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Inertial and gravitational mass in relativistic mechanics

By **Gunnar Nordstrøm**

Translated by D. H. Delphenich

In several recent works in the field of relativistic mechanics, the concept of the mass of the body has played a very subordinate role. The reason is easy to understand: As **Laue** ([1]) and **Herglotz** ([2]) have shown, one can construct the mechanics of extended bodies completely without introducing the concept of inertial mass anywhere. The concept of mass is therefore not absolutely necessary for mechanics, and on the other hand, that concept is also not sufficient to represent all inertial phenomena for matter when one considers bodies that are subject to arbitrary elastic stresses.

The question of the mass of matter is, however, of considerable importance for the theory of relativity, especially for the evaluation of the manner by which the theory of gravitation should be inserted into the theory of relativity. The inertia and the weight of matter must, in any event, have a close relationship to each other, and it would be simplest for one to address that essential property by calculating the mass that is based upon the two phenomena. One would seek to justify such a concept of mass whether or not one knows that there are also inertial phenomena in the theory of relativity that do not lead back to a mass in any way. In such cases, one must calculate with a special quantity of motion (viz., impulse) that does not depend upon the mass of the body, but, e.g., on the state of elastic stress itself.

In the present article, I would like to treat the relativistic mechanics of deformable bodies in such a way that the possibility of a general justification for a concept of mass will emerge clearly. In it, I will also examine the influence of the conduction of heat on mechanical processes. In conclusion, I will consider gravitation when I also ascribe the inertial mass to weight.

§ 1. The foundations of the relativistic mechanics of deformable bodies.

We consider a body in an arbitrary state of motion and stress. In addition to the elastic forces, a spatially-distributed ponderomotive force $\mathfrak{K}$ of arbitrary source might act upon the body. $\mathfrak{K}$ is a four-vector that will be referred to as the "external" ponderomotive force per unit volume or as the "external" moving force per unit rest volume ([3]).

([1]) **M. Laue**, *Das Relativitätsprinzip*, Braunschweig, 1911, VII; Ann. Phys. (Leipzig) **35** (1911), 524.
([2]) **G. Herglotz**, Ann. Phys. (Leipzig) **36** (1911), 493.
([3]) **H. Minkowski**, Gött. Nachr. (1908), pp. 107 and 108; cf., also equations (6) and (9) below.

For **Laue** ([1]), there is a symmetric four-dimensional tensor T whose components give the stresses, as well as the mechanical impulse and energy density. Thus, we can write the equations of motion of the body in the following form:

$$(1)\qquad \begin{cases} \mathfrak{K}_x = \dfrac{\partial T_{xx}}{\partial x} + \dfrac{\partial T_{xy}}{\partial y} + \dfrac{\partial T_{xz}}{\partial z} + \dfrac{\partial T_{xu}}{\partial u}, \\ \mathfrak{K}_y = \dfrac{\partial T_{yx}}{\partial x} + \dfrac{\partial T_{yy}}{\partial y} + \dfrac{\partial T_{yz}}{\partial z} + \dfrac{\partial T_{yu}}{\partial u}, \\ \mathfrak{K}_z = \dfrac{\partial T_{zx}}{\partial x} + \dfrac{\partial T_{zy}}{\partial y} + \dfrac{\partial T_{zz}}{\partial z} + \dfrac{\partial T_{zu}}{\partial u}, \\ \mathfrak{K}_u = \dfrac{\partial T_{ux}}{\partial x} + \dfrac{\partial T_{uy}}{\partial y} + \dfrac{\partial T_{uz}}{\partial z} + \dfrac{\partial T_{uu}}{\partial u}, \end{cases}$$

x, y, z, $u = i\,c\,t$ are the four coordinates; the speed of light c should be a universal constant.

We would like to ascribe a certain *rest-mass density* v to every space-time point in the matter. That quantity should be a four-dimensional scalar, but we shall leave it otherwise completely undetermined until later, such that we can have the concept of mass freely at our disposal. One determines the usual mass density μ from the rest density by the equation:

$$(2)\qquad v = \mu\sqrt{1 - \frac{\mathfrak{v}^2}{c^2}},$$

in which v means the (three-dimensional) velocity of the point considered. For the sake of simplicity, we set:

$$\mathfrak{q} = \frac{\mathfrak{v}}{c},$$

and then have:

$$v = \mu\sqrt{1 - \mathfrak{q}^2}.$$

We now regard the four-dimensional tensor T as the sum of two such tensors, when we set:

$$(3)\qquad \begin{cases} T_{xx} = p_{xx} + v\,\mathfrak{B}_x^2, \\ \dots\dots\dots\dots\dots\dots \\ T_{uu} = p_{uu} + v\,\mathfrak{B}_u^2, \\ T_{xy} = p_{xy} + v\,\mathfrak{B}_x\mathfrak{B}_y, \\ \dots\dots\dots\dots\dots\dots \\ T_{zu} = p_{zu} + v\,\mathfrak{B}_z\mathfrak{B}_u, \end{cases}$$

([1]) **M. Laue**, *Das Relativitätsprinzip*, pp. 149.

in which $\mathfrak{B}$ means the four-dimensional motion vector. As is known, it is connected to the velocity $\mathfrak{v}$ by the equations:

$$(4) \qquad \mathfrak{B}_x = \frac{\mathfrak{v}_x}{\sqrt{1-\mathfrak{q}^2}}, \ \ldots, \qquad \mathfrak{B}_u = \frac{ic}{\sqrt{1-\mathfrak{q}^2}}.$$

We call the four-dimensional tensor p that is introduced by equation (4) the *elastic stress tensor*. Like T, it is symmetric, so $p_{xy} = p_{yx}$, etc. One can call the second partial tensor of T the *material* tensor.

We set:

$$(5) \qquad \begin{cases} \mathfrak{K}_x^e = -\dfrac{\partial p_{xx}}{\partial x} - \dfrac{\partial p_{xy}}{\partial y} - \dfrac{\partial p_{xz}}{\partial z} - \dfrac{\partial p_{xu}}{\partial u}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathfrak{K}_u^e = -\dfrac{\partial p_{ux}}{\partial x} - \dfrac{\partial p_{uy}}{\partial y} - \dfrac{\partial p_{uz}}{\partial z} - \dfrac{\partial p_{uu}}{\partial u}, \end{cases}$$

and call the vector $\mathfrak{K}^e$ the *elastic* ponderomotive force. Our equations of motion (1) now read:

$$(6) \qquad \begin{cases} \mathfrak{K}_x + \mathfrak{K}_x^e = \dfrac{\partial}{\partial x} \nu \mathfrak{B}_x^2 + \dfrac{\partial}{\partial y} \nu \mathfrak{B}_x \mathfrak{B}_y + \dfrac{\partial}{\partial z} \nu \mathfrak{B}_x \mathfrak{B}_z + \dfrac{\partial}{\partial u} \nu \mathfrak{B}_x \mathfrak{B}_u, \\ \mathfrak{K}_y + \mathfrak{K}_y^e = \dfrac{\partial}{\partial x} \nu \mathfrak{B}_y \mathfrak{B}_x + \dfrac{\partial}{\partial y} \nu \mathfrak{B}_y^2 + \dfrac{\partial}{\partial z} \nu \mathfrak{B}_y \mathfrak{B}_z + \dfrac{\partial}{\partial u} \nu \mathfrak{B}_y \mathfrak{B}_u, \\ \mathfrak{K}_z + \mathfrak{K}_z^e = \dfrac{\partial}{\partial x} \nu \mathfrak{B}_z \mathfrak{B}_x + \dfrac{\partial}{\partial y} \nu \mathfrak{B}_z \mathfrak{B}_y + \dfrac{\partial}{\partial z} \nu \mathfrak{B}_z^2 + \dfrac{\partial}{\partial u} \nu \mathfrak{B}_z \mathfrak{B}_u, \\ \mathfrak{K}_u + \mathfrak{K}_u^e = \dfrac{\partial}{\partial x} \nu \mathfrak{B}_u \mathfrak{B}_x + \dfrac{\partial}{\partial y} \nu \mathfrak{B}_u \mathfrak{B}_y + \dfrac{\partial}{\partial z} \nu \mathfrak{B}_u \mathfrak{B}_z + \dfrac{\partial}{\partial u} \nu \mathfrak{B}_u^2. \end{cases}$$

