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The principle of relativity and gravitation

By **Gunnar Nordstrøm**

Translated by D. H. Delphenich

It emerged from the discussion between **Einstein** and **Abraham** ([1]) that **Einstein**’s hypothesis that the speed of light c depends upon the gravitational potential ([2]) leads to considerable difficulties in regard to the principle of relativity,. For that reason, one might ask whether it is possible to replace **Einstein**’s hypothesis with another one that keeps c constant and then adjust the principle of relativity of the theory of gravitation in such a way that gravitational and inertial mass will be equal ([3]). I believe that I have found such a hypothesis, and I would like to present it in what follows.

Let x, y, z, u be the four coordinates:

$$u = i\,c\,t.$$

With **Abraham** ([4]), I set:

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}+\frac{\partial^2\Phi}{\partial u^2}=4\pi f\,\gamma, \tag{1}$$

in which I have denoted the rest density of matter by γ and the gravitational potential by Φ. Φ, as well as γ, are four-dimensional quantities; f is the gravitational constant. In the gravitational field, we have a four-vector:

$$\mathfrak{F}_x=-\frac{\partial\Phi}{\partial x}, \qquad \mathfrak{F}_y=-\frac{\partial\Phi}{\partial y}, \qquad \mathfrak{F}_z=-\frac{\partial\Phi}{\partial z}, \qquad \mathfrak{F}_u=-\frac{\partial\Phi}{\partial u}, \tag{2}$$

from which, the acceleration of a mass-point in the field must originate. However, if one regards the four-vector $\mathfrak{F}$ as the *moving* force that acts upon an unvarying unit mass then one cannot maintain the constancy of the speed of light. Namely, in that case, $\mathfrak{F}$ would be equal to the four-dimensional acceleration vector of a mass-point, and it could not be

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([1]) See Ann. Phys. (Leipzig) **38** (1912), 355, 1056, 1059; *ibid.* **39** (1912), 444.

([2]) **A. Einstein**, Ann. Phys. (Leipzig) **35** (1911), 898.

([3]) However, I do not understand the equality of gravitational and inertial mass to mean that every inertial phenomenon will be required by the inertial and gravitational mass. Like **von Laue** (cf., *infra*), for elastically-stressed bodies, one will get a quantity of motion that does not imply a mass. I will come back to that question in a future publication.

([4]) **M. Abraham**, this Zeitschrift **13** (1912), 1.

perpendicular to the motion vector $\mathfrak{a}$ for an arbitrary direction of motion, as the constancy of the speed of light would require ([1]).

However, one can eliminate the difficulties when one keeps the speed of light constant, and in two different ways: Either one does not regard $\mathfrak{F}$ as the *moving force* ([2]), but only the part of it that is perpendicular to the motion vector, or one assumes that the mass of a mass-point is not constant, but depends upon the gravitational potential. The parallelism of the two four-vectors $\mathfrak{F}$ and $\mathfrak{a}$ will be abolished by these two assumptions; in the first case, by a supplementary force that enters into $\mathfrak{F}$, and in the second case, by the variability of mass. As we will see, the two methods will give the same law for the motion of a mass-point, but they correspond to two different ways of looking at the concept of force.

Corresponding to the viewpoint of most researchers in the field of relativity theory, I would next like to employ the second method, and therefore regard:

$$m\,\mathfrak{F}_x = -\,m\,\frac{\partial\Phi}{\partial x}, \qquad \text{etc.}$$

as the components of moving force that acts upon a mass-point, but I regard the rest mass m of the point as varying. If $\mathfrak{a}_x$, $\mathfrak{a}_y$, $\mathfrak{a}_z$, $\mathfrak{a}_u$ are the components of the motion vector and τ is the proper time then the equations of motion for the mass-point will read:

$$\left.\begin{aligned}
-m\frac{\partial\Phi}{\partial x} &= \frac{d}{d\tau}(m\,\mathfrak{a}_x) = m\frac{d\mathfrak{a}_x}{d\tau} + \mathfrak{a}_x\frac{dm}{d\tau},\\
-m\frac{\partial\Phi}{\partial y} &= \frac{d}{d\tau}(m\,\mathfrak{a}_y) = m\frac{d\mathfrak{a}_y}{d\tau} + \mathfrak{a}_y\frac{dm}{d\tau},\\
-m\frac{\partial\Phi}{\partial z} &= \frac{d}{d\tau}(m\,\mathfrak{a}_z) = m\frac{d\mathfrak{a}_z}{d\tau} + \mathfrak{a}_z\frac{dm}{d\tau},\\
-m\frac{\partial\Phi}{\partial u} &= \frac{d}{d\tau}(m\,\mathfrak{a}_u) = m\frac{d\mathfrak{a}_u}{d\tau} + \mathfrak{a}_u\frac{dm}{d\tau}.
\end{aligned}\right\} \tag{3}$$

