Faculté des Sciences de Paris

Cours complémentaire de Mécanique rationelle

LECTURES
ON THE INTEGRATION
OF THE DIFFERENTIAL EQUATIONS
OF MECHANICS AND APPLICATIONS

BY

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Maitre de Conférences à la Faculté des Sciences

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Presentations

on the integration of the equations of mechanics

(Programme d'Agrégation in 1891)

made to the Science Faculty at Lille

by P. Painlevé

(edited by Boulanger, alumnus of l'Ecole Polytechnique)

These presentations have the goal of presenting the work of Lagrange, Poisson, Hamilton, Jacobi, etc., on the integration of the equations of mechanics.

The first three are dedicated to briefly reviewing the definitions and fundamental principle of dynamics, as well as the general theorems that produce the first integrals of motion of material systems, and presenting some exercises as examples.

TRANSLATOR’S PREFACE

Paul Painlevé (1863-1933) was not only a French mathematician at the Sorbonne, but also a Prime Minister of the Third Republic on two separate occasions (a nine week term in 1917 and another brief stint in 1925). He was educated at L’École Normale Supérieure and was conferred a doctorate in 1887. He had also studied at Göttingen under Felix Klein and Hermann Schwarz. After first teaching that the University of Lille, we returned in 1892 to teach at the Sorbonne, L’École Polytechinque, Collège de France, and L’École Normale Supérieure.

Most of his research was concerned with the theory of ordinary differential equations. In particular, he is often associated with a certain class of transcendental functions that solve a certain class of differential equations, and those functions are now called the “Painlevé transcendents.” He also took an interest in the emerging theory of general relativity and defined a special set of coordinates for the Schwarzschild solution to Einstein’s field equations for gravitation that are referred to as “Gullstrand-Painlevé coordinates.” Another emerging topic in science and engineering of the era was due to the birth of heavier-than-air flight. Not only did Painlevé do research into the mathematical theory of flight, but in 1909, he was Wilbur Wright’s first French airplane passenger, and eventually established the first university course in aeronautics. As a result, one sees in Painlevé’s treatment of the issues that are concerned with integrating the differential equations of analytical mechanics that he clearly had roots in both the mathematical study of differential equations and its applications to physics and engineering.

The reader will note that there are occasional instances of the notation (?) appearing in the translation. That is because the original text was written in a printed form of cursive French and when combined with the scanning into an online PDF, there were times that characters were entirely illegible and not easily derived from the rest of the text. Nonetheless, overall, the translation is certainly more readable to English-language scholars who would rather deal with more modern fonts.

Spring Valley, Ohio, April 2022

David H. Delphenich

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LECTURE 1

Before embarking upon the study of systems, we shall commence with a summary of the postulates and axioms that one assumes from the outset in the dynamics of a material point. We shall refer to the *Cours de licence* for the developments that relate to that subject.

We shall suppose that the definitions that relate to length, time, and mass are known. In practice, length is measured in centimeters, time, in seconds, and mass, in grams.

Absolute axes. – Let Ox, Oy, Oz be three arbitrary material axes, and let M be a given material point that is placed at the point (x_0, y_0, z_0) at the instant t_0 with a velocity of (x_0', y_0', z_0') in the presence of a certain material medium P. All of the conditions for the phenomena that will happen will then be well-defined; in particular, the motion that the point M will exhibit with respect to the axes $Oxyz$. The acceleration (γ) of the point M at the instant t_0 (in the present of the medium P) will then depend upon only the initial conditions $(x_0, y_0, z_0, x_0', y_0', z_0')$.

Having said that, we agree to say *absolute* axes to mean any system of axes $Oxyz$ that satisfies the following conditions:

1. (Kepler's principle) – A material point (if there exists just one) will describe a straight line with respect to those axes with a constant velocity: In other words, its acceleration will be constantly zero.

2. (Newton's principle) – Let M be a given material point that is placed under given initial conditions at the instant t_0 . Let P be the external medium at that point and let (γ) be the acceleration of M at the instant t_0 . Imagine that P has the form of two distinct parts P' and P'', and that one suppresses the part P'' at the instant t_0 , while the point M and P' remain unchanged, moreover. The acceleration M will then be (γ'). Similarly, let (γ'') be the acceleration that M will have at the time t_0 if only P'' existed. The quantities (γ), (γ'), (γ'') will satisfy the equality:

$$(\gamma) = (\gamma') + (\gamma'') .$$

That principle includes Kepler's.

When a system of axes $Oxyz$ is absolute, all systems that are animated with respect to $Oxyz$ with a motion of uniform rectilinear translation will also be absolute systems of axes, and one will prove quite easily that there exist no others. If one says *absolute accelerations* to mean the accelerations of the material points with respect to the absolute axes then those accelerations will be independent of the absolute system of axes to which one refers the motion.

We assume that the existence of absolute axes has been established by experiment. Those axes are reasonably fixed with respect to the stars or animated with respect to the stars with a motion of uniform rectilinear translation ([1]).

Absolute force. – The absolute force or force that is exerted on a point M at a given instant t_0 is, by definition, the geometric magnitude that has its origin at the point M, and its direction and sense are the same as the absolute acceleration (Γ) of M, while its length is the length Γ of the acceleration times the mass M :

$$(F) = m\,(\Gamma)\,.$$

One says that the point M is subject to the action of several forces $F_1, F_2, \ldots, F_n$ when the medium P that is external to M is composed of n parts $P_1, P_2, \ldots, P_n$ that (if they alone exist) exert the forces $F_1, F_2, \ldots, F_n$ on that same point M at the instant t when it is placed under the same initial conditions. The equality:

$$(\Gamma) = (\Gamma_1) + (\Gamma_2) + \ldots + (\Gamma_n)$$

will imply the equality:

$$(F) = (F_1) + (F_2) + \ldots + (F_n)\,.$$

One can then state this proposition: Say that several forces $(F_1), (F_2), \ldots, (F_n)$ are exerted on M, i.e., that the force (F) that is exerted on M is the geometric sum of the magnitudes $(F_1), (F_2), \ldots, (F_n)$.

One will then deduce the following consequence from that:

Suppose that the medium P that acts upon the point M is a material ensemble. One would like to decompose the system P into n parts (where n is as large as one desires) and regard the absolute force that is exerted on M by the medium P as the geometric sum of the absolute forces that are exerted on that point by each of the n parts, while supposing that one can make them act separately without changing anything in either their state or that of the point M.

If f is the absolute force that is exerted by any of the n partial media under those conditions, and F is the absolute force that is exerted by the ensemble then one will have:

$$(F) = \sum (f)\,,$$

from the preceding postulate.

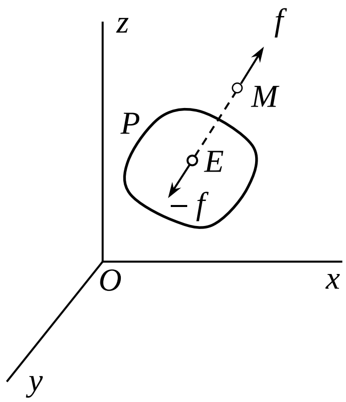
Therefore, that postulate will permit one to calculate the force that is exerted by an ensemble on a point when one knows the force that is exerted by any of elements of the ensemble on that point.

([1]) When one assumes the notion of absolute motion, one must regard the axes that one calls "absolute" to be absolutely fixed or animated with an absolute motion of uniform rectilinear translation. The accelerations that one calls "absolute" are the acceleration of the points in the course of their absolute motion.

Postulate of the equality of action and reaction.

We shall further assume that the following principle is verified by experiment in the cases that we shall treat:

Suppose that a material medium P acts upon a point M. One can decompose P into elements, and let E be one of them. The element E (if it alone exists with M without changing anything in either its state of that of M) will exert an absolute force f on the point M. One assumes that, conversely, the point M exerts an absolute force on E that is equal and directly opposite to the preceding f. That demands that the force f must be directed along the line that unites the points M and E.



Relative forces. Their link to absolute forces. – In all of the foregoing, one referred the motion to absolute axes. Now suppose that one studies the motion of a point M with respect to arbitrary axes (Ox, Oy, Oz).

At the time t, the point M occupies a position (x, y, z) and is animated with a velocity (x', y', z'). Under the action of a medium P, it can take an acceleration Γ_r with respect to the axes (Ox, Oy, Oz). We shall say the *relative force* that is exerted on the point M at the time t to mean the geometric quantity $(F_r) = m\,(\Gamma_r)$.

When one knows the motion of the axes (Ox, Oy, Oz) with respect to absolute axes, it will be easy to calculate the absolute force F_a when one knows the relative force. Indeed, one has:

$$(F_a) = (F_r) + m\,(\Gamma_e) + m\,(\Gamma_c)\,,$$

or

$$(F_a) = (F_r) + (\mathcal{A})\,,$$

in which $\mathcal{A}$ is the geometric sum of the two Coriolis terms. The quantity $\mathcal{A}$ is independent of the active medium.

One sees from this that one will obtain the absolute force by augmenting the relative force by a geometric quantity that depends upon time, the position and velocity of the point M, and the motion of the axes (Ox, Oy, Oz), but it is independent of the active medium.

Now regard the absolute force that acts on the point M as something that is produced by the action of several media $P_1, P_2, \ldots, P_n$ whose ensemble comprises the medium P. Let $(F_1)_a$, $(F_2)_a$, …, $(F_n)_a$ be the absolute forces that are exerted by those partial media on M. From Newton's principle regarding the action of the media, one will have:

$$(F_a) = (F_1)_a + (F_2)_a + \ldots + (F_n)_a \tag{1}$$

for the total absolute force.

Let $(F_1)_r$, $(F_2)_r$, …, $(F_n)_r$ be the relative forces that are exerted by the same partial media in isolation, and let (F_r) be the relative force that is exerted by the ensemble. One has:

$$(F_1)_a = (F_1)_r + (\mathcal{A}) ,$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$(F_n)_a = (F_n)_r + (\mathcal{A}) ,$$
$$(F_a) = (F_r) + (\mathcal{A}) .$$

As a result, the relation (1) will give, on the one hand:

$$(F_r) = (F_1)_r + (F_2)_r + \dots + (F_n)_r + (n-1)(\mathcal{A}) ,$$

while on the other hand (and this will be important for us):

$$(F_r) = (F_1)_r + (F_2)_a + \dots + (F_n)_a .$$

Therefore, since Newton's principle regarding the action of the media is allowable for absolute forces, one will reach the following conclusion for relative forces:

Let M be a material point that is placed under given initial conditions at the instant t, and let P be the medium external to M that is composed of several parts $P_1, P_2, \dots, P_n$. The relative force (with respect to an arbitrary system of axes) that is exerted on M by the medium P is the geometric sum of the relative force that is exerted by any one of the partial media and the absolute forces that are exerted by the other partial media (if each of those media exists alone with M, and is identical to itself, as well as M, at the instant t).

In particular, consider a material point M that is found in equilibrium under the influence of a given medium P with respect to arbitrary axes (Ox, Oy, Oz); terrestrial ones, for example. The relative force that acts on that point is zero: $(F_r) = 0$. Let a medium P' act upon it without modifying anything in the preceding medium. (For example, pull on a pendulum at rest with the aid of a string.) Under the simultaneous action of two media P and P', the point M will take on a relative acceleration Γ'_r, and the corresponding relative force will be $(F'_r) = m(\Gamma'_r)$. From the foregoing, one will have:

$$(F'_r) = (F_r) + (f_a) ,$$

in which (f_a) is the absolute force that is exerted by the a^{th} medium P'. Now, $(F_r) = 0$. Therefore:

$$(F'_r) = (f_a) .$$

Therefore, it is the absolute force (f_a) that is measured by the relative acceleration of the point M:

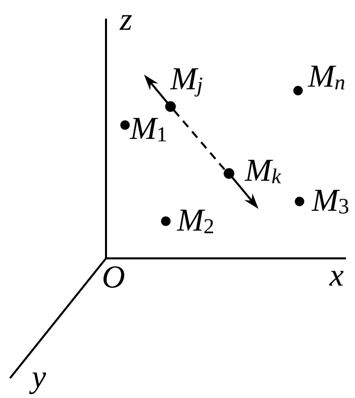
$$(f_a) = m(\Gamma'_r) .$$

One sees from this how the absolute forces are introduced most simply into physical experiments and correspond to the common notion of effort.

In all of the applications where one studies the relative motion of a material point M, when one says that several forces $F_1, F_2, \ldots, F_n$ act upon M, one intends that to mean that one of those forces is a relative force and the other one is an absolute force. That will then amount to saying that the relative force that is exerted upon the point M is the geometric sum of the quantities $F_1, F_2, \ldots, F_n$.

External and internal forces on a material system.

The principle of the equality of action and reaction naturally leads us to consider some important theorems concerning material systems.



Consider a system of absolute axes (Ox, Oy, Oz) and a certain number of material points $M_1, M_2, \ldots, M_j, M_k, \ldots, M_n$ with masses $m_1, m_2, \ldots, m_j, m_k, \ldots, m_n$, respectively. The ensemble is found to be placed under well-defined conditions, i.e., each of those points M_j has a given position (x_j, y_j, z_j) and a given velocity (x'_j, y'_j, z'_j) at the instant t. In the presence of a given external medium, each point M_j will take on an acceleration Γ_j, and the absolute force that is exerted on it at the instant t will be: $(F_j) = m\,(\Gamma_j)$.

From what we have seen, that force can be considered to be the geometric sum of the following two forces: One of them, $F_{j,e}$, is called *external* to the system, namely, the force that would be exerted on M_j by the external medium if it alone existed, while nothing changed in its state or that of M_j, moreover. The other one, $F_{j,i}$, is called *internal* to the system, namely, the force that would be exerted on M_j if the system existed alone and unchanged:

$$(F_j) = (F_{j,e}) + (F_{j,i}) .$$

That internal absolute force $F_{j,i}$ can be considered to be the geometric sum of the $(n-1)$ partial forces that are exerted in isolation by each of the $(n-1)$ points $M_1, M_2, \ldots, M_k, \ldots, M_n$ on the point M_j.

If one assumes the principle of the equality of action and reaction then the force that is exerted by one of those points M_k on the point M_j, namely, $f_{k,j}$, will be directed along the lines $M_j M_k$, and conversely, the point M_j will exert a force that is equal in magnitude and oppositely-directed to $f_{k,j}$:

$$(f_{j,k}) + (f_{k,j}) = 0 .$$

We then reach this conclusion: The internal forces that are exerted between all points of a material system can be regarded as the geometric sum of the partial internal forces that are pairwise equal and oppositely-directed.

That is a fundamental property of the internal forces.

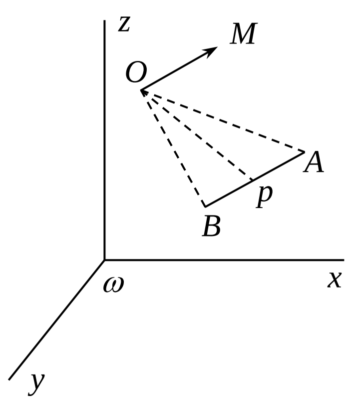
It is easy to extend the preceding results to the case of a motion that is referred to arbitrary axes (Ox, Oy, Oz). Let us repeat the same considerations. The point M_j has a relative acceleration Γ_j^r, and the relative force that acts upon it is $(F_j^r) = m(\Gamma_j^r)$. If $(F_{j,e})$ is the external relative force, i.e., the relative force that the external medium exerts upon M_j (if it existed alone with M_j) and the internal absolute force is $(F_{j,i})$ then an established proposition will permit us to write:

$$(F_j^r) = (F_{j,e}^r) + (F_{j,i}^r) .$$

The total relative force is the geometric sum of the external relative force and the internal absolute force.

As a result, whenever one knows the external relative forces, it will be to one's advantage to appeal to that decomposition, while taking into account the fact that the internal absolute forces form a system of forces that is pairwise-equal and oppositely-directed, and since those forces were eliminated due to that property in the theorems that were just established, those theorems will be true no matter what axes to which one refers the motion, provided that the external forces are relative to those axes.

Review of some definitions and properties concerned with geometric quantities. – In order to continue this presentation, it would be convenient to utilize some properties of geometric quantities that shall be rapidly recalled ([2]).



One says the *moment* of a geometric quantity *AB* with respect to a point *O* to mean a geometric quantity *OM* that is perpendicular to the plane (*O*, *AB*) and is directed in such a fashion that an observer whose feet are placed at *O* and whose head is at *M* will see the segment *AB* as if it pointed in the sense that is indicated by the trihedron of axes (ωx, ωy, ωz) once it is chosen (i.e., from left to right, here). Its absolute length is measured by the product of the two numbers, one of which measures the length of the segment and the other of which measures the distance from the point *O* to that segment, or if one prefers, its absolute length is twice the area of the triangle *OAB*.

The *geometric* sum of *n* geometric quantities *AB*, *CD*, … (which might or might not have a common origin) is a geometric quantity that is constructed as follows: Draw geometric quantities through an arbitrary point *O* that are equipollent to *AB*, *CD*, …, respectively, and determine the resultant *OR*. That geometric sum is defined in space only in magnitude and direction, but its origin is arbitrary.

If one is given a system of segments *AB*, *CD*, … then one says the *moment of a system with respect to a point O* to mean the geometric sum of the moments of the quantities *AB*, *CD*, … with respect to that point *O*.

Let (*AB*, *O*) denote the moment of *AB* with respect to the point *O*.

When one considers two points *O* and O', the following geometric relationship will exist between the moments of the same segment *AB* with respect to those two points:

([2]) Consult: J. Tannery, "Deux leçons de Cinématique," Annales de l'École Normale (3) **3** (1886), 43-80.

$$(AB, O) = (AB, O') + (O'B', O) ,$$

in which $O'B'$ is the segment equipollent to AB that is drawn through O'.

It results from this that if one is given an arbitrary system of segments then one can state this theorem:

The moment of a system of segments with respect to a point O is equal to the moment of the same system with respect to a point O', augmented geometrically by the moment with respect to the point O of the geometric sum of the system that is constructed with the point O' for its origin.

Having said that, one says that two systems of segments are *equivalent* when they have the same geometric sum and the same moment with respect to a point in space.

From the foregoing, when those conditions are fulfilled, two systems of segments will have the same moment with respect to an arbitrary point in space.

Furthermore, is easy to show that if one is given a system of segments then one can (and in an infinitude of ways) construct an equivalent system that is composed of either two segments or one segment and a couple.

In the case where one can form a system that is equivalent to a given system that is composed of a single segment, one says that the system *admits a resultant*.

When system of segments is composed of segments that are all parallel to each other, that system will always admit a resultant, unless one cannot form an equivalent system that is composed of only a couple.

Finally, when one combines an arbitrary system of segments that are pairwise equal in length and oppositely-directed with the system that is composed of the union of the two systems, it will be equivalent to the first system, because the geometric sum of the system that was introduced and its moment with respect to an arbitrary point in space will be zero.

Equivalence of the total system of forces that act upon a material ensemble and the system of external forces. – Let us return to dynamics. Let (Ox, Oy, Oz) be arbitrary axes, let M_1, M_2, …, M_n be a system of material points, and let m_1, m_2, …, m_n, resp., be the masses of those points.

Let Γ_j be the acceleration of the point M_j with respect to the axes (Ox, Oy, Oz) under the given conditions. The force that is exerted on that point will be $(F_j) = m\,(\Gamma_j)$. Let $(F_{j,e})$ be the external relative force that is exerted on M_j at the same instant.

Consider the system of segments that is formed from the forces (F_j) and the system that is formed from the forces $(F_{j,e})$. Those two systems are equivalent. Indeed, the system (F_j) differs from the system only by the introduction of internal forces $(F_{j,i})$. Now, the internal forces form a system of segments that are pairwise equal and oppositely-directed. Therefore, the geometric sum of the forces $(F_{j,e})$ will be the same as the geometric sum of the forces (F_j), and the moment of the system $(F_{j,e})$ with respect to an arbitrary point O is equipollent to the moment of the system (F_j) with respect to the same point.

The theorem can be stated thus:

The system of segments m (Γ_j) *is equivalent to the system of segments* $(F_{j,e})$.

That is the proposition from which we shall ultimately deduce the general theorems that we have in mind.

LECTURE 2

THEOREM ON THE MOTION OF THE CENTER OF GRAVITY

Let M be an arbitrary point of a system, let m be its mass, and let x, y, z be its coordinates with respect to any rectangular axes.

The acceleration Γ of that point will have components $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ along those axes.

Let X_e, Y_e, Z_e be the components of the external force F_e (relative to the axes $Oxyz$) that is exerted on that point.

Express the idea that the geometric sum of all the segments $m\,(\Gamma)$ and that of all segments (F_e) are equal. We will have:

$$\sum m\frac{d^2x}{dt^2} = \sum X_e\,, \quad \sum m\frac{d^2y}{dt^2} = \sum Y_e\,, \quad \sum m\frac{d^2z}{dt^2} = \sum Z_e\,.$$

The summations on the left-hand sides extend over all points of the system, while the summations of the right-hand sides extend over all external forces that act upon the system.

If one lets $\mathcal{M}$ denote the total mass $\sum m$ of the system and lets (ξ, η, ζ) denote the coordinates of the center of gravity then one will have:

$$\mathcal{M}\,\xi = \sum m\,x\,, \qquad \mathcal{M}\,\eta = \sum m\,y\,, \qquad \mathcal{M}\,\zeta = \sum m\,z\,,$$

from the definition of the center of gravity.

As a result, the preceding equations will take the form:

$$\mathcal{M}\frac{d^2\xi}{dt^2} = \sum X_e\,, \quad \mathcal{M}\frac{d^2\eta}{dt^2} = \sum Y_e\,, \quad \mathcal{M}\frac{d^2\zeta}{dt^2} = \sum Z_e\,.$$

Hence, the center of gravity of the system moves like a material point where the total mass of the system is concentrated, and which is subject to forces that are equipollent to all external forces.

That is what the theorem of the motion of the center of gravity consists of.

Theorem on the moments of the quantities of motion.

Let us express the idea that the moment with respect to the origin *O* of the system of all segments $m(\Gamma)$ is equal to the moment with respect to that point of the system of all segments (F_e). We shall do that by writing down the idea that the two moments have the same components along the three axes.

We have:

$$\sum m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right) = \sum(xY_e-yX_e)\,,$$
$$\sum m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right) = \sum(yZ_e-zY_e)\,,$$
$$\sum m\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right) = \sum(zX_e-xZ_e)\,.$$

One can put those equations into the form:

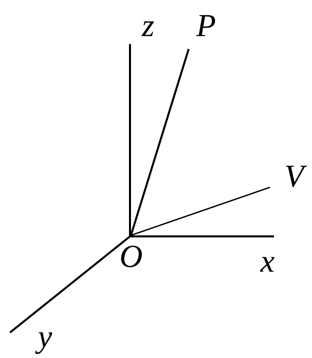
$$\frac{d}{dt}\sum m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right) = \sum(xY_e-yX_e)\,,$$
$$\frac{d}{dt}\sum m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right) = \sum(yZ_e-zY_e)\,,$$
$$\frac{d}{dt}\sum m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right) = \sum(zX_e-xZ_e)\,.$$

If one says the *quantity of motion* of the point *M* to mean the geometric quantity that has its origin at the point *M*, while its direction and sense are those of the velocity *V* of the point *M*, and its absolute magnitude is *m* (*V*) then the expressions $\sum m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$, … will represent the projections onto the axes of the sum of the moments of the quantities of motion of the points of the system with respect to the origin *O*.

Therefore, the derivatives with respect to time of the sum of the moments of the quantities of motion with respect to a fixed axis will be equal to the sum of the moments of the external forces with respect to that axis.

Geometric representation of the preceding theorem.

Construct the geometric sum *OV* of the quantities of motion of the points of the system at the origin and the moment *OP* of those quantities with respect to the origin.



Let (α, β, γ) and (a, b, c) be the coordinates of the points *V* and *P*. One has:

$$\alpha = \sum m \frac{dx}{dt} = M \frac{d\xi}{dt}, \qquad \beta = \ldots, \qquad \gamma = \ldots,$$

$$a = \sum m \left(y \frac{dz}{dt} - z \frac{dy}{dt} \right), \qquad b = \ldots, \qquad c = \ldots$$

The preceding theorems give:

$$\frac{d\alpha}{dt} = \sum X_e, \qquad \frac{d\beta}{dt} = \sum Y_e, \qquad \frac{d\gamma}{dt} = \sum Z_e,$$

$$\frac{da}{dt} = \sum (y Z_e - z Y_e), \qquad \frac{db}{dt} = \sum (z X_e - x Z_e), \qquad \frac{dc}{dt} = \sum (x Y_e - y X_e).$$

Therefore: The velocity of the point *V* is equipollent to the geometric sum of the external forces. The velocity of the point *P* is equipollent to the moment of the system of external forces with respect to the point *O*.

Integrals of motion of the given system from the preceding theorems. – The form of the equalities obtained shows one the cases in which those theorems will tell one about integrals of motion of the system.

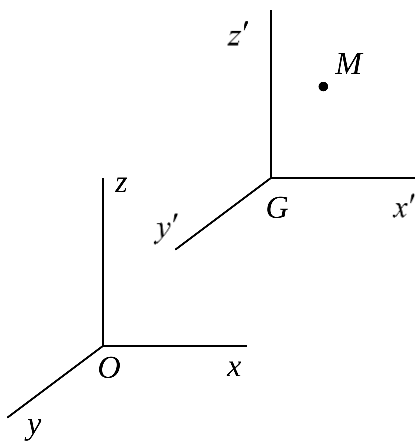
The theorem of the motion of the center of gravity will give an integral whenever the sum of the projections of the external forces onto an axis that is fixed with respect to (*Ox*, *Oy*, *Oz*) is zero.

The theorem of the moments of the quantities of motion will give an integral whenever the moment of the system of external forces with respect to an axis with a fixed position is zero; for example, when all of the external forces meet *Oz* or are parallel to it.

Extension of the theorem of the moments of quantities of motion. – One can extend the theorem of the moments of quantities of motion to the case of relative motion with respect to axes that are animated with respect to (*Ox*, *Oy*, *Oz*) with a certain motion that we shall define.

In particular, that theorem is applicable to the relative motion of the system with respect to axes with fixed directions that pass through the center of gravity *G* of the system.

Let (Gx', Gy', Gz') be three axes with fixed directions; for example, ones that are parallel to (*Ox*, *Oy*, *Oz*). Each point *M* of the system has a certain acceleration Γ^r with respect to those axes



and is subject to a certain force $m(\Gamma^r)$, which is the sum of the external relative force F_e^r and the internal absolute force F_i.

One knows that the system of segments $m(\Gamma_e^r)$ and the system of segments F_e^r are equivalent.

However, as one can show, the moment with respect to the point G of the system of segments (F_e^r) is equipollent to the moment with respect to that same point of the system of segments (F_e), i.e., one can apply the theorem of the moments of the quantities of motion to the axes Gx', Gy', Gz' without changing the external forces.

Indeed, from a theorem of Coriolis, one has the equality:

$$(F_e) = (F_e^r) + m(A),$$

in which (A) represents the acceleration of the center of gravity G with respect to the axes (Ox, Oy, Oz).

The system of segments (F_e^r) can be regarded as formed from the union of the system of segments (F_e) and the system of segments $- m\ (A)$. Now, the latter segments are all parallel to each other and admit a resultant that passes through the center of gravity G.

The moment of the system $- m\ (A)$ with respect to G is zero, and the moment of the system (F_e^r) with respect to G is equipollent to the moment of the system (F_e).

One can write:

$$\sum m\left(x'\frac{d^2y'}{dt^2} - y'\frac{d^2x'}{dt^2}\right) = \sum (x'Y_e - y'X_e)\ ,$$

....

or rather:

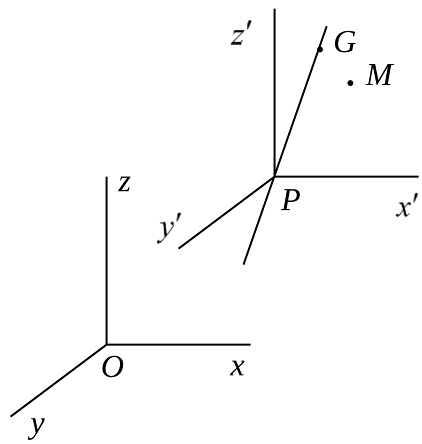
$$\frac{d}{dt}\sum m\left(x'\frac{dy'}{dt} - y'\frac{dx'}{dt}\right) = \sum (x'Y_e - y'X_e)\ ,$$

....

As a result, whenever there exists an axis with a direction that is fixed with respect to (Ox, Oy, Oz) and passes through the center of gravity of the system, such that all of the external forces either meet it or are parallel to it, one will get an integral of motion upon applying the theorem of the moments of the quantities of motion to that axis.

Rigorously speaking, the extension that was made of the theorem of the moments of the quantities of motion demands only that the point G must be the center of gravity of the system.

Indeed, suppose that one considers the relative motion with respect to the three axes (Px', Py', Pz') that are parallel to (Ox, Oy, Oz), where P is an arbitrary point whose acceleration with respect to (Ox, Oy, Oz) is (A). One will then have:



$$(F_e^r) = (F_e) + m(A).$$

In order for the system of segments (F_e^r) and the system of segments (F_e) to have equipollent moments with respect to the point P, it is necessary and sufficient that the system of segments m (A) should have a zero moment with respect to P. Now, the system m (A) admits a resultant that passes through the center of gravity G and is equal to $\mathcal{M}$ (A). In order for that system to have a zero moment, it is necessary and sufficient that this resultant must pass through the point P. Therefore, it is necessary and sufficient that the acceleration (A) of a point P should be directed along PG at each instant.

That condition is always realized when the point P coincides with G or when the acceleration of the point P is zero. However, in the latter case, P is animated with a uniform, rectilinear translation, and one knows that such a motion will not change the forces relative to the axes.

Remark that is useful for calculating the moment of the quantities of motion of a system. – The moment of the quantities of motion of a system with respect to a point O is equal to the moment of the quantities of motion of the system with respect to the center of gravity G of the system in its relative motion with respect to the center of gravity, geometrically augmented by the quantity of motion of the center of gravity (at which one supposes that the entire mass of the system is concentrated) with respect to the point O.

Let O be the origin of the axes, and let M_1, M_2, …, M_n be a system of points, while G is the center of gravity. By definition, one has:

$$\mathcal{M}\,\xi = \sum m\,x, \qquad \mathcal{M}\,\eta = \sum m\,y, \qquad \mathcal{M}\,\zeta = \sum m\,z.$$

Recall this property: The moment of a system of segments with respect to a point O is equal to the moment of that system with respect to a point G plus the moment with respect to O of the geometric sum of the segments that are constructed with G as its origin.

One applies that theorem by decomposing the moment of the quantities of motion of the system into two parts:

1. Moment of the sum of the quantities of motion, when constructed with G as the origin.

The projections of the sum of the quantities of motion onto the axes are $\sum m \frac{dx}{dt}$, $\sum m \frac{dy}{dt}$, $\sum m \frac{dz}{dt}$, or rather $\mathcal{M}\frac{d\xi}{dt}$, $\mathcal{M}\frac{d\eta}{dt}$, $\mathcal{M}\frac{d\zeta}{dt}$. The moment considered is then the moment with respect to the point *O* of the quantity of motion of the center of gravity where the total mass of the system $\mathcal{M}$ will be concentrated.

2. Moment of the quantities of motion of the system with respect to the point *G*.

I say that this moment is equipollent to the moment with respect to the same point *G* of the quantities of motion of the points of the system in their relative motion with respect to the center of gravity.

Indeed, let V_a and V_r be the velocities of an arbitrary point of the system with respect to (*Ox*, *Oy*, *Oz*) and (Gx', Gy', Gz'), resp., and let *W* be the velocity of the center of gravity *G* with respect to (*Ox*, *Oy*, *Oz*). One will have:

$$(V_r) = (V_a) - (W) .$$

From that, the set of segments $m\ (V_a)$ can be regarded as being formed from the system of segments $m\ (V_r)$ and the system of segments $m\ (W)$. In order for the moment of the system $m\ (V_a)$ and that of the system $m\ (W)$ with respect to *G* to be the same, it is necessary and sufficient that the moment of the system $m\ (V_r)$ with respect to *G* should be zero. Now, the segments $m\ (W)$ admit a resultant that passes through *G*, and as a result, the moment of their system with respect to *G* is zero. Therefore, the moment of system $m\ (V_a)$ and that of the system $m\ (V_r)$ with respect to *G* will coincide.