In order to see the meaning of the right-hand side, we convert it. We let dv denote the volume of a material particle in the body and let dv_0 denote its rest volume, so:

$$(7) \qquad dv_0 = \frac{dv}{\sqrt{1-\mathfrak{q}^2}}.$$

If we further let τ mean the proper time:

$$(8) \qquad d\tau = dt\sqrt{1-\mathfrak{q}^2}$$

then one will get:

$$\frac{\partial}{\partial x}\nu\mathfrak{B}_x^2+\frac{\partial}{\partial y}\nu\mathfrak{B}_x\mathfrak{B}_y+\frac{\partial}{\partial z}\nu\mathfrak{B}_x\mathfrak{B}_z+\frac{\partial}{\partial u}\nu\mathfrak{B}_x\mathfrak{B}_u=\frac{1}{dv_0}\frac{d}{d\tau}(\nu\mathfrak{B}_x\,dv_0)$$
$$=\frac{1}{dv}\frac{d}{dt}\left\{\frac{\mu\mathfrak{v}_x}{\sqrt{1-\mathfrak{q}^2}}dv\right\} \tag{9}$$

with the use of a known formula (1). One will get corresponding equations by permuting the index x with y, z, u. If one introduces the expressions that are found into (6) then one will get the equations of motion in a form that is similar to then ones for a material point.

As is known, the first three equations of motion express the law of impulse, while the fourth one expresses the law of energy. In order to study the first law more closely, we set:

$$\mathfrak{g}_x^e=-\frac{i}{c}p_{xu},\quad \mathfrak{g}_y^e=-\frac{i}{c}p_{yu},\quad \mathfrak{g}_z^e=-\frac{i}{c}p_{zu}, \tag{10}$$

and call the three-dimensional vector $\mathfrak{g}^e$ the *elastic impulse density*. The vector with the components:

$$\mathfrak{g}_x^m=-\frac{i}{c}\nu\mathfrak{B}_x\mathfrak{B}_u,\ \text{etc.}$$

that is defined in a similar way from the material tensor shall be referred to as the *material impulse density*. One finds:

$$\mathfrak{g}^m=\frac{\mu\mathfrak{v}}{\sqrt{1-\mathfrak{q}^2}}. \tag{11}$$

We further introduce the *relative stresses t* (2) by the following equations:

$$\left\{\begin{array}{l} t_{xx}=p_{xx}+\frac{i}{c}p_{xu}\mathfrak{v}_x, \\ t_{xy}=p_{xy}+\frac{i}{c}p_{xu}\mathfrak{v}_y, \\ \quad\text{etc.}\end{array}\right. \tag{12}$$

or, from (10):

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(1) $$\frac{d}{dt}\int\varphi\,dv=\int\left\{\operatorname{div}\varphi\mathfrak{v}+\frac{\partial\varphi}{\partial t}\right\}dv,$$

in which φ is an arbitrary function of x, y, z, t, and the integration on the left-hand side refers to a well-defined part of the matter. The vector-analytic symbols in this article are the ones that are explained in **Abraham**, *Theorie der Elektrizität*, Bd. I; they will always refer to *three-dimensional* vectors.

(2) **M. Abraham**, "Zur Elektrodynamik bewegten Körper," Rend. Circ. Matem. Palermo (1909), equation (10); **M. Laue**, *loc. cit.*, pp. 151.

$$\text{(12a)} \qquad \begin{cases} t_{xx} = p_{xx} - \mathfrak{g}_x^e\, \mathfrak{v}_x, \\ t_{xy} = p_{xy} - \mathfrak{g}_x^e\, \mathfrak{v}_y, \\ \quad \text{etc.} \end{cases}$$

The relative stresses define an asymmetric, three-dimensional tensor. As is obvious, these stresses are calculated in a manner that is similar to the way that the pressure on a moving surface is calculated in electrodynamics. It should be further remarked that one will get the same relative stresses when one writes T, in place of p, in (12), so the second (viz., material) tensor into which we have split T will give zero relative stresses. For that reason, the relative stresses that were defined by equation (12) will be identical with the ones that von **Laue** introduced (*loc. cit.*).

We can now convert the expressions for the spatial components of $\mathfrak{K}^e$. From (10) and (12a), we get:

$$\text{(13)} \qquad \mathfrak{K}_x^e = -\left\{\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z}\right\} - \left\{\frac{\partial}{\partial x}\mathfrak{g}_x^e\,\mathfrak{v}_x + \frac{\partial}{\partial y}\mathfrak{g}_x^e\,\mathfrak{v}_y + \frac{\partial}{\partial z}\mathfrak{g}_x^e\,\mathfrak{v}_z + \frac{\partial}{\partial t}\mathfrak{g}_x^e\right\},$$

and corresponding expressions for $\mathfrak{K}_y^e$ and $\mathfrak{K}_z^e$.

We multiply equation (13) by dv and integrate over a (three-dimensional) space v that is filled with mass. The integral of the expression in the first bracket can be converted into a surface integral by means of **Gauss**'s theorem. When we then apply the formula in the footnote on pp. 4 to the last bracketed expression, we will get:

$$\text{(14)} \qquad \int \mathfrak{K}_x^e\, dv = -\int \{t_{xx}\, d\mathfrak{f}_x + t_{xy}\, d\mathfrak{f}_y + t_{xz}\, d\mathfrak{f}_z\} - \frac{d}{dt}\int \mathfrak{g}_x^e\, dv.$$

Here, $d\mathfrak{f}_x$, $d\mathfrak{f}_y$, $d\mathfrak{f}_z$ are the components of a surface element of the boundary surface of the domain considered v ; $d\mathfrak{f}$ is regarded as a vector whose direction is that of the external normal. The symbol d / dt denotes the temporal change in a bounded part of the matter.

Corresponding expressions are true for the remaining spatial axis directions, and we see that the elastic force is determined, in part, by the relative elastic stresses that act as surface forces and in part, by the change in the elastic impulse.

We can now write the first of the equations of motion using (6), (9), and (14) in the following integral form:

$$\text{(15)} \qquad \int \mathfrak{K}_x\, dv - \int \{t_{xx}\, d\mathfrak{f}_x + t_{xy}\, d\mathfrak{f}_y + t_{xz}\, d\mathfrak{f}_z\} - \frac{d}{dt}\int \mathfrak{g}_x^e\, dv = \frac{d}{dt}\int \frac{\mu\mathfrak{v}}{\sqrt{1-\mathfrak{q}^2}}\, dv = \frac{d}{dt}\int \mathfrak{g}_x^m\, dv.$$

This equation and the two analogous ones for the remaining spatial axis direction express the law of impulse for a bounded part of the matter.

