mu)} = \sigma_{\mathrm{III}}/p = \sigma_{\mathrm{IV}}/p,\\
s_3 &= \frac{1}{p}(\sigma_v)_{v=0} = \frac{1}{p}(\sigma_w)_{v=0} = -\frac{3+15\nu-\mu}{2(7-5\nu+\mu)} = \sigma_{\mathrm{IV}}/p = \sigma_{\mathrm{V}}/p,\\
m_2 &= -\frac{1}{p}(m_{vw})_{v=\pi/2} = \frac{5\mu R}{3(3+b)(7-5\nu+\mu)} = m_{\mathrm{III}}/p = m_{\mathrm{IV}}/p,\\
m_3 &= \frac{1}{p}(m_{wv})_{v=0} = -\frac{1}{p}(m_{vw})_{v=0} = -(1-b)\,m_2 = m_{\mathrm{IV}}/p = m_{\mathrm{V}}/p.
\end{aligned} \tag{54}$$

The *maximum stress* occurs at the location $u = R$, $v = \pi/2$; referring to (18), one calculates the associated *equivalent stress:*

$$\sigma^* = p\sqrt{s_1^2 - s_1 s_2 + s_2^2 + 3m_2^2 R^2/(4l^2)}\,. \tag{55}$$

In the *limiting case* $l \ll R$, it follows that:

$$\begin{aligned}
&\mu \approx 18\,(1-\nu)\,(3+b)\,l^2/R^2,\\
&s_1 \approx \frac{3(9-5\nu)}{2(7-5\nu)}\left[1-\frac{60(2-\nu)(1-\nu)(3+b)\,l^2}{(9-5\nu)(7-5\nu)\,R^2}\right],\\
&s_2 \approx \frac{3(5\nu-1)}{2(7-5\nu)}\left[1+\frac{60(1-2\nu)(1-\nu)(3+b)\,l^2}{(5\nu-1)(7-5\nu)\,R^2}\right],\\
&s_3 \approx -\frac{3(5\nu-1)}{2(7-5\nu)}\left[1+\frac{60(1-2\nu)(1-\nu)(3+b)\,l^2}{(5\nu-1)(7-5\nu)\,R^2}\right],\\
&m_2 \approx \frac{30(1-\nu)\,l^2}{(7-5\nu)\,R^2},
\end{aligned} \tag{56}$$

$$\sigma^* \approx \frac{3p\sqrt{91-150\nu+75\nu^2}}{2(7-5\nu)}$$
$$\times\left\{1-\frac{30(1-\nu)[(3+b)(49-86\nu+45\nu^2)-5(1-\nu)(7-5)]\,l^2}{(7-5\nu)(91-150\nu+75\nu^2)\,R^2}\right\}.$$

In the *limiting case* $R \ll l$ one has:

$$\mu \approx 6\,(1-\nu)\,/\,(1+a)$$

$$s_1 \approx \frac{3[9-5\nu+(15-11\nu)\,a]}{2[7-5\nu+(13-11\nu)\,a]},$$
$$s_2 \approx \frac{3[5\nu-1+(5-\nu)\,a]}{2[7-5\nu+(13-11\nu)\,a]},$$
$$s_3 \approx -\frac{3[1+5\nu-(15-11\nu)\,a]}{2[7-5\nu+(13-11\nu)\,a]},$$
$$m_2 \approx \frac{10(1-\nu)\,a\,R}{(3+b)[7-5\nu+(13-11\nu)\,a]}, \tag{57}$$
$$\sigma^* \approx \frac{3\sqrt{91-150\nu+75\nu^2+2a(115-174\nu+75\nu^2)+a^2(175-250\nu+111\nu^2)}}{2[7-5\nu+(13-11\nu)a]}.$$

In this problem, as well, the *maximal stress decreases with increasing parameter* $l\,/\,R$ *according to (56).*