The stated proposition is thus found to have been proved.

The *vis viva* theorem. – We shall now establish a third theorem that also provides integrals of the motion of the system in certain cases.

Consider the equations of motion of a point of the system:

$$\left\{\begin{aligned} m\frac{d^2x}{dt^2} &= X &&= X_e + X_i, \\ m\frac{d^2y}{dt^2} &= Y &&= Y_e + Y_i, \\ m\frac{d^2z}{dt^2} &= Z &&= Z_e + Z_i. \end{aligned}\right.$$

Multiply them by dx / dt, dy / dt, dz / dt, resp., and add corresponding sides. Upon denoting the velocity of that point by v, that will give:

$$\frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) = \left(X_a\frac{dx}{dt}+Y_a\frac{dz}{dt}+Z_a\frac{dz}{dt}\right)+\left(X_i\frac{dx}{dt}+Y_i\frac{dz}{dt}+Z_i\frac{dz}{dt}\right) .$$

Take the sum of the analogous equations that relate to all of the points of the system, so:

$$\frac{d}{dt}\sum\tfrac{1}{2}mv^2 = \sum\left(X_a\frac{dx}{dt}+Y_a\frac{dz}{dt}+Z_a\frac{dz}{dt}\right)+\sum\left(X_i\frac{dx}{dt}+Y_i\frac{dz}{dt}+Z_i\frac{dz}{dt}\right) .$$

The forces (X_i, Y_i, Z_i) can be decomposed into partial forces that are pairwise equal and directly opposite. If one replaces X_i, Y_i, Z_i in that sum with the components of those partial forces (namely, $X_{j,k}$, $Y_{j,k}$, $Z_{j,k}$ for the points M_j and M_k) then one will know that the sum represents (up to the factor $1 / dt$), the elementary work done by all internal forces under an infinitely-small displacement of the system. Now, the work that is done by the force F_{jk} that is exerted between M_j and M_k under such a displacement is in fact $f_{j,k}\, dr_{jk}$, where r_{jk} is the distance between the two points M_j and M_k, and $f_{j,k}$ is the absolute value of the force F_{jk}, preceded by a + or – sign according to whether the force is repulsive or attractive.

The preceding relation can then be written:

$$d\sum\tfrac{1}{2}mv^2-\sum f_{j,k}\,dr_{jk} = \sum\left(X_e\,dx+Y_e\,dy+Z_e\,dz\right) .$$

In the case where the forces $f_{j,k}$ are given forces that are functions of only the distance between the points (which is what happens for the material points in our planetary system) then $\sum f_{j,k}\,dr_{jk}$ will be an exact total differential. Each function $f_{j,k}$ will depend upon only the variable r_{jk}.

When the internal forces are reactions due to the constraints that the system is subject to, one will know nothing about the forces $f_{j,k}$. However, in the case where the system is composed of a solid body, all of the dr_{jk} will be zero. Therefore, for a solid body, one will have:

$$d\sum\tfrac{1}{2}mv^2 = \sum\left(X_e\,dx+Y_e\,dy+Z_e\,dz\right) .$$

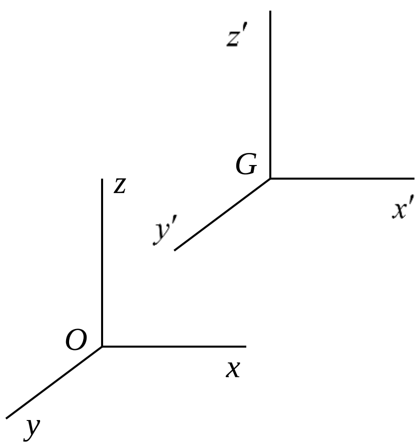
In those two distinct cases, in which f_{jk} depends upon just the variable r_{jk} and in which dr_{jk} is zero, if the expression $\sum\left(X_e\,dx+Y_e\,dy+Z_e\,dz\right)$ is an exact total differential of a function of the coordinates of the points of the system, namely, dU (x_j, y_j, z_j, …), then the relation that we have formed will tell us an integral of motion ([3]).

That integral is called the *vis viva* integral.

([3]) It even suffices that one has: $\sum X_e\,x'+Y_e\,y'+Z_e\,z' \equiv \frac{d}{dt}U$ (x_1, y_1, z_1, …, x_n, y_n, z_n, t), i.e., that X_e is equal to $dU / dt + X_e'$, Y_e is equal to $dU / dt + Y_e'$, and Z_e is equal to $dU / dt + Z_e'$, with the condition that: $\sum X_e'\,x'+Y_e'\,y'+Z_e'\,z' \equiv dU/ dt$. That will happen if, for example, the force X_e', Y_e', Z_e' is normal to the velocity x', y', z', and U does not depend upon t.

When there exists such a function U, one says that the external forces admit a force function U.

Useful remark for calculating the *vis viva* of a system. – One easily proves the following proposition: The *vis viva* of a system with respect to arbitrary axes (Ox, Oy, Oz) is equal to the sum of two terms: The *vis viva* of the center of gravity, where all of the mass of the system is concentrated, and the *vis viva* of the relative motion of the system with respect to axes with invariable directions that pass through the center of gravity.



Hence, let v be the velocity of the point M of mass m with respect to (Ox, Oy, Oz), let v' be its velocity with respect to (Gx', Gy', Gz'), let $\mathcal{M}$ be the total mass of the system, and let V be the velocity of the center of gravity.

One has:

$$\sum m v^2 = \mathcal{M} V^2 + \sum m v'^2 .$$

Moments of inertia. Calculating them. – When one deals with continuous systems (for example, homogeneous solid bodies), the application of the theorem of *vis viva* or the theorem of the moments of the quantities of motion will lead one to consider certain sums called *moments of inertia* that are expressed by triple, double, or single integrals according to whether the system is three, two, or one-dimensional, resp.

If one is given a system of material points at a given instant with masses m_1, m_2, … that occupy positions M_1, M_2, …, resp., that are at distances r_1, r_2, …, resp., from a line L then the quantity:

$$\sum m r^2 ,$$

when extended over all points of the system, is called the *moment of inertia* of the system with respect to the line L at the instant considered.

Here is how one introduces the consideration of such quantities:

Suppose that one must calculate the moment of the quantity of motion of a system with respect to an axis L under a motion such that all of the points are animated with the same rotation ω around the line L. For each point, the moment of the quantity of motion is $m\,\omega\, r \times r$ or $m\, r^2 \omega$. The desired moment is then $\omega \sum m r^2$.

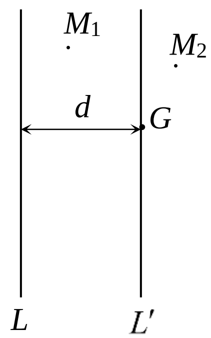
Similarly, if one must calculate the *vis viva* under such a motion then one will find $\sum m \omega^2 r^2$ or $\omega^2 \sum m r^2$.

Under the motion of a solid body around a fixed point, and notably under the motion of a solid around its center of gravity, the velocities of all points of the body at each instant are the same as if all points were animated with a rotation around a certain line that passes through the fixed point that is called the *instantaneous axis of rotation*. The calculation of the *vis viva* under such a motion will imply that one must calculate the moment of inertia of the system with respect to that instantaneous axis.

One can state certain general theorems on moments of inertia that facilitate the calculations.

The moment of inertia of a system with respect to a line L is equal to the moment of inertia of the system with respect to a line that is parallel to L and passes through the center of gravity, plus the moment of inertia with respect to L of the center of gravity where the total mass is concentrated.

That theorem is only a particular case of an analogous proposition that occurs in the context of the *vis viva* of the system. Indeed, suppose that the system is animated with a certain rotational motion ω around the line L. The *vis viva* of the system is:



$$\sum m r^2 \omega^2 .$$

Let r be the distance from the point M to the line L' that is parallel to L and drawn through the center of gravity of the system G. Let $\mathcal{M}$ be the total mass of the system, while d is the distance from L to L'.

The *vis viva* of the system in its motion with respect to the center of gravity is $\sum m \rho^2 \omega^2$.

The *vis viva* of the center of gravity where all of the mass is concentrated is $\mathcal{M} d^2 \omega^2$.

From the proposition that was just recalled, one will have:

$$\sum m r^2 \omega^2 = \sum m \rho^2 \omega^2 + \mathcal{M} d^2 \omega^2 ,$$

i.e.:

$$\sum m r^2 = \mathcal{M} d^2 + \sum m \rho^2 ,$$

which is precisely the theorem in question.

From it, when one knows how to calculate the moment of inertia of the system with respect to an arbitrary line OL that passes through a point O in space (for example, through its center of gravity), one will know how to calculate its moment of inertia with respect to an arbitrary line with no new quadrature, but on the condition that one must know the total mass of the system. Consequently, when one must calculate moments of inertia, one seeks the point O in space that is most advantageous for obtaining the moments of inertia with respect to the lines OL.

On the subject of moments of inertia with respect to lines that pass through a point O, one can prove the following proposition:

If one measures out a length of:

$$OA = \frac{1}{\sqrt{\sum m r^2}}$$

along each line OL, when one starts from O, in which $\sum m r^2$ *is the moment of inertia of the system with respect to OL, then the point A will describe an ellipsoid that has O for its center and is called the* ***ellipsoid of inertia*** *relative to the point O.*

Suppose that one has calculated the equation of that ellipsoid of inertia. In order to find the moment of inertia of the system with respect to a line that passes through *O*, it will suffice to find the intersection of the line with the ellipsoid.

If one knows the three axes of the ellipsoid of inertia and the moments of inertia relative to those axes then one can calculate the moment of inertia of the system with respect to an arbitrary line that passes through *O* algebraically (and as a result, with respect to an arbitrary line in space, on the condition that one must know the total mass of the system).

That will be the case for a solid with three symmetry planes when one knows how to calculate the moments of inertia with respect to the three symmetry axes.

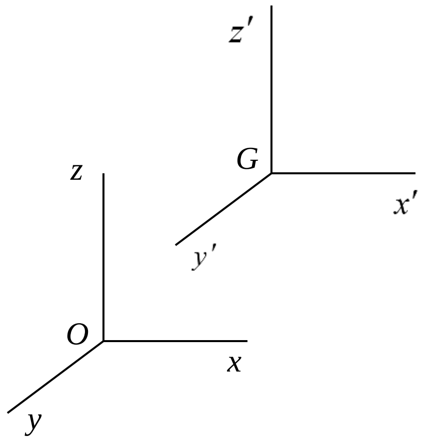
More generally, when one knows the mass of an arbitrary system and its moments of inertia relative to six concurrent lines that are not all situated in the same plane, one will know six points of the ellipsoid of inertia and its center. The ellipsoid of inertia will then be defined, and the moments of inertia with respect to arbitrary lines will be obtained algebraically.

The axes of the ellipsoid of inertia, which are called the *principal axes*, are characterized by the relations:

$$\sum m y z = 0 , \qquad \sum m z x = 0 , \qquad \sum m x y = 0$$

when one takes those axes to be the coordinate axes.

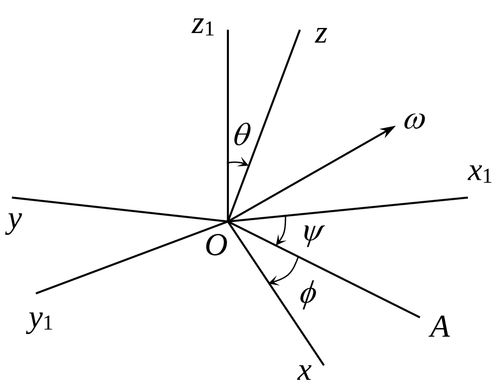
The preceding theorems are often useful in the calculation of the *vis viva* or moment (with respect to an axis or a point) of the quantity of motion of a system. For example, suppose that the system includes solid bodies, and let Σ be one of those solid bodies. One decomposes the *vis viva* *K* of Σ into two parts: The *vis viva* K' of the center of gravity *G* of Σ where the mass of Σ is concentrated and the *vis viva* K'' of the relative motion of Σ around the point *G*.



Let $G\omega$ be the instantaneous segment of rotation of Σ around *G* at time *t*, and let *p*, *q*, *r* be the components of $G\omega$ along the three principal axes of inertia *GX*, *GY*, *GZ* of Σ, which have well-defined positions at the instant *t*. One proves that K'' is equal to $Ap^2 + Bq^2 + Cr^2$ if *A*, *B*, *C* denote the moments of inertia relative to *GX*, *GY*, *GZ*.

Similarly, the calculation of the moment ($O\,\Gamma$) with respect to the point *O* of the quantity of motion of Σ (in its motion with respect to *Oxyz*) reduces to the calculation of the moment $(G\Gamma')$ with respect to the point *G* of the quantity of motion of Σ in its motion with respect to $G\,x'\,y'\,z'$. One proves that the projections of $(G\Gamma')$ onto *GX*, *GY*, *GZ* are equal to *Ap*, *Bq*, *Cr*, respectively. The calculation of the moments with respect to any axis is then deduced immediately from the foregoing.

LECTURE 3



Motion of a solid body that has a fixed point. – Let (Ox_1, Oy_1, Oz_1) be three arbitrary axes with respect to which one would like to study the motion of a solid body for which O is a fixed point.

One defines the position of the solid with the aid of the principal axes of inertia Ox, Oy, Oz, which are determined by the three Euler angles.

Let OA be the intersection of the plane xOy with the plane x_1Oy_1 . One takes a positive direction for OA arbitrarily along that line. The Euler angles are:

$\psi = x_1OA$	measured positively around	Oz	from left to right
$\theta = z_1Oz$	″ ″ ″	OA	″ ″
$\phi = AOx$	″ ″ ″	Oz	″ ″

One might need to express the coordinates x_1, y_1, z_1 of a point as functions of its coordinates x, y, z. In order to do that, one employs the well-known process: One makes (Ox_1, Oy_1, Oz_1) coincide with (Ox, Oy, Oz) by three successive rotations, one of which $\delta\psi$ is around Oz, the second of which $\delta\theta$ is around OA, and the third of which $\delta\phi$ is around Oz. Each partial rotation is effected by applying the transformation formulas for coordinates in plane geometry. That will then give:

$$\begin{cases} x_1 = (\cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi)\,x - (\sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi)\,y + (\sin\theta\sin\psi)\,z\,, \\ y_1 = (\sin\psi\cos\phi + \cos\theta\cos\psi\sin\phi)\,x + (\sin\phi\sin\psi - \cos\theta\cos\phi\cos\psi)\,y - (\sin\theta\cos\psi)\,z\,, \\ z_1 = (\sin\phi\sin\theta)\,x + (\cos\phi\sin\theta)\,y + (\cos\theta)\,z\,. \end{cases}$$

Let:

	Ox	Oy	Oz
Ox_1	α	α'	α''
Oy_1	β	β'	β''
Oz_1	γ	γ'	γ''

be the matrix of direction cosines of the angles between the axes $Oxyz$ and the axes $Ox_1\,y_1\,z_1$.

The transformation formulas are:

$$x_1 = \alpha\,x + \alpha'\,y + \alpha''\,z\,,$$
$$y_1 = \beta\,x + \beta'\,y + \beta''\,z\,,$$

$$z_1 = \gamma x + \gamma' y + \gamma'' z .$$

One will get the expressions for the direction cosines as functions of the Euler angles θ, ϕ, ψ upon comparing those formulas with the ones that were obtained above (†):

$$\begin{aligned}
\alpha &= \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi \;, \\
\beta &= \sin\psi\cos\phi + \cos\theta\cos\psi\sin\phi \;, \\
\gamma &= \sin\phi\sin\theta \,, \\
\alpha' &= - \sin\phi\cos\psi - \cos\theta\cos\phi\sin\psi \,, \\
\beta' &= - \sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi \,, \\
\gamma' &= - \cos\phi\sin\theta \,, \\
\alpha'' &= - \sin\theta\sin\psi \,, \\
\beta'' &= - \sin\theta\cos\psi \,, \\
\gamma'' &= - \cos\,\theta .
\end{aligned}$$

One knows that the velocities of all points of the solid at each instant are the same as if the entire system were animated with a rotation ω around the axis $O\omega$.

That instantaneous axis of rotation can be defined by either its projections p_1, q_1, r_1 onto $Ox_1y_1z_1$ or its projections onto $Oxyz$. Those quantities are functions of time t.

If one eliminates t from the expressions for p_1, q_1, r_1 then one will get the locus of points ω in the space $Ox_1y_1z_1$.

If one eliminates t from the expressions for the ratios p_1 / r_1, q_1 / r_1 then one will get the equation of the cone that is based in the motion of the solid, i.e., the locus of the instantaneous axis in the space $Ox_1y_1z_1$ during its motion.

If one eliminates t from the expressions for p, q, r (or from the ones for p / r, q/ r) then one will have the locus of the point in the solid (or the equation for the rolling cone, i.e., the locus of the instantaneous axis in the solid during its motion).

It is easy to calculate the expressions for p, q, r and p_1, q_1, r_1 as functions of the Euler angles and their derivatives with respect to time.

Indeed, x, y, z will remain invariable for a point in the solid body, and one will find that:

$$\frac{dx}{dt} = z_1\left(\gamma\frac{d\alpha}{dt} + \gamma'\frac{d\alpha'}{dt} + \gamma''\frac{d\alpha''}{dt}\right) - y_1\left(\alpha\frac{d\beta}{dt} + \alpha'\frac{d\beta'}{dt} + \alpha''\frac{d\beta''}{dt}\right) = q_1\, z_1 - r_1\, y_1 \,.$$

Therefore:

$$r_1 = \alpha\frac{d\beta}{dt} + \alpha'\frac{d\beta'}{dt} + \alpha''\frac{d\beta''}{dt} .$$

(†) Translator: The signs in these equations are clearly inconsistent with the ones in the previous equations.

Hence, one has p_1, q_1, r_1 as functions of ψ, θ, ϕ and $d\psi / dt$, $d\theta / dt$, $d\phi / dt$.

One will also have p, q, r as functions of the same quantities, because:

$$p = \alpha\, p_1 + \beta\, q_1 + \gamma\, r_1 = \alpha'' \frac{d\alpha'}{dt} + \beta'' \frac{d\beta'}{dt} + \gamma'' \frac{d\gamma'}{dt},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

That calculation is practicable, but long.

Instead of doing that, one can decompose $O\omega$ along OA, Oz, Oz_1 . The velocity of each point that is due to the instantaneous rotation $O\omega$ is the same as the one that will result from the composition of the three instantaneous rotations $d\psi / dt$, $d\theta / dt$, $d\phi / dt$ around Oz_1, OA, Oz, resp.

Furthermore, one will get p, q, r and p_1, q_1, r_1 by taking the sum of the projections of those three rotations onto Ox, Oy, Oz, Ox_1, Oy_1, Oz_1, respectively.

That will give:

$$p = \frac{d\psi}{dt}\gamma + \frac{d\theta}{dt}\cos\phi,$$
$$q = \frac{d\psi}{dt}\gamma' + \frac{d\theta}{dt}\sin\phi,$$
$$r = \frac{d\psi}{dt}\gamma'' + \frac{d\phi}{dt},$$

i.e.:

$$\text{(I)} \qquad \left\{\begin{aligned} p &= \frac{d\psi}{dt}\sin\phi\sin\theta + \frac{d\theta}{dt}\cos\phi, \\ q &= \frac{d\psi}{dt}\sin\theta\cos\phi - \frac{d\theta}{dt}\sin\phi, \\ r &= \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt}. \end{aligned}\right.$$

Similarly:

$$p_1 = \frac{d\theta}{dt}\cos\psi + \frac{d\phi}{dt}\alpha'',$$
$$q_1 = \frac{d\theta}{dt}\sin\psi + \frac{d\phi}{dt}\beta'',$$
$$r_1 = \frac{d\psi}{dt} + \frac{d\phi}{dt}\gamma'',$$

i.e.:

$$\begin{cases} p_1 = \dfrac{d\theta}{dt}\cos\psi + \dfrac{d\phi}{dt}\sin\theta\sin\psi, \\ q_1 = \dfrac{d\theta}{dt}\sin\psi - \dfrac{d\phi}{dt}\sin\theta\cos\psi, \\ r_1 = \dfrac{d\psi}{dt} + \dfrac{d\phi}{dt}\cos\theta. \end{cases}$$

It should be remarked that the quantities α'', β'', γ'', γ, γ' have simple expressions: They are the ones that enter into the problem, in general.

Having said that, one proves that the moment of the quantity of motion of the system with respect to the point O is a geometric quantity $O\Gamma$ whose projections onto (Ox, Oy, Oz) are Ap, Bq, Cr (where A, B, C are the moments of inertial relative to the axes Ox, Oy, Oz).

Euler's method for obtaining the equations of motion of the solid consists of expressing the idea that the velocity of the point Γ is a geometric quantity that is equipollent to the moment of the external forces with respect to the point O.

Since the reaction of the fixed point O has a zero moment, one will have three equations that are independent of the reaction of the point O for determining the motion of the system that depend upon the three parameters θ, ψ, ϕ, which are equations that one will obtain simply by writing that the projections onto the axes Ox, Oy, Oz of the velocity Γ with respect to Ox, Oy, Oz are equal to the moments with respect to Ox, Oy, Oz, respectively, of the external forces that are applied to the system.

The equations thus-obtained, which are called *Euler's equation*, are:

$$\text{(II)} \qquad \begin{cases} A\dfrac{dp}{dt} + (C-B)\,q\,r = L, \\ B\dfrac{dq}{dt} + (A-C)\,r\,p = M, \\ C\dfrac{dr}{dt} + (B-A)\,p\,q = N, \end{cases}$$

in which L, M, N are the projections onto Ox, Oy, Oz of the moment of the external forces with respect to O.

Those equations indeed give integrals of motion in some cases. For example, when the body is one of revolution around an axis that passes through O, that axis will be a principal axis of inertia, namely, the axis Ox. One will have $B = C$, and if the moment of the external forces with respect to that axis is zero ($L = 0$) then one will deduce that p = const. from the first Euler equation.

Generally, whenever the moment of the external forces with respect to a fixed direction (Oz, for example) is zero, one will have a first integral of the motion by expressing the idea that the moment of the quantity of motion with respect to that axis is a constant. Since the moment of the quantity of motion has Ap, Bq, Cr for its components along (Ox, Oy, Oz), one will have:

$$A\,p\,\gamma + B\,q\,\gamma' + C\,r\,\gamma'' = \text{const.} = \text{const.} = K\,,$$

if all of the external forces have a zero moment with respect to Oz_1 .

If they have a zero moment with respect to Ox_1, in addition, then one will have a second integral of motion.

One sees from this that it is indispensable in certain cases to know how to calculate α, α', α'', β, β', β'', γ, γ', γ'' as functions of the Euler angles.

The Euler equations and equations (I) form a system of six simultaneous first-order differential equations that define p, q, r, θ, ϕ, ψ.

If one replaces p, q, r with their values as functions of θ, ϕ, ψ in (II) then one will have a system of three simultaneous second-order equations that define θ, ϕ, ψ.

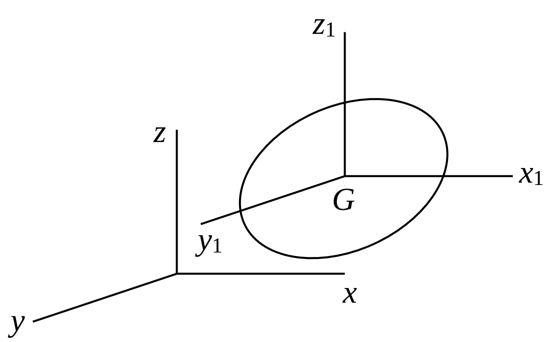
It is more advantageous to preserve the six equations, and above all, when L, M, N do not depend upon θ, ϕ, ψ. One can then integrate the system (II) separately, and when one has p, q, r, one can then integrate the system (I). The calculations will be simpler then.

The *vis viva* of the system is:

$$A\,p^2 + B\,q^2 + C\,r^2\,.$$

From the practical viewpoint, observe that the first integrals that one easily obtains are given by the *vis viva* theorem, along with the theorem of the moments of quantities of motion when the external forces meet a fixed axis or when the solid is one of revolution around a certain axis that passes through O and the external forces constantly have a zero moment with respect to that axis.

Motion of a solid that is entirely free. – Now let us consider the most-general motion of a solid body.



One appeals to the stated theorems in order to study:

1. The motion of the center of gravity G of the body, to which one supposes that all of the external forces that are exerted on the solid are applied, with respect to three fixed axes Ox, Oy, Oz.

2. The relative motion of the solid with respect to axes Gx_1, Gy_1, Gz_1 with fixed directions that are drawn through the center of gravity and parallel to Ox, Oy, Oz, resp.

In order to study the motion of the solid around its center of gravity, one appeals to the fact that the theorem of the moments of the quantities of motion that applies to the motion can be applied to the center of gravity without needing to change the external forces.

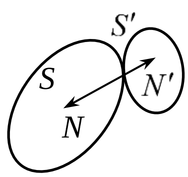
In the case where the solid body is not entirely free, less than six parameters will be required in order to determine its motion, but unknown reactions will be introduced. In certain cases, it will then be convenient to address the question differently: For example, if a fixed point P of the solid is subject to a certain constraint then it can be advantageous to study the relative motion around that point.

When all of the external forces pass through the center of gravity of the body, the motion of the solid around its center of gravity will be that of a body that has a fixed point and on which no external forces act.

Example. – Let a massive homogeneous sphere be launched in a vacuum. Its center will describe an arc of a parabola, and it will be animated about its center with a uniform rotation around an axis that is fixed in space and in the sphere.

Motion of an ensemble of solids. – When one has a system of solid bodies, their mutual reactions will be forces that are internal to the system.

In certain cases, there is some advantage to considering part of the system to be an isolated system, and one must then take care to regard all of the forces that act upon the partial system and do not cancel from the principle of action and reaction as external ones.



Therefore, let two surfaces S and S' slide without friction on each other. If one considers their ensemble then their mutual reactions will be internal to the system. If one regards one of them S as a separate system then one must consider the reaction of S' to be an external force.

Applications.

I. – *A massive, homogeneous, hollow sphere slides without friction on a horizontal plane. A massive point M slides without friction inside the sphere. Find the motion of the system.*

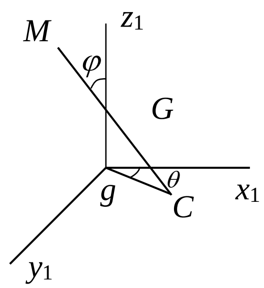
The motion of the system depends upon seven parameters: two to fix the position of the center of the sphere, three to fix the position of the sphere around its center, and two to fix the position of the point that moves on the sphere.

We first remark that the motion of the sphere around its center C (which is its center of gravity) is known immediately. The forces that act upon the sphere, when it is considered to be an isolated system, are indeed its weight, the reaction of the plane, and the reaction of the moving point, which all pass through the center C. From the theorem of generalized moments of quantities of motion, the total moment of the quantities of motion with respect to the point C is therefore zero, and the motion of the sphere around its center is a uniform rotation around an axis that passes through C and is fixed in the sphere and space.

Moreover, it will suffice to have four integrals in order to succeed in determining the motion.

Two integrals are given by the theorem of the motion of the center of gravity. All of the external forces that are applied to the system that consists of the sphere and the point are vertical, in such a way that if one refers the motion to three axes (*Ox*, *Oy*, *Oz*), one of which *Oz* is vertical, and if (ξ, η, ζ) are the coordinates of the center of gravity *G* of the system then one will have:

$$\frac{d^2\xi}{dt^2} = 0 , \qquad \frac{d^2\eta}{dt^2} = 0 .$$



As a result, the horizontal projection of the center of gravity of the system is animated with a uniform rectilinear translation.

Therefore, refer the system to three axes that are parallel to *Ox*, *Oy*, *Oz* and have their origin at the projection *g* of the center of gravity *G* onto the horizontal plane that contains the center of the sphere, namely (gx_1, gy_1, gz_1). In order to study the motion with respect to those new axes, one will not have to change the external forces that are applied to the system, because the new trihedron is animated with respect to the old one with a uniform rectilinear translational motion.

Let m' be the mass of the sphere and let *m* be that of the point. Upon letting *R* denote the radius of the sphere, one will have:

$$CG = \lambda = \frac{m'}{m'+m} R , \quad MG = \mu = \frac{m}{m'+m} R .$$

The position of the two points *M* and *G* is defined by the angle θ between *CG* and gx_1 and the angle φ between *CG* and gz_1 . The motion of the point *C* is the same as if that point were a material point of mass m' to which all of the external forces that are applied to the sphere will be applied (viz., weight, normal reaction of the horizontal plane, reaction of the point *M* on the sphere that points along *MC*). If one applies the theorem of the moments of quantities of motion with respect to *Oz* and the *vis via* theorem to the system of two points *M* and *C* then one will get first integrals that define θ and φ as functions of *t* : Indeed, the moment with respect to *Oz* of the external forces that are applied to the system is zero, and the work that they do reduces to the work done by gravity, when it is applied to the point *M*. As for the work done by internal forces, it will be zero because the distance *MC* is constant. Let us calculate those two integrals. The coordinates *x*, *y*, *z* of *C* and *M* are:

(*C*)	$\lambda \sin\varphi \cos\theta$,	$\lambda \sin\varphi \cos\theta$,	0
(*M*)	$-\mu \sin\varphi \cos\theta$,	$-\mu \sin\varphi \cos\theta$,	$R \cos\varphi$,

respectively.

Upon calculating the derivatives of the coordinates, one will see that the theorem of areas gives:

$$\theta' \sin\varphi = \text{const.},$$

and the *vis viva* theorem gives:

$$(m'\lambda^2 + m\mu^2)\sin^2\varphi\,\theta'^2 + [(m'\lambda^2 + m\mu^2)\cos^2\varphi + mR^2\sin^2\varphi]\varphi'^2 = -2\,m\,g\,R\cos\varphi + \text{const.}$$

If one eliminates θ' between the two equations then that will give (upon taking into account the values of λ and μ):

$$\left(1 - \frac{m}{m+m'}\cos^2\varphi\right)\varphi'^2 = h - \frac{2g}{R}\cos\varphi - \frac{K^2}{\sin^2\varphi} \;,$$

in which h and K denote arbitrary constants. The area constant is equal to $K\sqrt{\frac{m+m'}{m'}}$:

$$\theta'\sin^2\varphi = K\sqrt{\frac{m+m'}{m'}}\,.$$

One will thus obtain t and θ as functions of φ by two hyperelliptic quadratures:

$$(1)\qquad dt = \pm\sin\varphi\,d\varphi\sqrt{\frac{1-\frac{m}{m+m'}\cos^2\theta}{\left(h-\frac{2g}{R}\cos\varphi\right)\sin^2\varphi - K^2}} \quad\text{or}\quad \pm\,du\sqrt{\frac{1-\frac{m}{m+m'}u^2}{\left(h-\frac{2g}{R}\cos\varphi\right)(1-u^2) - K^2}}$$

if one sets $\cos\varphi = u$) and:

$$(2)\qquad \frac{1}{K}\sqrt{\frac{m'}{m+m'}}\,d\theta = \pm\frac{d\varphi}{\sin\varphi}\sqrt{\frac{1-\frac{m}{m+m'}\cos^2\theta}{\left(h-\frac{2g}{R}\cos\varphi\right)\sin^2\varphi - K^2}} = \pm\frac{du}{(1-u^2)}\sqrt{\frac{1-\frac{m}{m+m'}u^2}{\left(h-\frac{2g}{R}\cos\varphi\right)(1-u^2) - K^2}}\,.$$

Those equalities permit us to discuss the motion: Let $R(u) = P(u) / Q(u)$ be the quantity inside the radical. $P(u)$ will always be positive when u varies from – 1 to + 1. $Q(u)$ is a third-degree polynomial that is positive for $u = u_0$ and negative (if K^2 is non-zero) for $u = \pm 1$. Since the three roots u_1, u_2, u_3 of Q are real and the term in u^3 is positive, their orders are as follows:

$Q(u)$	–	0	+	0	–	0	+
u	– 1	u_1	u_0	u_2	+ 1	u_3	+ ∞

If du / dt is positive at the beginning of the motion then u will begin to increase up to the value u_2 that it attains. If du / dt is zero then u will begin to increase or decrease according to whether $Q'(u_0)$ is positive or negative, resp. ([4]):

$$Q'(u_0) = -\ 2u_0\left(h - \frac{2g}{R}u_0\right) - \left(1 - u_0^2\right)\frac{2g}{R},$$

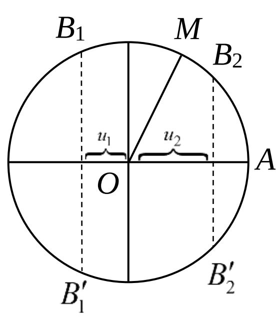
and upon deducing $\left(h - \frac{2g}{R}u_0\right)$ from the equation $Q(u_0) = 0$, one will see that $Q'(u_0)$ will be positive if one has:

$$\frac{k^2\,u_0}{(1-u_0)^2} + \frac{g}{R} < 0\,,$$

or rather (upon replacing K^2 as a function of θ_0' and u_0):

$$\frac{m}{m+m'}\,z_0\,\theta_0'^2 + g < 0\,,$$

z_0 denotes the initial value at M. If that condition is fulfilled then u will first increase (i.e., φ will decrease). In the opposite case, φ will begin by increasing. u_0 will coincide with u_2 under the first hypothesis and with u_1 under the second one.