We multiply these equations by $\mathfrak{a}_x$, $\mathfrak{a}_y$, $\mathfrak{a}_z$, $\mathfrak{a}_u$, resp., and add them. Since:

$$\frac{\partial\Phi}{\partial x}\mathfrak{a}_x + \frac{\partial\Phi}{\partial y}\mathfrak{a}_y + \frac{\partial\Phi}{\partial z}\mathfrak{a}_z + \frac{\partial\Phi}{\partial u}\mathfrak{a}_u = \frac{d\Phi}{d\tau},$$

$$\mathfrak{a}_x\frac{d\mathfrak{a}_x}{d\tau} + \mathfrak{a}_y\frac{d\mathfrak{a}_y}{d\tau} + \mathfrak{a}_z\frac{d\mathfrak{a}_z}{d\tau} + \mathfrak{a}_u\frac{d\mathfrak{a}_u}{d\tau} = 0,$$

([1]) **M. Abraham**, *loc. cit.*, eq. (5).

([2]) **Minkowski** proceeded in a corresponding way in his treatment of the electrodynamical force. Cf., Gött. Nachr. (1908), pp. 98, eq. (98).

$$\mathfrak{a}_x^2 + \mathfrak{a}_y^2 + \mathfrak{a}_z^2 + \mathfrak{a}_u^2 = -c^2,$$

we will get:

$$-m\,\frac{d\Phi}{d\tau} = -c^2\,\frac{dm}{d\tau} \qquad \text{or} \qquad \frac{1}{m}\frac{dm}{d\tau} = \frac{1}{c^2}\frac{d\Phi}{d\tau}. \tag{4}$$

Integration gives:

$$\ln m = \frac{1}{c^2}\Phi + \text{const.}$$

or

$$m = m_0\, e^{\Phi / c^2}. \tag{5}$$

This equation shows that the mass *m* depends upon the gravitational potential by a simple law.

From (4), one can also write the equations of motion (3) in the following form:

$$\left.\begin{aligned}
-\frac{\partial\Phi}{\partial x} &= \frac{d\mathfrak{a}_x}{d\tau} + \frac{\mathfrak{a}_x}{c^2}\frac{d\Phi}{d\tau}, \\
-\frac{\partial\Phi}{\partial y} &= \frac{d\mathfrak{a}_y}{d\tau} + \frac{\mathfrak{a}_y}{c^2}\frac{d\Phi}{d\tau}, \\
-\frac{\partial\Phi}{\partial z} &= \frac{d\mathfrak{a}_z}{d\tau} + \frac{\mathfrak{a}_z}{c^2}\frac{d\Phi}{d\tau}, \\
-\frac{\partial\Phi}{\partial u} &= \frac{d\mathfrak{a}_u}{d\tau} + \frac{\mathfrak{a}_u}{c^2}\frac{d\Phi}{d\tau}.
\end{aligned}\right| \tag{6}$$

As one sees, the mass *m* drops out of the equations. The laws by which a mass-point moves in a gravitational field are then completely independent of the mass of the point.

The considerations up to now have been based upon the assumption that $m\,\mathfrak{F}$ is the moving force. We would now like to assume, for the moment, that the moving force does not act upon $m\,\mathfrak{F}$ itself, but on the motion vector $\mathfrak{a}$. That part of $m\,\mathfrak{F}$ is a four-vector with the *x*-component ([1]):

$$m\,\mathfrak{F}_x + m\,\frac{\mathfrak{a}_x}{c^2}(\mathfrak{F}_x\,\mathfrak{a}_x + \mathfrak{F}_y\,\mathfrak{a}_y + \mathfrak{F}_z\,\mathfrak{a}_z + \mathfrak{F}_u\,\mathfrak{a}_u).$$

The second term is the *x*-component of a supplementary force that enters into $m\,\mathfrak{F}$. From (2), the expression can be converted into the following one:

$$-m\left\{\frac{\partial\Phi}{\partial x} + \frac{\mathfrak{a}_x}{c^2}\frac{d\Phi}{d\tau}\right\}.$$

Since we now regard *m* as constant, the first equation of motion of a mass-point will read:

([1]) Cf., **H. Minkowski**, *loc. cit.*

$$-m\left\{\frac{\partial\Phi}{\partial x}+\frac{\mathfrak{a}_x}{c^2}\frac{d\Phi}{d\tau}\right\}=m\,\frac{d\mathfrak{a}_x}{d\tau}.$$

However, that is nothing but the first of the equations of motion (6).

