"Über die Möglichkeit das elektromagnetische Feld und das Gravitationsfeld zu vereinigen," Phys. Zeit. **15** (1914), 504-506.

On the possibility of unifying the electromagnetic field and the gravitational field

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It is indeed one of the great services that the theory of relativity has provided that it made it possible to characterize the electromagnetic state of the ether by means of **one** vector, namely, **Minkowski**'s six-vector $\mathfrak{f}$, while two field vectors were required for that in the older way of looking at things. However, that possibility of characterizing the state of the ether will become untenable as soon as one assumes that there is also a gravitational field in the ether. In the theories of gravitation that **Mie** ([1]) and I ([2]) have developed, the gravitational field in the ether was given by a four-vector. If such a theory corresponds to reality then the state of the ether will be characterized by a six-vector and a four-vector.

We would like to denote the components of the electromagnetic six-vectors by:

$$\mathfrak{f}_{xy}\,,\ \mathfrak{f}_{yz}\,,\ \mathfrak{f}_{zx}\,,\ \mathfrak{f}_{xu}\,,\ \mathfrak{f}_{yu}\,,\ \mathfrak{f}_{zu}\,,$$

in which we have set $u = ict$, where c is the speed of light. For the components of the magnetic field strength $\mathfrak{H}$ and the electric field strength $\mathfrak{E}$, one will then have ([3]):

$$\left.\begin{aligned} \mathfrak{H}_x &= \mathfrak{f}_{yz} = -\mathfrak{f}_{zy}, \quad \text{etc.} \\ -i\,\mathfrak{E}_x &= \mathfrak{f}_{xu} = -\mathfrak{f}_{ux}, \quad \text{etc.} \end{aligned}\right\} \tag{1}$$

We now further introduce the following notations for the components of the four-vector of the gravitational field in a purely formal way:

$$\mathfrak{f}_{wx}\,,\ \mathfrak{f}_{wy}\,,\ \mathfrak{f}_{wz}\,,\ \mathfrak{f}_{wu}$$

(in which $\mathfrak{f}_{xw} = -\mathfrak{f}_{wx}$, etc.) and writing down the following system of equations:

([1]) **G. Mie**, Ann. Phys. (Leipzig) **40** (1913), 25.

([2]) **G. Nordstrøm**, this Zeit. **13** (1912), 1126; Ann. Phys. (Leipzig) **40** (1913), 872; *ibid.* **42** (1913), 533.

([3]) **H. Minkowski**, Gött. Nachr. (1908), 58.

$$\left.\begin{aligned}
\frac{\partial \mathfrak{f}_{xy}}{\partial y}+\frac{\partial \mathfrak{f}_{xz}}{\partial z}+\frac{\partial \mathfrak{f}_{xu}}{\partial u}+\frac{\partial \mathfrak{f}_{xw}}{\partial w}&=\frac{1}{c}\mathfrak{f}_{x},\\
\frac{\partial \mathfrak{f}_{yx}}{\partial x}\qquad+\frac{\partial \mathfrak{f}_{yz}}{\partial y}+\frac{\partial \mathfrak{f}_{yu}}{\partial u}+\frac{\partial \mathfrak{f}_{yw}}{\partial w}&=\frac{1}{c}\mathfrak{f}_{y},\\
\frac{\partial \mathfrak{f}_{xx}}{\partial x}+\frac{\partial \mathfrak{f}_{xy}}{\partial y}\qquad+\frac{\partial \mathfrak{f}_{zu}}{\partial u}+\frac{\partial \mathfrak{f}_{zw}}{\partial w}&=\frac{1}{c}\mathfrak{f}_{z},\\
\frac{\partial \mathfrak{f}_{uy}}{\partial y}+\frac{\partial \mathfrak{f}_{uz}}{\partial z}+\frac{\partial \mathfrak{f}_{uz}}{\partial z}\qquad+\frac{\partial \mathfrak{f}_{wu}}{\partial w}&=\frac{1}{c}\mathfrak{f}_{u},\\
\frac{\partial \mathfrak{f}_{wx}}{\partial x}+\frac{\partial \mathfrak{f}_{wy}}{\partial y}+\frac{\partial \mathfrak{f}_{wz}}{\partial z}+\frac{\partial \mathfrak{f}_{wu}}{\partial u}\qquad&=\frac{1}{c}\mathfrak{f}_{w},
\end{aligned}\right\} \tag{I}$$