The asymmetry of the relative stress tensor means that the elastic forces generally exert an rotational moment on any part of the body ([1]). From the theory of elasticity, the rotational moment that acts upon a unit volume around a direction that is parallel to the *x*-axis:

$$t_{yz} - t_{xy} = \mathfrak{v}_y \, \mathfrak{g}_z^e - \mathfrak{v}_z \, \mathfrak{g}_y^e .$$

As a result, one then has:

(16) $$\mathfrak{n} = [\mathfrak{v} \, \mathfrak{g}^e]$$

for the rotational moment $\mathfrak{n}$ per unit volume, when expressed vector-analytically. That rotational moment must always appear when the elastic impulse density has a component that is perpendicular to the velocity. The rotational moment is then also necessary for maintaining a uniform translational motion of the elastically-stressed body. As is known, in this, one finds an essential differential between classical and relativistic mechanics whose basis will emerge clearly when one presents the laws of surfaces ([2]). However, we would not like to go into that here.

Whereas the first three of the equations of motion express the law of impulse, the last of those equations expresses the law of energy. We set:

(17) $$\mathfrak{S}^e = c^2 \, \mathfrak{g}^e,$$

(18) $$\mathfrak{S}^m = c^2 \, \mathfrak{g}^m = \frac{c^2 \mu \, \mathfrak{v}}{\sqrt{1 - \mathfrak{q}^2}},$$

so

(17a) $$\mathfrak{S}_x^e = - i \, c \, p_{xu} , \text{ etc.},$$

(18a) $$\mathfrak{S}_x^m = - i \, c \, v \, \mathfrak{B}_x \, \mathfrak{B}_u , \text{ etc.},$$

and furthermore:

(19) $$\psi^e = - p_{uu} ,$$

(20) $$\psi^m = - v \, \mathfrak{B}_u^2 = \frac{c^2 \mu}{\sqrt{1 - \mathfrak{q}^2}} .$$

From (5), there is then an expression for $i \, c \, \mathfrak{K}_u^e$ that reads:

(21) $$i \, c \, \mathfrak{K}_u^e = \operatorname{div} \mathfrak{S}^e + \frac{\partial \psi^e}{\partial t},$$

when written vector-analytically. We can now write the last of the equations of motion (6), when multiplied by – *ic* as:

([1]) **M. Laue**, *loc. cit.*, pp. 168.
([2]) **M. Laue**, Ann. Phys. (Leipzig) **35** (1911), pp. 536.

$$-\,i\,c\,\mathfrak{K}_u = \mathrm{div}\,(\mathfrak{S}^e + \mathfrak{S}^m) + \frac{\partial}{\partial t}(\psi^e + \psi^m). \tag{22}$$

That is the energy equation. We see that the vectors $\mathfrak{S}^e$ and $\mathfrak{S}^m$ express the *elastic* (*material*, resp.) *energy current* and that the quantities ψ^e and ψ^m are the corresponding *energy densities*. The quantity $-\,i\,c\,\mathfrak{K}_u$ gives the energy supplied per unit volume and time by the external force $\mathfrak{K}$. The meaning of the right-hand side can be recognized with no further discussion. Upon integrating it over an arbitrary space and applying **Gauss**'s theorem, one will get the law of energy, when expressed for a fixed spatial region in the spatial reference system (x, y, z) that is employed.

§ 2. Approximate examination of the elastic state quantities.

In order to arrive at a clear presentation of the elastic quantities, we transform the stress tensor p to rest at the space-time point considered. The components of the tensor then give the matrix:

$$\left\{\begin{array}{cccc} p_{xx}^0 & p_{xy}^0 & p_{xz}^0 & 0 \\ p_{yx}^0 & p_{yy}^0 & p_{yz}^0 & 0 \\ p_{zx}^0 & p_{zy}^0 & p_{zz}^0 & 0 \\ 0 & 0 & 0 & p_{uu}^0 \end{array}\right. . \tag{23}$$

In the case of rest, the state of elastic stress might then give no energy current, and also no impulse ([1]).

We remark incidentally that the usual laws of the theory of elasticity will be valid in the case of rest. We can then relate the six spatial stress components p_{xx}^0, p_{xy}^0, … to the deformation quantities at rest ([2]). However, we would not like to go further into that topic.

One easily sees that p_{uu}^0 must be a four-dimensional scalar. One has:

$$-c^2\,p_{uu}^0 = p_{xx}\,\mathfrak{B}_x^{\,2} + p_{yy}\,\mathfrak{B}_y^{\,2} + p_{zz}\,\mathfrak{B}_z^{\,2} + p_u\,\mathfrak{B}_u^{\,2} + 2\,p_{xy}\,\mathfrak{B}_x\,\mathfrak{B}_y + 2\,p_{xz}\,\mathfrak{B}_x\,\mathfrak{B}_z + \ldots,$$

so the right-hand side is invariant under Lorentz transformations, and nine of the ten terms will vanish under the transformation to rest, which will then give one the identity $-c^2\,p_{uu}^0 = -c^2\,p_{uu}^0$.

Furthermore, from (19) and (20):

$$\Psi = -\,p_{uu}^0 + c^2\,\nu \tag{24}$$

([1]) If heat conduction is present then its influence under the action of the external force $\mathfrak{K}$ must be calculated. Cf., *infra*, § 5.

([2]) Cf., **G. Herglotz**, *loc. cit.*

is the *rest energy density* of matter, which is also a four-dimensional scalar. Since v is undetermined, for the moment, one can now define that quantity in such a way that $p_{uu}^0 = 0$. For that reason, we would not like to make any such assumption until later.

Upon transforming to rest, one will also see the validity of the following system of equations:

$$(25)\qquad \begin{cases} p_{xx}\mathfrak{B}_x + p_{xy}\mathfrak{B}_y + p_{xz}\mathfrak{B}_z + p_{xu}\mathfrak{B}_u = p_{uu}^0\mathfrak{B}_x, \\ p_{yx}\mathfrak{B}_x + p_{yy}\mathfrak{B}_y + p_{yz}\mathfrak{B}_z + p_{yu}\mathfrak{B}_u = p_{uu}^0\mathfrak{B}_y, \\ p_{zx}\mathfrak{B}_x + p_{zy}\mathfrak{B}_y + p_{zz}\mathfrak{B}_z + p_{zu}\mathfrak{B}_u = p_{uu}^0\mathfrak{B}_z, \\ p_{ux}\mathfrak{B}_x + p_{uy}\mathfrak{B}_y + p_{uz}\mathfrak{B}_z + p_{uu}\mathfrak{B}_u = p_{uu}^0\mathfrak{B}_u. \end{cases}$$

We get expressions for the components of the elastic energy current and the impulse density from the first three of these equations. Namely, by employing equations (4), we will find that:

$$-i\,c\,p_{ux} = -p_{uu}^0\,\mathfrak{v}_x + p_{xx}\,\mathfrak{v}_x + p_{xy}\,\mathfrak{v}_y + p_{xz}\,\mathfrak{v}_z\,,$$

so

$$(26)\qquad \begin{cases} \mathfrak{S}_x^e = c^2\mathfrak{g}_x^e = -p_{uu}^0\mathfrak{v}_x + p_{xx}\mathfrak{v}_x + p_{xy}\mathfrak{v}_y + p_{xz}\mathfrak{v}_z, \\ \mathfrak{S}_y^e = c^2\mathfrak{g}_y^e = -p_{uu}^0\mathfrak{v}_y + p_{yx}\mathfrak{v}_x + p_{yy}\mathfrak{v}_y + p_{yz}\mathfrak{v}_z, \\ \mathfrak{S}_z^e = c^2\mathfrak{g}_z^e = -p_{uu}^0\mathfrak{v}_z + p_{zx}\mathfrak{v}_x + p_{zy}\mathfrak{v}_y + p_{zz}\mathfrak{v}_z. \end{cases}$$

We can also express these vector components in terms of the relative stresses when we eliminate p by means of (12a). We will get:

$$(26a)\qquad \mathfrak{S}_x^e\,(1-\mathfrak{q}^2) = -p_{uu}^0\,\mathfrak{v}_x + t_{xx}\,\mathfrak{v}_x + t_{xy}\,\mathfrak{v}_y + t_{xz}\,\mathfrak{v}_z\,,$$

and correspondingly for the two remaining components.