We will then get precisely the same laws for the motion of a mass-point in a gravitational field by both of the alternative assumptions, except that the force and the mass will be different in the two cases. The latter way of looking at things that we considered corresponds to the original one of **Minkowski**, while the former one was the one that **von Laue** and **Abraham** maintained ([1]).

Up to now, we have considered an isolated mass-point. We would now like to examine the motion of arbitrary bodies in a gravitational field and present the energy theorem for the process. We assume only that there actually is a real meaning to the mass of each particle in the bodies, such that we can speak of the rest density γ at a space-time point. That is certainly the case when no tangential stresses are present in the bodies ([2]). Naturally, the rest-density γ is a function of the four coordinates:

$$\gamma=\gamma(x, y, z, u).$$

We again regard the mass as variable and accept the concept of force that is based upon eq. (3). The components of the force that gravitation exerts upon the *unit volume* of matter ([3]) are then:

$$\mathfrak{K}_x=-\gamma\frac{\partial\Phi}{\partial x},\quad \mathfrak{K}_y=-\gamma\frac{\partial\Phi}{\partial y},\quad \mathfrak{K}_z=-\gamma\frac{\partial\Phi}{\partial z},\quad \mathfrak{K}_u=-\gamma\frac{\partial\Phi}{\partial u}. \tag{7}$$

For the sake of generality, we assume that in addition to gravitation, yet another "external" force $\mathfrak{K}'$ with the components:

$$\mathfrak{K}'_x,\ \mathfrak{K}'_y,\ \mathfrak{K}'_z,\ \mathfrak{K}'_u$$

acts upon a unit volume of matter. We can then write the equations of motion of matter in the following general form ([4]):

([1]) Cf., also, the discussion between **Abraham** and the author in this Zeitschrift **10** (1909), 681, 737.; *ibid.* **11** (1910), 440, 527. I shall now assume the viewpoint that **Abraham** held in that.

([2]) Cf., **M. Laue**, *Das Relativitäsprinzip,* Braunschweig, 1911, pp. 151, *et seq.*; **G. Nordstrøm**, this Zeitschrift **12** (1911), 854.

([3]) If the four-vector $\mathfrak{K}$ were regarded as the *moving* force then it would have to be referred to as the *moving force per unit rest volume*.

([4]) **G. Nordstrøm**, this Zeitschrift **11** (1910), 441, eq. (4′).

$$\left.\begin{array}{l} -\gamma\dfrac{\partial\Phi}{\partial x}+\mathfrak{K}'_x=\dfrac{\partial}{\partial x}\gamma\mathfrak{a}_x^{\,2}+\dfrac{\partial}{\partial y}\gamma\mathfrak{a}_x\mathfrak{a}_y+\dfrac{\partial}{\partial z}\gamma\mathfrak{a}_x\mathfrak{a}_z+\dfrac{\partial}{\partial u}\gamma\mathfrak{a}_x\mathfrak{a}_u, \\ \dotfill \\ \dotfill \\ -\gamma\dfrac{\partial\Phi}{\partial u}+\mathfrak{K}'_u=\dfrac{\partial}{\partial x}\gamma\mathfrak{a}_u\mathfrak{a}_x+\dfrac{\partial}{\partial y}\gamma\mathfrak{a}_u\mathfrak{a}_y+\dfrac{\partial}{\partial z}\gamma\mathfrak{a}_u\mathfrak{a}_z+\dfrac{\partial}{\partial u}\gamma\mathfrak{a}_x^{2}. \end{array}\right\} \tag{8}$$

If we would like to introduce the usual three-dimensional velocity $\mathfrak{v}$ and the usual mass-density ρ then we would have to set:

$$\mathfrak{a}_x=\frac{\mathfrak{v}_x}{\sqrt{1-\mathfrak{q}^2}},\ \ldots,\qquad \mathfrak{a}_u=\frac{ic}{\sqrt{1-\mathfrak{q}^2}},\qquad \gamma=\rho\,\sqrt{1-\mathfrak{q}^2},$$

in which we have set $\mathfrak{q}=\mathfrak{v}\,/\,c$, for the sake of simplicity. We multiply the last of equations (8) by $-\,ic$ and introduce the expression above on the right-hand side. If we further employ the notations of three-dimensional vector analysis then the equation will read:

$$\gamma\frac{\partial\Phi}{\partial t}-ic\,\mathfrak{K}'_u=c^2\operatorname{div}\frac{\rho\mathfrak{v}}{\sqrt{1-\mathfrak{q}^2}}+c^2\frac{\partial}{\partial t}\frac{\rho}{\sqrt{1-\mathfrak{q}^2}}. \tag{9}$$