From that, if one marks out the point M that defines the angle φ on the trigonometric circle then one will see that the point will constantly oscillate between the points M_1 and M_2 . $d\varphi / dt$ will keep the same absolute value whenever M passes through the same point of the arc B_1B_2, but it will alternate between positive and negative. φ and $d\varphi / dt$ will take the same values again after a time $T = 2\int_{u_1}^{u_2} du\sqrt{R(u)}$.

On the other hand, $d\theta / dt$ will always keep the sign of K. For example, suppose that K is positive. θ will constantly increase with t. Let ω denote the variation of θ during the time T :

$$\omega = 2\int_{u_1}^{u_2} du\sqrt{R(u)}\ .$$

([4]) We remark that $u \equiv u_0$ is an integral of (1) in this case. The equality $u \equiv u_0$ implies the equality $\theta = a\ t + b$. However, those integrals of equations (1) and (2) do not satisfy the equations of motion of the system, since the further consequences of that discussion will at least show that u_0 is not a double root of Q. Those solutions are parasitic solutions that were introduced by the transformations that take the equations of motion to equations (1) and (2): It would be easy to account for them directly, but that is a point to which we shall return in the context of the Lagrange equations.

The values of θ at times t and $t + T$ have a constant difference, which is ω. If one turns the axes gx_1, gy_1 through the angle ω around gz_1 then the motion of the system at time $t + T$ with respect to the new axes is the same as its motion at time t is with respect to the first axes $g\,x_1\,y_1\,z_1$.

The foregoing supposes that u_1 and u_2 are distinct. They can coincide only if u_0 is annulled along with Q and its derivatives, which is a condition that can be written as:

$$Q\,(u_0) = 0 \qquad \text{or} \qquad \varphi_0' = 0\,, \qquad \text{and} \qquad \frac{m'}{m+m'}\,z_0\,\theta_0'^2 + g = 0\,.$$

If those conditions are fulfilled then u must constantly coincide with u_0, and φ must coincide with φ_0. The area integral will then give:

$$\theta' = \theta_0'\,,$$

so

$$\theta = \theta_0'\,t + \alpha\,.$$

Conversely, the equalities $\varphi \equiv \varphi_0$, $\theta = \theta_0'\,t$ can be true only if $u_0 = \cos\varphi_0$ is the double root of $Q\,(u)$. Indeed, the point M will then describe a horizontal circle of radius $\mu \sin \varphi$ and center at P (P being the foot of the perpendicular to gz_1 that is based at M) and an angular velocity of θ_0'. The force that is exerted upon it then points along MP and is equal to $m'\,\lambda\,\theta_0'^2 \sin\varphi_0$ or $\frac{m\,m'}{m+m'}R\,\theta_0'^2 \sin\varphi_0$. However, that force is the resultant of the forces that are applied to the point M, and as a result, it will coincide with the horizontal component $T \sin \varphi_0$ of the reaction (T) of the sphere to M. (T) must then point from M to G, and its absolute value must be $\frac{m\,m'}{m+m'}R\,\theta_0'^2$. One will find the same value for T by arguing with G one does with M. On the other hand, it is necessary that the vertical component of (T) must be equal and directly opposite to the weight that is applied to the point M, so one must have the equality:

$$\frac{m\,m'}{m+m'}R\,\theta_0'^2 + m\,g = 0\,,$$

which can be written:

$$\frac{m\,m'}{m+m'}\,z_0\,\theta_0'^2 + g = 0\,.$$

The equations:

$$\varphi_0' = 0\,, \qquad \frac{m\,m'}{m+m'}\,z_0\,\theta_0'^2 + g = 0$$

are equivalent to the following ones:

$$Q\,(u_0) = 0\,, \qquad Q'(u_0) = 0\,.$$

Thus, u_0 is a double root of Q. Having fulfilled those conditions, we will know that the motion of the system is indeed a uniform rotation around gz_1. It is easy to see that directly, moreover. Let N be the normal reaction to the horizontal plane on the sphere, when counted positively in the sense of gz_1. One must then have:

$$M - m'g - T\,u_0 = 0\,, \quad \text{i.e.,} \qquad N = (m+m')\,g\,.$$

If one writes the three equations of motion of M and C then one will find that those equations are verified if one sets:

$$x_1 = \lambda \sin \varphi_0 \cos \theta\,, \quad y_1 = \lambda \sin \varphi_0 \sin \theta\,, \quad z_1 = 0 \qquad \text{for the point } C\,,$$
$$x_1 = \lambda \sin \varphi_0 \cos \theta\,, \quad y_1 = \lambda \sin \varphi_0 \sin \theta\,, \quad z_1 = 0 \qquad \text{for the point } M\,,$$

respectively (with $\theta = \theta_0' t + \alpha$), and:

$$T = \frac{m\,m'}{m+m'} R\,\theta_0'^2\,, \qquad N = (m+m')\,g\,,$$

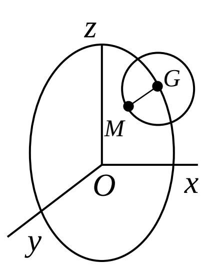
provided that the condition $\frac{m}{m+m'} R\,\theta_0'^2 \cos\theta_0 + g = 0$ is fulfilled.

By definition, if the point M is released with a horizontal velocity at a point such that its z-coordinate is equal to $-\frac{g\,(m+m')}{m\,\theta_0'^2}$ then the motion of MC will be a uniform rotational motion around gz_1.

Ignoring that case, the hyperelliptic integrals (1) and (2) can reduce (for $K \neq 0$) only if u_3 is also a root of P, which will demand that u_3 must be equal to $\sqrt{1+\frac{m'}{m}}$. t and θ are then given in terms of u by elliptic quadratures, but the preceding discussion will not be modified.

If $K = 0$, or rather, if $\sin \varphi$ is constantly zero, then MC will coincide with gz_1, or rather $\theta = \theta_0$, and MC will oscillate in a vertical plane.

II. – *Motion of a massive homogeneous sphere that slides without friction on an ellipsoid of revolution with a vertical axis.*



The external forces that are applied to the sphere are weight and the reaction of the ellipsoid (normal to the surface): Those forces pass through the center C of the sphere, the motion of the sphere around that point is a uniform rotational motion around an axis that is fixed in space and in the sphere.

As for the motion of the center of gravity C, it is that of a massive point that moves on a surface of revolution that is parallel to the given ellipsoid. The *vis viva* theorem and the theorem of moments, when applied to Oz, will give two first integrals that determine the motion. Let us express the coordinates of the ellipsoid as functions of the angle θ, φ:

$$x = a \cos \varphi \cos \theta, \qquad y = a \cos \varphi \sin \theta, \qquad x = b \sin \varphi .$$

If R denotes the radius of the sphere then the coordinates of C that correspond to a point M will be:

$$x = a\cos\varphi\cos\theta + \frac{bR\sin\varphi\cos\theta}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}},$$

$$y = a\cos\varphi\sin\theta + \frac{bR\sin\varphi\sin\theta}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}},$$

$$z = b\sin\varphi + \frac{aR\cos\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}}.$$

The desired integrals are written:

$$\left(a\cos\varphi + \frac{bR\sin\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}}\right)\frac{d\theta}{dt} = \text{const.}$$

and

$$\left(a\cos\varphi + \frac{bR\sin\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}}\right)^2 \left(\frac{d\theta}{dt}\right)^2$$

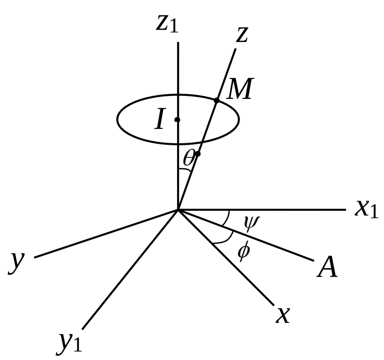
$$+ \left\{\frac{d}{d\varphi}\left[a\cos\varphi + \frac{bR\sin\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}}\right]^2 + \frac{d}{d\varphi}\left[b\sin\varphi + \frac{aR\cos\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}}\right]^2\right\}\left(\frac{d\varphi}{dt}\right)^2$$

$$= A\left[b\sin\varphi + \frac{aR\cos\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}}\right] + \text{const.}$$

The elimination of $d\theta / dt$ from those equations will give $dt = F(\varphi)\, d\varphi$, and as a result $d\theta = G(\varphi)\, d\varphi$.

The problem is reduced to quadratures.

III. – *A massive homogeneous solid body admits a symmetry axis. That axis has a fixed point O and is constrained to slide without friction on a fixed horizontal circle whose center is on the vertical through O. Find the motion of the system.*



The symmetry axis Oz is a principal axis of inertia relative to the point O.

Let Ox and Oy be two other principal axes of inertia relative to O, while Oz_1 is the vertical at the point O, and Ox_1 and Oy_1 are two fixed rectangular horizontals, while OA is the trace of the plane yOx onto the plane y_1Ox_1.

The z_1Oz is constant as a result of the constraints. The position of the solid then depends upon two parameters:

$$x_1OA = \psi \qquad \text{and} \qquad AOx = \phi .$$

The center of gravity G of the solid is along Oz. Its z_1 is constant and as a result, the work done by gravity will be zero.

The *vis viva* theorem and the theorem of the moment of the quantities of motion, when applied to Oz, which the reaction of O meets, and to which gravity is parallel, give two first integrals of the motion:

$$A\,p^2 + B\,q^2 + C\,r^2 = h ,$$

$$A\,p \sin \theta_0 \sin \phi + B\,q \sin \theta_0 \cos \phi + C\,r \cos \theta_0 = K .$$

Moreover, one has:

$$p = \frac{d\psi}{dt}\sin\theta_0 \sin\phi , \quad q = \frac{d\psi}{dt}\sin\theta_0 \cos\phi , \quad r = \frac{d\psi}{dt}\cos\theta_0 + \frac{d\phi}{dt} .$$

Our two integrals can then be written:

$$(1) \qquad \left(\frac{d\psi}{dt}\right)^2 \left[(A\sin^2\phi + B\cos^2\phi)\sin^2\theta_0 + C\cos^2\theta_0\right] + 2C\,\frac{d\psi}{dt}\frac{d\phi}{dt}\cos\theta_0 + C\left(\frac{d\phi}{dt}\right)^2 = h ,$$

$$(2) \qquad \frac{d\psi}{dt}\left[(A\sin^2\phi + B\cos^2\phi)\sin^2\theta_0 + C\cos^2\theta_0\right] + C\,\frac{d\phi}{dt}\cos\theta_0 = k .$$

The elimination of $d\psi / dt$ from those two equations gives:

$$\left[(A\sin^2\phi + B\cos^2\phi)\sin^2\theta_0 + C\cos^2\theta_0\right]\left[h - C\left(\frac{d\phi}{dt}\right)^2\right] = K^2 - C^2\cos^2\theta_0\left(\frac{d\phi}{dt}\right)^2 .$$

Hence:

$$dt = \frac{d\phi}{\sqrt{\dfrac{h}{C} - \dfrac{K^2 - hC\cos^2\theta_0}{C\sin^2\theta_0\,(A\sin^2\phi + B\cos^2\phi)}}}\,.$$

Upon setting tan $\phi = u$, one will see that t is given as a function of u by an elliptic integral. Upon replacing dt with its value in equation (2), one will see that the same thing will be true for ψ. When $A = B$, $d\phi / dt$ and $d\psi / dt$ will be constants.

In the general case, ψ will always vary in the same sense, because if $d\psi / dt$ is annulled then equations (1) and (2) will give:

$$h - C\left(\frac{d\phi}{dt}\right)^2 = 0\,, \qquad K - C\frac{d\phi}{dt}\cos\theta_0 = 0\,,$$

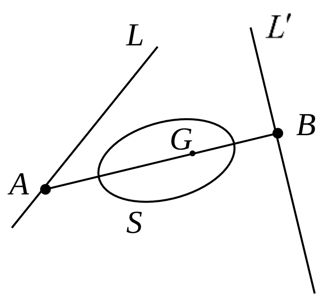
and as a result:

$$C\,h\cos^2\theta_0 - K^2 = 0\,,$$

which is a condition that is not verified by given initial data, in general. If it is verified then $d\psi / dt$ will be identically zero, so ψ will be constant.

IV. – *A massive homogeneous solid of revolution is traversed along its axis by a needle that is fixed in it and whose extremities slide without friction along two non-parallel straight lines L and* L'. *Find the motion of the body.*

The position of the system depends upon two parameters: one to determine the position of the needle of constant length *AB* and one to fix the orientation of the solid around that line.



The *vis viva* gives a first integral of motion.

On the other hand, the external forces, namely, the weight and the reactions of the lines, have zero moment with respect to the axis of revolution *AB*. If one studies the motion of the solid around its center of gravity *G*, which is a point for which *AB* is a principal axis of inertia, then one of the Euler equations will show that the component r along *AB* of the instantaneous rotation of the solid will be a constant of its motion. Thus, one has a further first integral of the motion.

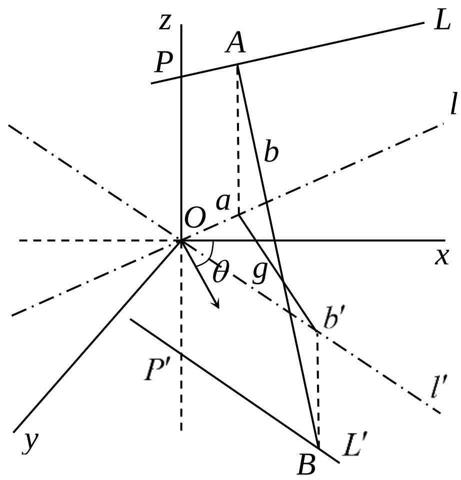
Hence, the motion will depend upon two parameters, and one will have two first integrals.

Now take the z-axis to be the common perpendicular to two lines L and L', and take the origin to be the midpoint of the shortest distance between them. Take the xy-plane to be a plane perpendicular to Oz. Take the x and y axes to be the bisectors of the projections of L and L' onto that plane.

First of all, we can make a few geometric remarks.

Any point of the line *AB*, and in particular, the center of gravity *G*, will remain in a plane parallel to *xOy*.

The line *AB* makes a constant angle α with the direction *Oz*.



The projection *ab* of *AB* onto *xOy* is a segment of constant length whose extremities describe the projections l and l' of the lines *L* and L'. Therefore, any point of *ab*, and in particular, the projection *g* of *G*, will describe an ellipse with center *O*. As a result, the center of gravity of the solid describes an ellipse that is situated in a plane that is parallel to *xOy* whose center is situated on *Oz* and whose two conjugate diameters are parallel to *Ox* and *Oy*.

In addition, one can fix the position of the line *AB* by the angle θ between its projection *ab* and *Ox*, and it is easy to obtain the coordinates of the center of gravity (ξ, η, ζ) as functions of θ, moreover.

Let: $\overline{a\,g} = a$, $\overline{b\,g} = b$, and let $\pm K$ be the angular coefficients of l and l'.

The equations of *ab*, *l*, and l' are:

$$y - \eta = \tan\theta\,(x - \xi)\,, \qquad y = K\,x\,, \qquad y = -K\,x\,.$$

One then deduces that:

$$\text{Proj. of } ag \text{ onto } Ox = \frac{\eta - K\,\xi}{K - \tan\theta} = a\cos\theta\,,$$

$$\text{Proj. of } bg \text{ onto } Ox = \frac{\eta - K\,\xi}{K + \tan\theta} = b\cos\theta\,.$$

Hence:

$$\xi = -\frac{a+b}{2}\cos\theta + \frac{a-b}{2}\cdot\frac{1}{K}\sin\theta\,,$$

$$\eta = -\frac{a-b}{2}K\cos\theta - \frac{a+b}{2}\sin\theta\,.$$

Furthermore:

$$\zeta = \text{const.}$$

From that, we can deduce the expressions for the *vis viva* of the center of gravity, where all of the mass is concentrated, and the force function.

For the first expression, if *M* is the mass of the solid then one will have:

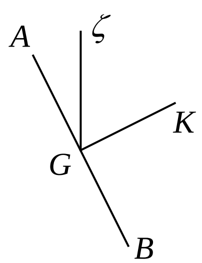
$$M\,V^2 = M\left\{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2\right\}$$

$$= \frac{M}{h}\left\{(a+b)^2+(a-b)^2\left(\frac{\cos^2\theta}{K^2}+K^2\sin^2\theta\right)+2(a^2-b^2)\left(\frac{1}{K}-K\right)\sin\theta\cos\theta\right\}\theta'^2 .$$

For the second one, if λ, μ, γ are the direction cosines of the direction of gravity then one will have:

$$U = Mg\left\{-\left[\frac{a\,r\,b}{2}\lambda - r\frac{a-b}{2}\mu\right]\cos\theta + \left[\frac{a-b}{2}\frac{\lambda}{K} - \frac{a\,r\,b}{2}\mu\right]\sin\theta\right\} + \text{const.}$$

Having established those preliminaries, we see how to form the two first integrals of motion that we indicated by taking the second parameter that defines the position of the system to be the angle ψ that the half-plane *AGz* makes with the half-plane *AGK* in the solid (*GK* is perpendicular to *AB*, $G\zeta$ is parallel *Oz*).



The elementary displacement of the solid can be defined by the translation $G\Gamma$ of the center of gravity and the rotation $G\omega$ around the instantaneous axis that passes through the center of gravity.

Let *p*, *q*, *r* be the components of $G\omega$ along *GK*, GK' (which are perpendicular to the plane *AGK*) and *GB*. *A* is the moment of inertia relative to *GK* and GK' (the solid is one of revolution), and *C* is the moment of inertia relative to *GB*.

The first integrals of motion are:

$$(\gamma) \qquad \left\{ \begin{array}{l} MV^2 + A(p^2+q^2) + Cr^2 = 2U + h, \\ r = r_0 . \end{array} \right.$$

Therefore, everything comes down to calculating $\sqrt{p^2+q^2}$ and *r*, i.e., the projections of $G\omega$ onto the plane KGK' and onto *GA*, resp., as functions of θ, ψ, θ', ψ'.

To that effect, we remark that the rotation $G\omega$ and the translation $G\,\Gamma$ can be replaced with two rotations, one of which is around *AB* and the other of which is around a certain axis *CD* that we shall determine. The velocity of the point *A* points along *L*. The rotation around *AB* does not displace the point, since that velocity is due to the rotation around *CD*, so *CD* will be in the plane perpendicular to *L* at *A*. As a result, the axis *CD* is the intersection of the planes that are perpendicular to *L* and L' and *A* and *B*, respectively. That axis is parallel to *Oz*.

Let (ω_1) and (ω_2) be the two segments that define the instantaneous rotations around *AB* and *CD*. The geometric sum $(\omega_1) + (\omega_2)$ is equal to $(G\,\omega)$. One then concludes that:

$$\sqrt{p^2+q^2} = |\,\omega_2 \sin\alpha\,|$$

and

$$\omega = \omega_1 + \omega_2 \cos\alpha ,$$

if ω_1, ω_2 are the lengths (ω_1) and (ω_2), which are regarded as positive when ω_1 has the sense of *AB* and the sense of *Oz* for ω_2 . On the other hand, the rotation (ω_1) will leave θ constant and will vary ψ by $\omega_1\, dt$ (if one regards ψ as positive from left to right around the direction *AB*).

Similarly, the rotation (ω_2) will leave ψ constant and vary θ by $\omega_1\, dt$. One will then have:

$$\omega_2 = \frac{d\theta}{dt} = \theta', \qquad \omega_1 = \frac{d\psi}{dt} = \psi' .$$

Equations (γ) will then become:

$$\psi' + \theta' \cos\alpha = \text{const.},$$
$$MV^2 + A\theta'^2 \sin^2\alpha = 2U + \text{const.}$$

The quantities *V* and *U* depend upon θ . *t* is given as a function of θ by a quadrature that has the form (as one sees immediately upon replacing *V* and *U* with their values):

$$t = \int \sqrt{\frac{\mathcal{A}\cos 2\theta + B\sin 2\theta + C}{\mathcal{A}'\cos\theta + B'\sin 2\theta + C'}}\, d\theta ,$$

which is a hyperelliptic integral.

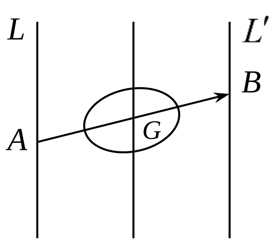
The first equation gives:

$$\psi + \theta\cos\alpha = \lambda\, t + \mu ,$$

in which λ and μ are constant.

The preceding applies to the case in which the lines *L* and L' meet each other. We shall now assume the hypothesis that they are parallel.

Special case in which the two lines are parallel. – In that case, the center of gravity *G* describes a line that is parallel to the given lines and is situated in their plane. The elementary displacement of the system results from a translation (with a velocity that is equal to the velocity of *G*) and a rotation around *G*. The velocities of the points *C*, *A*, *B* are the same, so that rotation must leave *A* and *B* fixed, and as a result, the axis of rotation will be *AB*.



One has the same integrals as before, but their calculation is simpler. Let ζ be the *z*-coordinates of the point *G* (with the *z*-axis being parallel to *L* and L') and let ψ be the angle between the plane of *L* and L' and a fixed plane in the solid that passes through *AB*. Finally, let *i* be the angle between the direction of gravity and *Oz*. That will give:

$$\begin{cases} M\,\xi'^2 + M\,K^2\psi'^2 = 2M\,g\,\zeta\cos i + \text{const.}, \\ \psi' = \text{const.} \end{cases}$$

Thus:

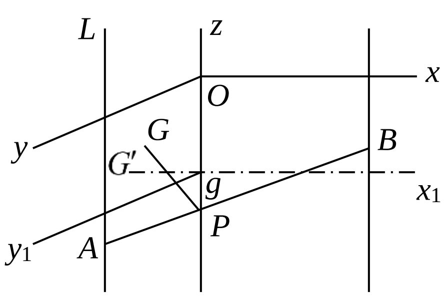
$$\begin{cases} \psi = \lambda t + \mu, \\ \zeta = \frac{1}{2}(g\cos i)t^2 + \lambda' t + \mu'. \end{cases}$$

The center of gravity describes a line like a massive point that falls vertically, while the acceleration of gravity is the projection of g onto that line. All of the points of AB have the same motion, and the solid is animated with a uniform rotational motion around AB. One can also utilize the theorem of the motion of the center of gravity relative to Oz. That is what we shall do by treating the preceding problem without supposing that the solid is one of revolution as a last application.

V. – *A massive solid body has two points A and B that slide without friction along two fixed parallel lines. Find the motion of the system.*

The center of gravity G of the solid will no longer be only the line, in general. Let P be the foot of the perpendicular to AB that is based at G. The point P will describe a line that is parallel to L, L' and which we will take to be the z-axis. The axis Ox is a perpendicular to L and L' that is drawn in the plane of those lines.

The two parameters by which we define the position of the system are ζ, which is the z-coordinate of G, and ψ, which is the angle between the half-plane ABG and the half-plane ABz, which is regarded as positive from left to right around the direction AB.



When the theorem of the motion of the center of gravity is applied to Oz, that will give us a first integral of the motion: Indeed, since the reaction at A and B are normal to $L$and L', we will have:

$$M\frac{d^2\zeta}{dt^2} = M\gamma$$

(upon letting α, β, γ are the components along Ox, Oy, Oz of the weight that is exerted on a unit of mass). Therefore:

$$\xi = \tfrac{1}{2}\gamma t^2 + \lambda t + \mu .$$

The *vis viva* theorem will give us another integral into which ψ, ψ', ζ, and ζ' will enter. If one replaces ζ and ζ' as functions of t then one will find that t has been eliminated and that ψ will depend upon t by a quadrature. However, one can account for that fact without calculation in the following manner:

Let G' denote the projection of G onto the plane xOy, and let g denote its projection onto Oz. The acceleration of g is the component (γ) of the weight. In what follows, one refers the motion of the solid to axes gx_1, gy_1, gz_1 that are parallel to the first ones. The points A and B will again describe fixed lines L and L' under that relative motion. The relative external forces that are applied to the solid are the reactions at A and B (which are normal to the relative velocities at A

and B) and the weight, diminished by its component along Oz, which is the guiding force F_e here. If ξ_1, η_1, ζ_1 are the new coordinates of G and T_1 is the relative semi-*vis viva* of the solid then one will have:

$$T_1 = M\,(\alpha\,\xi_1 + \beta\,\eta_1) + \text{const.},$$

$$\xi_1 = -\,l\cos\psi\sin\omega\,,$$
$$\eta_1 = -\,l\sin\psi\,,$$
$$\zeta_1 = 0\,,$$

in which ω denotes the angle between AB and gx_1 .

As for T_1, it is a function of only ψ and ψ'.

In order to calculate it, observe that the elementary displacement of the system decomposes into a translation whose velocity is the velocity of A and a rotation around A that leaves B fixes and is effected around AB with the angular velocity $d\psi / dt$. The segment of rotation remains the same under all analogous decompositions, so one will see that the motion of the solid around G is a rotational motion (with an angular velocity $d\psi / dt$) around the direction GB' that is parallel to AB, as well as fixed in space and the solid. Let MK^2 be the moment of inertia of the solid with respect to GB'. From the foregoing:

$$2T_1 = M\,l^2(\cos^2\psi + \sin^2\omega\sin^2\psi)\psi'^2 + M\,K^2\,\psi'^2.$$

One then deduces that:

$$\psi'^2\,[A + \cos^2\psi] = B\cos\psi + C\sin\psi + h\,,$$

in which one sets:

$$A = \frac{l^2\sin^2\omega + K^2}{l^2\cos^2\omega}\,, \qquad B = \frac{-2\alpha}{l^2\cos^2\omega}\,, \qquad C = \frac{-2\beta}{l^2\cos^2\omega}\,.$$

That equation will permit one to discuss the motion with respect to the axes $g\,x_1\,y_1\,z_1$, which is a periodic motion. If one sets $\tan\gamma / 2 = u$ then one will see that t depends upon u by way of a hyperelliptic integral of degree eight.

What is the motion of the line AB with respect to the former axes? The z-coordinate of the point P is equal to $\zeta + l\cos\omega\cos\psi$, so one concludes that the velocities of each point of the line are equal to $g\,t + \xi_0' - l\cos\omega\sin\psi\,(d\psi / dt)$, i.e., to $\gamma\,t + W$, where W denotes a function of t that is periodic and oscillates between constants W_0 and W_1 .

When $l = 0$, ψ' will be constant, so the rotation of the solid around AB will be uniform.

When α and β are zero (i.e., when gravity is parallel to the lines L, L'), one will have:

$$\sqrt{h}\,dt = d\psi\sqrt{A + \cos^2\psi}\,,$$

in which t depends upon ψ by an elliptic integral. ψ varies constantly in the same sense, and the period of relative motion with respect to g will be:

$$T = \frac{-1}{\sqrt{h}} \int_0^{2\pi} d\psi \sqrt{A + \cos^2 \psi} \ .$$

When $h = 0$, $\psi = \psi_0$, and each point of the solid will move like a massive point.

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LECTURE 4

GENERAL EQUATIONS OF MOTION OF SYSTEMS. SYSTEMS WITH AND WITHOUT FRICTION.

We shall indicate the methods of Lagrange that will permit us to determine the motion of a material system with the least-possible number of equations and with the least-possible number of givens.

Let a system be composed of n points M_i (x_i, y_i, z_i) (i = 1, 2, …, n). Its points are supposed to be subject to some constraints that are expressed by some relations between their coordinates and time:

(1)
$$\left\{\begin{array}{l} f_1(x_1, y_1, z_1, \ldots, x_i, y_i, z_i, \ldots, x_n, y_n, z_n, t) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_p(x_1, y_1, z_1, \ldots, x_i, y_i, z_i, \ldots, x_n, y_n, z_n, t) = 0. \end{array}\right.$$

Those p equations (p being necessarily less than the number n of coordinates) are supposed to be distinct, i.e., one can express p of the quantities x_i, y_i, z_i as functions of the other $3n - p$ and time, so those other $3n - p$ will be independent then. For example, we assume that we can infer the last p quantities z_n, y_n, x_n , z_{n-1}, … as functions of the first $3n - p$ and time from equations (1). That amounts to assuming that the functional determinant of f_1, f_2, …, f_p, when considered as functions of the p variables z_n, y_n, x_n , z_{n-1}, …, is not zero.

The position of the system depends upon $3n - p = k$ independent parameters in this case.

One says that the constraints depend upon time when equations (1) depend upon t: I intend that to mean that t enters into f_1, f_2, …, f_p, and that the systems of relations (1) that correspond to two arbitrary values of t are not equivalent. More precisely, the derivatives $\frac{\partial f_1}{\partial t}$, $\frac{\partial f_2}{\partial t}$, …, $\frac{\partial f_p}{\partial t}$ are not zero for any t for an arbitrary system of values x_i, y_i, z_i that satisfies equations (1), in which one gives the value t_0 to t. In the contrary case where t does not appear in (1), it will suffice to replace it with a constant.

One says *a virtual displacement* of a system at the instant t to means any displacement that is compatible with the constraints at the instant t. We shall consider only infinitely-small virtual displacements.

If δx_i, δy_i, δz_i are the variations of the coordinates x_i, y_i, z_i under such a displacement then those relations will satisfy the relations:

$$
(2) \qquad \begin{cases}
\sum\left(\dfrac{\partial f_1}{\partial x}\delta x + \dfrac{\partial f_1}{\partial y}\delta y + \dfrac{\partial f_1}{\partial z}\delta z\right) = 0, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\sum\left(\dfrac{\partial f_1}{\partial x}\delta x + \dfrac{\partial f_1}{\partial y}\delta y + \dfrac{\partial f_1}{\partial z}\delta z\right) = 0,
\end{cases}
$$

in which (x, y, z) represent an arbitrary point of the system, and the sums $\sum$ are extended over the n points of the system.

Conversely, any set of values δx_i, δy_i, δz_i that satisfies equations (2) will define an elementary virtual displacement. One can infer the last p of the δ from the first k, because the determinant of the linear equations to be solved is nothing but the functional determinant of the $f_1, f_2, \dots, f_p$, when they are regarded as functions of the p variables z_n, y_n, x_n , z_{n-1}, etc.

The real displacement of the system will coincide with one of the virtual displacements for only particular positions of the system and particular values of time ([5]).

Let (X, Y, Z) be the force that is exerted on the point (x, y, z).

The virtual work done by the forces (X, Y, Z) is the work done by those forces under a virtual displacement. It is represented by the expression:

$$\sum\left(X\,\delta x + Y\,\delta y + Z\,\delta z\right),$$

in which the sum $\sum$ extends over the n points of the system, and the quantities δx, δy, δz are constrained to verify the relations (2).