$$\left.\begin{aligned}
\frac{\partial \mathfrak{f}_{yz}}{\partial x}+\frac{\partial \mathfrak{f}_{zx}}{\partial y}+\frac{\partial \mathfrak{f}_{xy}}{\partial z}&=0,\\
\frac{\partial \mathfrak{f}_{zu}}{\partial y}+\frac{\partial \mathfrak{f}_{uy}}{\partial z}+\frac{\partial \mathfrak{f}_{yz}}{\partial u}&=0,\\
\frac{\partial \mathfrak{f}_{xu}}{\partial z}+\frac{\partial \mathfrak{f}_{uz}}{\partial x}+\frac{\partial \mathfrak{f}_{zx}}{\partial u}&=0,\\
\frac{\partial \mathfrak{f}_{yu}}{\partial x}+\frac{\partial \mathfrak{f}_{ux}}{\partial y}+\frac{\partial \mathfrak{f}_{xy}}{\partial u}&=0,\\
\frac{\partial \mathfrak{f}_{zw}}{\partial y}+\frac{\partial \mathfrak{f}_{wy}}{\partial z}+\frac{\partial \mathfrak{f}_{yz}}{\partial w}&=0,\\
\frac{\partial \mathfrak{f}_{xw}}{\partial z}+\frac{\partial \mathfrak{f}_{wz}}{\partial x}+\frac{\partial \mathfrak{f}_{zx}}{\partial w}&=0,\\
\frac{\partial \mathfrak{f}_{yw}}{\partial x}+\frac{\partial \mathfrak{f}_{wx}}{\partial y}+\frac{\partial \mathfrak{f}_{xy}}{\partial w}&=0,\\
\frac{\partial \mathfrak{f}_{uw}}{\partial x}+\frac{\partial \mathfrak{f}_{wx}}{\partial u}+\frac{\partial \mathfrak{f}_{xu}}{\partial w}&=0,\\
\frac{\partial \mathfrak{f}_{uw}}{\partial y}+\frac{\partial \mathfrak{f}_{wy}}{\partial u}+\frac{\partial \mathfrak{f}_{yu}}{\partial w}&=0,\\
\frac{\partial \mathfrak{f}_{uw}}{\partial x}+\frac{\partial \mathfrak{f}_{wz}}{\partial y}+\frac{\partial \mathfrak{f}_{zu}}{\partial w}&=0.
\end{aligned}\right\} \tag{II}$$

These systems of equations are both completely symmetric relative to *x*, *y*, *z*, *u*, *w*. Naturally, they have no physical meaning to begin with. However, if one sets all partial derivatives with respect to *w* equal to zero then one will find that they go to the field equations for the electromagnetic field when $\mathfrak{i}_x$, $\mathfrak{i}_y$, $\mathfrak{i}_z$, $\mathfrak{i}_u$ are the components of the fourcurrent, and – $(1/c)\ \mathfrak{i}_w$ is the rest density of the gravitating mass ([1]). The first four equations in the two systems are **Maxwell**'s equations in the form that **Minkowski** gave them, moreover. The last equation (I) is the fundamental equation of gravitation, and the remaining six equations (II) express the irrotationality of the gravitation vector.

That meaning for the equations (I), (II) shows that it is justified for us to regard the four-dimensional space-time as a surface that is embedded in a five-dimensional world. In that five-dimensional world, the $\mathfrak{i}_m$ are the components of a **five-vector**, and the $\mathfrak{f}_{mn}$ components of a **ten-vector**; the latter vector characterizes the physical state of the ether completely. The five-dimensional world has a distinguished axis – viz., the *w*-axis. The four-dimensional space-time is perpendicular to that axis and the derivatives of all components of $\mathfrak{f}$ with respect to *w* are equal to zero at all of its points.