We would like to convert the first term on the left. Eq. (1) gives:

$$4\pi f\,\gamma=\operatorname{div}\nabla\Phi-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2},$$

so

$$\begin{aligned}\gamma\frac{\partial\Phi}{\partial t}&=\frac{1}{4\pi f}\left\{\frac{\partial\Phi}{\partial t}\operatorname{div}\nabla\Phi-\frac{1}{c^2}\frac{\partial\Phi}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial t}\right)\right\}\\ &=\frac{1}{4\pi f}\left\{\operatorname{div}\left(\frac{\partial\Phi}{\partial t}\nabla\Phi\right)-\nabla\Phi\cdot\frac{\partial}{\partial t}\nabla\Phi-\frac{1}{2c^2}\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial t}\right)^2\right\},\end{aligned}$$

$$\gamma\frac{\partial\Phi}{\partial t}=\frac{1}{4\pi f}\operatorname{div}\left(\frac{\partial\Phi}{\partial t}\nabla\Phi\right)-\frac{1}{8\pi f}\frac{\partial}{\partial t}\left\{(\nabla\Phi)^2+\frac{1}{c^2}\left(\frac{\partial\Phi}{\partial t}\right)^2\right\}.$$

We substitute this expression in eq. (9) and thus obtain the following equation, which expresses the energy theorem:

$$\begin{aligned}&-ic\,\mathfrak{K}'_u\\ &=-\frac{1}{4\pi f}\operatorname{div}\left(\frac{\partial\Phi}{\partial t}\nabla\Phi\right)+\frac{1}{8\pi f}\frac{\partial}{\partial t}\left\{(\nabla\Phi)^2+\frac{1}{c^2}\left(\frac{\partial\Phi}{\partial t}\right)^2\right\}+c^2\operatorname{div}\frac{\rho\mathfrak{v}}{\sqrt{1-\mathfrak{q}^2}}+c^2\frac{\partial}{\partial t}\frac{\rho}{\sqrt{1-\mathfrak{q}^2}}.\end{aligned}$$

The quantity $-ic\,\mathfrak{K}'_u$ gives the energy supplied per unit volume and unit time by the external force $\mathfrak{K}'$. The first two of the terms on the right refers to the gravitational field, while the second two refer to the matter of the body. If we set:

$$\mathfrak{S}^K = -\frac{1}{4\pi f}\frac{\partial\Phi}{\partial t}\nabla\Phi\,, \tag{11}$$

$$\psi^K = \frac{1}{8\pi f}\left\{(\nabla\Phi)^2 + \frac{1}{c^2}\left(\frac{\partial\Phi}{\partial t}\right)^2\right\}, \tag{12}$$

$$\mathfrak{S}^m = \frac{c^2\rho\mathfrak{v}}{\sqrt{1-\mathfrak{q}^2}}\,, \tag{13}$$

$$\psi^m = \frac{c^2\rho}{\sqrt{1-\mathfrak{q}^2}}\,. \tag{14}$$

ψ^K is the *energy density* of the gravitational field, $\mathfrak{S}^K$ is the *energy current* in that field, while ψ^m and $\mathfrak{S}^m$ are the energy density and convective energy current of matter. We have already found the previous expression (13) and (14) for these quantities ([1]).

We remark that, from (12), the energy density of the field is always positive.

The energy equation now ultimately reads:

$$-ic\,\mathfrak{K}'_u = \mathrm{div}\,(\mathfrak{S}^K + \mathfrak{S}^m) + \frac{\partial}{\partial t}(\psi^K + \psi^m). \tag{15}$$

We then see that the law of conservation of energy is fulfilled.

The quantities $\mathfrak{S}^K$ and ψ^K depend upon a four-dimensional tensor that also gives fictitious stresses for the gravitational force $\mathfrak{K}$. That tensor is precisely the same one that **Abraham** obtained under different assumptions ([2]). The ten components of the gravitational tensor are:

([1]) **G. Nordstrøm**, *loc. cit.*, eq. (11) and (12); **M. Laue**, *loc. cit.*, § 24.
([2]) **M. Abraham**, *loc. cit.*, pp. 3.