We just saw that we can choose $(3n - p)$ of the variations δ arbitrarily, and that the other p will then be determined.

One must show that knowing the total work done by the forces that are exerted on each point of the system under an arbitrary virtual displacement will suffice to determine the motion of the system.

First of all, it is clear that if one knows the total force that is exerted on each point of the system then one will have for $3n$ equations such as:

([5]) Indeed, the real displacement of the system will satisfy the p relations:

$$\frac{\partial f}{\partial t}\delta t + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}dx_i + \frac{\partial f}{\partial y_i}dy_i + \frac{\partial f}{\partial z_i}dz_i\right) = 0 \qquad (f = f_1, f_2, \dots, \text{or } f_p).$$

That displacement will be a virtual displacement only if $\dfrac{\partial f_1}{\partial t}, \dfrac{\partial f_2}{\partial t}, \dots, \dfrac{\partial f_p}{\partial t}$ are zero: The constraints will be independent of time when those conditions are realized at an arbitrary instant for an arbitrary position of the system.

$$(\alpha) \qquad \begin{cases} m\dfrac{d^2x}{dt^2} = X\,, \\ m\dfrac{d^2y}{dt^2} = Y\,, \\ m\dfrac{d^2z}{dt^2} = Z \end{cases}$$

for determining the motion of the system.

There are too many of those equations. They must be compatible with equations (1). In order for the motion of the system to be determined, it will suffice to preserve k of them ($k = 3n - p$) that include only known quantities, along with x, y, z.

Having said that, consider the sum:

$$\sum \left\{ \left(m\frac{d^2x}{dt^2} - X \right) \delta x + \left(m\frac{d^2y}{dt^2} - Y \right) \delta y + \left(m\frac{d^2z}{dt^2} - Z \right) \delta z \right\}.$$

That sum will be zero for any δx, δy, δz, and in particular, for any virtual displacement.

If one uses the relations (2) to express the last p variations δ as functions of the first ones, which are independent, then one will immediately explain the fact that knowing the right-hand side of the preceding equality will imply k distinct relations between the x, y, z, x', y', z', x'', y'', z'', and the given quantities.

In order to construct those relations conveniently, we shall first establish a lemma.

Lemma:

When a system of forces is such that the total virtual work done by the forces is zero for any virtual displacement, if (X, Y, Z) *is the force that is exerted on the point* (x, y, z) *then one will have:*

$$(4) \qquad \begin{cases} X = \lambda_1 \dfrac{\partial f_1}{\partial x} + \lambda_2 \dfrac{\partial f_2}{\partial x} + \cdots + \lambda_p \dfrac{\partial f_p}{\partial x}\,, \\ Y = \lambda_1 \dfrac{\partial f_1}{\partial y} + \lambda_2 \dfrac{\partial f_2}{\partial y} + \cdots + \lambda_p \dfrac{\partial f_p}{\partial y}\,, \\ Z = \lambda_1 \dfrac{\partial f_1}{\partial z} + \lambda_2 \dfrac{\partial f_2}{\partial z} + \cdots + \lambda_p \dfrac{\partial f_p}{\partial z}\,, \end{cases}$$

in which $\lambda_1, \lambda_2, \ldots, \lambda_p$ *are coefficients that are the same for all points of the system.*

The hypothesis is that one has:

$$(5) \qquad \sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = 0$$

for any δx, δy, δz that verify (2).

It is initially clear that if X, Y, Z have the form (4) then the relation (5) will be satisfied. It will suffice to make that substitution in order for the left-hand side of (5) to become the sum of the left-hand sides of equations (2), multiplied by λ_1, λ_2, …, λ_p, respectively. Consequently, the relations (4) will be sufficient conditions for the forces to enjoys the indicated property.

They are also necessary conditions. Indeed, if the forces X, Y, Z verify the relation (5) for any virtual displacement then the following relation will be verified for the same displacements:

$$(5') \qquad \sum\left[\left(X-\lambda_1\frac{\partial f_1}{\partial x}-\cdots-\lambda_p\frac{\partial f_p}{\partial x}\right)\delta x+\left(Y-\lambda_1\frac{\partial f_1}{\partial y}-\cdots-\lambda_p\frac{\partial f_p}{\partial y}\right)\delta y+\left(Z-\lambda_1\frac{\partial f_1}{\partial z}-\cdots-\lambda_p\frac{\partial f_p}{\partial z}\right)\delta z\right]=0,$$

in which the coefficients λ_1, λ_2, …, λ_p are arbitrary. Let us determine the manner by which the coefficients of the last p variations δ will be annulled. We will get:

$$Z_n-\lambda_1\frac{\partial f_1}{\partial z_n}-\cdots-\lambda_p\frac{\partial f_p}{\partial z_n}=0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Those p equations, which are linear in λ_1, λ_2, …, λ_p, define those quantities. Their determinant will be non-zero, as we have pointed out before.

Having thus chosen the λ, only the $3n - p$ independent variations δ will still remain in the relation (5′).

As a result, that relation will imply the conditions:

$$X_1-\lambda_1\frac{\partial f_1}{\partial x_1}-\cdots-\lambda_p\frac{\partial f_p}{\partial x_1}=0,$$
$$Y_1-\lambda_1\frac{\partial f_1}{\partial y_1}-\cdots-\lambda_p\frac{\partial f_p}{\partial y_1}=0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

which are $3n - p$ in number.

If one combines that system of equations with the previous one then one will see that if X, Y, Z verify the relation (5) for any virtual displacement then there will exist a system of values λ_1, λ_2, …, λ_p such that X, Y, Z will be identical to the expressions (4).

The lemma is then true.

From that, if one considers two systems of forces (X, Y, Z), (X', Y', Z'), where (X, Y, Z) and (X', Y', Z') are both applied to the point (x, y, z), and if those two systems of forces do the same virtual work for any virtual displacement then one can conclude that:

$$X = X' + \lambda_1 \frac{\partial f_1}{\partial x} + \cdots + \lambda_p \frac{\partial f_p}{\partial x},$$
$$Y = Y' + \lambda_1 \frac{\partial f_1}{\partial y} + \cdots + \lambda_p \frac{\partial f_p}{\partial y},$$
$$Z = Z' + \lambda_1 \frac{\partial f_1}{\partial z} + \cdots + \lambda_p \frac{\partial f_p}{\partial z}.$$

Indeed, it will suffice to observe that the system of forces $X - X'$, $Y - Y'$, $Z - Z'$ does zero virtual work and then apply the preceding lemma.

Having said that, suppose that one knows a form for the virtual work done that is by the forces on the system for an arbitrary virtual displacement, namely:

$$\sum (A\delta x + B\delta y + C\delta z) \ .$$

If one regards A, B, C as the projections of a segment whose origin is x, y, z, or rather as the components of a force that is exerted on the point x, y, z, then the virtual work done by that force will be:

$$A\ \delta x + B\ \delta y + C\ \delta z\ .$$

By hypothesis, the sum of the analogous works will coincide with the virtual work that is done by the total force X, Y, Z that is applied to each point of the system, so one will have:

$$\sum (A\delta x + B\delta y + C\delta z) = \sum (X\ \delta x + Y\ \delta y + Z\ \delta z)$$

for any virtual displacement of the system.

It will result from the last remark that:

$$(6) \qquad \left\{ \begin{aligned} X &= A + \lambda_1 \frac{\partial f_1}{\partial x} + \lambda_2 \frac{\partial f_2}{\partial x} + \cdots + \lambda_p \frac{\partial f_p}{\partial x} = m \frac{d^2 x}{dt^2}, \\ Y &= B + \lambda_1 \frac{\partial f_1}{\partial y} + \lambda_2 \frac{\partial f_2}{\partial y} + \cdots + \lambda_p \frac{\partial f_p}{\partial y} = m \frac{d^2 y}{dt^2}, \\ Z &= C + \lambda_1 \frac{\partial f_1}{\partial z} + \lambda_2 \frac{\partial f_2}{\partial z} + \cdots + \lambda_p \frac{\partial f_p}{\partial z} = m \frac{d^2 z}{dt^2}. \end{aligned} \right.$$

It is clear, moreover, that knowing the quantities A, B, C as functions of the points of the system, their velocities, and time will suffice to determine the motion of the system.

Indeed, when the $3n$ equations (6) are combined with the p equations (1), that will permit us to calculate the $(3n + p)$ unknowns x_i, y_i, z_i, λ_1, …, λ_p as functions of t. More precisely, if we differentiate equations (1) twice with respect to t then we will define the p relations:

$$(1')\qquad \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}x_i''+\frac{\partial f}{\partial y_i}y_i''+\frac{\partial f}{\partial z_i}z_i''\right)+\left[\frac{\partial f}{\partial t}+\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}x_i'+\frac{\partial f}{\partial y_i}y_i'+\frac{\partial f}{\partial z_i}z_i'\right)\right]_2 = 0 \quad (f = f_1, f_2, \ldots, \text{or } f_p)\,.$$

The index 2 defines a symbolic square. The $(3n + p)$ equations (1) and (6) are linear with respect to the $(3n + p)$ unknowns x_i'', y_i'', z_i'', $\lambda_1, \ldots, \lambda_p$, and the determinant Δ of those unknowns is not zero. Otherwise, the $(3n + p)$ homogeneous equations in δx_i, δy_i, δz_i, and $\lambda_1, \ldots, \lambda_p$:

$$(2)\qquad \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\delta x_i+\frac{\partial f}{\partial y_i}\delta y_i+\frac{\partial f}{\partial z_i}\delta z_i\right) = 0 \qquad (f = f_1, f_2, \ldots, \text{or } f_p)$$

and

$$(6')\qquad \delta x_i = \lambda_1\frac{\partial f_1}{\partial x_i}+\cdots+\lambda_p\frac{\partial f_p}{\partial x_i}\,,\qquad \delta y_i = \lambda_1\frac{\partial f_1}{\partial y_i}+\cdots+\lambda_p\frac{\partial f_p}{\partial y_i}\,,\qquad \delta z_i = \lambda_1\frac{\partial f_1}{\partial z_i}+\cdots+\lambda_p\frac{\partial f_p}{\partial z_i}$$

would admit solutions for which all of the unknowns are non-zero. Now, multiply equations (6′) by δx_i, δy_i, δz_i, respectively, and take the sum. From (2), that will give:

$$\sum (\delta x_i)^2+(\delta y_i)^2+(\delta z_i)^2 = 0\,,$$

i.e., that all of the δ must be zero, and as a result, all of the λ. From (6), the determinant Δ cannot be identically zero then, and equations (6) and (1′) will define the x'', y'', z'', and the λ as functions of the x, y, z, x', y', z', and t. In order to get the λ, it will suffice to substitute the values of x'', y'', z'' that are inferred from (6) in (1′). Having thus calculated the λ, the first k of equations (6), in which one replaces the p quantities z_n, y_n, … as functions of the other k and t, will define the x, y, z as functions of t and $2k$ initial constants.

All of the foregoing results from the definition of virtual work. We shall now introduce some dynamical definitions that will show the importance of the results that were obtained.

We decompose the absolute force (F) that is exerted on the point M of the system at the instant t into two other ones, namely, the active force and the reactive force or the reaction.

Let V be a small, closed volume that surrounds the point M. Consider all of the material elements E interior to V that do not permit one to displace the point M arbitrarily in V at the instant t. We can regard the absolute force (F) that is exerted on M at the time t as the geometric sum of the force (φ) that is exerted on M by the elements E and the force (φ') that is exerted by the elements external to M other than E. If V tends to zero then we assume that (φ) and (φ') tend to limits (R) and (F'), respectively. (R) is the absolute reaction or reactive force, and (F') is the absolute active force.

$$(F) = (F') + (R)\,.$$

(F') is (in more concise language) the absolute force that is exerted on the point M if one can make the point M free at the instant t, while changing nothing in the other conditions.

Now suppose that one studies the motion of the point with respect to arbitrary axes $Oxyz$. Let F_r' be the relative active force, let R_a be the absolute reactive force, and let F_r be the relative total forces that are exerted on M at the instant t. From a property that was established before, one will have:

$$m\,(\Gamma_r) = (F_r) = (F_r') + (R_a)\,.$$

The relative force that the point M is subject to at time t is the geometric sum of the relative active force and the absolute reaction.

Those definitions have led us to subdivide systems into systems with friction and systems without friction.

We know that (F_r) will be determined at the instant t for a given system when the external conditions are given when one knows the positions of the points of the system and their velocities at the instant t.

As a result, R_a will also be determined as a function of the x, y, z, x', y', z' at the instant t.

When one considers a system in reality, one can study its motion with respect to the axes $Oxyz$ under the influence of a given medium. One then measures the force (F_r) that is exerted on each point. One can make each point free and measure F_r' under the same conditions. The absolute reaction will be $(F_r) - (F_r')$.

Two cases present themselves.

On the one hand, when one calculates the virtual work done by all of the reactions, that virtual work will be zero for an arbitrary virtual displacement, no matter what the instant t considered, the position of the system, its velocities, and the force F_r'. In that case, one says that the system is *frictionless*.

On the other hand, that virtual work will always be zero under those conditions. One then says that the system has friction in it.

In reality, the work done by the reactions is never rigorously zero. However, it is frequently negligible, and one consider the case of frictionless systems to be the limit of a large number of cases that one encounters experimentally.

We now consider the case in which the system is frictionless.

Let R_x, R_y, R_z be the projections of the reaction R that is exerted on the point x, y, z. From a previously-established lemma, one has:

$$\begin{cases} R_x = \lambda_1 \dfrac{\partial f_1}{\partial x} + \cdots + \lambda_p \dfrac{\partial f_p}{\partial x}, \\[2ex] R_y = \lambda_1 \dfrac{\partial f_1}{\partial y} + \cdots + \lambda_p \dfrac{\partial f_p}{\partial y}, \\[2ex] R_z = \lambda_1 \dfrac{\partial f_1}{\partial z} + \cdots + \lambda_p \dfrac{\partial f_p}{\partial z}, \end{cases}$$

the coefficients $\lambda_1, \ldots, \lambda_p$ have the same values for the various points x, y, z.

When the system is frictionless, knowing the active forces will suffice to determine the motion of the system. That also results from a theorem that was just proved. Indeed, the virtual work done by active forces is the same as the work done by total forces that are exerted at each point of the system.

If X', Y', Z' is the active force that is exerted on the point (x, y, z) then one will have:

$$
(7) \qquad \left\{
\begin{aligned}
X &= X' + \lambda_1 \frac{\partial f_1}{\partial x} + \cdots + \lambda_p \frac{\partial f_p}{\partial x} = m \frac{d^2 x}{dt^2}, \\
Y &= Y' + \lambda_1 \frac{\partial f_1}{\partial y} + \cdots + \lambda_p \frac{\partial f_p}{\partial y} = m \frac{d^2 y}{dt^2}, \\
Z &= Z' + \lambda_1 \frac{\partial f_1}{\partial z} + \cdots + \lambda_p \frac{\partial f_p}{\partial z} = m \frac{d^2 z}{dt^2}.
\end{aligned}
\right.
$$

By definition, when there is no friction, the virtual work:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z)$$

will be equal to:

$$\sum (X'\,\delta x + Y'\,\delta y + Z'\,\delta z)$$

for any virtual displacement, which is the virtual work done by the active forces that are exerted on the various points of the system.

One often refers to the active forces as the *given forces*. They are the ones that are provided directly by experiment and knowing them will suffice to determine the motion of the system when there is no friction.

When one knows the active forces, one will know not only the motion of the system, but also the reactions that are exerted on each point of the system, because one can calculate the coefficients $\lambda_1, \ldots, \lambda_p$.

On the subject of those coefficients, we repeat that equations (7) allow us to express them as functions of the positions of the points of the system, their velocities, and time, i.e., as functions of the x_i, y_i, z_i, x_i', y_i', z_i', and t (and also X_i', Y_i', Z_i').

Example. – Let us study the motion of a point that is constrained to move on a fixed or moving surface:

$$f(x, y, z, t) = 0 .$$

Let X, Y, Z be the force relative to the axes $Oxyz$ that is exerted on M :

$$m\frac{d^2x}{dt^2} = X,$$
$$\ldots\ldots\ldots\ldots\ldots$$

It is not necessary to know *X*, *Y*, *Z*, but only the value of the quantity:

$$X\,\delta x + Y\,\delta y + Z\,\delta z\,.$$

If X', Y', Z' are such that one has:

$$X'\,\delta x + Y'\,\delta y + Z'\,\delta z = X\,\delta x + Y\,\delta y + Z\,\delta z$$

for any virtual displacement then it will result from the theorems that were proved that:

$$X = X' + \lambda\frac{\partial f}{\partial x},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

If R_x, R_y, R_z denote the projections of the reactions onto the axes, i.e., of the absolute force that is exerted on the point by the surface element that contacts it, then in order for there to be no friction, it is necessary and sufficient that the virtual work done by *R* should be zero or that:

$$R_x = \lambda\frac{\partial f}{\partial x},$$
$$R_y = \lambda\frac{\partial f}{\partial y},$$
$$R_z = \lambda\frac{\partial f}{\partial z}.$$

R must be normal to the surface, which is geometrically obvious, since a virtual displacement is an arbitrary displacement that is tangent to the surface. There will be friction if *R* is oblique to the surface.

When there is no friction, knowing the active force relative to *Oxyz*, namely, X', Y', Z', will suffice to determine the motion of the system. One will have:

$$m\frac{d^2x}{dt^2} = X' + \lambda\frac{\partial f}{\partial x},$$
$$m\frac{d^2y}{dt^2} = Y' + \lambda\frac{\partial f}{\partial y},$$
$$m\frac{d^2z}{dt^2} = Z' + \lambda\frac{\partial f}{\partial z},$$

$$f(x, y, z, t) = 0 .$$

Those four equations will give x, y, z, and λ as functions of t.

In order to have λ as a function of x, y, z, x', y', z', t, it will suffice to differentiate the equation of the surface twice with respect to time t :

$$\frac{\partial f}{\partial x} x' + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial z} z' + \frac{\partial f}{\partial t} = 0 ,$$

$$\frac{\partial f}{\partial x} x'' + x' \left(\frac{\partial^2 f}{\partial x^2} x' + \frac{\partial^2 f}{\partial x \partial y} y' + \frac{\partial^2 f}{\partial x \partial z} z' \right) + \cdots + \frac{\partial^2 f}{\partial t^2} = 0 ,$$

and to replace x'', y'', z'' in the last equation with their values that one infers from the equations of motion.

LECTURE 5

SYSTEMS WITH FRICTION

Let M or (x, y, z) be a point in the system, let (X, Y, Z) be the force that is exerted on it at the instant t, let (X', Y', Z') be the active force, and let (R_x, R_y, R_z) be the reaction.

The equations of motion of that will then be:

$$(1) \qquad \begin{cases} m\dfrac{d^2x}{dt^2} = X = X' + R_x, \\ m\dfrac{d^2y}{dt^2} = Y = Y' + R_y, \\ m\dfrac{d^2y}{dt^2} = Z = Z' + R_z. \end{cases}$$

By definition, the system is frictionless if:

$$\sum R_x\,\delta x + R_y\,\delta y + R_z\,\delta z = 0$$

no matter what virtual displacement is given to the system, or the instant t considered, or the positions of the points of the system and their velocities.

In that case, one sees that knowing the active forces will suffice to determine the motion of the system. The $3n$ equations:

$$(2) \qquad \begin{cases} m\dfrac{d^2x}{dt^2} = X' + \lambda_1\dfrac{\partial f_1}{\partial x} + \cdots + \lambda_p\dfrac{\partial f_p}{\partial x}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \end{cases}$$

combined with the p constraint equations:

$$(3) \qquad f_1 = 0\,, \qquad f_2 = 0\,, \ \ldots, \qquad f_p = 0\,,$$

will define the motion of the system and permit one to express $\lambda_1, \lambda_2, \ldots, \lambda_p$ as functions of the quantities x, y, z, x', y', z', and time.

Instead of appealing to equations (2) and (3), one can express p of the quantities x_i, y_i, z_i as functions of the other $(3n - p) = k$, and upon expressing the latter as functions of k independent parameters $q_1, q_2, \ldots, q_k$, one will put the x_i, y_i, z_i into the form:

$$(4) \qquad \begin{cases} x_i = \varphi_i(q_1, q_2, \cdots, q_k, t), \\ y_i = \psi_i(q_1, q_2, \cdots, q_k, t), \\ z_i = \chi_i(q_1, q_2, \cdots, q_k, t). \end{cases}$$

Conversely, observe that when x_i, y_i, z_i can be expressed in that way, the system will be subject to $(3n - k) = p$ distinct constraints, because if one infers $q_1, q_2, \ldots, q_k$ as functions of the k quantities x_i, y_i, z_i k of the equations (4) then upon substituting those values in the other p equations (4), one will get p distinct relations between the x_i, y_i, z_i . That nonetheless supposes that k of the equations can be solved with respect to $q_1, q_2, \ldots, q_k$. In other words, one assumes that if one suppresses any p rows in the matrix:

$$\begin{matrix} \dfrac{\partial x_1}{\partial q_1} & \dfrac{\partial x_1}{\partial q_2} & \cdots & \dfrac{\partial x_1}{\partial q_k} \\ \dfrac{\partial y_1}{\partial q_1} & \dfrac{\partial y_1}{\partial q_2} & \cdots & \dfrac{\partial y_1}{\partial q_k} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial z_p}{\partial q_1} & \dfrac{\partial z_p}{\partial q_2} & \cdots & \dfrac{\partial z_p}{\partial q_k} \end{matrix}$$

then at least one of the determinants D thus-obtained will not be zero identically. In the contrary case, there will exist $(p + 1)$ distinct relations between the x_i, y_i, z_i that can be expressed as functions of a lower number of parameters.

When the φ, ψ, χ do not include t, the constraints will depend upon time unless the relations that is obtained by eliminating $q_1, q_2, \ldots, q_k$ from any $(k + 1)$ of equations (4) are not independent of t: That exceptional case presents itself when all of the determinants obtained by suppressing any $(p - 1)$ rows in the matrix:

$$\begin{matrix} \dfrac{\partial x_1}{\partial q_1} & \cdots & \dfrac{\partial x_1}{\partial q_k} & \dfrac{\partial x_1}{\partial t} \\ \vdots & \ddots & \vdots & \vdots \\ \dfrac{\partial z_p}{\partial q_1} & \cdots & \dfrac{\partial z_p}{\partial q_k} & \dfrac{\partial z_p}{\partial t} \end{matrix}$$

are zero. It will then suffice to set $t = t_0$ in the φ, ψ, χ.

For example, if the system reduces to a material point whose coordinates are expressed as:

$$\begin{aligned} x &= \varphi(q_1, q_2, t), \\ y &= \psi(q_1, q_2, t), \\ z &= \chi(q_1, q_2, t) \end{aligned}$$

then the point will be subject to a constraint, i.e., it moves on a surface Σ, unless the three determinants:

$$\left(\frac{\partial x}{\partial q_1}\frac{\partial y}{\partial q_2}-\frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_1}\right), \qquad \left(\frac{\partial y}{\partial q_2}\frac{\partial z}{\partial q_1}-\frac{\partial y}{\partial q_1}\frac{\partial z}{\partial q_2}\right), \qquad \left(\frac{\partial z}{\partial q_1}\frac{\partial x}{\partial q_2}-\frac{\partial z}{\partial q_2}\frac{\partial x}{\partial q_1}\right)$$

are not identically zero: In that case, the point *M* will move on a curve. The surface Σ varies with time if the determinant:

$$\begin{vmatrix} \dfrac{\partial x}{\partial q_1} & \dfrac{\partial x}{\partial q_2} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial q_1} & \dfrac{\partial y}{\partial q_2} & \dfrac{\partial y}{\partial t} \\ \dfrac{\partial z}{\partial q_1} & \dfrac{\partial z}{\partial q_2} & \dfrac{\partial z}{\partial t} \end{vmatrix}$$

is non-zero.

When one puts the x_i, y_i, z_i into that form, an arbitrary virtual displacement δx_i, δy_i, δz_i can be represented in the form:

$$\delta x_i = \frac{\partial \varphi_i}{\partial q_1}\delta q_1 + \frac{\partial \varphi_i}{\partial q_2}\delta q_2 + \cdots + \frac{\partial \varphi_i}{\partial q_k}\delta q_k \ ,$$

$$\delta y_i = \frac{\partial \psi_i}{\partial q_1}\delta q_1 + \frac{\partial \psi_i}{\partial q_2}\delta q_2 + \cdots + \frac{\partial \psi_i}{\partial q_k}\delta q_k \ ,$$

$$\delta z_i = \frac{\partial \chi_i}{\partial q_1}\delta q_1 + \frac{\partial \chi_i}{\partial q_2}\delta q_2 + \cdots + \frac{\partial \chi_i}{\partial q_k}\delta q_k \ ,$$

in which the δq are arbitrary. Observe that the δx_i, δy_i, δz_i cannot all be zero at once unless all of the δq are zero. In other words, all of the determinants *D* will be zero.

There is no friction if the *k* sums:

$$\sum \left(R_x \frac{\partial x}{\partial q_j} + R_y \frac{\partial y}{\partial q_j} + R_z \frac{\partial z}{\partial q_j} \right),$$

which extend over all points of the system, are zero for (j = 1, 2, …, or k).

When one expresses the x_i, y_i, z_i in equations (2) and their derivatives as functions of the q_j and their derivatives, one will define 3n equations that determine the $\dfrac{d^2 q_j}{dt^2}$ and the λ as functions of the q_j, $\dfrac{dq_j}{dt}$, and t.

In order to eliminate the λ, it suffices to form the sum:

$$\sum \left(X \frac{\partial x}{\partial q_j} + Y \frac{\partial y}{\partial q_j} + Z \frac{\partial z}{\partial q_j} \right) = \sum \left(X' \frac{\partial x}{\partial q_j} + Y' \frac{\partial y}{\partial q_j} + Z' \frac{\partial z}{\partial q_j} \right) .$$

One will then get:

$$\sum m \left(\frac{d^2 x}{dt^2} \frac{\partial x}{\partial q_j} + \frac{d^2 y}{dt^2} \frac{\partial y}{\partial q_j} + \frac{d^2 z}{dt^2} \frac{\partial z}{\partial q_j} \right) = \sum \left(X' \frac{\partial x}{\partial q_j} + Y' \frac{\partial y}{\partial q_j} + Z' \frac{\partial z}{\partial q_j} \right) ,$$

i.e. (upon setting $j = 1, 2, \ldots, k$), k relations between t, the k functions q, and their first and second derivatives that determine the motion of the system.

In the next lecture, we will see the form that one can give to those equations. Before we do that, we shall indicate the modifications that the existence of friction will imply in the preceding considerations.

Definition of the force of friction. – Suppose that the point M of the system has a well-defined position and velocity at the instant t, and let R or (R_x, R_y, R_z) be the reaction that is exerted on M at the time t.

For an arbitrary virtual displacement, the total work done by the reactions τ is equal to:

$$\sum (R_x\, \delta x + R_y\, \delta y + R_z\, \delta z) .$$

There exists an infinitude of systems of forces R or (R'_x, R'_y, R'_z) that are applied to each point M and are such that the work done:

$$\tau' = \sum (R'_x\, \delta x + R'_y\, \delta y + R'_z\, \delta z)$$

is equal to τ for any virtual displacement of the system. From what was proved, in order for that to be true, it is necessary that one should have:

$$R'_x = R_x + \lambda_1 \frac{\partial f_1}{\partial x} + \cdots + \lambda_p \frac{\partial f_p}{\partial x} ,$$
$$R'_y = R_y + \lambda_1 \frac{\partial f_1}{\partial y} + \cdots + \lambda_p \frac{\partial f_p}{\partial y} ,$$
$$R'_z = R_z + \lambda_1 \frac{\partial f_1}{\partial z} + \cdots + \lambda_p \frac{\partial f_p}{\partial z} ,$$

or what amounts to the same thing (for $j = 1, 2, \ldots$, or k):

$$\sum (R'_x - R_x)\frac{\partial x}{\partial q_j} + (R'_y - R_y)\frac{\partial y}{\partial q_j} + (R'_z - R_z)\frac{\partial z}{\partial q_j} = 0 ,$$

in which the sum extends over all points of the system.

Among the systems (R'), *there exists one and only one of them* [*namely, the system of forces* (f)] *such that the segments* (f) δt *will define a virtual displacement: In other words, such that the displacement* $\delta x = \rho_x\, \delta t$, $\delta y = \rho_y\, \delta t$, $\delta z = \rho_z\, \delta t$ *is a virtual displacement.*

In order to prove that, observe that if τ' is zero for any virtual displacement then there will exist no other system (f) than the one for which all of the segments (ρ) are zero. Indeed, the work done τ' by the forces (ρ) under the virtual displacement (ρ) δt is equal to $\delta t \sum \rho^2$, which is a quantity that cannot be zero unless all of the ρ are zero.

Having said that, the $3n$ quantities ρ_x, ρ_y, ρ_z must satisfy the k equations (α) and the p equations (β):

$$(\alpha) \qquad \sum \rho_x \frac{\partial x}{\partial q_j} + \rho_y \frac{\partial y}{\partial q_j} + \rho_z \frac{\partial z}{\partial q_j} = \sum R_x \frac{\partial x}{\partial q_j} + R_y \frac{\partial y}{\partial q_j} + R_z \frac{\partial z}{\partial q_j} ,$$

$$(\beta) \qquad \sum \rho_x \frac{\partial f}{\partial x} + \rho_y \frac{\partial f}{\partial y} + \rho_z \frac{\partial f}{\partial z} = 0$$

(in which one sets f equal to $f_1, f_2, \ldots, f_p$, in succession). Those $3n$ equations are linear with respect to the $3n$ unknowns ρ_x, ρ_y, ρ_z , and the determinant of the unknowns is non-zero: In other words, there will exist systems of forces (ρ) that are not all zero and whose virtual work is zero, which is impossible. There will then exist one and only one system of quantities ρ_x, ρ_y, ρ_z .

Theorem:

Among all of the systems (R'), *the system* (ρ) *is the one for which the sum* $\sum R'^2$ *is minimal: Indeed, one has:*

$$R'_x = \rho_x + \lambda_1 \frac{\partial f_1}{\partial x} + \lambda_2 \frac{\partial f_2}{\partial x} + \cdots + \lambda_p \frac{\partial f_p}{\partial x} = \rho_x + \rho'_x ,$$

$$R'_y = \rho_y + \lambda_1 \frac{\partial f_1}{\partial y} + \lambda_2 \frac{\partial f_2}{\partial y} + \cdots + \lambda_p \frac{\partial f_p}{\partial y} = \rho_y + \rho'_y ,$$

$$R'_z = \rho_z + \lambda_1 \frac{\partial f_1}{\partial z} + \lambda_2 \frac{\partial f_2}{\partial z} + \cdots + \lambda_p \frac{\partial f_p}{\partial z} = \rho_z + \rho'_z .$$

If one calculates $\sum R'^2 = \sum (R'^2_x + R'^2_y + R'^2_z)$, while taking into account the fact that the sums $\sum \left(\rho_x \frac{\partial f_j}{\partial x} + \rho_y \frac{\partial f_j}{\partial y} + \rho_z \frac{\partial f_j}{\partial z} \right)$ are zero, then that will give (upon setting $\rho'^2 = \rho'^2_x + \rho'^2_y + \rho'^2_z$):

$$\sum R'^2 = \sum \rho^2 + \sum \rho'^2 .$$

That sum is minimal when $\sum \rho'^2$ is zero, i.e., when the ρ' are zero, which proves the proposition.

By definition, let an arbitrary system of forces (R) be applied to various points of the system. One can decompose each force (R) into two forces (ρ) and (ρ') that satisfy the following conditions:

1. The virtual work done by the forces (ρ') is zero for any virtual displacement.

2. The segments (ρ) δt define a virtual displacement.

That decomposition is possible in only one way. Among all of the systems of forces (R') whose virtual work is equal to that of the forces (R), the system of forces (ρ) is the one for which the sum $\sum R'^2$ is minimal.

If the force R is applied to each point and M is the reaction that is exerted on that point then one gives the name of *force of constraint* to the force (ρ') and the name of *force of friction* to the force (ρ).