One gets the expression:

$$(27)\qquad \psi^e = -\ p_{uu}^0 + \mathfrak{g}^e\,\mathfrak{v}$$

from the last equation (25).

§ 3. The changes in mass and rest energy.

In order to obtain the law for the variation of mass, we multiply equations (6) by $\mathfrak{B}_x$, $\mathfrak{B}_y$, $\mathfrak{B}_z$, $\mathfrak{B}_u$, in succession, and add them. In that way, we observe that:

$$\mathfrak{B}_x\left\{\frac{\partial}{\partial x}v\mathfrak{B}_x^2 + \frac{\partial}{\partial y}v\mathfrak{B}_x\mathfrak{B}_y + \frac{\partial}{\partial z}v\mathfrak{B}_x\mathfrak{B}_z + \frac{\partial}{\partial u}v\mathfrak{B}_x\mathfrak{B}_u\right\}$$

$$= \mathfrak{B}_x^2\left\{\frac{\partial}{\partial x}v\mathfrak{B}_x + \frac{\partial}{\partial y}v\mathfrak{B}_y + \frac{\partial}{\partial z}v\mathfrak{B}_z + \frac{\partial}{\partial u}v\mathfrak{B}_u\right\}$$

$$+\ \tfrac{1}{2}\nu\left\{\mathfrak{B}_x\frac{\partial\mathfrak{B}_x^2}{\partial x}+\mathfrak{B}_y\frac{\partial\mathfrak{B}_x^2}{\partial y}+\mathfrak{B}_z\frac{\partial\mathfrak{B}_x^2}{\partial z}+\mathfrak{B}_u\frac{\partial\mathfrak{B}_x^2}{\partial u}\right\},\ \text{etc.}$$

Since we further have:

$$\mathfrak{B}_x^2+\mathfrak{B}_y^2+\mathfrak{B}_z^2+\mathfrak{B}_u^2=-c^2, \tag{28}$$

from the foundations of the theory of relativity, we will get:

$$\left\{\begin{aligned}&\mathfrak{B}_x(\mathfrak{K}_x+\mathfrak{K}_x^e)+\mathfrak{B}_y(\mathfrak{K}_y+\mathfrak{K}_y^e)+\mathfrak{B}_z(\mathfrak{K}_z+\mathfrak{K}_z^e)+\mathfrak{B}_u(\mathfrak{K}_u+\mathfrak{K}_u^e)\\&\qquad=-c^2\left\{\frac{\partial}{\partial x}\nu\mathfrak{B}_x+\frac{\partial}{\partial y}\nu\mathfrak{B}_y+\frac{\partial}{\partial z}\nu\mathfrak{B}_z+\frac{\partial}{\partial u}\nu\mathfrak{B}_u\right\}.\end{aligned}\right. \tag{29}$$

That equation gives the variation of mass in time when $d\upsilon_0$ is the rest volume of a material particle [cf., equation (9)]:

$$\frac{\partial}{\partial x}\nu\mathfrak{B}_x+\frac{\partial}{\partial y}\nu\mathfrak{B}_y+\frac{\partial}{\partial z}\nu\mathfrak{B}_z+\frac{\partial}{\partial u}\nu\mathfrak{B}_u=\frac{1}{d\tau_0}\frac{d}{d\tau}(\nu\, d\upsilon_0), \tag{30}$$

in which $\nu\, d\upsilon_0 = \mu\, d\upsilon$ is the mass of the particle.

If the sum of the external and elastic forces is orthogonal to the motion vector $\mathfrak{B}$ then the mass of the matter will be unchanging in time, but otherwise, it will not.

We would like to present a formula for the elastic force, and to that end, we differentiate equations (25) with respect to x, y, z, u, and add them. After a simple conversion and observing (5), we will then get:

$$\left\{\begin{aligned}&\mathfrak{B}_x\mathfrak{K}_x^e+\mathfrak{B}_y\mathfrak{K}_y^e+\mathfrak{B}_z\mathfrak{K}_z^e+\mathfrak{B}_u\mathfrak{K}_u^e\\&\quad=p_{xx}\frac{\partial\mathfrak{B}_x}{\partial x}+p_{yy}\frac{\partial\mathfrak{B}_y}{\partial y}+p_{zz}\frac{\partial\mathfrak{B}_z}{\partial z}+p_{uu}\frac{\partial\mathfrak{B}_u}{\partial u}\\&\quad+p_{xy}\left\{\frac{\partial\mathfrak{B}_x}{\partial y}+\frac{\partial\mathfrak{B}_y}{\partial x}\right\}+p_{xz}\left\{\frac{\partial\mathfrak{B}_x}{\partial z}+\frac{\partial\mathfrak{B}_z}{\partial x}\right\}+p_{xu}\{\}+p_{yz}\{\}+p_{yx}\{\}+p_{zu}\{\}\\&\quad-\left\{\frac{\partial}{\partial x}p_{uu}^0\mathfrak{B}_x+\frac{\partial}{\partial y}p_{uu}^0\mathfrak{B}_y+\frac{\partial}{\partial z}p_{uu}^0\mathfrak{B}_z+\frac{\partial}{\partial u}p_{uu}^0\mathfrak{B}_u\right\}.\end{aligned}\right. \tag{31}$$

When (29) is subtracted from (31), and one observes (24), that will yield:

$$
(32)\quad \left\{
\begin{aligned}
&\frac{\partial}{\partial x}\Psi\mathfrak{B}_x + \frac{\partial}{\partial y}\Psi\mathfrak{B}_y + \frac{\partial}{\partial z}\Psi\mathfrak{B}_z + \frac{\partial}{\partial u}\Psi\mathfrak{B}_u \\
&\qquad = -\{\mathfrak{B}_x\,\mathfrak{K}_x + \mathfrak{B}_y\,\mathfrak{K}_y + \mathfrak{B}_z\,\mathfrak{K}_z + \mathfrak{B}_u\,\mathfrak{K}_u\} \\
&\qquad - p_{xx}\frac{\partial\mathfrak{B}_x}{\partial x} + p_{yy}\frac{\partial\mathfrak{B}_y}{\partial y} + p_{zz}\frac{\partial\mathfrak{B}_z}{\partial z} + p_{uu}\frac{\partial\mathfrak{B}_u}{\partial u} \\
&\qquad + p_{xy}\left\{\frac{\partial\mathfrak{B}_x}{\partial y} + \frac{\partial\mathfrak{B}_y}{\partial x}\right\} + p_{xz}\left\{\frac{\partial\mathfrak{B}_x}{\partial z} + \frac{\partial\mathfrak{B}_z}{\partial x}\right\} + p_{xu}\{\} + p_{yz}\{\} + p_{yx}\{\} + p_{zu}\{\}.
\end{aligned}
\right.
$$

This equation expresses the law of energy for a rest unit volume that moves with the matter, as opposed to equation (22), which refers to a unit volume that is fixed in the spatial reference system employed. Equation (32) is completely symmetric relative to x, y, z, u; one will easily find that it actually expresses the law of energy upon transforming to rest.

§ 4. Definition of the inertial mass.

Up to now, we have considered the rest mass density to be a completely arbitrary function of the four coordinates of the space-time point of matter. We would now like to remove that indeterminacy, and for that reason, we shall next direct our attention to the various possibilities for doing that.