XI. The spherical cavity in a homogeneous stress field

An arbitrary homogeneous stress field with no moment-stresses might arise in the distant neighborhood of a cavity, moreover – i.e., a stress tensor whose components are not only constant on the grounds of equilibrium, but also symmetric. Let the associated principal stresses be denoted by p_1, p_2, p_3; their directions might lie parallel to the axes of the Cartesian coordinate system $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$. For the twelve stress values that are possible in the outer surface of the spherical cavity at the piercing points of the coordinate axes (cf., Fig. 5) one obtains the results that were obtained in the previous section by superposition:

$$\begin{aligned}
\sigma_{\mathrm{I}} &= p_1\,s_1 + p_2\,s_3 + p_3\,s_2\,, & m_{\mathrm{I}} &= p_1\,m_1 + p_2\,m_3 + p_3\,m_2\,,\\
\sigma_{\mathrm{II}} &= p_1\,s_1 + p_2\,s_2 + p_3\,s_3\,, & m_{\mathrm{II}} &= p_1\,m_1 + p_2\,m_2 + p_3\,m_3\,,\\
\sigma_{\mathrm{III}} &= p_1\,s_2 + p_2\,s_1 + p_3\,s_3\,, & m_{\mathrm{III}} &= p_1\,m_2 + p_2\,m_1 + p_3\,m_3\,,\\
\sigma_{\mathrm{IV}} &= p_1\,s_3 + p_2\,s_1 + p_3\,s_2\,, & m_{\mathrm{IV}} &= p_1\,m_3 + p_2\,m_1 + p_3\,m_2\,,
\end{aligned} \tag{58}$$

$\sigma_{\rm V} = p_1\, s_3 + p_2\, s_2 + p_3\, s_1$, $\quad m_{\rm V} = p_1\, m_3 + p_2\, m_2 + p_3\, m_1$,
$\sigma_{\rm VI} = p_1\, s_2 + p_2\, s_3 + p_3\, s_1$, $\quad m_{\rm VI} = p_1\, m_2 + p_2\, m_3 + p_3\, m_1$.

For an *everywhere uniform tensile load* with $p_1 = p_2 = p_3 = p$, one obtains the (classical) value of $3p/2$ for stresses – as one would expect – on the basis of formula (54), while all moment-stresses vanish. The reason for this lies in the fact that the associated deformation can be generated by a pure radial displacement that depends upon only the radius, and therefore can have no rotations as a consequence.

In the case $p_1 = p_2 = p$, $p_3 = 0$, one is dealing with a *uniform tensile load in the mutually perpendicular direction;* it follows that:

$$\begin{aligned}
\sigma_{\rm I} &= \sigma_{\rm IV} = p\,(s_1 + s_3),\\
\sigma_{\rm II} &= \sigma_{\rm III} = p\,(s_1 + s_2) = \sigma_{\max},\\
\sigma_{\rm V} &= \sigma_{\rm VI} = p\,(s_2 + s_3),\\[1ex]
m_{\rm I} &= m_{\rm IV} = p\,(m_1 + m_3),\\
m_{\rm II} &= m_{\rm III} = p\,(m_1 + m_2),\\
m_{\rm V} &= m_{\rm VI} = p\,(m_2 + m_3).
\end{aligned} \tag{59}$$

At the location where one finds the highest stress, there will be the equivalent stress:

$$\sigma^* = \frac{p}{7-5\nu+\mu}\sqrt{(12+\mu)^2 + \frac{25(1-b)\mu^2 R^2}{6(3+b)^2 l^2}}\,. \tag{60}$$

Another case that is important for the theory of solids is the *pure shear stress;* with $p_1 = -p_2$, $p_3 = 0$, one obtains:

$$\begin{aligned}
&\sigma_{\rm I} = -\,\sigma_{\rm IV} = p\,(s_1 - s_3) = \frac{15p}{7-5\nu+\mu} = \sigma_{\max},\\
&\sigma_{\rm II} = -\,\sigma_{\rm III} = (1-\nu)\sigma_{\rm I},\quad \sigma_{\rm VI} = -\,\sigma_{\rm V} = \nu\,\sigma_{\rm I},\\[1ex]
&m_{\rm I} = -\,m_{\rm IV} = p\,(m_1 - m_3) = p\,(1-2b)\,m_2,\\
&m_{\rm II} = -\,m_{\rm III} = -\,p\,(1+b)\,m_2,\\
&m_{\rm VI} = -\,m_{\rm V} = p\,(2-b)\,m_2.
\end{aligned} \tag{61}$$

At the location where one finds the highest stress (for $\sigma_{\rm II}$), one finds the equivalent stress:

$$\sigma^* = \frac{15p}{7-5\nu+\mu}\sqrt{3(1-\nu)^2 + \frac{(1+b)\mu^2 R^2}{54(3+b)^2 l^2}}\,. \tag{62}$$

For small values of the parameter b, one further shows that the stress decreases with the characteristic length l for COSSERAT bodies.

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