The components of the force of friction (ρ) have the form:

$$\rho_x = \mu_1 \frac{\partial x}{\partial q_1} + \mu_2 \frac{\partial x}{\partial q_2} + \cdots + \mu_k \frac{\partial x}{\partial q_k} ,$$

$$\rho_y = \mu_1 \frac{\partial y}{\partial q_1} + \mu_2 \frac{\partial y}{\partial q_2} + \cdots + \mu_k \frac{\partial y}{\partial q_k} ,$$

$$\rho_z = \mu_1 \frac{\partial z}{\partial q_1} + \mu_2 \frac{\partial z}{\partial q_2} + \cdots + \mu_k \frac{\partial z}{\partial q_k} ,$$

which amounts to saying that they satisfy the p equalities:

$$\sum \rho_x \frac{\partial f}{\partial x} + \rho_y \frac{\partial f}{\partial y} + \rho_z \frac{\partial f}{\partial z} = 0 \qquad \text{(in which } f = f_1, f_2, \ldots, \text{ or } f_p\text{).}$$

The components of the force of constraint (ρ') have the form:

$$\rho'_x = \lambda_1 \frac{\partial f_1}{\partial x} + \lambda_2 \frac{\partial f_2}{\partial x} + \cdots + \lambda_p \frac{\partial f_p}{\partial x},$$
$$\rho'_y = \lambda_1 \frac{\partial f_1}{\partial y} + \lambda_2 \frac{\partial f_2}{\partial y} + \cdots + \lambda_p \frac{\partial f_p}{\partial y},$$
$$\rho'_z = \lambda_1 \frac{\partial f_1}{\partial z} + \lambda_2 \frac{\partial f_2}{\partial z} + \cdots + \lambda_p \frac{\partial f_p}{\partial z},$$

which amounts to saying that they satisfy the k equalities:

$$\sum \left(\rho'_x \frac{\partial x}{\partial q_j} + \rho'_y \frac{\partial y}{\partial q_j} + \rho'_z \frac{\partial z}{\partial q_j} \right) = 0 .$$

Finally, it is appropriate to remark that one has:

$$\sum R^2 = \sum \rho^2 + \sum \rho'^2 .$$

When the virtual work done by the forces R is zero, all of the forces ρ will be zero. The reaction will coincide with the force of constraint.

Suppose, for example, that the system is composed of only one point that is subjected to a constraint:

$$f(x, y, z, t) = 0 .$$

One can decompose the reaction (R) into a force (ρ') that does zero virtual work, i.e., one that is normal to the surface $f = 0$, and a force (ρ) such that (ρ) δt is a virtual displacement, i.e., one that is tangent to the surface $f = 0$. In this particular case, the force of friction and the force of constraint are then the component of the reaction that is tangent to the surface and the one that is normal to it, respectively.

It is important to make the following remark: Suppose that at the instant t, each point M of the system has a given position and velocity and is subject to a given active force (X', Y', Z'). The force of constraint (ρ') that is exerted on M will be the same whether the system does or does not include friction.

Indeed, write the equations:

$$m\frac{d^2x}{dt^2} = X' + \rho_x + \rho'_x = X' + \mu_1 \frac{\partial x}{\partial q_1} + \cdots + \mu_k \frac{\partial x}{\partial q_k} + \lambda_1 \frac{\partial f_1}{\partial x} + \lambda_2 \frac{\partial f_2}{\partial x} + \cdots + \lambda_p \frac{\partial f_p}{\partial x},$$

$$(\gamma) \qquad m\frac{d^2y}{dt^2} = Y' + \rho_y + \rho'_y = Y' + \mu_1 \frac{\partial y}{\partial q_1} + \cdots + \mu_k \frac{\partial y}{\partial q_k} + \lambda_1 \frac{\partial f_1}{\partial x} + \lambda_2 \frac{\partial f_2}{\partial y} + \cdots + \lambda_p \frac{\partial f_p}{\partial y},$$

$$m\frac{d^2z}{dt^2} = Z' + \rho_z + \rho'_z = Z' + \mu_1\frac{\partial z}{\partial q_1} + \cdots + \mu_k\frac{\partial z}{\partial q_k} + \lambda_1\frac{\partial f_1}{\partial z} + \lambda_2\frac{\partial f_2}{\partial z} + \cdots + \lambda_p\frac{\partial f_p}{\partial z}.$$

In order to eliminate the *m* from those equations, it is sufficient to multiply the first one by $\partial f / \partial x$, the second one by $\partial f / \partial y$, the third one by $\partial f / \partial z$, and add them, and then take the sum over all points of the system. Upon setting *f* equal to $f_1, f_2, \ldots, f_p$, one will get *p* linear equations (γ') that determine the λ, because they are the *p* distinct combinations of equations (γ) that are compatible and determinate, since they define one and only one system of quantities λ, μ.

On the other hand, one has:

$$\frac{d^2x}{dt^2} = \frac{\partial x}{\partial q_1}q''_1 + \frac{\partial x}{\partial q_2}q''_2 + \cdots + \frac{\partial x}{\partial q_k}q''_k + A,$$

$$\frac{d^2y}{dt^2} = \frac{\partial y}{\partial q_1}q''_1 + \frac{\partial y}{\partial q_2}q''_2 + \cdots + \frac{\partial y}{\partial q_k}q''_k + B,$$

$$\frac{d^2z}{dt^2} = \frac{\partial z}{\partial q_1}q''_1 + \frac{\partial z}{\partial q_2}q''_2 + \cdots + \frac{\partial z}{\partial q_k}q''_k + C.$$

A, *B*, *C* depend upon only $q_1, q_2, \ldots, q_k, q'_1, q'_2, \ldots, q'_k$, and *t*.

It follows from this that equations (γ') determine the λ as functions of the quantities X', Y', Z', $q_1, q_2, \ldots, q_k, q'_1, q'_2, \ldots, q'_k$, and *t* ([6]). The force of constraint (ρ') that is exerted on each point *M* will then be determinate at the instant *t* when one knows the position and the velocities of the points of the system and the active forces that are exerted on them (and that will be true when one makes no hypothesis about friction in the system).

One can further say then that the force of constraint is the reaction (ρ') *that is exerted on the point M at the instant t if the system is frictionless (when every point of the system has the same position and velocity and is subject to the same active force at time t).*

The force of friction is the geometric difference $(\rho) = (R) - (\rho')$. The force thus-defined enjoys the following properties: The segments $(\rho)\ \delta t$ represent a virtual displacement of the system. Among all systems of forces (R') that do the same virtual work as the forces (R), the system of forces (ρ) is the one for which the sum $\sum R'^2$ is minimal.

From that, the force of friction is the geometric quantity $m\,[(\Gamma) - (\Gamma_1)]$, where (Γ) represents the acceleration of the point *M* at the time *t*, and (Γ_1) is the acceleration that the point would have if the system were placed with the same conditions at the instant *t*, but with no friction.

Indeed, one has:

$$m\,(\Gamma) = (F') + (R),$$

([6]) One can also say that the equations (γ') are no different from the equations that are obtained by differentiating the equations of constraint twice with respect to *t* and replacing x'', y'', z'' with their values that one infers from (γ): In the last lecture, we saw that those equations in λ are determinate.

$$m\,(\Gamma_1) = (F') + (\rho').$$

Hence:

$$m\,[(\Gamma) - (\Gamma_1)] = (R) - (\rho') = (\rho)\,.$$

Furthermore, the preceding theorems are deduced from the following kinematical proposition:

Let two systems Σ *and* Σ' *of n points* (x, y, z) *and* (ξ, η, ζ) *be subject to the same constraints and placed under the same initial conditions at the instant* t_0 . *If* (Δ) *represents the geometric difference between the accelerations* (Γ) *and* (γ) *of the points* (x, y, z) *and* (ξ, η, ζ) *at time* t_0 *then the segments* $(\Delta)\,\delta t$ *will define the virtual displacements of the systems* Σ *and* Σ'.

In order to see that, express x, y, z and ξ, η, ζ as functions of the same parameters $q_1, q_2, \ldots, q_k$. Under the motion of the system Σ, $q_1, q_2, \ldots, q_k$ will be certain functions $q_1(t), q_2(t), \ldots, q_k(t)$. Under the motion of the system Σ', they will be other ones $\bar{q}_1(t)$, $\bar{q}_2(t)$, …, $\bar{q}_k(t)$. At time t_0, one has: $q_1 = \bar{q}_1, \ldots, q_k = \bar{q}_k$ and $q_1' = \bar{q}_1', \ldots, q_k' = \bar{q}_k'$. If one calculates $\dfrac{d^2x}{dt^2} - \dfrac{d^2\xi}{dt^2}$ then one will find (while taking those conditions into account) that at time t_0, one will have:

$$\Delta_x = \frac{d^2x}{dt^2} - \frac{d^2\xi}{dt^2} = \frac{\partial x}{\partial q_1}(q_1'' - \bar{q}_1'') + \frac{\partial x}{\partial q_2}(q_2'' - \bar{q}_2'') + \cdots + \frac{\partial x}{\partial q_k}(q_k'' - \bar{q}_k'')\,,$$

$$\Delta_y = \frac{d^2y}{dt^2} - \frac{d^2\eta}{dt^2} = \frac{\partial y}{\partial q_1}(q_1'' - \bar{q}_1'') + \frac{\partial y}{\partial q_2}(q_2'' - \bar{q}_2'') + \cdots + \frac{\partial y}{\partial q_k}(q_k'' - \bar{q}_k'')\,,$$

$$\Delta_z = \frac{d^2z}{dt^2} - \frac{d^2\zeta}{dt^2} = \frac{\partial z}{\partial q_1}(q_1'' - \bar{q}_1'') + \frac{\partial z}{\partial q_2}(q_2'' - \bar{q}_2'') + \cdots + \frac{\partial z}{\partial q_k}(q_k'' - \bar{q}_k'')\,,$$

which shows us that the displacement $\delta x = \Delta_x\,\delta t$, $\delta y = \Delta_y\,\delta t$, $\delta z = \Delta_z\,\delta t$ is a virtual displacement.

Having established that theorem, we shall call the geometric quantity $m\,[(\Gamma) - (\Gamma_1)]$ that was introduced above the force of friction that is exerted on M and we will call the reaction (ρ') that is exerted on M if the system were frictionless the force of constraint. We will have:

$$(R) = (\rho_1) + (\rho')\,.$$

The virtual work done by the forces (ρ') is zero, and the segments $(\rho_1)\,\delta t$ represent a virtual displacement. Such a decomposition of the (R) is possible in only one way, since the (ρ_1) are no different from the (ρ). That new definition of friction will coincide with the first one then.

If one supposes, for example, that the system is composed of just one point that moves on a surface then the preceding remarks can be stated as follows: The component of the reaction that is normal to the surface will be the same when the point is placed under the same given initial conditions and is subjected to a given active force (say, gravity) whether or not the surface is frictionless. If two points that move on the same surface have the same position and velocity at the instant t_0 then the geometric difference between their acceleration at time t_0 will be tangent to the surface.

It is now easy for us to prove the theorem that Gauss stated regarding the deviation of a system that he felt was at the root of the dynamics of systems.

Suppose that each point M (of mass m) of the system has a given position (x_0, y_0, z_0) and velocity (x_0', y_0', z_0') at the instant t, and let it be subjected to a given active force (X', Y', Z'). If the point M is free then after a time dt, it will occupy a certain position (x_1, y_1, z_1):

$$x_1 = x_0 + x_0'\,dt + \frac{X'}{2m}dt^2 + \cdots,$$
$$y_1 = y_0 + y_0'\,dt + \frac{Y'}{2m}dt^2 + \cdots,$$
$$z_1 = z_0 + z_0'\,dt + \frac{Z'}{2m}dt^2 + \cdots$$

In reality, after a time dt, it will occupy the position (x, y, z):

$$x = x_0 + x_0'\,dt + \frac{X}{2m}dt^2 + \cdots,$$
$$y = y_0 + y_0'\,dt + \frac{Y}{2m}dt^2 + \cdots,$$
$$z = z_0 + z_0'\,dt + \frac{Z}{2m}dt^2 + \cdots$$

Let d denote the distance between the two points (x, y, z) and (x_1, y_1, z_1), and consider the sum $E = \sum m^2 d^2$, which is extended over all points of the system. Upon neglecting the higher-order infinitesimals, one will have:

$$E = \sum m^2 d^2 = \frac{dt^4}{4}\sum (X - X')^2 + (Y - Y')^2 + (Z - Z')^2 = \frac{dt^4}{4}\sum R^2 .$$

That is the quantity E that Gauss called the *deviation* of a system at time t_0 over the interval of time dt. Gauss proved that this deviation is a minimum at each instant when the system is frictionless.

In order to see that, it will suffice to write the equality:

$$E = \frac{dt^4}{4}\sum R^2 = \frac{dt^4}{4}\left(\sum \rho^2 + \sum \rho'^2\right).$$

The quantities (ρ') are the reactions that would be exerted on the points M if there were no friction. The sum $\sum \rho^2 + \sum \rho'^2$ will be a minimum if all the quantities (ρ) are zero, i.e., if there is no friction. In order for the deviation E to constantly be a minimum, it is therefore necessary and sufficient that the system should be frictionless.

Study of the motion of a system with friction. – When there is no friction, knowing the active forces will suffice for one to be able to determine the motion of the system. However, when there is friction, that will no longer be true. In addition to the active forces, one must know the forces of friction, or at least the virtual work that they do.

Experience shows us that for a given system whose points have given positions and velocities at the instant t, the forces of friction will be determined when we know the forces of constraint. In other words, for a given system, the ρ_x, ρ_y, ρ_z are functions of the ρ'_x, ρ'_y, ρ'_z whose coefficients depend upon $q_1, q_2, \ldots, q_k, q'_1, q'_2, \ldots, q'_k, t$.

In particular, if the constraints are independent of time and the elements of the system remain identical to themselves then one can empirically determine the pressures (which are independent of t) of the ρ_x, ρ_y, ρ_z as functions of the ρ'_x, ρ'_y, ρ'_z by placing the system under variable initial conditions and subjecting it to some simple active forces (such as gravity).

In other words, one sets:

$$\begin{aligned} \rho_x &= \mu_1 \frac{\partial x}{\partial q_1} + \cdots + \mu_k \frac{\partial x}{\partial q_k}, & \qquad \rho'_x &= \lambda_1 \frac{\partial f_1}{\partial x} + \cdots + \lambda_p \frac{\partial f_p}{\partial x}, \\ \rho_y &= \mu_1 \frac{\partial y}{\partial q_1} + \cdots + \mu_k \frac{\partial y}{\partial q_k}, & \text{and} \qquad \rho'_y &= \lambda_1 \frac{\partial f_1}{\partial y} + \cdots + \lambda_p \frac{\partial f_p}{\partial y}, \\ \rho_z &= \mu_1 \frac{\partial z}{\partial q_1} + \cdots + \mu_k \frac{\partial z}{\partial q_k}, & \qquad \rho'_z &= \lambda_1 \frac{\partial f_1}{\partial z} + \cdots + \lambda_p \frac{\partial f_p}{\partial z}, \end{aligned}$$

in which the $\mu_1, \mu_2, \ldots, \mu_k$ are functions of $\lambda_1, \lambda_2, \ldots, \lambda_p$ and $q_1, q_2, \ldots, q_k, q'_1, q'_2, \ldots, q'_k$ that one calculates empirically for the given system. The $\lambda_1, \lambda_2, \ldots, \lambda_p$ are expressed with the aid of the active forces X', Y', Z', which amounts to saying that the forces of friction at the instant t will be determined when the system is placed under given initial conditions and subjected to given active forces (no matter what the medium might be that exerts those active forces).

More generally, when one studies the motion of a system with friction, one supposes that the expressions for the $\mu_1, \mu_2, \ldots, \mu_k$ as functions of the λ, the q (and t if the constraints depend upon time) are known (in addition to the active forces X', Y', Z'). One can then write the equations of motion of a point of the system:

$$(\gamma)\qquad \begin{cases} m\dfrac{d^2x}{dt^2} = X = X' + \mu_1\dfrac{\partial x}{\partial q_1} + \cdots + \mu_k\dfrac{\partial x}{\partial q_k} + \lambda_1\dfrac{\partial f_1}{\partial x} + \cdots + \lambda_p\dfrac{\partial f_p}{\partial x}, \\ m\dfrac{d^2y}{dt^2} = Y = Y' + \mu_1\dfrac{\partial y}{\partial q_1} + \cdots + \mu_k\dfrac{\partial y}{\partial q_k} + \lambda_1\dfrac{\partial f_1}{\partial y} + \cdots + \lambda_p\dfrac{\partial f_p}{\partial y}, \\ m\dfrac{d^2z}{dt^2} = Z = Z' + \mu_1\dfrac{\partial z}{\partial q_1} + \cdots + \mu_k\dfrac{\partial z}{\partial q_k} + \lambda_1\dfrac{\partial f_1}{\partial z} + \cdots + \lambda_p\dfrac{\partial f_p}{\partial z}. \end{cases}$$

We can replace that system of $3n$ equations with the following ones:

1. The p equations (γ') that are obtained by multiplying the three equations (γ) by $\dfrac{\partial f_i}{\partial x}$, $\dfrac{\partial f_i}{\partial y}$, $\dfrac{\partial f_i}{\partial z}$, resp., adding them, and taking the sum over all points of the system. (As we saw, those p equations determine the $\lambda_1, \lambda_2, \ldots, \lambda_p$ as functions of the q', of t, and of X', Y', Z'.)

2. The k equations (γ'') that are obtained by multiplying the three equations (γ) by $\dfrac{\partial x}{\partial q_j}$, $\dfrac{\partial y}{\partial q_j}$, $\dfrac{\partial z}{\partial q_j}$, respectively, adding them, and taking the sum over all points of the system.

Those equations have the form:

$$(\gamma'')\qquad \sum m\left(\frac{d^2x}{dt^2}\frac{\partial x}{\partial q_j} + \frac{d^2y}{dt^2}\frac{\partial y}{\partial q_j} + \frac{d^2z}{dt^2}\frac{\partial z}{\partial q_j}\right)$$
$$= \sum\left(X'\frac{\partial x}{\partial q_j} + Y'\frac{\partial y}{\partial q_j} + Z'\frac{\partial z}{\partial q_j}\right) + \sum\left(\rho_x\frac{\partial x}{\partial q_j} + \rho_y\frac{\partial y}{\partial q_j} + \rho_z\frac{\partial z}{\partial q_j}\right).$$

The sum $\sum\left(\rho_x\dfrac{\partial x}{\partial q_j} + \rho_y\dfrac{\partial y}{\partial q_j} + \rho_z\dfrac{\partial z}{\partial q_j}\right)$ has the form:

$$\mu_1\sum\left(\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_1}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_1}\frac{\partial z}{\partial q_j}\right) + \cdots + \mu_k\sum\left(\frac{\partial x}{\partial q_k}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_k}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_k}\frac{\partial z}{\partial q_j}\right).$$

The $\mu_1, \mu_2, \ldots, \mu_k$ are given functions of λ, the q, the q', and t. One replaces the λ (in the μ) with their values that are inferred from equations (γ''). The k equations (γ''), whose right-hand sides are then known functions of the q, the q', and t, will then determine the motion of the system.

Let us apply those generalities to the study of the motion of a point *M* on a fixed surface. Let:

$$z = F(x, y)$$

be the equation of the surface. (Here $q_1 = x$, $q_2 = y$). We have:

$$\begin{aligned}
m\frac{d^2x}{dt^2} &= X' + \mu_1 - \lambda\frac{\partial F}{\partial x}, \\
m\frac{d^2y}{dt^2} &= Y' + \mu_2 - \lambda\frac{\partial F}{\partial y}, \\
m\frac{d^2z}{dt^2} &= Z' + \mu_1\frac{\partial F}{\partial x} + \mu_2\frac{\partial F}{\partial y} + \lambda.
\end{aligned}$$

Experience shows that when the point *M* is placed at a well-defined point on the surface, the force of friction (ρ) will point in the opposite sense to the velocity of *M* and will be reasonably independent of that velocity and proportional to the normal component of the reaction, which is the force of constraint (ρ') in this particular case. We will then have:

$$\rho = f\,\rho'.$$

The coefficient of friction depends upon only the position of *M* on the surface (which is *x*, *y*, here). It will be constant if the surface is equally rough everywhere.

When the velocity of *M* is zero, (ρ) will be directly opposite to the component F_T' of the active force that is tangential to the surface. However, there are two cases to distinguish in regard to its absolute value according to whether one has $F_T' > f\,\rho'$ or $F_T' < f\,\rho'$.

In the first case, ρ will be equal to $f\,\rho'$. In the second case, ρ will be equal to F_T', and the acceleration of *M* will be zero.

From that, since ρ' is equal to:

$$+\sqrt{\lambda^2\left[1+\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2\right]},$$

μ_1 and μ_2, which are equal to ρ_x and ρ_y, resp., can be expressed as follows:

$$\mu_1 = \frac{f\,x'\,|\,\lambda\,|\sqrt{1+\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2}}{\sqrt{x'^2+y'^2+\left(\frac{\partial F}{\partial x}x'+\frac{\partial F}{\partial y}y'\right)^2}},$$

$$\mu_2 = \frac{f\, y' \,|\,\lambda\,|\sqrt{1+\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2}}{\sqrt{x'^2+y'^2+\left(\frac{\partial F}{\partial x}x'+\frac{\partial F}{\partial y}y'\right)^2}}\,.$$

λ is given by the equality:

$$m\left[\frac{\partial^2 F}{\partial x^2}x'^2+2\frac{\partial^2 F}{\partial x\,\partial y}x'y'+\frac{\partial^2 F}{\partial y^2}y'^2\right] = -\;X'\frac{\partial F}{\partial x}-Y'\frac{\partial F}{\partial y}+Z'+\lambda\left[1+\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2\right],$$

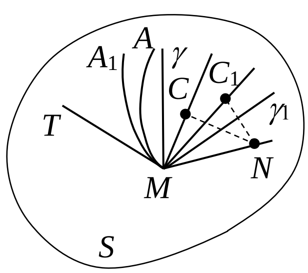
and if one replaces λ with that value in the expressions for μ_1 and μ_2 then the motion of the point M will be determined by the equations:

$$m\left[x''+\frac{\partial F}{\partial x}z''\right] = X'+Z'\frac{\partial F}{\partial x}+\mu_1\left[1+\left(\frac{\partial F}{\partial x}\right)^2\right]+\mu_2\frac{\partial F}{\partial y}\frac{\partial F}{\partial x},$$

$$m\left[y''+\frac{\partial F}{\partial y}z''\right] = Y'+Z'\frac{\partial F}{\partial y}+\mu_1\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}+\mu_2\left[1+\left(\frac{\partial F}{\partial y}\right)^2\right],$$

which will have second order in x and y if one replaces z with $F(x, y)$ in them.

In the particular case of a point that moves on a fixed surface, the general theorems that we proved can be easily obtained by a geometric argument. For example, in order to establish that the normal reaction to the surface will be the same whether there is or is not friction, it will suffice to prove that if two points move on that same surface and have the same position and velocity at the instant t then the geometric difference of their accelerations will be tangent to the surface.



Now let MA be the trajectory of the first moving point, while MA_1 is that of second. Their respective accelerations γ and γ_1 are situated in the osculating planes to MA and MA_1 at M. Those accelerations have projections onto the tangent and principal normal to the trajectory that are $\frac{dv}{dt}\cdot\frac{v^2}{R}$, in the one case, and $\left(\frac{dv}{dt}\right)_1\cdot\frac{v^2}{R_1}$, in the other. The initial velocity is the same for the two points. The trajectories are then tangent at M, and it will be proved that the geometric difference $(\gamma) - (\gamma_1)$ is not the normal component to the surface if one proves that the quantities v^2/R and v^2/R_1, which are carried by the principal normals MC and MC_1, respectively, towards the centers of curvature, have the same projection MN onto the normal to the surface at M. From Meusnier's theorem, if one draws planes perpendicular to MC and MC_1 through the centers of curvatures C and C_1, respectively, then they will intersect the normal to the surface at the same point N, i.e., C and C_1 lie on a circle of diameter MN. If one transforms the figure by reciprocal vectors, the pole being M and the modulus being v^2, then the inverse of the circle will be a line perpendicular to MN at P. Therefore, the magnitudes v^2 / R, v^2 / R_1, which are carried by MC and MC_1, have the same projection MP onto the normal to the surface, which proves the proposition.

One can further appeal to the intrinsic equations of motion of a point on a surface. Those equations are written:

$$\left\{\begin{aligned} m\frac{dv}{dt} &= F_t = F_t' + R_t , \\ m\frac{v^2}{R}\sin\theta &= F_p = F_p' + R_p , \\ m\frac{v^2}{R}\cos\theta &= F_R = F_R' + R_R . \end{aligned}\right.$$

F_t	is	the component of the total force	in the direction Mt of the tangent to the trajectory in the sense of motion
F_p	″	″	in the direction Mp of the tangent to the surface perpendicular to Mt
F_R	″	″	along the normal MR to the surface
θ	″	the angle between the binormal to the trajectory and the normal to the surface	

If r denotes the radius of curvature MC' of the section of the surface by the normal plane at M to the surface at M that passes through Mt then one will have:

$$R_n = \frac{mv^2}{r} - F_R'$$

(on choosing the direction of Mm to be MC'). That shows that the force of constraint $\rho' = (R_n)$ is determined when one knows the position of the point, its velocity, and the force F'.

Furthermore, those intrinsic equations are suitable for the study of motion of M in any case. Whether there is or is not friction, R_p will be zero, because the component (ρ) of the reaction that is tangential to the surface points in the opposite direction to Mt, i.e., it is normal to Mp.

On the other hand, ρ is equal to $f\,\rho'$, i.e., to:

$$f\left|\frac{mv^2}{r} - F_n'\right| .$$

The motion is then determined by the equations:

$$m\frac{dv}{dt} = F_t' - f\left|\frac{mv^2}{r} - F_R'\right| ,$$

$$m\frac{v^2}{R}\sin\theta = F_R'.$$

R is the radius of the trajectory, and r is the radius of curvature of the section that is normal to the surface and tangent to the trajectory.

For example, if F' is zero then one will have:

$$\sin\theta = 0 ,$$

i.e., the trajectory will again be a geodesic of the surface. However, the point will traverse that trajectory with a velocity that is constantly diminished by the force of friction.

That is a general fact that is shown by experience: When the constraints do not depend upon time, the forces of friction will always reduce the *vis viva* of the system. In other words, the work done by the forces of friction is essentially negative.

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LECTURE 6

LAGRANGE'S EQUATIONS

D'ALEMBERT'S EQUATION

Everything that was said before can be reduced to this: When the sum:

$$\sum\left[\left(m\frac{d^2x}{dt^2}-X\right)\delta x+\left(m\frac{d^2y}{dt^2}-Y\right)\delta y+\left(m\frac{d^2z}{dt^2}-Z\right)\delta z\right]$$

extends over all points in the system, it will be identically zero. [δx, δy, δz denote an arbitrary virtual displacement, and (X, Y, Z) is the force that is exerted upon the point (x, y, z).]

The equality that expresses that fact will persist when there is no friction if one replaces (X, Y, Z) by the active force (X', Y', Z') and when there is friction when one replaces (X, Y, Z) with the geometric sum of the active force and the force of friction that is exerted upon the point (x, y, z).

When one lets (X', Y', Z') denote the active force or the geometric sum of the active force and the force of friction, according to the situation, that equality will lead to equations of the form:

$$m\frac{d^2x}{dt^2}= X'+\lambda_1\frac{\partial f_1}{\partial x}+\cdots+\lambda_p\frac{\partial f_p}{\partial x},$$

and those equations, when combined with the constraint equations:

$$f_1 = 0 , \; \ldots, f_p = 0 , \tag{1}$$

will suffice to determine the motion of the system, and as a result, the constraint forces.

The equation:

$$\sum\left[\left(m\frac{d^2x}{dt^2}-X\right)\delta x+\left(m\frac{d^2y}{dt^2}-Y\right)\delta y+\left(m\frac{d^2z}{dt^2}-Z\right)\delta z\right] = 0 , \tag{2}$$

which one can regard as the most general equation of mechanics, is often called the ***d'Alembert** equation.*

Principle of virtual velocities. – In particular, in order for a system whose constraints do not depend upon time, and in which there is no friction, to be in equilibrium, it is necessary and sufficient that the virtual work done by the active forces should be zero.

1. The condition is necessary: Indeed, since there is no friction, the work done by the constraint force will be zero, and the total force (X, Y, Z) that is exerted on (x, y, z) will be:

$$X = X' + \lambda_1 \frac{\partial f_1}{\partial x} + \cdots + \lambda_p \frac{\partial f_p}{\partial x},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

in which (X', Y', Z') is the active force at that point. If there is equilibrium then $X = Y = Z = 0$. Therefore, for any virtual displacement:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = 0 .$$

As a result, for any virtual displacement, one will also have:

$$\sum (X'\,\delta x + Y'\,\delta y + Z'\,\delta z) = 0 .$$

Q.E.D.

2. The condition is sufficient: Indeed, if it is fulfilled, and if the points of the system occupy positions x_0, y_0, z_0 at the instant t with zero velocities then the equations of motion of the system will be verified when one sets $x \equiv x_0$, $y \equiv y_0$, $z \equiv z_0$. Furthermore, there exists only one system of integrals of the equations of motion that satisfy the initial conditions $x(t_0) = x_0$, $y(t_0) = y_0$, $z(t_0) = z_0$, $x'(t_0) = 0$, $y'(t_0) = 0$, $z'(t_0) = 0$, unless, however, the values t_0, x_0, y_0, $z_0 = 0$, $x'_0 = 0$, $y'_0 = 0$, $z'_0 = 0$ define a system of singular values of the equations of motion. Ignoring that exceptional case, the system will then necessarily remain in equilibrium.

One can prove that proposition by appealing to the following necessary and sufficient condition for a material point to remain in equilibrium if one releases it with no initial velocity at the point x_0, y_0, z_0, which is that one must have:

$$X(x_0, y_0, z_0, 0, 0, 0) = 0 , \qquad Y(x_0, y_0, z_0, 0, 0, 0) = 0 , \qquad Z(x_0, y_0, z_0, 0, 0, 0) = 0 .$$

That condition is rigorously sufficient only if X, Y, Z are regular functions in the neighborhood of the values of x_0, y_0, z_0, $x'_0 = 0$, $y'_0 = 0$, $z'_0 = 0$.

Once that lemma has been assumed, suppose that the virtual work done by the active forces X', Y', Z' that are exerted upon the different points of a frictionless system is zero. The force X, Y, Z that is exerted at each point of the system is necessarily zero. Indeed, let (F) be the force that is exerted on the point M. That point must put into motion with the same sense of (F) and the work that is done by (F) under the real displacement ds of the point M is equal to $F\,ds$. The sum $\sum F\,ds$ is essentially positive. The virtual work that is done by the forces (F), and as a result, the virtual work that is done by active forces, is therefore non-zero for the virtual displacement $\delta s = ds$, at least when all of the forces (F) are not zero. Q.E.D.

The equilibrium positions of a frictionless system that is subjected to given active forces are then determined by the equality:

$$\sum (X'\,\delta x + Y'\,\delta y + Z'\,\delta z) = 0 ,$$

which is equivalent to the $3n$ equations:

$$X' + \lambda_1 \frac{\partial f_1}{\partial x} + \lambda_2 \frac{\partial f_2}{\partial x} + \cdots + \lambda_p \frac{\partial f_p}{\partial x} = 0 ,$$
$$Y' + \lambda_1 \frac{\partial f_1}{\partial y} + \lambda_2 \frac{\partial f_2}{\partial y} + \cdots + \lambda_p \frac{\partial f_p}{\partial y} = 0 ,$$
$$Z' + \lambda_1 \frac{\partial f_1}{\partial z} + \lambda_2 \frac{\partial f_2}{\partial z} + \cdots + \lambda_p \frac{\partial f_p}{\partial z} = 0 .$$

In the most general case, those $3n$ equations, when combined with the p equations of constraint, will determine isolated values for x, y, z, and λ. In certain cases, those equations will be incompatible or indeterminate.