In equation (24):

$$\Psi = -\,p_{uu}^0 + c^2\,\nu,$$

the rest energy Ψ is a well-defined quantity; however, the quantities p_{uu}^0 and ν are freely at our disposal. We demand of p_{uu}^0 that it must be zero when no elastic stresses appear in the body considered; the tensor p shall represent the state of elastic stress, and only that one. Hence, if we transform to rest (matrix 23) then all of the spatial components of p shall be zero, and therefore p_{uu}^0 shall also be zero. However, that can be achieved in different ways.

When one considers only bodies in which an omnidirectional normal pressure is present, one can easily define the rest density ν in such a way that (when no heat conduction is present) the total inertia of the body will be determined by its mass. In order to do that, one needs only to set ([1]):

$$
\begin{aligned}
p_{xx}^0 &= p_{yy}^0 = p_{zz}^0 = p_{uu}^0, \\
0 &= p_{xy}^0 = p_{xz}^0 = p_{xu}^0 = \dots
\end{aligned}
$$

in the matrix (23), so ν will then be determined by (24).

([1]) **G. Nordstrøm**, Phys. Zeit. **12** (1911), pp. 854; **M. Laue**, *Das Relativitätsprinzip*, pp. 151.

The stress tensor *p* will then degenerate into a scalar, and the elastic impulse density $\mathfrak{g}^e$ will be equal to zero independently of the motion.

As far as I can see, that is why that way of looking at mass cannot be casually extended to the general case in which (relative) tangential stresses also appear in the body. It seems to me that the simplest and most preferable definition in the general case is to set:

(33) $$c^2\,\nu = \Psi.$$

The rest mass density will then be set to something proportional to the rest energy density. From (24), one will then have:

(34) $$p^0_{xu} = 0,$$

and several of our previous equations will be simplified by that.

Naturally, the factual content of relativistic mechanics will not be touched upon by that at all, namely, how we define the inertial mass. Our definition will first take on a more than formal definition later when we also ascribe gravity to the inertial mass in § 6.

If we now establish the concept of mass by the definition (33) then we will have to remark that an impulse $\mathfrak{g}^e$ will appear in any moving and elastically-stressed body that is determined, not by the mass of the body, but by the state of elastic stress. From (26) and (26a), we get:

(35) $$\left\{\begin{aligned} \mathfrak{g}^e_x &= \frac{1}{c^2}(p_{xx}\mathfrak{v}_x + p_{xy}\mathfrak{v}_y + p_{xz}\mathfrak{v}_z) \\ &= \frac{1}{c^2(1-\mathfrak{q}^2)}(t_{xx}\mathfrak{v}_x + t_{xy}\mathfrak{v}_y + t_{xz}\mathfrak{v}_z), \text{ etc.}\end{aligned}\right.$$

That impulse will also appear when an omnidirectional normal pressure is present in the body considered. In that case, $\mathfrak{g}^e$ can be derived from an *apparent* inertial mass that is added to the one that is defined the equation (33).

§ 5. Influence of heat conduction.

All of the equations that were presented will also be valid when heat condition is present in the bodies in question, so the influence of heat conduction can be ascribes to a ponderomotive force $\mathfrak{K}^w$ that appears in the heat conduction field, which is a force that is counted along with the external force $\mathfrak{K}$. The energetic components of the heat conduction force $\mathfrak{K}^w$ are the ones that play the essential role. Like all ponderomotive forces, from our basic assumptions, $\mathfrak{K}^w$ shall also be derived from a symmetric, four-dimensional tensor. We denote the heat conduction tensor by *w*, so we will have:

$$(36)\qquad \left\{\begin{array}{l} \mathfrak{K}_x^w = -\dfrac{\partial w_{xx}}{\partial x} - \dfrac{\partial w_{xy}}{\partial y} - \dfrac{\partial w_{xz}}{\partial z} - \dfrac{\partial w_{xu}}{\partial u}, \\ \dotfill \\ \dotfill \\ \mathfrak{K}_u^w = -\dfrac{\partial w_{ux}}{\partial x} - \dfrac{\partial w_{uy}}{\partial y} - \dfrac{\partial w_{uz}}{\partial z} - \dfrac{\partial w_{uu}}{\partial u}, \end{array}\right.$$

in which, one has, e.g.:

$$w_{zu} = w_{uz}, \qquad \text{etc.}$$

In the case of rest, the tensor w will give the following matrix:

$$(37)\qquad \left\{\begin{array}{cccc} 0 & 0 & 0 & w_{xu}^0 \\ 0 & 0 & 0 & w_{yu}^0 \\ 0 & 0 & 0 & w_{zu}^0 \\ w_{ux}^0 & w_{uy}^0 & w_{uz}^0 & 0, \end{array}\right.$$

so in the rest state, the total spatial stresses will be given by the tensor p (matrix 23), and the total energy density of matter will be given by $\Psi = c^2\, \nu$. The real components of w must then be zero in the rest case.

If we consider the conduction of heat then we will have three four-dimensional tensors that relate to the matter: viz., the heat conduction, elastic, and material tensors. In the case of rest, all three of them can be summarized in the following common matrix:

$$(38)\qquad \left\{\begin{array}{ccc|c} p_{xx}^0 & p_{xy}^0 & p_{xz}^0 & w_{xu}^0 \\ p_{yx}^0 & p_{yy}^0 & p_{yz}^0 & w_{yu}^0 \\ p_{zx}^0 & p_{zy}^0 & p_{zz}^0 & w_{zu}^0 \\ \hline w_{ux}^0 & w_{uy}^0 & w_{uz}^0 & -c^2\nu \end{array}\right. .$$

Since we have hit upon the convention (33), the decomposition of the total tensor into three parts is unique.

We can introduce a four-vector $\mathfrak{B}$ by the system of equations:

$$(39)\qquad \left\{\begin{array}{l} w_{xx}\mathfrak{B}_x + w_{xy}\mathfrak{B}_y + w_{xz}\mathfrak{B}_z + w_{xu}\mathfrak{B}_u = -\mathfrak{W}_x, \\ \dotfill \\ \dotfill \\ w_{ux}\mathfrak{B}_x + w_{uy}\mathfrak{B}_y + w_{uz}\mathfrak{B}_z + w_{uu}\mathfrak{B}_u = -\mathfrak{W}_x. \end{array}\right.$$

As we would like to show, that vector is referred to as the *rest heat current.* The law of energy for the heat conduction will indeed be expressed [cf., equation (21)] by the equation:

$$ic\,\mathfrak{K}_u^w = -\,i\,c\,\left\{\frac{\partial w_{ux}}{\partial x}+\frac{\partial w_{uy}}{\partial y}+\frac{\partial w_{uz}}{\partial z}+\frac{\partial w_{uu}}{\partial u}\right\},$$

and for the case of rest, one will get from (39) that:

$$\begin{aligned} i\,c\;w_{xu}^0 &= -\,\mathfrak{W}_x^0, \\ i\,c\;w_{yu}^0 &= -\,\mathfrak{W}_y^0, \\ i\,c\;w_{zu}^0 &= -\,\mathfrak{W}_z^0, \\ 0 &= -\,\mathfrak{W}_u^0, \end{aligned}$$

from which, the stated interpretation for the vector $\mathfrak{W}$ will emerge.

One also sees from the last equations that the four-vector $\mathfrak{W}$ is orthogonal to the motion vector $\mathfrak{B}$, such that:

(40) $$\mathfrak{W}_x\,\mathfrak{B}_x + \mathfrak{W}_y\,\mathfrak{B}_y + \mathfrak{W}_z\,\mathfrak{B}_z + \mathfrak{W}_u\,\mathfrak{B}_u = 0,$$

so the left-hand side of this will be invariant under Lorentz transformations and will equal zero when it is transformed to rest.