The components of $\mathfrak{f}$ can be expressed in terms of a five-potential with the components Φ_x, Φ_y, Φ_z, Φ_u, Φ_w such that for each component of $\mathfrak{f}$, one will have:

$$\mathfrak{f}_{mn} = \frac{\partial \Phi_n}{\partial m} - \frac{\partial \Phi_m}{\partial n}. \tag{2}$$

Upon differentiating equations (I), one will find the following relation for the "five-current" $\mathfrak{i}$:

$$\frac{\partial \mathfrak{i}_x}{\partial x} + \frac{\partial \mathfrak{i}_y}{\partial y} + \frac{\partial \mathfrak{i}_z}{\partial z} + \frac{\partial \mathfrak{i}_u}{\partial u} + \frac{\partial \mathfrak{i}_w}{\partial w} = 0, \tag{3}$$

and for that reason, one can prescribe the following six partial differential equations for the five-potential:

$$\frac{\partial \Phi_x}{\partial x} + \frac{\partial \Phi_y}{\partial y} + \frac{\partial \Phi_z}{\partial z} + \frac{\partial \Phi_u}{\partial u} + \frac{\partial \Phi_w}{\partial w} = 0, \tag{4}$$

$$\left.\begin{aligned} \frac{\partial^2 \Phi_x}{\partial x^2} + \frac{\partial^2 \Phi_x}{\partial y^2} + \frac{\partial^2 \Phi_x}{\partial z^2} + \frac{\partial^2 \Phi_x}{\partial u^2} + \frac{\partial^2 \Phi_x}{\partial w^2} &= -\frac{1}{c}\mathfrak{i}_x, \\ \frac{\partial^2 \Phi_w}{\partial x^2} + \frac{\partial^2 \Phi_w}{\partial y^2} + \frac{\partial^2 \Phi_w}{\partial z^2} + \frac{\partial^2 \Phi_w}{\partial u^2} + \frac{\partial^2 \Phi_w}{\partial w^2} &= -\frac{1}{c}\mathfrak{i}_w. \end{aligned}\right\} \tag{5}$$

(For the sake of space-saving, only the first and last of the five equations in (5) were written out.) When those six conditions are fulfilled, the expressions (2) will satisfy the field equations (I), (II) identically.

([1]) I understand that to mean the quantity that I denoted by $g \cdot v$ in the cited paper; one then has:

$$-\tfrac{1}{c}\,\mathfrak{i}_w = g \cdot v.$$

One can indeed introduce the notations $a\ \mathfrak{f}_{wx}$, $a\ \mathfrak{f}_{wy}$, $a\ \mathfrak{f}_{wz}$, $a\ \mathfrak{f}_{wu}$, in which *a* is an arbitrary real or imaginary constant. One would then have $-\frac{a}{c}\,\mathfrak{i}_w = g\ v$. However, in the presentation of the impulse-energy theorem, it will be shown that *a* must be equal to + 1 or – 1 if the last equation (I) is truly to behave like the other ones.

Our formulas (I), (II) also make it possible to present the impulse-energy theorem for the combined electromagnetic and gravitational field in a unified way. In order to get that theorem for the *x*-direction, one multiplies those four equations (I) that relate to the remaining axis directions by $\mathfrak{f}_{xy}$, $\mathfrak{f}_{xz}$, $\mathfrak{f}_{xw}$, $\mathfrak{f}_{xw}$, resp., and the six equations (II) that contain x by $\mathfrak{f}_{mn}$, where m, n mean the two remaining indices, in their correct sequence, that enter into the equation in question. The ten equations thus-obtained are added, and a simple conversion of that result will give the desired theorem.

We would like to present the theorem for the *u*-direction – hence, the energy theorem – and must then multiply the first three equations (I) by $\mathfrak{f}_{ux}$, $\mathfrak{f}_{uy}$, $\mathfrak{f}_{uz}$, resp., and the last one by $\mathfrak{f}_{uw}$. Furthermore, of equations (II), the second, third, fourth, eighth, ninth, and tenth must be multiplied by $\mathfrak{f}_{yz}$, $\mathfrak{f}_{zx}$, $\mathfrak{f}_{xy}$, $\mathfrak{f}_{wx}$, $\mathfrak{f}_{wy}$, $\mathfrak{f}_{wz}$, resp. Upon adding the terms, and after some conversions, we will then get:

$$\begin{aligned}
&\mathfrak{f}_{ux}\left(\frac{\partial\mathfrak{f}_{ux}}{\partial y}+\frac{\partial\mathfrak{f}_{xz}}{\partial z}+\frac{\partial\mathfrak{f}_{xw}}{\partial w}\right)+\mathfrak{f}_{uy}\left(\frac{\partial\mathfrak{f}_{yx}}{\partial x}+\frac{\partial\mathfrak{f}_{yz}}{\partial z}+\frac{\partial\mathfrak{f}_{yw}}{\partial w}\right)+\mathfrak{f}_{uw}\left(\frac{\partial\mathfrak{f}_{wx}}{\partial x}+\frac{\partial\mathfrak{f}_{wy}}{\partial y}+\frac{\partial\mathfrak{f}_{wz}}{\partial z}\right)\\
&+\mathfrak{f}_{yz}\left(\frac{\partial\mathfrak{f}_{zu}}{\partial y}+\frac{\partial\mathfrak{f}_{uy}}{\partial z}\right)+\mathfrak{f}_{zx}\left(\frac{\partial\mathfrak{f}_{xu}}{\partial z}+\frac{\partial\mathfrak{f}_{uz}}{\partial x}\right)+\mathfrak{f}_{xy}\left(\frac{\partial\mathfrak{f}_{yu}}{\partial x}+\frac{\partial\mathfrak{f}_{ux}}{\partial y}\right)\\
&+\mathfrak{f}_{wx}\left(\frac{\partial\mathfrak{f}_{uw}}{\partial x}+\frac{\partial\mathfrak{f}_{xu}}{\partial w}\right)+\mathfrak{f}_{wy}\left(\frac{\partial\mathfrak{f}_{uw}}{\partial y}+\frac{\partial\mathfrak{f}_{yu}}{\partial w}\right)+\mathfrak{f}_{wz}\left(\frac{\partial\mathfrak{f}_{uw}}{\partial z}+\frac{\partial\mathfrak{f}_{zu}}{\partial w}\right)\\
&+\mathfrak{f}_{ux}\frac{\partial\mathfrak{f}_{xu}}{\partial u}+\mathfrak{f}_{uy}\frac{\partial\mathfrak{f}_{yu}}{\partial u}+\mathfrak{f}_{uz}\frac{\partial\mathfrak{f}_{zu}}{\partial u}+\mathfrak{f}_{uw}\frac{\partial\mathfrak{f}_{wu}}{\partial u}\\
&+\mathfrak{f}_{yz}\frac{\partial\mathfrak{f}_{yz}}{\partial u}+\mathfrak{f}_{zx}\frac{\partial\mathfrak{f}_{zx}}{\partial u}+\mathfrak{f}_{xy}\frac{\partial\mathfrak{f}_{xy}}{\partial u}+\mathfrak{f}_{wx}\frac{\partial\mathfrak{f}_{wx}}{\partial u}+\mathfrak{f}_{wy}\frac{\partial\mathfrak{f}_{wy}}{\partial u}+\mathfrak{f}_{wz}\frac{\partial\mathfrak{f}_{wz}}{\partial u}\\
&=\frac{1}{c}(\mathfrak{f}_{ux}\,\mathfrak{i}_x+\mathfrak{f}_{uy}\,\mathfrak{i}_y+\mathfrak{f}_{uz}\,\mathfrak{i}_z+\mathfrak{f}_{uw}\,\mathfrak{i}_w).
\end{aligned}$$

A simple conversion will then give the desired equation:

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}(\mathfrak{f}_{uy}\,\mathfrak{f}_{yx}+\mathfrak{f}_{uz}\,\mathfrak{f}_{zx}+\mathfrak{f}_{uw}\,\mathfrak{f}_{wx})+\frac{\partial}{\partial y}(\mathfrak{f}_{ux}\,\mathfrak{f}_{xy}+\cdots)+\frac{\partial}{\partial z}(\mathfrak{f}_{ux}\,\mathfrak{f}_{xz}+\cdots)+\frac{\partial}{\partial w}(\mathfrak{f}_{ux}\,\mathfrak{f}_{xw}+\cdots)\\
&+\frac{1}{2}\frac{\partial}{\partial u}(-\mathfrak{f}_{ux}^2-\mathfrak{f}_{uy}^2-\mathfrak{f}_{uz}^2-\mathfrak{f}_{uw}^2+\mathfrak{f}_{yz}^2+\mathfrak{f}_{zx}^2+\mathfrak{f}_{xy}^2+\mathfrak{f}_{wx}^2+\mathfrak{f}_{wy}^2+\mathfrak{f}_{wz}^2)\\
&=\frac{1}{c}(\mathfrak{f}_{ux}\,\mathfrak{i}_x+\mathfrak{f}_{uy}\,\mathfrak{i}_y+\mathfrak{f}_{uz}\,\mathfrak{i}_z+\mathfrak{f}_{uw}\,\mathfrak{i}_w).
\end{aligned}\right\} \quad (6)$$