If one expresses the x, y, z as functions of k parameters $q_1, q_2, \ldots, q_k$ then that will give:

$$\sum (X'\,\delta x + Y'\,\delta y + Z'\,\delta z)$$

$$= \delta q_1 \sum \left(X' \frac{\partial x}{\partial q_1} + Y' \frac{\partial y}{\partial q_1} + Z' \frac{\partial z}{\partial q_1} \right) + \cdots + \delta q_k \sum \left(X' \frac{\partial x}{\partial q_k} + Y' \frac{\partial y}{\partial q_k} + Z' \frac{\partial z}{\partial q_k} \right)$$

$$= Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_k\,\delta q_k .$$

The necessary and sufficient conditions are:

$$Q_1 (q_1, q_2, \ldots, q_k, 0, \ldots, 0) = 0,\ Q_2 (q_1, q_2, \ldots, q_k, 0, \ldots, 0) = 0,\ \ldots,\ Q_k (q_1, q_2, \ldots, q_k, 0, \ldots, 0) = 0,$$

which are then k equations between the $q_1, q_2, \ldots, q_k$.

When $Q_1, Q_2, \ldots, Q_k$ are the partial derivatives of the same function $U (q_1, q_2, \ldots, q_k)$, those equalities will be the necessary conditions for the function to U to have a minimum. In the next lecture, we shall see that if U is truly a minimum then the equilibrium will be stable, and we will study the small oscillations of the system about its equilibrium position.

In particular, in order for a massive system to be in equilibrium, it is necessary and sufficient that the vertical ordinate ζ of its center of gravity must satisfy the conditions:

$$\frac{\partial \zeta}{\partial q_1} = 0 , \qquad \ldots, \qquad \frac{\partial \zeta}{\partial q_k} = 0 .$$

In other words, its center of gravity is the highest or lowest one possible.

When there is friction, but the external material elements that exert reactions on the system are fixed, the conditions of equilibrium that were found before will again be sufficient, but they will no longer be necessary.

Indeed, under the hypothesis that we just assumed, experience will show us that the work done by the forces of friction will always be negative. If the stated equilibrium conditions are fulfilled, while the system is supposed to be frictionless, then the *vis viva* will be zero at an arbitrary instant. Friction can only reduce that *vis viva*, so it will again be zero *a fortiori* if the system is frictionless, i.e., the system will remain in equilibrium.

However, those conditions are no longer necessary. For example, let M be a point that moves without friction on a fixed material surface. Any point P where the active force (F') that is exerted upon M is normal to the surface will be an equilibrium position. However, consider the points P' of the surface such that one has: $F'_t < f\, F'_n$, in which f is the coefficient of friction, while F'_t and F'_n are the components of (F') that are tangent and normal to the surface, resp. Those points P' form a continuous zone around P whose points are all equilibrium positions.

Nonetheless, we shall not belabor the principle of virtual velocities here but move on to a study of the motions of systems.

LAGRANGE'S EQUATIONS

We saw in the fifth lecture that one can define the motion of a system with the aid of k independent equations of the constraint forces that can be written:

$$(1) \qquad \sum m\left(\frac{d^2x}{dt^2}\frac{\partial x}{\partial q_j}+\frac{d^2y}{dt^2}\frac{\partial y}{\partial q_j}+\frac{d^2z}{dt^2}\frac{\partial z}{\partial q_j}\right) = \sum\left(X'\frac{\partial x}{\partial q_j}+Y'\frac{\partial y}{\partial q_j}+Z'\frac{\partial z}{\partial q_j}\right) = Q_j\,.$$

The sums must be extended over all points of the system; as for j, it is equal to 1, 2, …, k.

If one replaces the x, y, z and their derivatives as functions of the q, the q', and the q'', then the preceding equations will be linear with respect to the q'', and will be soluble for those variables. Indeed, those k equations form a system of k distinct combinations of the $3n$ equations (γ) of the fifth lecture (see page 56), which are distinct. However, that is a point that we shall soon prove directly.

The quantities X', Y', Z' are the projections of the active force that is exerted upon M when there is no friction or the geometric sum of the active force and the force of friction when there is friction.

Lagrange gave a form to the left-hand sides of equations (1) that permits one to easily calculate them. He was led to that result by the following induction: The right-hand side Q_j of an equation (1) is the coefficient of the δq_j in the expression for the virtual work that relates to the displacement δq_j. If one makes the arbitrary change of rectangular axes (x, y, z), while keeping the same parameters q, then the Q_j will not change, and the left-hand side of equation (1) must also remain invariant then. Now, the *vis viva* of a system is a function of t, the q, and the q' that remains invariant under a change of rectangular axes. Based upon that idea, Lagrange expressed the left-hand side of equations (1) in terms of only the *vis viva* and its derivatives.

Set:

$$T = \tfrac{1}{2}\sum m v^2 = \tfrac{1}{2}\sum m(x'^2 + y'^2 + z'^2),$$

and on the other hand, write the left-hand side of equation (1) as:

$$\frac{d}{dt}\sum m\left(x'\frac{\partial x}{\partial q_j} + y'\frac{\partial y}{\partial q_j} + z'\frac{\partial z}{\partial q_j}\right) - \sum m\left(x'\frac{d}{dt}\frac{\partial x}{\partial q_j} + y'\frac{d}{dt}\frac{\partial y}{\partial q_j} + z'\frac{d}{dt}\frac{\partial z}{\partial q_j}\right)$$

or

$$\frac{d}{dt}\cdot P_1 - P_2 \ .$$

Recall that one has:

$$x' = \frac{dx}{dt} = \frac{\partial x}{\partial q_1}q_1' + \frac{\partial x}{\partial q_2}q_2' + \cdots + \frac{\partial x}{\partial q_k}q_k' + \frac{\partial x}{\partial t} \ ,$$

$$y' = \frac{dy}{dt} = \frac{\partial y}{\partial q_1}q_1' + \frac{\partial y}{\partial q_2}q_2' + \cdots + \frac{\partial y}{\partial q_k}q_k' + \frac{\partial y}{\partial t} \ ,$$

$$z' = \frac{dz}{dt} = \frac{\partial z}{\partial q_1}q_1' + \frac{\partial z}{\partial q_2}q_2' + \cdots + \frac{\partial z}{\partial q_k}q_k' + \frac{\partial z}{\partial t} \ ,$$

and as a result:

$$\frac{\partial x}{\partial q_j} = \frac{\partial x'}{\partial q_j'}, \quad \frac{\partial y}{\partial q_j} = \frac{\partial y'}{\partial q_j'}, \quad \frac{\partial z}{\partial q_j} = \frac{\partial z'}{\partial q_j'},$$

and therefore:

$$P_1 \equiv \sum m\left(x'\frac{\partial x'}{\partial q_j} + y'\frac{\partial y'}{\partial q_j} + z'\frac{\partial z'}{\partial q_j}\right) \equiv \frac{\partial T}{\partial q_j'}.$$

In order to calculate P_2, we remark that $\frac{d}{dt}\frac{\partial x}{\partial q_j}$ does not differ from $\frac{\partial}{\partial q_j}\frac{dx}{dt}$. Indeed:

$$\frac{d}{dt}\frac{\partial x}{\partial q_j} = \frac{\partial^2 x}{\partial q_j\,\partial q_1}q_1' + \frac{\partial^2 x}{\partial q_j\,\partial q_2}q_2' + \cdots + \frac{\partial^2 x}{\partial q_j\,\partial q_k}q_k' + \frac{\partial^2 x}{\partial q_j\,\partial t},$$

and

$$\frac{\partial}{\partial q_j}x' = \frac{\partial^2 x}{\partial q_1\,\partial q_j}q_1' + \frac{\partial^2 x}{\partial q_2\,\partial q_j}q_2' + \cdots + \frac{\partial^2 x}{\partial q_k\,\partial q_j}q_k' + \frac{\partial^2 x}{\partial t\,\partial q_j}.$$

Similarly:

$$\frac{d}{dt}\frac{\partial y}{\partial q_j} \equiv \frac{\partial y'}{\partial q_j}, \qquad \frac{d}{dt}\frac{\partial z}{\partial q_j} \equiv \frac{\partial z'}{\partial q_j}.$$

One can then write:

$$P_2 \equiv \sum m\left(x'\frac{\partial x'}{\partial q_j} + y'\frac{\partial y'}{\partial q_j} + z'\frac{\partial z'}{\partial q_j}\right) \equiv \frac{\partial T}{\partial q_j}.$$

The Lagrange equations will then become:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} = Q_1,$$

……………………..

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_j'}\right) - \frac{\partial T}{\partial q_j} = Q_j,$$

………………………

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_k'}\right) - \frac{\partial T}{\partial q_k} = Q_k.$$

Before employing those equations in the study of the motion of a system, we shall extend the preceding considerations to continuous systems.

CONTINUOUS SYSTEMS WHOSE POSITION DEPENDS UPON ONLY A FINITE NUMBER OF PARAMETERS

Up to now, we have addressed only those systems that are composed of a finite number of material points. Now suppose that the system in question includes continuous bodies, but whose positions depend upon only a finite number of parameters (for example, homogeneous solids that are subject to certain constraints).

The coordinates x, y, z of a well-defined material element of the system are expressed as functions of the k parameters $q_1, q_2, \ldots, q_k$, and t :

$$\text{(1)} \qquad \begin{cases} x = \varphi(q_1, q_2, \ldots, q_k, t), \\ y = \psi(q_1, q_2, \ldots, q_k, t), \\ z = \chi(q_1, q_2, \ldots, q_k, t). \end{cases}$$

Each continuous body in the system corresponds to a system of functions φ, ψ, χ that depends upon the constants a_1, b_1, …, which will be 3, 2, or 1 in number according to whether the body is a volume, an area, or a length, resp. The coordinates x, y, z of the points of a free (three-dimensional) solid can then be expressed as functions of six parameters (and three constants, for example, the coordinates a, b, c of the point considered when referred to the principal axes of inertia Ga, Gb, Gc, where G denotes the center of gravity of the solid body).

If the system includes isolated material points, in addition, then each material point will correspond to a well-defined system of functions φ, ψ, χ.

We suppose that the k parameters q are independent, i.e., that for any instant t, one can give arbitrary values to $q_1, q_2, \ldots, q_k$ and $\frac{dq_1}{dt}, \frac{dq_2}{dt}, \ldots, \frac{dq_k}{dt}$. That again signifies that one can give the positions x, y, z and the velocities x', y', z' of all elements of the system arbitrarily, provided that those initial conditions are compatible with equations (1).

There exists a relationship between $(k + 1)$ coordinates x, y, z of the well-defined points of the system in which t might appear that one obtains by eliminating $q_1, q_2, \ldots, q_k$, from the corresponding $(k + 1)$ equations (1). We assume that such a relation does not exist between k arbitrary coordinates x, y, z: In other words, we can choose the parameters q (for arbitrary t) in such a fashion as to give arbitrary values to the k coordinates x, y, z of certain points of the system. If we write the k corresponding equations (1) then those k can be solved for the $q_1, q_2, \ldots, q_k$. (Under the opposite hypothesis, one can leave one or more of the parameters q constant, and the coordinates of the points of the system will be expressed as functions of a lower number of parameters.) We shall then say that the position of the system depends upon k distinct parameters. Analytically, that amounts to assuming that when the equations in $\delta q_1, \delta q_2, \ldots, \delta q_k$:

$$\frac{\partial\varphi}{\partial q_1}\delta q_1 + \frac{\partial\varphi}{\partial q_2}\delta q_2 + \cdots + \frac{\partial\varphi}{\partial q_k}\delta q_k = 0\,,$$
$$\frac{\partial\psi}{\partial q_1}\delta q_1 + \frac{\partial\psi}{\partial q_2}\delta q_2 + \cdots + \frac{\partial\psi}{\partial q_k}\delta q_k = 0\,,$$
$$\frac{\partial\chi}{\partial q_1}\delta q_1 + \frac{\partial\chi}{\partial q_2}\delta q_2 + \cdots + \frac{\partial\chi}{\partial q_k}\delta q_k = 0$$

are written for all points of the system, they will admit no other solutions than $\delta q_1 = 0$, $\delta q_2 = 0$, …, $\delta q_k = 0$ (for arbitrary $q_1, q_2, \ldots, q_k$, and t). If things were otherwise then the functional determinant of any k of equations (1) (in terms of $q_1, q_2, \ldots, q_k$) would be identically zero.

The constraints will be independent of time when the φ, ψ, χ do not include t. If t enters into φ, ψ, χ then the constraints will depend upon time, at least when all of the relations that are obtained by eliminating the parameters q from the $(k + 1)$ equations (1) are not independent of t.

In order for those exceptional conditions to be fulfilled, it is necessary and sufficient that the equations in $\delta q_1, \delta q_2, \ldots, \delta q_k, \delta t$:

$$(a) \qquad \left\{ \begin{aligned} \frac{\partial\varphi}{\partial q_1}\delta q_1 + \frac{\partial\varphi}{\partial q_2}\delta q_2 + \cdots + \frac{\partial\varphi}{\partial q_k}\delta q_k + \frac{\partial\varphi}{\partial t}\delta t = 0\,, \\ \frac{\partial\psi}{\partial q_1}\delta q_1 + \frac{\partial\psi}{\partial q_2}\delta q_2 + \cdots + \frac{\partial\psi}{\partial q_k}\delta q_k + \frac{\partial\psi}{\partial t}\delta t = 0\,, \\ \frac{\partial\chi}{\partial q_1}\delta q_1 + \frac{\partial\chi}{\partial q_2}\delta q_2 + \cdots + \frac{\partial\chi}{\partial q_k}\delta q_k + \frac{\partial\chi}{\partial t}\delta t = 0\,, \end{aligned} \right.$$

when written for all points of the system, must admit other solutions besides $\delta q_1 = 0$, $\delta q_2 = 0$, …, $\delta q_k = 0$, $\delta t = 0$ (for arbitrary q_1, q_2, …, q_k, and t). In that particular case, it is legitimate to let t have a constant value in φ, ψ, χ.

A *virtual displacement* of the system is an elementary displacement that is compatible with the constraints at time t. When the k parameters q are distinct and independent, the most general virtual displacement will depend upon k arbitrary variations δq_1, δq_2, …, δq_k. For each material element x, y, z of the system, one will have:

$$\begin{aligned} \delta x &= \frac{\partial\varphi}{\partial q_1}\delta q_1 + \frac{\partial\varphi}{\partial q_2}\delta q_2 + \cdots + \frac{\partial\varphi}{\partial q_k}\delta q_k\ , \\ \delta y &= \frac{\partial\psi}{\partial q_1}\delta q_1 + \frac{\partial\psi}{\partial q_2}\delta q_2 + \cdots + \frac{\partial\psi}{\partial q_k}\delta q_k\ , \\ \delta z &= \frac{\partial\chi}{\partial q_1}\delta q_1 + \frac{\partial\chi}{\partial q_2}\delta q_2 + \cdots + \frac{\partial\chi}{\partial q_k}\delta q_k\ . \end{aligned}$$

The δx, δy, δz cannot all be zero simultaneously (for arbitrary q_1, q_2, …, q_k, and t) unless all of the δq are.

The real displacement of the system satisfies the equations:

$$\begin{aligned} dx &= \frac{\partial\varphi}{\partial q_1}dq_1 + \frac{\partial\varphi}{\partial q_2}dq_2 + \cdots + \frac{\partial\varphi}{\partial q_k}dq_k + \frac{\partial\varphi}{\partial t}dt\ , \\ dy &= \frac{\partial\psi}{\partial q_1}dq_1 + \frac{\partial\psi}{\partial q_2}dq_2 + \cdots + \frac{\partial\psi}{\partial q_k}dq_k + \frac{\partial\psi}{\partial t}dt\ , \\ dz &= \frac{\partial\chi}{\partial q_1}dq_1 + \frac{\partial\chi}{\partial q_2}dq_2 + \cdots + \frac{\partial\chi}{\partial q_k}dq_k + \frac{\partial\chi}{\partial t}dt\ . \end{aligned}$$

When the constraints are independent of time, the real displacement will be one of the virtual displacements. That will no longer be true when the constraints depend upon time. Indeed, one must have:

$$(\alpha') \qquad \begin{cases} \dfrac{\partial\varphi}{\partial q_1}(dq_1-\delta q_1)+\dfrac{\partial\varphi}{\partial q_2}(dq_2-\delta q_2)+\cdots+\dfrac{\partial\varphi}{\partial q_k}(dq_k-\delta q_k)+\dfrac{\partial\varphi}{\partial t}dt=0, \\ \dfrac{\partial\psi}{\partial q_1}(dq_1-\delta q_1)+\dfrac{\partial\psi}{\partial q_2}(dq_2-\delta q_2)+\cdots+\dfrac{\partial\psi}{\partial q_k}(dq_k-\delta q_k)+\dfrac{\partial\psi}{\partial t}dt=0, \\ \dfrac{\partial\chi}{\partial q_1}(dq_1-\delta q_1)+\dfrac{\partial\chi}{\partial q_2}(dq_2-\delta q_2)+\cdots+\dfrac{\partial\chi}{\partial q_k}(dq_k-\delta q_k)+\dfrac{\partial\chi}{\partial t}dt=0 \end{cases}$$

for convenient values of $\delta q_1, \delta q_2, \ldots, \delta q_k$. However, when the constraints depend upon time, equations (α') will admit no other solutions than $(dq_1 - \delta q_1) = 0, \ldots, (dq_k - \delta q_k) = 0$, $dt = 0$ (except for some exceptional values of $q_1, q_2, \ldots, q_k$, and t).

Having established those preliminaries, decompose the system into a finite number n of very-small-dimensional elements. If we regard those n elements as material points then we can apply Lagrange's equations to the system, and the approximate equations thus-obtained will tend to rigorous equalities when we make the number of elements increase indefinitely while the dimensions of each element tend to zero. In order to account for what those equalities will become, we consider a continuous body Σ of the system at the instant t, and an element dv of Σ that surrounds the point M or (x, y, z) and an element of volume, area, or arc according to whether Σ is a volume, surface, or line, resp. To fix ideas, suppose that Σ is three-dimensional. Let m be the mass of the element dv. If we make the dimensions of dv tend to zero then m / dv will tend to a limit ρ, that we can call the *density* of the body at the point M, which will be a well-defined function of x, y, z at the instant t for a given position of the system $\rho = f(t, q_1, q_2, \ldots, q_k, x, y, z)$. There can still be exceptions for isolated points M' of finite mass, or for points M'' that form surfaces σ or lines λ. At those points M'', m will have the same order as the element of σ or λ that is intercepted by the material mass dv. By definition, we assume that the system in question is composed of isolated points and continuous lines, surfaces, or volumes that have a linear, superficial, or volumetric density, respectively, at each point.

We now regard each of the n elements dv as a material point: The force (F), relative $Oxyz$, that is exerted on the material point M or dv at an instant will be equal to m (γ), in which γ is the acceleration of the point M with respect to $Oxyz$. One then has:

$$F = (\rho + \varepsilon)\, \gamma\, dv\,,$$

in which ε tends to zero with the dimensions of dv. As a result:

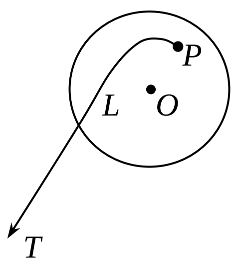
$$(F) = [(f) + (\eta)]\, dv,$$

in which (f) is a well-defined geometric quantity, whose length can be zero, and η tends to 0 with dv. If one decomposes (F) into an active force (F') and an absolute reaction (R) then, in general (F') and (R) will have the same order as dv:

$$(F') = [(f') + (\eta')]\,dv\,, \qquad (R) = [(r) + (\eta'')]\,dv\,.$$

The quantities $(f)\,dv$, $(f')\,dv$, $(r)\,dv$ are the total, active, and reactive forces, resp., that are exerted upon the element dv at the instant t. Nevertheless, there can be exceptions at certain isolated points M' (where F' and R are finite) and points M'' that define volumes, surfaces σ, or lines λ according to the body Σ. At those points M'', (F'), and (R) will have the same order as the element of σ or λ that is intercepted by the material mass considered dv in question. In any case, the sum $(F') + (R)$ has the form $[(f) + (\eta)]\,dv$.

For example, suppose that Σ is a massive homogeneous sphere of density $\rho = 1$ that is submerged in a liquid, and on which a traction is exerted by an elastic string L that is fixed to a point P of the sphere. Suppose, moreover, that the sphere is in equilibrium under those conditions. The active force (F') that is exerted on each element dv is equal and directly opposite to the reaction (R). For any element dv that is interior to the sphere, (F') will be equal to $(g)\,dv$. For an element dv that is situated on the surface σ of the sphere, (F') will be equal to $(g)\,dv + (p)\,d\sigma$, where $(p)\,d\sigma$ denotes the pressure of the liquid on the element $d\sigma$. For an element dv in contact with the string PL, (F') will be equal to $(g)\,dv + (\mu)\,d\sigma + (l)\,d\lambda$, where $(l)\,d\lambda$ denotes the tension in the string, which has the same order as the arc element $d\lambda$ of the string. Finally, at the point P, (F') will tend to T (viz., the tension in the string) when the dv tend to zero.



Similarly, when the continuous body is a surface σ, the active force and the reaction will, in general, have the same order as the two-dimensional material element $d\sigma$.

Those remarks apply to the decomposition of the reaction into a force of constraint and a force of friction.

Having said that, one writes the Lagrange equations that define the motion of the system that is composed of the n material points dv. One will have:

$$\text{(1)} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q'_j}\right) - \frac{\partial T}{\partial q_j} = Q_i\,,$$

with

$$T = \tfrac{1}{2}\sum m(x'^2 + y'^2 + z'^2)\,.$$

If one expresses x', y', z' as functions of t, the q, and the q', then that will give:

$$2T = A_1^1\,q_1'^2 + A_2^2\,q_2'^2 + \cdots + A_k^k\,q_k'^2 + 2A_1^2\,q_1'\,q_2' + \cdots + 2A_{k-1}^k\,q_{k-1}'\,q_k' + B_1\,q_1' + \cdots + B_k\,q_k' + C_1\,,$$

with:

$$A_1^1 = \sum m\left[\left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2\right], \qquad A_2^2 = \text{etc.},$$

$$A_1^2 = \sum m \left(\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2} + \frac{\partial y}{\partial q_1}\frac{\partial y}{\partial q_2} + \frac{\partial z}{\partial q_1}\frac{\partial z}{\partial q_2} \right), \quad A_1^3 = \text{etc.},$$

$$B_1 = \sum m \left(\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial q_1}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial q_1}\frac{\partial z}{\partial t} \right), \qquad B_2 = \text{etc.},$$

$$C = \sum m \left[\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2 \right].$$

In each of the coefficients A, B, C, say A_1^1, consider the part a_1^1 that relates to one of the continuous bodies Σ of the system and suppose, to fix ideas, that Σ is a volume. For each element dv (affixed to x, y, z) of Σ, one will have:

$$m\,[\rho\,(t, q_1, \ldots, q_k, x, y, z) + \varepsilon]\,dv\,,$$

and furthermore, one can write:

$$\left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2 = \alpha_1^1\,(t, q_1, q_2, \ldots, q_k, x, y, z)$$

since the derivatives $\frac{\partial x}{\partial q_1}$, $\frac{\partial y}{\partial q_1}$, $\frac{\partial z}{\partial q_1}$ are well-defined at each point x, y, z for given values of t, q_1, q_2, …, q_k. As a result, when the sum $a_1^1 = \sum (\rho + \varepsilon)\,\alpha_1^1\,dv$ is extended over the elements dv of Σ and one makes the dimensions of those elements tend to zero, that sum will tend to the integral:

$$\int \rho\,\alpha_1^1\,(t, q_1, q_2, \ldots, q_k, x, y, z)\,dx\,dy\,dz$$

which extends over the entire volume Σ.

The function T that figures in the defining equations will then be a polynomial of degree two with respect to q' whose coefficients, which are functions of t and the q, are calculated with the aid of triple, double, or single integrals that extend over the continuous bodies of the system that define volumes, surfaces, or lines, resp.

One similarly sees that the right-hand sides of the defining equations are calculated with the aid of triple, double, or single integrals that extend over volumes v, surfaces σ, and lines λ, resp., such that the active force that is exerted on each material element will have the same order as the volume dv of v, the area $d\sigma$ of σ, or the arc-length $d\lambda$ of λ, resp., that is intercepted by the element.

Before developing those generalities, we shall indicate a certain number of systems that are frictionless.

ENUMERATING SOME SYSTEMS WITHOUT FRICTION

We shall first point out the free solid body: The reactions that are exerted on the points of that system are internal forces that are pair-wise equal and directly opposite, from the principle of action and reaction. The work done t by those forces under a virtual displacement of the system will be equal to:

$$\sum f_{jk}\,\delta x_{jk}\ ,$$

in which f_{jk} denotes the force that is exerted on the points M_j, M_k, and r_{jk} is the distance between those points. Under a virtual displacement, all of the r_{jk} will remain constant; t will then be zero.

In order for that proposition to persist, it is likewise sufficient to assume that the reactions of the solid for a system of segments that are geometrically to a single segment of length zero. Indeed, under an arbitrary virtual displacement of the solid, the variations δx, δy, δz of x, y, z of an arbitrary point will have the form:

$$\begin{aligned}\delta x &= (a + q\,z - r\,y)\ \delta t\ ,\\ \delta y &= (b + r\,x - p\,z)\ \delta t\ ,\\ \delta z &= (c + p\,y - q\,x)\ \delta t\ ,\end{aligned}$$

and the virtual work τ done by the reactions (R_x, R_y, R_z) will be equal to:

$$a\sum R_x + b\sum R_y + c\sum R_z + p\sum(y\,R_z - z\,R_y) + q\sum(z\,R_x - x\,R_z) + r\sum(x\,Ry - y\,R_x)\ .$$

In order for τ to be zero for any virtual displacement, it is necessary and sufficient that the coefficients of a, b, c, p, q, r should be zero: That is precisely the stated condition.

What we just said will remain true if the dimensions of the solid vary with time, in other words, if the distances between the different points of the system are constrained to remain invariable at each instant t, but are given functions of t.

Before enumerating some other constraints, we shall make the following remark: Let M be a point of the system and let (R) be the reaction that is exerted upon it. If one decomposes the material elements E that exert that reaction into two parts E' and E'' then (R) will be the geometric sum of the reactions that are exerted by (E') and (E''). For example, let a solid sphere be constrained to remain in contact with a plane. The reaction that is exerted at the instant t on the element M of the sphere in contact with the plane is the geometric sum of two reactions that are exerted on M, on the one hand, by the elements of the sphere, and on the other hand, by the elements of the plane that are close to M.

With that, consider a system whose elements are subject to several distinct constraints. If the partial reactions that are due to each constraint do zero work for any displacement that is compatible with the constraint then that work itself will be zero *a fortiori* for any virtual displacement of the system, and the system will be frictionless.

Having said that, let a solid body S be subject to slide on a given surface σ (i.e., to remain in contact with σ). The reactions that are exerted on S are the internal reactions (R') of the solid and the reaction (R'') that is exerted by σ on S. In order for the work done (R'') under an arbitrary displacement for which the solid remains tangent to σ to be zero, it is necessary and sufficient (as one will soon see) that (R'') should be normal to S and σ. That is the usual definition of the absence of friction in this particular case. One sees that it coincides with the general definition. If that condition is verified, we will say that the surface σ is *perfectly polished*. Moreover, there can be contact between S and σ at several points, and even along a line.

More generally, a solid body S that is not free will define a frictionless system $\mathcal{S}$ if it is constrained by one of the following conditions:

1. It slides on a perfectly-polished surface σ.
2. It slides on a perfectly-polished curve σ.
3. It has a fixed point P.
4. It rolls on an arbitrary surface σ or a curve γ ([7]).

That will again be true when the surface σ or the curve γ or the point P varies with t according to a given law (as well as the distance to the points of the body S). One can likewise suppose that S reduces to a planar area or a line or a point.

Now let a system be composed of two solids S and Σ. That system will be frictionless if S and Σ roll on each other or if their perfectly-polished surfaces are constrained to slide on each other or if the bodies are articulated around a common point, etc. The two solids can reduce to a planar area, a curve, or a point, respectively.

Finally, if two elements of the system are coupled by a flexible, inextensible string that is also massless that passes through either a given point or a curve γ or a perfectly-polished surface σ with a given position then the reactions that result from that constraint will do zero virtual work. The same thing will be true if the surface σ, the curve γ, or the point P are a surface, a curve, or a point of the system, resp. The same thing will again be true if one of the extremities or both extremities of the string are attached to points with a given position and not to elements of the system.

If one now considers a system that is composed of a finite number of material points and solid bodies whose constraints result from a combination of the preceding constraints then it will be clear that this system is frictionless.

In the case where there is friction, one must borrow from experience to get the data necessary to calculate the forces of friction. For example, let a solid body S slide on a fixed surface σ: There will be friction if the reaction (R'') is not normal to the common tangent plane to S and σ. One easily verifies that the force of friction, as we have defined it in general, will coincide with the component R_t'' of (R'') that is tangent to S and σ in this particular case. Experiments show that R_t'' points in the opposite direction to the material point of S in contact with σ and which is

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([7]) One intends that to mean that S remains in contact with σ or (γ) and that the arcs traversed by the point of contact between the surface of S and σ (or γ) are equal.

proportional to the normal component of R'': $R''_t = f\, R''_n$ (for a given position of the system). Knowing the coefficient of friction *f* will suffice for one to be able to determine the motion of *S* when *S* is subjected to given active forces.

In all of what follows, we will suppose that the systems under study are systems without friction.

LECTURE 7

LAGRANGE EQUATIONS (CONT.)

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The motion of a system without friction whose position depends upon k parameters q_1, q_2, …, q_k is determined by the k Lagrange equations:

(1)
$$\frac{d}{dt}\frac{\partial T}{\partial q'_j} - \frac{\partial T}{\partial q_j} = Q_j ,$$

in which T represents the semi-*vis viva* of the system, and Q_j represents the coefficient of δq_j in the expression for the virtual work done by the active forces. The motion of the system is thus found to be determined with the aid of the least-possible number of equations and givens.

Equations (1) include the second derivatives of the parameters q linearly. We shall prove directly that those equations can be solved for the q''. In order to do that, we recall the form of T:

$$T = \tfrac{1}{2}\sum m(x'^2 + y'^2 + z'^2) ,$$

which is a polynomial of second degree in the q'. One can set:

$$T = T_2 + T_1 + T_0 ,$$

in which T_2 and T_1 are two homogeneous polynomials of degree two and one, respectively. T_0 does not depend upon the q'.

When the coordinates x, y, z of each point of the system are expressed as functions of q_1, q_2, …, q_k without t entering explicitly, T will be homogeneous and reduce to T_2. When the constraints are independent of time, one can always choose the parameters q in such a fashion that the condition is realized.

In any case, one will have:

$$T_2 = \sum m\left[\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2 + \left(\frac{\delta z}{\delta t}\right)^2\right] ,$$

when one agrees to set:

$$\frac{\delta x}{\delta t} = \frac{\partial x}{\partial q_1} q'_1 + \frac{\partial x}{\partial q_2} q'_2 + \cdots + \frac{\partial x}{\partial q_k} q'_k ,$$

$$\frac{\delta y}{\delta t} = \frac{\partial y}{\partial q_1} q'_1 + \frac{\partial y}{\partial q_2} q'_2 + \cdots + \frac{\partial y}{\partial q_k} q'_k ,$$

$$\frac{\delta z}{\delta t} = \frac{\partial z}{\partial q_1} q_1' + \frac{\partial z}{\partial q_2} q_2' + \cdots + \frac{\partial z}{\partial q_k} q_k' \ .$$

Recall that from a remark that was made in the previous lecture, the $\frac{\delta x}{\delta t}$, $\frac{\delta y}{\delta t}$, $\frac{\delta z}{\delta t}$ can all be zero at once only if q_1', q_2', …, q_k' are all zero (except perhaps for particular values of q_1, q_2, …, q_k, t). It follows from this that T_2 cannot be zero (for arbitrary values of q_1, q_2, …, q_k, t) unless all of the q' are zero.