The tensor w can be expressed as a "tensor product" ([1]) of the two four-vectors $\mathfrak{W}$ and $\mathfrak{B}$. As one finds upon transforming to rest, one will have the following expressions:

(41) $$\left\{\begin{aligned} w_{xx} &= \frac{2}{c^2}\mathfrak{W}_x\,\mathfrak{B}_x, \\ w_{uu} &= \frac{2}{c^2}\mathfrak{W}_u\,\mathfrak{B}_u, \\ w_{xy} &= \frac{1}{c^2}\{\mathfrak{W}_x\,\mathfrak{B}_y + \mathfrak{W}_y\,\mathfrak{B}_x\}, \\ w_{zu} &= \frac{1}{c^2}\{\mathfrak{W}_x\,\mathfrak{B}_u + \mathfrak{W}_u\,\mathfrak{B}_x\}, \\ &\text{etc.} \end{aligned}\right.$$

for the components of w.

Naturally, one has:

(42) $$\psi^w = -\,w_{uu}\,,$$

(43) $$\mathfrak{S}_x^w = -\,i\,c\,w_{uu}\,, \qquad \mathfrak{S}_y^w = -\,i\,c\,w_{yu}\,, \qquad \mathfrak{S}_z^w = -\,i\,c\,w_{zu}$$

([1]) **W. Voigt**, Gött. Nachr. (1904), pp. 500.

for the energy density ψ^w and the energy current $\mathfrak{S}^w$ of the heat conduction field. These quantities can also be expressed in terms of the vector $\mathfrak{W}$. We initially find from (40) that:

$$-i\,c\,\mathfrak{W}_u = \mathfrak{W}\,\mathfrak{v}, \tag{44}$$

in which the scalar product of two three-dimensional vectors is on the right-hand side. From (41), we further get:

$$\psi^w = \frac{2}{c^2\sqrt{1-\mathfrak{q}^2}}\,\mathfrak{W}\,\mathfrak{v}, \tag{42a}$$

$$\left\{\begin{aligned} \mathfrak{S}^w &= \frac{1}{\sqrt{1-\mathfrak{q}^2}}\left\{\mathfrak{W} + \frac{\mathfrak{v}}{c^2}(\mathfrak{W}\,\mathfrak{v})\right\}, \\ &= \frac{\mathfrak{W}}{\sqrt{1-\mathfrak{q}^2}} + \mathfrak{v}\frac{\psi^w}{2}. \end{aligned}\right. \tag{43a}$$

Naturally, the energy current $\mathfrak{S}^w$ corresponds to the impulse density:

$$\mathfrak{g}^w = \frac{1}{c^2}\,\mathfrak{S}^w.$$

We can write the last of equations (36), when multiplied by *ic*, as:

$$i\,c\;\mathfrak{K}_u^w = \operatorname{div}\mathfrak{S}^w + \frac{\partial\psi^w}{\partial t}, \tag{45}$$

which can be introduced into the energy equation (22). When $\mathfrak{K}^w$ is the only "external" force that is present, one must naturally set $\mathfrak{K} = \mathfrak{K}^w$ in all equations of the previous paragraphs, and in particular, one must set $\mathfrak{K}_u = \mathfrak{K}_u^w$ in (22).

We would like to exhibit a few formulas for $\mathfrak{K}^w$. If we introduce the expressions (41) into the system of equations (36) then, after a simple conversion ([1]), we will get:

([1]) If φ is an arbitrary function of four coordinates then one will indeed have:

$$\frac{d\varphi}{d\tau} = \mathfrak{B}_x\frac{\partial\varphi}{\partial x} + \mathfrak{B}_y\frac{\partial\varphi}{\partial y} + \mathfrak{B}_z\frac{\partial\varphi}{\partial z} + \mathfrak{B}_u\frac{\partial\varphi}{\partial u}.$$

$$
(46)\quad \left\{
\begin{aligned}
c^2 \mathfrak{K}_x^w = & -\frac{d\mathfrak{W}}{d\tau} - \mathfrak{B}_x \left\{ \frac{\partial \mathfrak{W}_x}{\partial x} + \frac{\partial \mathfrak{W}_y}{\partial y} + \frac{\partial \mathfrak{W}_z}{\partial z} + \frac{\partial \mathfrak{W}_u}{\partial u} \right\} \\
& - \left\{ \mathfrak{W}_x \frac{\partial \mathfrak{B}_x}{\partial x} + \mathfrak{W}_y \frac{\partial \mathfrak{B}_y}{\partial y} + \mathfrak{W}_z \frac{\partial \mathfrak{B}_z}{\partial z} + \mathfrak{W}_u \frac{\partial \mathfrak{B}_u}{\partial u} \right\} \\
& - \mathfrak{W}_x \left\{ \frac{\partial \mathfrak{B}_x}{\partial x} + \frac{\partial \mathfrak{B}_y}{\partial y} + \frac{\partial \mathfrak{B}_z}{\partial z} + \frac{\partial \mathfrak{B}_u}{\partial u} \right\},
\end{aligned}
\right.
$$

and corresponding expressions for the remaining components of $\mathfrak{K}^w$. If we multiply these expressions by $\mathfrak{B}_x$, $\mathfrak{B}_y$, $\mathfrak{B}_z$, $\mathfrak{B}_u$, resp., and add them then, upon observing (40) and (28), we will further get:

$$
(47)\quad \left\{
\begin{aligned}
& \mathfrak{B}_x \mathfrak{K}_x^w + \mathfrak{B}_y \mathfrak{K}_y^w + \mathfrak{B}_z \mathfrak{K}_z^w + \mathfrak{B}_u \mathfrak{K}_u^w \\
& = \frac{\partial \mathfrak{W}_x}{\partial x} + \frac{\partial \mathfrak{W}_y}{\partial y} + \frac{\partial \mathfrak{W}_z}{\partial z} + \frac{\partial \mathfrak{W}_u}{\partial u} + \frac{1}{c^2} \left\{ \mathfrak{W}_x \frac{d\mathfrak{B}_x}{d\tau} + \mathfrak{W}_y \frac{d\mathfrak{B}_y}{d\tau} + \mathfrak{W}_z \frac{d\mathfrak{B}_z}{d\tau} + \mathfrak{W}_u \frac{d\mathfrak{B}_u}{d\tau} \right\}.
\end{aligned}
\right.
$$

This formula allows one to consider heat conduction in formulas (29) and (32)

§ 6. Gravitation.

The treatment of gravitational phenomena from the standpoint of the theory of relativity has been attempted from several angles. In particular, the theories of **Einstein** ([1]) and **Abraham** ([2]) should be mentioned. However, in those two theories, the speed of light was not constant, but depended upon the gravitational field, and that situation would demand at least a completely radical change in the foundations of the theory of relativity up to now.

However, as I have shown in another place ([3]), one can keep the constancy of the speed of light by altering **Abraham**'s theory and develop a theory that is compatible with the theory of relativity in its form up to now. Since I would like to generalize that theory at some point, its foundations might be recalled here.

I introduce a gravitational potential Φ and set:

$$
(48)\qquad \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial u^2} = g\, \nu,
$$

in which I have employed rational units. Here, ν is the rest density of matter that is defined by equation (33). The gravitational potential Φ and the quantity g are also four-dimensional scalars; we call g the *gravitational factor.*

([1]) **A. Einstein**, Ann. Phys. (Leipzig) **35** (1911), pp. 898.
([2]) **M. Abraham**, Phys. Zeit. **13** (1912), 1.
([3]) **G. Nordstrøm**, Phys. Zeit. **13** (1912), 1126.