$$
\left.\begin{aligned}
X_x &= \frac{1}{4\pi f}\left\{-\left(\frac{\partial\Phi}{\partial x}\right)^2 + \Psi\right\}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
U_u &= \frac{1}{4\pi f}\left\{-\left(\frac{\partial\Phi}{\partial u}\right)^2 + \Psi\right\}, \\
X_y &= Y_x = -\frac{1}{4\pi f}\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
Z_u &= U_z = -\frac{1}{4\pi f}\frac{\partial\Phi}{\partial z}\frac{\partial\Phi}{\partial u},
\end{aligned}\right\} \tag{16}
$$

in which Ψ is the following four-dimensional scalar:

$$
\Psi = \frac{1}{2}\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2 + \left(\frac{\partial\Phi}{\partial u}\right)^2\right\} . \tag{16a}
$$

One easily finds that one actually has:

$$
-\gamma\frac{\partial\Phi}{\partial x} = \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + \frac{\partial X_u}{\partial u},
$$

etc.,

$$
\mathfrak{S}_x^K = i\,c\,U_x , \qquad \mathfrak{S}_y^K = i\,c\,U_y , \qquad \mathfrak{S}_z^K = i\,c\,U_z , \qquad \psi^K = U_u .
$$

Since the gravitational tensor is symmetric, the impulse density will be equal to the energy-current, divided by c^2.

Equation (4), which expresses the variability of the mass of a mass-point, can be easily generalized to extended masses. To that end, we must treat the system of equations (8) in the same way that we previously did with the system of equations (3). We multiply eqs. (8) by $\mathfrak{a}_x$, $\mathfrak{a}_y$, $\mathfrak{a}_z$, $\mathfrak{a}_u$, in succession, and add them. If no other agencies besides the gravitational field imply a variability of the mass then the external force $\mathfrak{K}'$ will be perpendicular to $\mathfrak{a}$, and after some conversions, one will get:

$$
\frac{\partial}{\partial x}\gamma\mathfrak{a}_x + \frac{\partial}{\partial y}\gamma\mathfrak{a}_y + \frac{\partial}{\partial z}\gamma\mathfrak{a}_z + \frac{\partial}{\partial u}\gamma\mathfrak{a}_u = \frac{\gamma}{c^2}\frac{d\Phi}{d\tau} \tag{17}
$$

or

$$
\operatorname{div}\rho\,\mathfrak{v} + \frac{\partial\rho}{\partial t} = \frac{\rho}{c^2}\left\{\mathfrak{v}\,\nabla\Phi + \frac{\partial\Phi}{\partial t}\right\}, \tag{18}
$$

or rather:

$$\frac{d}{dt}(\rho\, dv) = \frac{\rho\, dv}{c^2}\frac{d\Phi}{d\tau} \tag{18a}$$

(*dv* is a volume element). These three mutually-identical equations express the general law of the variation of mass due to the gravitational field.

Equation (1) can be integrated in a well-known way. One gets the known formula for the retarded potential:

$$\Phi\,(x_0, y_0, z_0, t_0) = -f \cdot \int \frac{dx\,dy\,dz}{r}\gamma_{t-r/c} + \text{const.} \tag{19}$$

in which

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}\ .$$

The integration relates to three-dimensional space, and the value for γ is taken at time $t - r / c$.

One sees from (5) and (19) that there can be no true point-like masses, since one would then have $\Phi = -\infty$ at such a point, and the mass would then be zero. If a body contracts then its mass will diminish, and for a vanishing volume, the mass would also zero. As far as I can see, those consequences of the theory do not lead to any contradictions.

As one sees, the theory that was developed here has much in common with the one that **Abraham** gave in this Zeitschrift **13** (1912), 1, but later discarded ([1]). However, the theory that was developed here is free of all the inconveniences that the theories of **Einstein** and **Abraham** on the variability of the speed of light bring with them.

Added in proof. I have learned from a written communication by Herrn Prof. Dr. **A. Einstein** that he has already addressed the possibility that was employed above of treating gravitational phenomena in a simpler way, but he came to the conclusion that the consequences of such a theory could not correspond to reality. He showed, by a simple example, that according to this theory, a rotating system in a gravitational field would take on a smaller acceleration that a non-rotating one.

I do not find that consequence to be disturbing in its own right, since the difference is too small to give a contradiction with experiments. However, the stated consequence probably shows that my theory is not consistent with **Einstein**'s equivalence hypothesis, by which an unaccelerated reference system in a homogeneous gravitational field would be equivalent to an accelerated reference system in a space that is free of gravitation.

However, I do not see sufficient grounds for rejecting the theory because of that fact, because although **Einstein**'s hypothesis is exceptionally ingenious, it does raise great difficulties. For that reason, other attempts to treat gravitation would also be desirable, and I would like to think that this communication has made a contribution to them.

Helsingfors, 20 October 1912.

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([1]) **M. Abraham**, this Zeitschrift **13** (1912), 793.