Having said that, consider the quantity:

$$\frac{\partial T}{\partial q_j'} = \frac{\partial T_2}{\partial q_j'} + \frac{\partial T_1}{\partial q_j'} .$$

The term $\partial T_2 / \partial q_j'$ depends upon only the q', and the determinant of the q'' in equations (1) coincides with the determinant of the k equations that are linear and homogeneous with respect to the q':

(2) $$\frac{\partial T_2}{\partial q_1'} = 0 , \qquad \frac{\partial T_2}{\partial q_2'} = 0 , \qquad \ldots, \qquad \frac{\partial T_2}{\partial q_k'} = 0 .$$

I say that this determinant is not identically zero. In other words, there exist values of q_1', q_2', …, q_k' (q_1, q_2, …, q_k, t being chosen arbitrarily) that are not all zero and satisfy equations (2). However, from Euler's theorem, one will have:

$$2T_2 = q_1' \frac{\partial T_2}{\partial q_1'} + q_2' \frac{\partial T_2}{\partial q_2'} + \cdots + q_k' \frac{\partial T_2}{\partial q_k'} \ .$$

T_2 would then be annulled for values of q' that are not all zero, which is impossible.

Equations (1) can then be solved for the q'', and will form a system of k distinct second-order differential equations. Those equations will admit one and only one system of integrals $q_1(t)$, $q_2(t)$, …, $q_k(t)$ that satisfy the initial conditions: $q_1(t_0) = q_1^0$, $q_1'(t_0) = q_1'^0$, …, $q_k(t_0) = q_k^0$, $q_k'(t_0) = q_k'^0$. Nonetheless, there can be exceptions for certain singular systems of values t_0, q_1^0, $q_1'^0$, …, q_k^0, $q_k'^0$.

Calculating *T*. – In order to write out the Lagrange equations, one must calculate, on the one hand, the *vis viva* of the system, and on the other hand, the virtual work done by the active forces. When the system includes continuous bodies, that calculation will be performed with the help of integrals that extend over the continuous body. The only continuous bodies that appear in the applications are solids. In order to calculate their *vis viva*, the simplest way will be, in general, to decompose that *vis viva* into two parts: The *vis viva* of the center of gravity, where one supposes that all of the mass is concentrated, and the *vis viva* of the solid in its relative motion around the center of gravity. In any case, the calculation will reduce to the calculation of the total mass of the solid and its three principal axes of inertia relative to an arbitrarily-chosen point.

The parameters q must be chosen in such a fashion as to give the simplest form to T. In particular, one seeks to make the products $q'_j\, q'_k$ disappear from T. For example, if one is studying the motion of a point on a fixed surface then T will have the form:

$$T = \tfrac{1}{2} m \, (A_1^1 \, q_1'^2 + 2A_1^2 \, q' \, q_2' + A_2^2 \, q_2'^2) \ ,$$

in which

$$A_1^1 = \left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2, \qquad A_2^2 = \left(\frac{\partial x}{\partial q_2}\right)^2 + \left(\frac{\partial y}{\partial q_2}\right)^2 + \left(\frac{\partial z}{\partial q_2}\right)^2,$$

$$A_1^2 = \frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2} + \frac{\partial y}{\partial q_1}\frac{\partial y}{\partial q_2} + \frac{\partial z}{\partial q_1}\frac{\partial z}{\partial q_2}$$

in the simplest case.

If the curvilinear coordinates q_1 = const., q_2 = const. are orthogonal then A_1^2 will be zero, and conversely.

One can even choose q_1 and q_2 in such a fashion that the ds^2 of the surface has the form:

$$ds^2 = A(q_1, q_2)(dq_1^2 + dq_2^2) \, ,$$

and T will then have the form:

$$T = \tfrac{1}{2} m \, A(q_1'^2 + q_2^2) \ .$$

Similarly, if one studies the motion of a free point in curvilinear coordinates then the necessary and sufficient conditions for T to not include products are:

$$\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2} + \frac{\partial y}{\partial q_1}\frac{\partial y}{\partial q_2} + \frac{\partial z}{\partial q_1}\frac{\partial z}{\partial q_2} = 0 \, ,$$

$$\frac{\partial x}{\partial q_2}\frac{\partial x}{\partial q_3} + \frac{\partial y}{\partial q_2}\frac{\partial y}{\partial q_3} + \frac{\partial z}{\partial q_2}\frac{\partial z}{\partial q_3} = 0 \, ,$$

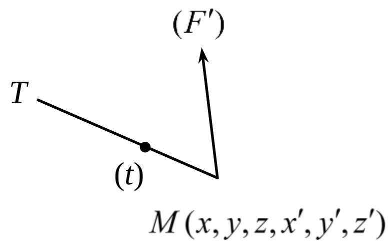
$$\frac{\partial x}{\partial q_3}\frac{\partial x}{\partial q_1} + \frac{\partial y}{\partial q_3}\frac{\partial y}{\partial q_1} + \frac{\partial z}{\partial q_3}\frac{\partial z}{\partial q_1} = 0 \, ,$$

which are equalities that express the idea that the curvilinear coordinates q_1 = const., q_2 = const., q_3 = const. must be orthogonal.

Calculating the Q_j . – The coefficients Q_j are given by the equalities:

$$Q_j = \sum \left(X' \frac{\partial x}{\partial q_j} + Y' \frac{\partial y}{\partial q_j} + Z' \frac{\partial z}{\partial q_j} \right) .$$

One can sometimes simplify their calculation with the aid of the following remarks:



Give q_1 the variation δq_1, while $q_2, \ldots, q_k$ remain constant. The virtual work done by that displacement will be $Q_1\ \delta q_1$. Consider a point M of the system. Let (x, y, z) be its position, let (x', y', z') be its velocity, and let (F') be the active force that is exerted on it at the instant t. Under the displacement δq_1, the various points of the system will describe elementary arcs. Let δs be the arc that is described by M, and let f' be the projection of (F') onto the direction of δs. $f'\,\delta s$ will represents the work done by (F') under the displacement δq_1 . It will then result that:

$$Q_1\ \delta q_1 = \sum f' \frac{\delta s}{\delta q_1} \delta q_1$$

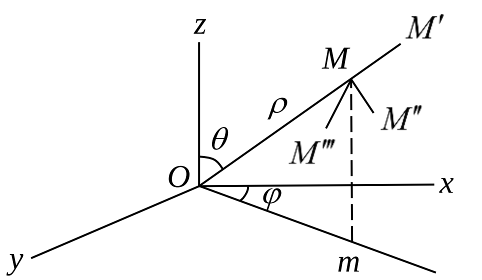
or

$$Q_1 = \sum f' \frac{\delta s}{\delta q_1} ,$$

in which the summation extends over all points of the system.

That form of Q_1 is useful in certain cases where one can easily perceive the displacement at each point for the each variation $\delta q_1, \ldots, \delta q_k$.

Example. – Let us study the motion of a point M in polar coordinates in space.



The *vis viva* is calculated immediately. It is:

$$T = \tfrac{1}{2} m(\rho'^2 + \rho^2\,\theta'^2 + \rho^2 \sin^2\theta\,\varphi'^2) .$$

Let:

$$q_1 = \rho , \qquad q_2 = \theta , \qquad q_3 = \varphi .$$

One must calculate Q_1, Q_2, Q_3 .

One then gives, in succession:

an increase	$+\delta\rho$	to	ρ:	M will then go to	M'	along the prolongation of OM.
″	$+\delta\theta$	to	θ:	″	M''	The displacement MM'' is in the plane zOM, which is perpendicular to OM. It makes an angle of $\theta + \pi/2$ with Oz.
″	$+\delta\varphi$	to	φ:	″	M''',	and the direction MM''' is perpendicular to the plane zOM in the sense of increasing φ. MM''' makes an angle of $\varphi + \pi/2$ with Ox.

Having done that, let F be the force that is exerted on the point M, which is supposed to be free. Let R, Θ, Φ be its components along the directions MM', MM'', MM''', resp. From what was just said:

$$Q_1 = R\,\frac{\delta s'}{\delta\rho} = R\,,$$

$$Q_2 = \Theta\,\frac{\delta s''}{\delta\theta} = \Theta\,\rho\,,$$

$$Q_3 = \Phi\,\frac{\delta s'''}{\delta\varphi} = \Phi\,\rho\sin\theta\,.$$

Moreover, the Lagrange equations for the motion of a point in polar coordinates in space are:

$$m\,\rho'' - m\,\rho\,(\theta'^2 + \sin^2\theta\,\varphi'^2) = R\,,$$

$$m\frac{d}{dt}(\rho\,\theta') - m\,\rho^2\,\varphi'^2\sin\theta\cos\theta = \rho\,\Theta\,,$$

$$m\frac{d}{dt}(\rho^2\sin^2\theta\,\varphi') = \rho\,\Phi\sin\theta\,.$$

In the case where Φ is zero, one will have:

$$\rho^2\sin^2\theta\,\varphi' = 0\,.$$

In order for that to be the case, it is necessary and sufficient that the force should be in the plane MOz. Furthermore, the equality is given by the theorem of moments of quantities of motion, when it is applied to the axis Oz.

When the system includes solid bodies, in order to calculate the virtual work done by the active forces that are applied to the solids, one can replace those forces with any other system of geometrically-equivalent segments, in particular, with two conveniently-chosen forces [or by just one in the case where the forces (F') admit a resultant]. If one would like, the calculation of the virtual work will come down to the calculation of the quantities $\sum X'$, $\sum Y'$, $\sum Z'$, and

$\sum(yZ'-zY')$, $\sum(zX'-xZ')$, $\sum(xY'-yX')$. Once those sums are calculated, the virtual work τ done by the applied forces on the solid under an arbitrary virtual displacement of the solid will be given by the formula:

$$\tau = \delta a \sum X' + \delta b \sum Y' + \delta c \sum Z' + \delta u \sum (yZ'-zY') + \delta v \sum (zX'-xZ') + \delta w \sum (xY'-yX').$$

δa, δb, δc represent the virtual displacement of the solid that coincides with the origin O at the instant t, and δu, δv, δw represent the components of the quantity $(\omega)\, dt$, where (ω) is the instantaneous virtual rotation.

In particular, if the virtual displacement is a translation (parallel to Ox, for example) then the virtual work done by the forces (F') applied to the solid will be equal to $\delta x \sum X'$. If the displacement is a rotation of the solid around Oz then the virtual work done by the forces (F') will be equal to $\delta\theta \sum (xY'-yX')$. θ denotes the angle that the plane MOz that contains the material point M makes with the plane xOz.

More generally, if one supposes that Oz coincides with the axis of the virtual helicoidal displacement then the virtual work done by the forces F' under that displacement will be equal to:

$$\delta c \sum Z' + \delta\theta \sum (xY'-yX') .$$

Application to the motion of a free solid. – The position of a solid body (whether that body is continuous or composed of a finite number of points) depends upon six parameters: For example, the coordinates ξ, η, ζ of the center of gravity G of the solid and the three Euler angles θ, φ, ψ that define the direction cosines of the three principal axes of inertia of the solid relative to the point G. The active forces (F') coincide with the external forces for that system.

First express the *vis viva* as a function of those six parameters. We know that we have:

$$2T = M(\xi'^2 + \eta'^2 + \zeta'^2) + A\,p^2 + B\,q^2 + C\,r^2 . \tag{3}$$

M is the total mass of the solid, A, B, C are the moments, and $G\alpha\beta\gamma$ are the principal axes of inertia of the solid relative to the point G. p, q, r are given by the formulas:

$$p = \psi' \sin\theta \sin\varphi + \theta' \cos\varphi, \quad q = \psi' \sin\theta \cos\varphi - \theta' \sin\varphi, \quad r = \psi' \cos\theta + \varphi' .$$

If we set $\xi = q_1$ then the first Lagrange equation can be written:

$$\frac{d}{dt} M\,\xi' = Q_1 .$$

Furthermore, one has $Q_1\ \delta q_1 = \sum X' \cdot \delta\xi$. One will then have:

$$M\,\xi'' = \sum X',$$
$$M\,\eta'' = \sum Y',$$
$$M\,\zeta'' = \sum Z'.$$

Now construct the equation that relates to the parameter φ. One finds from (3) that:

$$\frac{\partial T}{\partial \varphi'} = C\,r',$$

$$\frac{\partial T}{\partial \varphi} = A\rho\frac{\partial p}{\partial \varphi} + B\,q\frac{\partial q}{\partial \varphi} = A\,p\,(\psi'\sin\theta\cos\varphi - \theta'\sin\varphi) + B\,q\,(-\psi'\sin\theta\sin\varphi - \theta'\cos\varphi)$$
$$= (A - B)\,p\,q\,.$$

Furthermore, the virtual displacement $\delta\theta$ is a rotation around the axis of inertia $G\gamma$. If one lets N denote the moment of the forces (F') with respect to $G\gamma$ then the Lagrange equation can be written:

(4) $$C\frac{dr}{dt} + (B - A)\,p\,q = N\,.$$

That equation coincides with the third Euler equation. The Lagrange equations that relate to θ and ψ will be combinations of the Euler equations. However, can deduce the first two Euler equations from equation (4) by permutation. One thus, in fact, recovers the equations for the motion of a solid that were studied before.

First integrals of the Lagrange equations. – When the system is a free solid, one will know one integral of the equations of motion.

1. When the projection of the geometric sum of the forces (F') onto a fixed axis is zero or constant.

2. When the system of forces (F') has a zero or constant moment relative to a fixed axis, say *Oz*, or to a fixed direction that passes through the center of gravity, say GZ'.

One will further obtain a first integral quite easily if the work done by the forces (F') is zero for a well-defined helicoidal displacement of the solid no matter what the position of the solid and the instant considered. I intend the term *well-defined helicoidal displacement* to mean a displacement for which each point *x*, *y*, *z*, when regarded as a point of the solid, is subjected to a well-defined displacement. The axis of the helicoidal displacement is a fixed line L. The displacement (v) δt of a point on that axis and the instantaneous of virtual rotation (ω) are two constant segments that have L for their line of action. In order for the stated condition to be fulfilled, it is necessary and sufficient there should exist constant values of δa, δb, δc, δu, δv, δw, such that one will have:

$$\delta a \sum X' + \delta b \sum Y' + \delta c \sum Z' + \delta u \sum (y Z' - z Y') + \delta v \sum (z X' - x Z') + \delta w \sum (x Y' - y X') \equiv 0 ,$$

no matter what the instant considered, or the positions and velocities of the points of the solid. Having fulfilled that condition, one can take the line L to be the z-axis, and write down the equality:

$$\delta c \sum m \frac{d^2 z}{dt^2} + \delta w \sum m \left(x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right) = \delta c \sum Z' + \delta w \sum (y X' - x Y') = 0 ,$$

or rather, upon setting $\delta w / \delta c = h$:

$$M \frac{d^2 z}{dt^2} + h \frac{d}{dt} \sum m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = 0 ,$$

and thus one will have the integral:

$$M \frac{dz}{dt} + h \sum m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = \text{const.}$$

Those equalities can apply to the motion of an arbitrary system in certain cases. More generally, if one applies the theorem on the projection of the quantities of motion to an arbitrary system then one will obtain an equality such as:

(4) $$\sum m \frac{d^2 x}{dt^2} = M \frac{d^2 \xi}{dt^2} = \sum X' + \sum R_x ,$$

in which one lets R_x denote the component of one of the external reactions (i.e., the force exerted by the elements external to the system) along Ox.

The sum $\sum R_x$ will be zero only in some particular cases.

Similarly, the theorem of the moment of the quantities of motion, when applied to the axis Oz or to the axis OZ', for example, will lead to an equation that is independent of the external reactions only if those reactions have a zero moment with respect to that axis.

That is what happens in the following special cases:

1. The system admits a translation parallel to a fixed axis (say, *Oz*) for a virtual displacement (for an arbitrary position and instant). From d'Alembert's principle, one has:

$$\sum m \frac{d^2 x}{dt^2} = \sum X'.$$

That equation must be a consequence of the Lagrange equations, which are equivalent to d'Alembert's principle. It indeed coincides with the equality (4), because the sum $\sum R_x$ is zero since the virtual work $\delta x \sum R_x$ is zero, by hypothesis.

In order for the system to enjoy that kinematical property, it is necessary and sufficient that the equations of constraint should include the coordinates x_j of the points M_j of the system only by their differences $x_j - x_k$. That is the case for a point that moves on a cylinder whose generators are parallel to *Ox*.

If the sum $\sum X'$ is zero, in addition, then one will get an integral of the Lagrange equations:

$$\sum m x = M \xi = a t + b.$$

It even suffices that $\sum X'$ should be a constant or a simple function of *t*.

2. The system that admits a collective rotation around a fixed axis (say *Oz*) or a fixed direction that passes through the center of gravity (say GZ') for a virtual displacement (for an arbitrary position and instant). One can then write the equality:

$$\frac{d}{dt} \sum m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = \sum (x Y' - y X') \qquad \text{(for the axis } Oz\text{)}$$

or the equality:

$$\frac{d}{dt} \sum m \left(x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right) = \sum (x' Y' - y' X') \qquad \text{(for the axis } GZ'\text{)},$$

in which $x' = x - \xi$, $y' = y - \eta$.

In order for the system to enjoy that kinematic property, it is necessary and sufficient that the equations of constraint, in which *x*, *y* (or rather x', y') are expressed with the aid of polar coordinates *r* and θ, should include the angles θ_j only by their differences ($\theta_j - \theta_k$). That is the case for a point that moves on a surface of revolution around *Oz*.

If the moment of the forces (F') with respect to *Oz* (or to GZ') is zero or constant or a function of only *t* then one will obtain an integral of the Lagrange equations.

3. The system admits a well-defined helicoidal displacement as a virtual displacement (for an arbitrary position and instant). If one takes the axis to be the z-axis of the displacement then one can write down the equality:

$$M\frac{d^2\zeta}{dt^2}+h\frac{d}{dt}\sum m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=\sum Z'+h\sum(xY'-yX'),$$

in which h denotes a certain constant.

In order for that system to enjoy that kinematical property, it is necessary and sufficient that the equations of constraint should include the last two of the coordinates θ, z only by their differences $\theta_j - \theta_k$, $z_j - z_k$, $z_j - h\ \theta_k$. That is the case for a point that moves on a helicoid that is skew to the director plane of the axis Oz.

If, in addition, the quantity $\sum Z'+h\sum(xY'-yX')$ is zero then one will obtain a first integral:

$$M\frac{d\zeta}{dt}+h\sum m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=\text{const.}$$

It will even suffice that $\sum Z'+h\sum(xY'-yX')$ is a constant or a simple function of t.

Case in which T does not include one of the parameters q explicitly. – If the parameter q_j does not appear explicitly in the function T then the Lagrange equation that relates to q_j will be written:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_j}\right)=Q_j\ .$$

That will imply a first integral of the motion whenever Q_j is zero, constant, or simply a function of t.

The first integrals that then come to light will often coincide with the ones that we pointed out above.

For example, let a point M move on the cylindrical surface $z = f(y)$. Set $x = q_1$, $y = q_2$. The expression for T is:

$$T=\tfrac{1}{2}m[x'^2+y'^2(1+f_y'^{\,2})]\ .$$

The Lagrange equation that relates to x is:

$$m x'' = X',$$

so if X' is zero then:

$$x = a\,t + b\ .$$

That equality coincides with the one that one obtains by applying the theorem on the projection of the quantity of motion onto the axis *Ox*.

Similarly, let *M* be a point that moves on a surface of revolution $z = \varphi(r)$. Set $r = q_1$, $\theta = q_2$. The expression for *T* is:

$$T = \tfrac{1}{2} m[r'^2(1+\varphi_r'^2) + r^2\theta'^2] .$$

Hence, one has the equality:

$$\frac{d}{dt} m r^2 \theta' = xY' - y X',$$

and if $xY' - y X'$ is zero then:

$$m r^2 \theta' = \text{const.}$$

That equality coincides with the one that one obtains by applying the theorem on the moment of the quantity of motion to *Oz*.

Finally, let *M* be a point that moves on a helicoid with a director plane $x = r \cos \theta$, $y = r \sin \theta$, $z = k\,\theta$. Let $r = q_1$, $\theta = q_2$. The expression for *T* will become:

$$T = \tfrac{1}{2} m[r'^2 (r^2 + k^2)\theta'^2] .$$

Hence, one has the equality:

$$\frac{d}{dt} m (r^2 + k^2)\theta' = xY' - y X' + k Z',$$

and if $xY' - y X' + k Z'$ is zero then:

$$m (r^2 + k^2)\theta' = \text{const.}$$

That equality coincides with the one that one obtains by applying the remark on the helicoidal displacement to a system.

However, the integral that presents itself most frequently in the applications is the *vis viva*.

***Vis viva* integral.** – When the constraints are independent of time, we said that the real displacement of the system will coincide with a virtual displacement (no matter what the position of the system and the instant considered): That is not exactly true when the constraints depend upon time. For a particular virtual displacement *dx*, *dy*, *dz* (when the system is supposed to be frictionless), d'Alembert's principle gives:

$$\sum m\left(\frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy + \frac{d^2z}{dt^2}dz\right) = \sum (X'\,dx + Y'\,dy + Z'\,dz) ,$$

or rather:

$$d \sum \tfrac{1}{2} m v^2 = \sum (X'\,dx + Y'\,dy + Z'\,dz) .$$

When the constraints are independent of time, one can always choose the parameters $q_1, \ldots, q_k$ in such a fashion that the x, y, z are expressed as functions of those parameters without t entering into them explicitly. Under those conditions, one will have:

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \cdots + \frac{\partial x}{\partial q_k} dq_k \ , \quad \text{etc.,}$$

and

$$\sum (X'\,dx + Y'\,dy + Z'\,dz) = Q_1\,dq_1 + Q_2\,dq_2 + \ldots + Q_k\,dq_k \ .$$

The equality of *vis viva* is then written:

$$\frac{d}{dt} \sum \tfrac{1}{2} m v^2 = Q_1\,q_1' + Q_2\,q_2' + \cdots + Q_k\,q_k' \ .$$

That equality, which is a consequence of d'Alembert's principle, must be a consequence of the Lagrange equations. One easily verifies that in the following manner: In the case that concerns us, T is a homogeneous second-degree polynomial in the q'. One calculates the quantity $Q_1\,q_1' + Q_2\,q_2' + \ldots + Q_k\,q_k'$ from the Lagrange equations. One finds that:

$$\begin{aligned} Q_1\,q_1' + Q_2\,q_2' + \cdots + Q_k\,q_k' &= q_1' \frac{d}{dt}\frac{\partial T}{\partial q_1'} + \cdots + q_k' \frac{d}{dt}\frac{\partial T}{\partial q_k'} - q_1' \frac{\partial T}{\partial q_1} - \cdots - q_k' \frac{\partial T}{\partial q_k} \\ &= \frac{d}{dt}\left(q_1' \frac{\partial T}{\partial q_1'} + \cdots + q_k' \frac{\partial T}{\partial q_k'} \right) - q_1'' \frac{\partial T}{\partial q_1'} - \cdots - q_k'' \frac{\partial T}{\partial q_k'} - q_1' \frac{\partial T'}{\partial q_1'} - \cdots - q_k' \frac{\partial T'}{\partial q_k'} \ . \end{aligned}$$

From Euler's theorem: $q_1' \frac{\partial T}{\partial q_1'} + q_2' \frac{\partial T}{\partial q_2'} + \cdots + q_k' \frac{\partial T}{\partial q_k'} = 2T$. On the other hand, T does not contain t explicitly, so:

$$\frac{dT}{dt} = \frac{\partial T}{\partial q_1'} q_1'' + \cdots + \frac{\partial T}{\partial q_k'} q_k'' + \frac{\partial T}{\partial q_1} q_1' + \cdots + \frac{\partial T}{\partial q_k} q_k' \ .$$

From that:

$$Q_1\,q_1' + Q_2\,q_2' + \cdots + Q_k\,q_k' = 2\frac{dT}{dt} - \frac{dT}{dt} = \frac{dT}{dt} \, .$$

That supposes essentially that the constraints are independent of time: Otherwise, the work done by the reactions would enter into the variation of the *vis viva*.

That equation for the *vis viva* provides an integral for the equations of motion whenever the quantity $Q_1\,dq_1 + \ldots + Q_k\,dq_k$ is the exact total differential of the same function U of $q_1, q_2, \ldots, q_k$. One can then write:

$$T = U + h \ .$$

In order for $Q_1, Q_2, \ldots, Q_k$ to be the partial derivatives of the same function $U(q_1, q_2, \ldots, q_k)$, as one knows, it is necessary and sufficient that those quantities should depend upon only $q_1, q_2, \ldots, q_k$ and satisfy the $k(k-1)/2$ conditions:

$$\frac{\partial Q_j}{\partial q_k} = \frac{\partial Q_k}{\partial q_j} .$$

Those conditions are always fulfilled whenever the quantity $\sum (X'\,dx + Y'\,dy + Z'\,dz)$ is the exact total differential of a function $u(x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$. However, the latter condition is not necessary.

In the case where there exists a force function $U(q_1, q_2, \ldots, q_k)$, the Lagrange equations will be written:

$$\frac{d}{dt}\frac{\partial T}{\partial q'_j} - \frac{\partial (T+U)}{\partial q_j} = 0 .$$

One can give them the same form whenever the Q_j (which are functions of the q, the q', and t) are the derivatives of the same function $U(q_1, \ldots, q_k, q'_1, \ldots, q'_k, t)$ with respect to the q_j :

$$Q_j(q_1, \ldots, q_k, q'_1, \ldots, q'_k, t) = \frac{\partial}{\partial q_j} U(q_1, \ldots, q_k, q'_1, \ldots, q'_k, t) ,$$

but one can no longer write the *vis viva* integral then.

That is especially the case when the constraints depend upon time, so the sum $\sum (X'\,dx + Y'\,dy$ $Z'\,dz)$ is the total differential of a function $u(x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$. The Q_j are then the derivatives of a function U that depends upon q_j and t with respect to the q_j .

However, it happens that the theorem of *vis viva* provides another *vis viva* integral in a great number of cases in which the forces depend upon time.

In order for one to be able to write out the *vis viva* integral, it is indeed necessary and sufficient that one should have (when the constraints do not depend upon time):

$$Q_1\, q'_1 + Q_2\, q'_2 + \ldots + Q_k\, q'_k \equiv \frac{dU}{dt} ,$$

in which U is a function of $q_1, q_2, \ldots, q_k, t$ that can depend upon the q'. In other words, dU / dt can depend upon the q''.

That condition can be further written:

$$Q_1\, q'_1 + Q_2\, q'_2 + \ldots + Q_k\, q'_k \equiv \frac{\partial U}{\partial q_1} q'_1 + \frac{\partial U}{\partial q_2} q'_2 + \cdots + \frac{\partial U}{\partial q_k} q'_k + \frac{\partial U}{\partial t}$$

or

$$Q_1 \equiv \frac{\partial U}{\partial q_1} + Q_1', \qquad Q_2 \equiv \frac{\partial U}{\partial q_2} + Q_2', \qquad \ldots, \qquad Q_k \equiv \frac{\partial U}{\partial q_k} + Q_k',$$

with the condition:

$$Q_1\, q_1' + Q_2\, q_2' + \ldots + Q_k\, q_k' \equiv \frac{\partial U}{\partial t}.$$

If the Q_j do not depend upon the q_i' then those equalities will demand that $Q_1' = Q_2' = \ldots = Q_k' = \partial U / \partial t \equiv 0$. The same thing is true then the Q_j depend upon the q'. In particular, in a great number of cases, it can happen that those conditions are verified when $\partial U / \partial t$ is zero, i.e., that one has:

$$Q_1 \equiv \frac{\partial U}{\partial q_1} + Q_1', \qquad \ldots, \qquad Q_k \equiv \frac{\partial U}{\partial q_k} + Q_k',$$

with the condition that:

$$Q_1\, q_1' + Q_2\, q_2' + \ldots + Q_k\, q_k' \equiv 0.$$

U denotes a function of $q_1, q_2, \ldots, q_k$.

Suppose, for example, that the force (F') is the geometric sum of two forces (F_1') and (F_2'): The forces (F_1') admit a force function $u\,(x_1, y_1, \ldots, z_n)$, and each force F_2' is normal to the velocity of the material point to which it is applied:

$$x'\,X_2' + y'\,Y_2' + z'\,Z_2' = 0.$$

Now, in that case:

$$\sum (X'\,x' + Y'\,y' + Z'\,z') = \frac{du}{dt} = \frac{dU}{dt}(q_1, q_2, \ldots, q_k),$$

and

$$T = U + h.$$

That remark finds an application in the study of the relative motion of the system. Let M be a point of the system Σ, and let (F') be the force that is exerted on it relative to the axes $Oxyz$. One can study the motion of Σ with respect to the axes $O_1\,x_1\,y_1\,z_1$, which are animated with respect to $Oxyz$ by a given motion. The active force (F_r') relative to the axes $O_1\,x_1\,y_1\,z_1$ is equal to $(F') - m(\Gamma_a) - m(\Gamma_c)$. The force $(F_c) = m\,(\Gamma_c)$ is normal to the relative velocity of the point M. If the coordinates x_1, y_1, z_1 are expressed as functions of the q without t entering explicitly, and if the forces $(F') - m\,(\Gamma_c)$ admit a force function $U\,(q_1, \ldots, q_k)$, while T_1 denotes the relative semi-*vis viva* then the integral of the applied *vis viva* will be $T_1 = U + h$.

Thus, let M be a massive point that moves without friction on a surface that turns with a uniform motion around a vertical axis Oz. If one keeps the axes $Oxyz$ then the constraint will depend upon time, and one cannot employ the *vis viva* theorem.

However, take axes $O_1\, x_1\, y_1\, z_1$ that are coupled to the surface, and as a result, they will turn around Oz with the constant angular velocity ω.

Suppose that $T_1 = \frac{1}{2} m(x_1'^2 + y_1'^2 + z_1'^2)$ and $U = [-g\,z + \frac{1}{2}\omega^2(x_1'^2 + y_1'^2)]$. One has $T_1 = U + h$.

That process will be useful when one can make the constraints independent of time by a change of axes. However, we shall return to the theory of relative motion in the next lecture.

Case in which the constraints depend upon time. – One can form an integral that is analogous to the *vis viva* integral in certain cases where the constraints depend upon time. First of all, if T has the form $T = T_2 + T_0$, where T_0 depends upon only t and T_2 does not, then one will have:

$$\frac{dT_2}{dt} = Q_1\, q_1' + \cdots + Q_k\, q_k' ,$$

which will give the integral $T_2 = U + h$ if the right-hand side $Q_1\, q_1' + \cdots + Q_k\, q_k'$ is equal to dU / dt.

More generally, let $T = T_2 + T_1 + T_0$. One can write:

$$Q_1\, q_1' + \cdots + Q_k\, q_k' = \frac{d}{dt}\left(q_1 \frac{\partial T}{\partial q_1'} + \cdots + q_k \frac{\partial T}{\partial q_k'}\right) - q_1'' \frac{\partial T}{\partial q_1'} - \cdots - q_k'' \frac{\partial T}{\partial q_k'} - q_1' \frac{\partial T}{\partial q_1} - \cdots - q_k' \frac{\partial T}{\partial q_k} .$$

From Euler's theorem:

$$q_1' \frac{\partial T}{\partial q_1'} + \cdots + q_k' \frac{\partial T}{\partial q_k'} = 2T_2 + T_1 .$$

As a result, that will give:

$$Q_1\, q_1' + \cdots + Q_k\, q_k' = 2\frac{dT_2}{dt} + \frac{dT_1}{dt} - \frac{dT_2}{dt} - \frac{dT_1}{dt} - \frac{dT_0}{dt} + \frac{\partial T}{\partial t} ,$$

or rather:

$$\frac{d}{dt}(T_2 - T_0) = Q_1\, q_1' + \cdots + Q_k\, q_k' - \frac{\partial T}{\partial t} .$$

If the quantity $Q_1\, q_1' + \cdots + Q_k\, q_k' - \frac{\partial T}{\partial t}$ is equal to $\frac{d}{dt} U(q_1, \ldots, q_k, t)$ then one will have:

$$T_2 - T_0 = U + h .$$

That is what happens, for example, when $\partial T / \partial t$ depends upon only t and $Q_1\, dq_1 + \ldots + Q_k\, dq_k$ is an exact total differential $dU(q_1, \ldots, q_k)$.