The quantities in the parentheses on the left are the components of a five-dimensional tensor, and the equation will then have the form:

$$\frac{\partial P_{ux}}{\partial x}+\frac{\partial P_{uy}}{\partial y}+\frac{\partial P_{uz}}{\partial z}+\frac{\partial P_{uu}}{\partial u}+\frac{\partial P_{uw}}{\partial w}=\mathfrak{K}_u \,. \tag{6a}$$

If one multiplies this by ic and sets $\partial P_{uw} / \partial w$ equal to zero then one will get the energy equation in its usual form. One then finds that x-component of the energy-current is:

$$\mathfrak{S}_x = ic\,(\mathfrak{f}_{uy}\,\mathfrak{f}_{yx}+\mathfrak{f}_{uz}\,\mathfrak{f}_{zx}+\mathfrak{f}_{uw}\,\mathfrak{f}_{wx}) = c\,(\mathfrak{E}_y\,\mathfrak{H}_z-\mathfrak{E}_z\,\mathfrak{H}_y)-\frac{\partial\Phi_w}{\partial t}\frac{\partial\Phi_w}{\partial x}\,. \tag{7}$$

The last expression is obtained by means of formulas (1) and (2), and the derivatives with respect to w that appear in them are set to zero. In vector-analytic notation, the expression for $\mathfrak{S}$ reads:

$$\mathfrak{S} = c\,[\mathfrak{E}\,\mathfrak{H}]-\frac{\partial\Phi_w}{\partial t}\nabla\Phi_w\,. \tag{7a}$$

One likewise obtains an expression for the energy density from (6) that reads:

$$\psi=\frac{1}{2}\left\{\mathfrak{E}^2+\mathfrak{H}^2+(\nabla\Phi_w)^2+\frac{1}{c^2}\left(\frac{\partial\Phi_w}{\partial t}\right)^2\right\}, \tag{8}$$

when written vector-analytically.

That expression for the energy-current and energy density is composed additively from known expressions for the relevant quantities in the electromagnetic and gravitational fields, and that is, in fact, the desired result of our considerations.

One easily sees that the statement in the footnote on pp. 3 is correct, and that of the components of the ten-vector $\mathfrak{f}$ and the five-vector $\mathfrak{i}$, only the ones that are provided with an index of u are regarded as imaginary.

One will get four other equations by permuting the indices in equation (6). The three that refer to spatial axis-directions naturally express the impulse theorem in a well-known way. After introducing the field strengths and the gravitational potential Φ_w, the equation for the w-direction will read:

$$-\operatorname{div}\left\{[\mathfrak{H},\nabla\Phi_w]+\frac{1}{c}\mathfrak{E}\frac{\partial\Phi_w}{\partial t}\right\}+\frac{1}{c}\frac{\partial}{\partial t}(\mathfrak{E}\nabla\Phi_w)+\frac{1}{2}\frac{\partial}{\partial w}\left\{\mathfrak{H}^2-\mathfrak{E}^2-(\nabla\Phi_w)^2+\frac{1}{c^2}\left(\frac{\partial\Phi_w}{\partial t}\right)^2\right\}$$
$$=-\frac{1}{c}\left\{\mathfrak{i}\,\nabla\Phi_w+\mathfrak{i}_u\frac{\partial\Phi_w}{\partial u}\right\},$$

when written vector-analytically. For the time being, there is no reason for this equation to be worthy of a physical meaning.

As we have seen, the way of looking at things that was presented above offers a formal advantage in that the electromagnetic and gravitational field can be regarded as a single field with it. Naturally, no new physical content for the equations is given by it.

However, I do not exclude the possibility that the formal symmetry that was found might have a deeper basis. Nevertheless, I would not like to go into the possibilities that one can imagine in regard to that here.

Summary. It was shown that a unified treatment of the electromagnetic and gravitational field is possible when one regards the four-dimensional space-time as a surface that is embedded in a five-dimensional world.

Helsingfors, 30 March 1914.

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