The gravitational field exerts forces on the material bodies that are found in it. I set:

$$\mathfrak{K}_x^g = -g\,v\frac{\partial\Phi}{\partial x}, \quad \mathfrak{K}_y^g = -g\,v\frac{\partial\Phi}{\partial y}, \quad \mathfrak{K}_z^g = -g\,v\frac{\partial\Phi}{\partial z}, \quad \mathfrak{K}_u^g = -g\,v\frac{\partial\Phi}{\partial u} \tag{49}$$

for the ponderomotive gravitational force $\mathfrak{K}^g$ per unit volume.

Equations (48) and (49), along with the principle of the constancy of c:

$$c = \text{universal constant}, \tag{50}$$

give the complete foundations for my theory of gravitation. Those equations also determine the rational units of Φ and g. I initially regard the gravitational factor g as a universal constant. However, I remark here that nothing prevents me from taking g to depend upon the internal state of the matter, since g occurs only as a factor of v.

The basic equations (48), (49), (50) demand that the mass of a material particle depend upon the gravitational potential at the same point. In order to obtain the law of that dependency, we conveniently consider the motion of a mass point of mass m in an arbitrary gravitational field. No other forces than gravitation can act on the mass point. We can then write the equations of motion of the mass point as follows [cf., equation (6) and (9)]:

$$\left\{\begin{aligned} -gm\frac{\partial\Phi}{\partial x} &= m\frac{d\mathfrak{B}_x}{d\tau} + \mathfrak{B}_x\frac{dm}{d\tau}, \\ -gm\frac{\partial\Phi}{\partial y} &= m\frac{d\mathfrak{B}_y}{d\tau} + \mathfrak{B}_y\frac{dm}{d\tau}, \\ -gm\frac{\partial\Phi}{\partial z} &= m\frac{d\mathfrak{B}_z}{d\tau} + \mathfrak{B}_z\frac{dm}{d\tau}, \\ -gm\frac{\partial\Phi}{\partial u} &= m\frac{d\mathfrak{B}_u}{d\tau} + \mathfrak{B}_u\frac{dm}{d\tau}. \end{aligned}\right. \tag{51}$$

We multiply the equations by $\mathfrak{B}_x$, $\mathfrak{B}_y$, $\mathfrak{B}_z$, $\mathfrak{B}_u$, in succession, and add them. Upon observing (28), since:

$$\frac{d\Phi}{d\tau} = \mathfrak{B}_x\frac{\partial\Phi}{\partial x} + \mathfrak{B}_y\frac{\partial\Phi}{\partial y} + \mathfrak{B}_z\frac{\partial\Phi}{\partial z} + \mathfrak{B}_u\frac{\partial\Phi}{\partial u},$$

$$-g\,m\,\frac{d\Phi}{d\tau} = -c^2\,\frac{dm}{d\tau},$$

or

$$\frac{1}{m}\frac{dm}{d\tau} = \frac{g}{c^2}\frac{d\Phi}{d\tau}. \tag{52}$$

If g is taken to be constant then integration will give:

$$m = m_0\,e^{(g/c^2)\Phi}, \tag{53}$$

and that equation will give the dependency of the mass upon the gravitational potential.

From (52), the equations of motion of a mass-point can also be written in the following form:

$$(54)\qquad \begin{cases} -g\dfrac{\partial\Phi}{\partial x} = \dfrac{d\mathfrak{B}_x}{d\tau} + \dfrac{g}{c^2}\mathfrak{B}_x\dfrac{d\Phi}{d\tau}, \\ -g\dfrac{\partial\Phi}{\partial y} = \dfrac{d\mathfrak{B}_y}{d\tau} + \dfrac{g}{c^2}\mathfrak{B}_y\dfrac{d\Phi}{d\tau}, \\ -g\dfrac{\partial\Phi}{\partial z} = \dfrac{d\mathfrak{B}_z}{d\tau} + \dfrac{g}{c^2}\mathfrak{B}_z\dfrac{d\Phi}{d\tau}, \\ -g\dfrac{\partial\Phi}{\partial u} = \dfrac{d\mathfrak{B}_u}{d\tau} + \dfrac{g}{c^2}\mathfrak{B}_u\dfrac{d\Phi}{d\tau}, \end{cases}$$

in which the mass *m* drops out of the equations of motion.

The variability of the mass has its basis in the fact that the gravitational force $\mathfrak{K}^g$ is not orthogonal to the motion vector $\mathfrak{B}$ (cf., pp. 9). If we multiply equations (49) by $\mathfrak{B}_x$, $\mathfrak{B}_y$, $\mathfrak{B}_z$, $\mathfrak{B}_u$, and add them then we will get:

$$(55)\qquad \mathfrak{B}_x\,\mathfrak{K}_x^g + \mathfrak{B}_y\,\mathfrak{K}_y^g + \mathfrak{B}_z\,\mathfrak{K}_z^g + \mathfrak{B}_u\,\mathfrak{K}_u^g = -\,g\,v\,\frac{d\Phi}{d\tau}.$$

We can introduce that expression into equation (29) for the variation of mass; naturally, the gravitational $\mathfrak{K}^g$ belongs to the "external" force $\mathfrak{K}$.

The gravitational force $\mathfrak{K}^g$ is derived from a symmetric, four-dimensional tensor *G* by way of:

$$(56)\qquad \begin{cases} \mathfrak{K}_x^g = -\dfrac{\partial G_{xx}}{\partial x} - \dfrac{\partial G_{xy}}{\partial y} - \dfrac{\partial G_{xz}}{\partial z} - \dfrac{\partial G_{xu}}{\partial u}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \mathfrak{K}_u^g = -\dfrac{\partial G_{ux}}{\partial x} - \dfrac{\partial G_{uy}}{\partial y} - \dfrac{\partial G_{uz}}{\partial z} - \dfrac{\partial G_{uu}}{\partial u}. \end{cases}$$

One gets equations of that form when one introduces the expression (48) for *g v* into (49) and then performs a conversion. One also finds the following expressions ([1]) for the tensor components then:

([1]) **Abraham** obtained precisely the same expression in his aforementioned theory; **M. Abraham**, *loc. cit.*, pp. 3.

$$(57)\qquad \begin{cases} G_{xx} = \dfrac{1}{2}\left\{\left(\dfrac{\partial\Phi}{\partial x}\right)^2 - \left(\dfrac{\partial\Phi}{\partial y}\right)^2 - \left(\dfrac{\partial\Phi}{\partial z}\right)^2 - \left(\dfrac{\partial\Phi}{\partial u}\right)^2\right\}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ G_{uu} = \dfrac{1}{2}\left\{-\left(\dfrac{\partial\Phi}{\partial x}\right)^2 - \left(\dfrac{\partial\Phi}{\partial y}\right)^2 - \left(\dfrac{\partial\Phi}{\partial z}\right)^2 + \left(\dfrac{\partial\Phi}{\partial u}\right)^2\right\}, \\ G_{xy} = \dfrac{\partial\Phi}{\partial x}\dfrac{\partial\Phi}{\partial y}, \\ \dots\dots\dots\dots \\ G_{zu} = \dfrac{\partial\Phi}{\partial z}\dfrac{\partial\Phi}{\partial u}. \end{cases}$$

Those quantities give the fictitious gravitational stresses (pressure stresses, measured positively), as well as the impulse density, energy current, and energy density in a gravitational field. One has:

$$\mathfrak{S}_x^g = c^2\,\mathfrak{g}_x^g = -\,i\,c\,G_{xu}\text{ , etc.}$$

for the energy current $\mathfrak{S}^g$ and the impulse density $\mathfrak{g}^g$, and:

$$\psi^g = -\,G_{uu}$$

for the energy density ψ^g, and thus, from (57):

$$(58)\qquad \mathfrak{S}^g = c^2\mathfrak{g}^g = -\frac{\partial\Phi}{\partial t}\,\nabla\Phi,$$

$$(59)\qquad \psi^g = \frac{1}{2}\left\{(\nabla\Phi)^2 + \frac{1}{c^2}\left(\frac{\partial\Phi}{\partial t}\right)^2\right\},$$

in vector-analytic notation. One sees that ψ^g is always positive.

The last of equations (56), when multiplied by ic, now reads:

$$(60)\qquad ic\,\mathfrak{K}_u^g = \operatorname{div}\mathfrak{S}^g + \frac{\partial\psi^g}{\partial t},$$

which is an expression that expresses the law of energy for the gravitational field. Naturally, one has $\mathfrak{K}_u^g = 0$ for regions outside of the material body. Equation (60) combines with equation (22) for regions inside of the body.

Equation (48) can indeed be regarded as a four-dimensional **Poisson** equation, and its integration can be performed accordingly ([1]). However, the form of equation (48) also shows that one can calculate Φ from the known formula for the retarded potential. When one considers the possibility that g can be variable, one will have:

$$(61)\qquad \begin{cases} \Phi(x_0, y_0, z_0, u_0) = -\dfrac{1}{4\pi}\displaystyle\int \dfrac{dx\,dy\,dz}{r}(g\,v)_{x,y,z,t} + \text{const.} \\ \qquad\qquad = -\dfrac{1}{4\pi}\displaystyle\int \dfrac{dx\,dy\,dz}{r}\left(g\,\mu\sqrt{1-\mathfrak{q}^2}\right)_{x,y,z,t} + \text{const.}, \end{cases}$$

in which

$$(61a)\qquad \begin{cases} r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}, \\ t = t_0 - \dfrac{r}{c}. \end{cases}$$

The integration extends over three-dimensional space.

§ 7. Falling motion.

We would next like to present an equation for the motion of a mass-point in an arbitrary *static* gravitational field. We have two remarks to make about that: First of all, the fact that our theory does not admit any true point-like masses, since from (61), one would have $\Phi = -\infty$ at such a point, and therefore, from (53), the mass of the point would be zero. A "mass point" must always have a certain extension then. Secondly, it should be remarked that in order for the field to be regarded as static, the particle that moves in the field must be arranged such that its own field is vanishingly weak in comparison to the external field, even in the immediate vicinity of the particle.

Indeed, one will have:

$$\frac{\partial\Phi}{\partial t} = 0$$

in the static field.

We multiply the first three of equations (54) by $\mathfrak{v}_x$, $\mathfrak{v}_y$, $\mathfrak{v}_z$, resp., and add them. We then get $-g\,\mathfrak{v}\,\nabla\Phi$ on the left-hand side. We further have, in general:

$$(62)\qquad \frac{d\mathfrak{B}_x}{d\tau} = \frac{1}{1-\mathfrak{q}^2}\frac{d\mathfrak{v}_x}{dt} + \frac{\mathfrak{v}_x}{c^2(1-\mathfrak{q}^2)^2}\frac{d\mathfrak{v}_x}{dt}, \text{ etc.;}$$

hence:

$$\mathfrak{v}_x\frac{d\mathfrak{B}_x}{d\tau} + \mathfrak{v}_y\frac{d\mathfrak{B}_y}{d\tau} + \mathfrak{v}_z\frac{d\mathfrak{B}_z}{d\tau} = \frac{1}{(1-\mathfrak{q}^2)^2}\,\mathfrak{v}\frac{d\mathfrak{v}}{dt}.$$

([1]) **M. Abraham**, Phys. Zeit. **13** (1912), pp. 5; **A. Sommerfeld**, Ann. Phys. (Leipzig) **33** (1910), pp. 665.

Since we further have:

$$\frac{d\Phi}{d\tau} = \frac{1}{\sqrt{1-\mathfrak{q}^2}}\,\mathfrak{v}\,\nabla\Phi\,,$$

in our case, we will get:

$$-\,g\ \mathfrak{v}\ \nabla\Phi = \frac{1}{(1-\mathfrak{q}^2)^2}\,\mathfrak{v}\,\frac{d\mathfrak{v}}{dt} + g\,\frac{\mathfrak{q}^2}{1-\mathfrak{q}^2}\,\mathfrak{v}\,\nabla\Phi\,,$$

and ultimately:

(63) $$-\,g\ \mathfrak{v}\ \nabla\Phi = \frac{1}{1-\mathfrak{q}^2}\,\mathfrak{v}\,\frac{d\mathfrak{v}}{dt}\,.$$

We would now like to assume, in particular, that the gravitational field is homogeneous and parallel to the *z*-axis, so:

$$\frac{\partial\Phi}{\partial z} = \text{const.},\quad \frac{\partial\Phi}{\partial x} = \frac{\partial\Phi}{\partial y} = \frac{\partial\Phi}{\partial u} = 0,$$

and examine the motion of a mass-point in that field. The third of equations (54) gives:

$$-\,g\ \frac{\partial\Phi}{\partial z} = \frac{1}{1-\mathfrak{q}^2}\,\frac{d\mathfrak{v}_x}{dt} + \frac{\mathfrak{v}_x}{c^2(1-\mathfrak{q}^2)^2}\,\mathfrak{v}\,\frac{d\mathfrak{v}}{dt} + g\,\frac{\mathfrak{v}_x}{c^2(1-\mathfrak{q}^2)}\,\frac{\partial\Phi}{\partial z}\,.$$

If we observe (63) then we will find that the last two terms cancel each other, and we will get:

$$\frac{d\mathfrak{v}_x}{dt} = -\,(1-\mathfrak{q}^2)\,g\,\frac{\partial\Phi}{\partial z}\,.$$

The first of equations (54) gives:

$$0 = \frac{1}{1-\mathfrak{q}^2}\,\frac{d\mathfrak{v}_x}{dt} + \frac{\mathfrak{v}_x}{c^2(1-\mathfrak{q}^2)^2}\,\mathfrak{v}\,\frac{d\mathfrak{v}}{dt} + g\,\frac{\mathfrak{v}_x\mathfrak{v}_z}{c^2(1-\mathfrak{q}^2)}\,\frac{\partial\Phi}{\partial z}$$

in a similar way. The last two terms also cancel here; one will then have $d\mathfrak{v}_x / dt = 0$. Since the same thing must be true for $d\mathfrak{v}_y / dt$, we will get the equations of motion:

(64) $$\begin{cases} \dfrac{d\mathfrak{v}_x}{dt} = -\left(1-\dfrac{\mathfrak{v}^2}{c^2}\right) g\,\dfrac{\partial\Phi}{\partial z}, \\[2ex] \dfrac{d\mathfrak{v}_x}{dt} = \dfrac{d\mathfrak{v}_x}{dt} = 0 \end{cases}$$

for a mass-point in a homogeneous gravitational field.

These equations state the following:

The component of the motion that is perpendicular to the field direction is uniform. The acceleration of falling will get smaller as the velocity gets larger, but independently of the direction of the velocity. A body that is thrown in a horizontal direction will fall slower than one that falls vertically with no initial velocity.

One also sees that a rotating body must fall slower than a non-rotating one. Naturally, for attainable rotational velocities, the difference is much too small to be accessible to observation.

That result raises the question of whether the molecular motions of a falling body will have any influence on the acceleration of falling. At the very least, one cannot reject the possibility that this is the case. One must then modify the theory of gravitation simply by considering the gravitational factor g, not to be constant, but as something that depends upon the molecular motion of the body. For that reason, we have left that possibility open in the foregoing development. In connection with that, let us suggest that the mass density of a body also depends upon the molecular motions such that the rest energy density, which determines v by equation (33), will be influences by those motions.

Nonetheless, the questions in the theory of gravitation that are connected with the atomic structure of matter lie beyond the scope of this article.

Helsingfors, January 1913.

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