In that case, the integral is written:

$$T_2 - T_0 = U + F(t) + h ,$$

so $\partial T / \partial t$ is equal to $F'(t)$.

As an application, we treat the following problem:

A massive point M moves without friction on a cylinder of revolution with a vertical axis that dilates in proportion to time while remaining homothetic to itself.

We can write:

$$x = R\,t\cos\frac{s}{t}, \quad y = R\,t\sin\frac{s}{t}, \quad z = z\,,$$

in which R is a constant. As a result:

$$T = \tfrac{1}{2}m v^2 = \tfrac{1}{2}m\left[R^2\left(s' - \frac{s}{t}\right)^2 + R^2 + z'^2\right].$$

Apply the equality:

$$\frac{d}{dt}(T_2 - T_0) = Q_1\,q_1' + Q_2\,q_2' - \frac{\partial T}{\partial t}.$$

Here $q_1 = s$, $q_2 = r$. That will give:

$$\frac{d}{dt}\tfrac{1}{2}m\left(R^2 s'^2 + z'^2 - R^2\frac{s^2}{t^2}\right) = -\,m\,g\,z' - m\,R^2\left(\frac{s\,s'}{t^2} - \frac{s^2}{t^3}\right) = -\,m\,g\,z' - \tfrac{1}{2}m\,R^2\,\frac{d}{dt}\frac{s^2}{t^2}.$$

Therefore:

$$R^2\,s'^2 + z'^2 - R^2\,\frac{s^2}{t^2} = -\,2g\,z - R^2\,\frac{s^2}{t^2} + h$$

or

$$R^2\,s'^2 + z'^2 = -\,2g\,z + h\,.$$

Furthermore, the Lagrange equation that relates to z will give:

$$z'' = -\,g\,, \qquad \text{so} \qquad z = -\,\tfrac{1}{2}g\,t^2 + at + b,$$

and as a result, $R^2\,s'^2 = k$, $s = ct + d$. (a, b, c, d are constants.)

If g is zero then one will see that $z = \alpha\,s + \beta$.

One will arrive at the same results by expressing x, y, z as functions of $\theta = q_1$, $z = q_2$ and using the formulas:

$$x = R\,t\cos\theta, \quad y = R\,t\sin\theta, \quad z = z\,,$$

and upon writing out the Lagrange equations that relate to θ and z, in which:

$$T = \tfrac{1}{2} m[R^2(t^2\,\theta'^2 + 1) + z'^2]$$

here, so one has the equations:

$$\frac{d}{dt} m\,R^2\,t^2\,\theta' = 0\,,$$

$$\frac{d}{dt} m\,z' = -\,m\,g\,,$$

which give:

$$z = -\,\tfrac{1}{2} g\,t^2 + at + b, \quad \text{and} \quad t^2\frac{d\theta}{dt} = \text{const.}, \qquad \text{or} \qquad \theta = c + \frac{d}{t}\,,$$

which indeed coincide with the equations that were obtained above.

Case in which the parameters q are not independent. – In the foregoing, we supposed that the coordinates x, y, z were expressed as functions of the least possible number of parameters. In certain cases, it is convenient to express x, y, z as functions of a larger number of parameters that are coupled by certain relations. For example, one can study the motion of a point on a surface by appealing to polar coordinates in space r, θ, φ, which are then restricted by one relation $f(r, \theta, \varphi) = 0$.

Hence, let $q_1, q_2, \ldots, q_k$ be k distinct parameters in terms of which the x, y, z are expressed as functions:

$$x = \varphi(q_1, q_2, \ldots, q_k, t)\,, \qquad y = \psi(q_1, q_2, \ldots, q_k, t)\,, \qquad z = \chi(q_1, q_2, \ldots, q_k, t)\,,$$

and let:

$$(\alpha) \qquad \begin{cases} f_1(q_1, q_2, \ldots, q_k, t) = 0\,, \\ f_2(q_1, q_2, \ldots, q_k, t) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_k(q_1, q_2, \ldots, q_k, t) = 0 \end{cases}$$

be p distinct relations to which those parameters are subjected. One can infer p of the quantities q_j as functions of the other $k - p$ from those p relations. In other words, at least one of the determinants δ that are obtained by suppressing $k - p$ columns in the matrix:

$$\begin{matrix} \dfrac{\partial f_1}{\partial q_1} & \dfrac{\partial f_1}{\partial q_2} & \cdots & \dfrac{\partial f_1}{\partial q_k} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_p}{\partial q_1} & \dfrac{\partial f_p}{\partial q_2} & \cdots & \dfrac{\partial f_p}{\partial q_k} \end{matrix}$$

is not identically zero.

Under those conditions, the position of the system will depend upon $(k - p)$ distinct parameters. A virtual displacement of the system is a displacement for which the variations $\delta q_1, \ldots, \delta q_k$ of the $q_1, \ldots, q_k$ satisfy the p relations:

$$\frac{\partial f}{\partial q_1}\delta q_1 + \frac{\partial f}{\partial q_2}\delta q_2 + \cdots + \frac{\partial f}{\partial q_k}\delta q_k = 0 \qquad (f = f_1, f_2, \ldots, \text{or } f_p) .$$

The virtual work done by a system of forces X, Y, Z that is applied to the points (x, y, z) of the system has the form:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = Q_1\;\delta q_1 + Q_2\;\delta q_2 + \ldots + Q_k\;\delta q_k .$$

However, in order for the work done to be zero under an arbitrary virtual displacement, it is no longer necessary that the k coefficients Q_j should be zero. It is necessary and sufficient that one should have:

$$Q_1 = \lambda_1 \frac{\partial f_1}{\partial q_1} + \cdots + \lambda_p \frac{\partial f_p}{\partial q_1} ,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$Q_k = \lambda_1 \frac{\partial f_1}{\partial q_k} + \cdots + \lambda_p \frac{\partial f_p}{\partial q_k} .$$

That proposition is proved in absolutely the same way as the analogous proposition in Lecture 4.

With that, let (X, Y, Z) be the total forces, and let (X', Y', Z') be the active force that is exerted on each point (x, y, z) of a frictionless system. Set:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = Q_1\;\delta q_1 + Q_2\;\delta q_2 + \ldots + Q_k\;\delta q_k ,$$

$$\sum (X'\,\delta x + Y'\,\delta y + Z'\,\delta z) = Q_1'\,\delta q_1 + Q_2'\,\delta q_2 + \cdots + Q_k'\,\delta q_k .$$

One has:

$$Q_j = Q_j' + \lambda_1 \frac{\partial f_1}{\partial q_k} + \lambda_2 \frac{\partial f_2}{\partial q_k} + \cdots + \lambda_p \frac{\partial f_p}{\partial q_k} \qquad (j = 1, 2, \ldots, \text{or } k).$$

On the other hand, one deduces from d'Alembert's principle:

$$\sum m\left(\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + \frac{d^2z}{dt^2}\,\delta z\right) = \sum (X\,\delta x + Y\,\delta y + Z\,\delta z) ,$$

that:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_j'}\right)-\frac{\partial T}{\partial q_j} = Q_j \qquad (j = 1, 2, \ldots, \text{or } k).$$

Thus:

$$(\beta) \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_j'}\right)-\frac{\partial T}{\partial q_j} = Q_j' + \lambda_1\frac{\partial f_1}{\partial q_k}+\lambda_2\frac{\partial f_2}{\partial q_k}+\cdots+\lambda_p\frac{\partial f_p}{\partial q_k}.$$

Those k equations (β), when combined with the p constraint equations (α), will determine the k parameters q and the p indeterminates λ as functions of the time and the initial constraints. In order to see that rigorously, differentiate each equation of constraint twice with respect to t. That will give:

$$(\gamma) \qquad \frac{\partial f}{\partial q_k}q_1''+\cdots+\frac{\partial f}{\partial q_k}q_k''+\left(\frac{\partial f}{\partial q_k}q_1'+\frac{\partial f}{\partial q_k}q_2'+\cdots+\frac{\partial f}{\partial q_k}q_k'\right)_2 = 0 .$$

The system of ($k + p$) equations (β) and (γ), which is linear with respect to the q'' and the λ, can be solved for those variables. Indeed, the determinant of the (q'', λ) is the same as the determinant Δ of the (q', λ) in the homogeneous equations:

$$(\beta') \qquad \frac{\partial T_2}{\partial q_j'}-\lambda_1\frac{\partial f_1}{\partial q_k}-\lambda_2\frac{\partial f_2}{\partial q_k}-\cdots-\lambda_p\frac{\partial f_p}{\partial q_k} = 0 \qquad (j = 1, 2, \ldots, \text{or } k)$$

and

$$(\gamma') \qquad \frac{\partial f}{\partial q_k}q_1'+\frac{\partial f}{\partial q_k}q_2'+\cdots+\frac{\partial f}{\partial q_k}q_k' = 0 \qquad (f = f_1, f_2, \ldots, \text{or } f_p).$$

Now, Δ is not identically zero, since otherwise there would exist values of the q' and the λ that would not all be zero and verify the equations (β'), (γ'), and as a result, the equation:

$$q_1'\frac{\partial T_2}{\partial q_1'}+q_2'\frac{\partial T_2}{\partial q_2'}+\cdots+q_k'\frac{\partial T_2}{\partial q_k'} = 0$$

or

$$T_2 = 0 .$$

That can happen (for arbitrary $q_1, q_2, \ldots, q_k$) only if the q' are all zero, and as a result, all of the λ.

One will then infer the q_j'' and the λ as functions of t, the q_j, and q_j' the from equations (β) and (γ).

The differential equations (β), when combined with the constraint equations, will then determine the motion of the system when one knows the positions and the velocities of its points at the instant t_0 .

One can apply the *vis viva* theorem whenever x, y, z are expressed as functions of the q_j without the time entering explicitly into those expressions or the constraint equations (α). That theorem will give an integral when the quantity $Q_1' \, dq_1 + \cdots + Q_k' \, dq_k$ is the total differential of a function $U(q_1, q_2, \ldots, q_k)$, moreover. Nonetheless, one should observe that those conditions are not necessary.

LECTURE 8

APPLICATIONS OF THE LAGRANGE EQUATIONS

We shall now apply the Lagrange equations to the study of the motion of some particular systems. We make the following remark in regard to that subject: Whenever one perceives a simple combination of Lagrange equations, and in particular, an integrable combination, it will be appropriate to replace one of the Lagrange equations with that combination. In particular, whenever the *vis viva* theorem gives an integral, one might substitute that integral for one of the equations.

For example, let us study the motion of a point M that moves without friction on a surface of revolution and is subject to an active force that admits a force function U. Let:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = \chi(r)$$

be the coordinates of the point M ($r = q_1$, $\theta = q_2$). One has:

$$T = \tfrac{1}{2} m\left[(1+\chi'^2)\,r'^2 + r^2\,\theta'^2\right],$$

in which χ' denotes $d\chi / dr$ and the equations of motion are:

$$(1) \qquad \begin{cases} m\,r'' - m\,(\chi'\,\chi''\,r'^2 + r\,\theta'^2) = \dfrac{\partial U}{\partial r}, \\[2mm] \dfrac{d}{dt}\cdot m\,r^2\,\theta = \dfrac{\partial U}{\partial r}, \end{cases}$$

One can replace the first equation with the *vis viva* integral:

$$(2) \qquad \tfrac{1}{2} m\left[(1+\chi'^2)\,r'^2 + r^2\,\theta'^2\right] = U + b\,.$$

Suppose, moreover, that $\partial U / \partial\theta$ is zero: In order for that to be true, it is necessary and sufficient that the work done by (F') must be zero under the virtual displacement $\delta\theta$, and as a result, that F' must be in the plane zOM. The second equation is then integrated, and it will become:

$$(3) \qquad m\,r^2\,\theta' = \text{const.} \qquad \text{or} \qquad r^2\,\theta' = K,$$

which is an integral that one can also obtain by applying the moment theorem to the axis Oz.

Upon replacing θ' as a function of t in equation (2) using (3), one will see that t and θ are given as functions of r by a quadrature ([8]):

$$(4) \qquad \pm\, dt = \frac{r\sqrt{m(1+\chi'^2)}}{\sqrt{2(U+b)r^2 - mK^2}}\,dr\,, \qquad \pm\, d\theta = m\,\frac{K\sqrt{m(1+\chi'^2)}}{r\sqrt{2(U+b)r^2 - mK^2}}\,dr\,.$$

Nonetheless, observe that such transformations can introduce parasitic solutions that do not satisfy the original equations of motion. The new equations that one substitutes for the Lagrange equations are consequences of those equations, but the converse is not necessarily true.

Therefore, in the preceding case, equations (2) and (3) are equivalent to the Lagrange equations only if r' is non-zero.

More generally, there exists one and only one system of functions $q_j(t)$ that satisfy the Lagrange equations and the initial conditions $q_j(t_0) = q_j^0$, $q_j'(t_0) = q_j'^0$. The only exception is when the functions q_j'' of $q_1, q_2, \ldots, q_1', q_2', \ldots, q_k'$, and t that are defined by the Lagrange equations are not regular in the neighborhood of the values $q_1^0, \ldots, q_k^0, q_1'^0, \ldots, q_k'^0, t_0$ of the variables. In that case, we say that *initial conditions are singular*. With that, suppose that the system is released under initial conditions that are not singular. If the system of equations (1) that one substitutes for the Lagrange equations (1) admits only one system of integrals that satisfies the initial conditions then it will be clear that those integrals satisfy equations (1) and define the motion of the system. However, if equations (1′) admit several systems of integrals for given initial conditions then only one of those systems will satisfy equations (1). The other ones were introduced by the transformations in the calculation. Finally, when the initial conditions are singular, it is necessary to verify whether the integrals of equations (1′) are integrals of the Lagrange equations.

In order fix ideas, we return to the preceding example: Equations (4) are equivalent to equations (1) if r' is non-zero. Suppose that one has $r = r_0$, $r' = 0$, $\theta = \theta_0$, $\theta' = \theta_0'$ at the instant t_0, where r_0 is non-zero and does not correspond to a singular value of $U(r)$. r_0 is a root of $\Phi(r)$ or $2(U+b)r^2 - mK^2$. Equations (4) are verified by the two systems:

$$r(t) \equiv r_0\,, \qquad \theta = \theta_0 + \theta_0'(t - t_0)$$

or

$$t - t_0 = \int_{r_0}^{r} \frac{r\sqrt{m(1+\chi'^2)}}{\sqrt{2(U+b)r^2 - mK^2}}\,dr\,, \quad \theta - \theta_0 = \int_{r_0}^{r} \frac{K\sqrt{m(1+\chi'^2)}}{r\sqrt{2(U+b)r^2 - mK^2}}\,dr\,.$$

([8]) This is a special case of a general proposition that we prove as follows: Let S be a system without friction whose constraints are independent of time and whose position depends upon two parameters. If the active forces that are exerted on the system depend upon neither time nor velocity then knowing two first integrals (which do not contain t explicitly) will always permit one to reduce the equations of motion of the system to quadratures.

In those equalities, K is equal to $r_0^2\,\theta_0'$, and the sign that one takes for the coefficient of dr is positive if the derivative of $(U+b)\,r^2$ is positive for $r = r_0$ and negative in the contrary case. One of those systems of integrals will not verify the original equations of motion: It is necessarily the first one for which r' is zero. Nonetheless, if r_0 is a multiple root of $\Phi\,(r)$ then the equations will admit no other integrals that necessarily satisfy equations (1) than:

$$r\,(t) \equiv r_0\,, \qquad \theta = \theta_0 + \theta_0'\,(t - t_0)\,.$$

That can be verified, moreover, by differentiating the first equation in (4) with respect to t :

$$\left(\frac{dr}{dt}\right)^2 = \frac{\Phi\,(r)}{m\,r^2(1+\chi'^2)} = V\,(r)\,.$$

Upon dividing both sides by r', that will give:

$$2\,r'' = V'\,(r)\,, \tag{4′}$$

and that equation, combined with equation (3), is equivalent to equations (1). In order for $r \equiv r_0$ to satisfy equation (4′), it is necessary that $V'(r_0)$ should be zero: $r \equiv r_0$ will then be a parasitic solution unless one has both $V\,(r_0)$ = and $V'(r_0) = 0$, or what amounts to the same thing $\Phi\,(r_0)$ = 0, $\Phi'(r_0) = 0$.

We shall further make this simple remark: Let S and S' be two frictionless systems that depend upon the same number of parameters, and whose motion under the action of given forces we will study. If we can choose the parameters in such a fashion that the expressions for T, Q_1, …, Q_k are identical for the two systems then it will be clear that the two problems will be equivalent. The motion of one of the two systems is deduced from the motion of the other. For example, let Σ and Σ' be two surfaces that are applicable to each other. One knows that one can choose the parameters q_1, q_2 in such a fashion that the ds^2 has the same expression for the two surfaces:

$$ds^2 = A_1^1\,dq_1^2 + 2A_1^2\,dq_1\,dq_2 + A_2^2\,dq_2^2\,.$$

If the two points M and M' (which have the same mass) move without friction on the surfaces Σ and Σ', respectively, and they are subject to an active force that admits a force function $U\,(q_1, q_2)$ that is the same for the two points then the motion of each of those points will be defined by the same equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} = \frac{\partial U}{\partial q_1}\,,$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_2'}\right) - \frac{\partial T}{\partial q_2} = \frac{\partial U}{\partial q_2},$$

in which:

$$T = \tfrac{1}{2} m (A_1^1 q_1'^2 + 2A_1^2 q_1' q_2' + A_2^2 q_2'^2).$$

For example, let us study the motion of a point that slides without friction on a skew helicoid with a director plane:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = K\,\theta.$$

We have:

$$ds^2 = (r^2 + K^2)\,\theta'^2 + r'^2.$$

On the other hand, the ds^2 of a surface of revolution:

$$\xi = \rho \cos \omega, \quad \eta = \rho \sin \omega, \quad \zeta = \chi(\rho)$$

will have the form:

$$ds^2 = (1 + \chi'^2)\,d\rho^2 + \rho^2 d\omega^2.$$

One can reduce the two ds^2 to the same form by setting:

$$\theta = \omega, \qquad r = F(\rho), \qquad \text{with} \quad F^2 + K^2 = \rho^2, \qquad 1 + \chi'^2 = F'^2.$$

One immediately deduces from those equations that:

$$\frac{d\chi}{d\rho} = \frac{K}{\sqrt{\rho^2 - K^2}},$$

so:

$$\frac{\chi}{iK} = \frac{\zeta}{iK} = \arccos \frac{\rho}{K} + \text{const.},$$

and upon annulling the constant (which amounts to changing the origin of ζ):

$$\rho = \tfrac{1}{2} K (e^{\zeta/K} + e^{-\zeta/K}),$$

i.e., the helicoid is applicable to the surface of revolution that is generated by a catenary that rotates around its base (or an *alysseide*).

On the other hand, we know that the equations of motion of a point on a surface of revolution are integrated by quadrature when the component of the active force that is tangent to the surface admits a force function $U(r)$ that does not depend upon θ. That property will then apply to the motion of a point on a skew helicoid with a director plane whenever there exists a force function

U whose only variable is *r*. In order for that to be true, it will suffice that the component of the active force that is tangent to the surface must point along the rectilinear generator that passes through the point *M* and be a function of only the distance from that point to the axis of the helicoid. For example, that is what happens when the point *M* is attracted (or repelled) by all of the elements pp' of the axis *Oz* in proportion to the their length PP' and in inverse proportion to R^n, where *R* denotes the distance *MP*, and *n* is a number that is greater than 1.

Furthermore, it is easy to reduce the equations of motion to quadratures immediately in the case that we are dealing with. Those equations are:

$$m\,r'' - mr\,\theta'^2 = \frac{\partial U}{\partial r},$$

$$\frac{d}{dt}[m(r^2 + K^2)\,\theta'] = 0 .$$

Replace the first one with the *vis viva* integral:

$$m\,[(r^2 + K^2)\,\theta'^2 + r'^2] = 2U + h ,$$

and integrating the second one will give:

$$(r^2 + K^2)\,\theta' = C .$$

In order to get *t*, it will suffice to infer θ' from the last equality and substitute it into the preceding one, and as a result, one will get θ as a function of *r* by a quadrature. In particular, if *U* is zero then the geodesics of the helicoid will be given by an elliptic quadrature: They correspond to the geodesics of the alysseide.

The integral $(r^2 + K^2)\,d\theta / dt$ = const. can also be obtained by applying the remark to the helicoidal virtual displacement. In analogy with the theorem of moments for surfaces of revolution, that remark will provide an integral of motion of a point on a helicoid of the most general type when the active force is normal to the helix on the surface that passes through the point *M*. Moreover, one knows that an arbitrary helicoid is applicable to a surface of revolution.

Similarly, let *M* be a point that moves on a cone. The cone is applicable to a plane. If one lets σ denote the arc-length of the spherical curve that is cut out by the cone on the sphere of radius 1, and one lets *r* denote the distance from *M* to the summit then one will have:

$$ds^2 = r^2\,d\sigma^2 + dr^2 .$$

Upon setting $\sigma = \theta$, one will see that the problem is the same as that of the motion a point in a plane that is referred to polar coordinates *r* and θ. Whenever there exists a force function *U*, which is a function of only *r*, the problem will be solved by quadratures.

The motion of a point on a cylinder likewise reduces to a planar motion. In order to put ds^2 into the form:

$$ds^2 = d\sigma^2 + dx^2 ,$$

it will suffice to let σ denote the arc-length of the orthogonal trajectory to the rectilinear generator of the cylinder, while x is the distance from the point M to a well-defined cross-section.

If one sets $\sigma = y$ then one will get the same equations that one gets for the motion in Cartesian coordinates x, y. Whenever there exists a force function U, and the component of (F') that is tangent to the cylinder makes a constant angle with the rectilinear generators, the problem can be solved by quadratures.

Notably, if (F') is normal to the generators then one of the first integrals can be obtained with the aid of the theorem of the projection of the quantities of motion when it is applied to the direction of those generators.

We shall not belabor these very simple and well-known applications any further, but we shall now study some examples of the motion of systems that are more complicated.

APPLICATIONS

I. – *Two points M and M_1 are constrained to slide without friction on two helices:*

$$x = R \cos \theta, \quad y = R \sin \theta, \quad z = K \theta \quad \text{and} \quad x_1 = R_1 \cos \theta_1 , \quad y_1 = R_1 \sin \theta_1 , \quad z_1 = K \theta_1 .$$

The two points repel each other in proportion to the distance between them. Find the motion of the system. Examine the special case in which the two helices reduce to two circles ($K = 0$).

The position of the system depends upon two parameters θ and θ_1 . The *vis viva* theorem gives one first integral. In order to obtain another one, observe that the virtual displacement $\delta\theta = \delta\theta_1$, which corresponds to the variations:

$$\delta x = - y \ \delta\theta, \quad \delta y = x \ \delta\theta, \quad \delta z = K \ \delta\theta, \quad \text{and} \quad \delta x_1 = - y_1 \ \delta\theta, \ \delta y_1 = x_1 \ \delta\theta, \quad \delta z_1 = K \ \delta\theta,$$

is a constant helicoidal displacement. The work done by the active forces (which are internal forces here) under that displacement is zero. One will then have:

$$\frac{d}{dt}\left[m\left(R^2 \frac{d\theta}{dt} + K \frac{dz}{dt} \right) + m_1 \left(R_1^2 \frac{d\theta_1}{dt} + K \frac{dz_1}{dt} \right) \right] = 0 ,$$

or rather:

$$m(R^2+K^2)\frac{d\theta}{dt}+m_1(R_1^2+K^2)\frac{d\theta_1}{dt} = 0 .$$

That integral can also be obtained with the aid of the Lagrange equations. Here, one has:

$$2T = m(R^2+K^2)\theta'^2+m_1(R_1^2+K^2)\theta_1'^2 .$$

On the other hand, the force function U is equal to $-\frac{1}{2}\mu r^2+h$, in which r denotes the distance MM_1, and μ is a certain constant (which is positive since the force F' is repulsive). Let us calculate r^2:

$$r^2 = R^2+R_1^2-2RR_1\cos(\theta_1-\theta)+K^2(\theta_1-\theta)^2 .$$

As a result:

$$U = -\nu\cos(\theta_1-\theta)+\tfrac{1}{2}\mu K^2(\theta_1-\theta)^2+\text{const.} \qquad (\nu=\mu R R_1) .$$

Set $\theta_1 - \theta = \varphi$ and replace the parameter θ_1 with the parameter φ. That will give:

$$2T = \theta'^2[mR^2+m_1R_1^2+K^2(m+m_1)]+m_1(R^2+R_1^2)[2\theta'\varphi'+\varphi'^2] ,$$

and

$$U = -\nu\cos\varphi+\tfrac{1}{2}\mu K^2\varphi^2+\text{const.}$$

The Lagrange equation that relates to θ is written:

$$\frac{d}{dt}\{[mR^2+m_1R_1^2+(m+m_1)K^2]\theta'+m_1(R^2+R_1^2)\varphi'\} = 0 ,$$

or rather:

$$(\alpha) \qquad [mR^2+m_1R_1^2+(m+m_1)K^2]\theta'+m_1(R^2+R_1^2)\varphi' = C ,$$

which is equivalent to the integral that was found above:

$$m(R^2+K^2)\theta'+m_1(R^2+R_1^2)\theta_1' = \text{const.}$$

Now substitute the *vis viva* integral:

$$(\beta) \qquad \theta'^2[mR^2+m_1R_1^2+(m+m_1)K^2]+m_1[K^2+K_1^2][2\theta'\varphi'+\varphi'^2] = \mu K^2\varphi^2-2\nu\cos\varphi$$

for the second Lagrange equation.

If we replace θ' with its value that is inferred from (α) then that will give:

$$(\gamma) \qquad \varphi'^2 = A\varphi^2 - 2B\cos\varphi + b\,,$$

in which b denotes a constant, and A, B are the coefficients:

$$A = \mu R^2 \left[\frac{1}{m(R^2+K^2)} + \frac{1}{m_1(R_1^2+K^2)}\right], \qquad B = \nu\left[\frac{1}{m(R^2+K^2)} + \frac{1}{m_1(R_1^2+K^2)}\right].$$

Equation (γ) will give us t as a function of φ by the quadrature:

$$t - t_0 = \int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{A\varphi^2 - 2B\cos\varphi + b}}\,,$$

and from equation (a), one will have:

$$\theta = ct - \frac{m_1(R_1^2+K_1^2)}{mR^2 + m_1R_1^2 + (m+m_1)K^2}\,\varphi + c' = c' + ct - a\varphi\,,$$

$$\theta_1 = ct + \frac{m(R^2+K^2)}{mR^2 + m_1R_1^2 + (m+m_1)K^2}\,\varphi + c' = c' + ct + (1-a)\varphi\,,$$

moreover, in which c, c' are constants, with $c = mR^2\,\theta_0' + m_1\,R_1^2\,\theta_{1,0}'$.

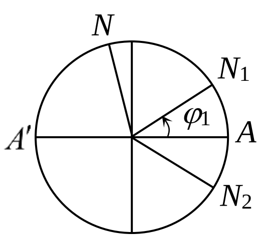
In particular, suppose that K is zero. The two helices reduce to two circles with their centers at O. Equation (α) will then become: $-\ mR^2\,\theta' + m_1\,R_1^2\,\theta_1' = c$, which is an integral that one could have likewise obtained by applying the theorem of moments to the axis Oz. t is given as a function of φ by the quadrature:

$$t - t_0 = \int_{\varphi_1}^{\varphi} \frac{d\varphi}{\sqrt{-2B\cos\varphi + b}}\,,$$

in which $B = \left[\frac{1}{mR^2} + \frac{1}{m_1\,R_1^2}\right]$. That is the equation that determines the motion of a pendulum of length $l = B / g$ if φ denotes the angle between the pendulum and the vertical that is drawn from bottom to top. (If the force F' is attractive then B will be negative, and φ will the angle between the pendulum and the direction of gravity). cos φ is a doubly-periodic function of t. One agrees to distinguish two cases according to whether the absolute value of b / 2B is or is not greater than 1, resp. In the former case, φ will always vary in the same sense, and it will vary by 2π during the time T:

$$T = \int_0^{2\pi} \frac{d\varphi}{\sqrt{b - 2B\cos\varphi}}\,.$$

The system will return to its initial conditions after a time T, except that the entire system will have turned about O through an angle of $\omega = c\ T \pm 2a\pi$ according to whether φ' is negative or positive, resp. On the contrary, if $| b / 2B | < 1$ then mark out the two extremities N_1, N_2 on the trigonometric circle of the angles φ_1, φ_2 whose cosines are equal to $b / 2B$, and the extremity N of the angle φ. N will traverse the arc $N_2A'N_2$ twice in opposite senses during a time $T = \int_{\varphi_1}^{2\pi-\varphi_1} \frac{d\varphi}{\sqrt{b-2B\cos\varphi}}$. The system will return to the same initial condition after a time T, except that it will have turned around O through angle of cT. In particular, if the system is released with no initial velocity then it will return to the same position with a zero velocity after a time $T = \int_{\varphi_0}^{2\pi-\varphi_0} \frac{d\varphi}{\sqrt{b-2B\cos\varphi}}$. Finally, when $| b / 2B | = 1$, φ will tend to 0 or π (according to whether $b/2B$ is equal to + 1 or – 1, resp.) when t increases indefinitely, and $\theta - c\ t$ will tend to 0 or $-a\pi$. However, in the case where $\varphi_0 = b / aB = \pm 1$, φ will remain constantly equal to 0 or to π, and θ and θ_1 will vary in proportion to time.



If the force (F') were attractive then nothing in the preceding discussion would change, except that under the hypothesis that $| b / 2B | < 1$, N will traverse the arc N_1AN_2 .

Let us now study some other systems that include continuous bodies. First of all, it is easy to apply the Lagrange equations to the problems that we have treated in Lecture Three. For example, in Problem III (see page 31), one will have (while preserving the same notation):

$$2T = \psi'^2[(A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta_0 + C\cos^2\theta_0] + 2C\psi'\theta'\cos\theta_0 + C\varphi'^2$$

and $U = 0$. The *vis viva* integral and the Lagrange equation that relates to ψ will provide us with two integrals that will allow us to study the motion.

Similarly, in Problem IV (see page 33), one has:

$$2T = [L\cos 2\theta + M\sin 2\theta]\theta'^2 + A\sin^2\alpha\,\theta'^2 + C[\psi' + \cos\alpha\,\theta']^2,$$

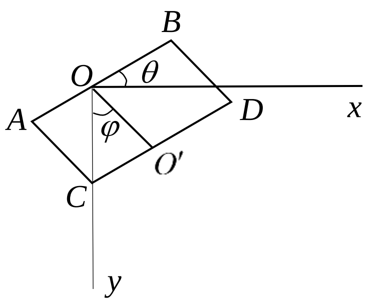
and

$$U = l\cos\theta + m\sin\theta,$$

in which L, M, A, C, l, m are well-defined coefficients (see page 34). The *vis viva* integral and the Lagrange equation that relate to ψ coincide with the equations that we shall appeal to. However, here are some new applications:

II. – *Two massive homogeneous bars AB, CD of equal length and the same density have their extremities A, C and B, D linked with two massless, flexible, inextensible strings. The midpoint O of AB is fixed. Find the motion of the system when it is released with zero initial velocity in the vertical plane xOy.*

The motion of the system takes place in the *xy*-plane. The position of the system in that plane depends upon two parameters, namely, φ, which is the angle that OO' makes with the direction Oy of gravity (O' is the midpoint of CD), and θ, which is the angle xOB between the horizontal Ox and AB.



The *vis viva* of *AB* is equal to:

$$2\int_0^l \rho\,\lambda^2\,\theta'^2\,d\lambda = \tfrac{2}{3}\rho l^3\,\theta'^2 = \tfrac{1}{3}M\,l^2\,\theta'^2,$$

in which $2l$ denotes the length of each bar, ρ is its density, and M is its mass.

On the other hand, the velocity of rotation is CD around its center of gravity O' is equal to θ', so the *vis viva of* CD is equal to:

$$M\,R^2\varphi'^2 + \tfrac{1}{3}M\,l^2\,\theta'^2 \qquad \text{if one sets } R = \overline{OO'}.$$

Therefore:

$$T = \tfrac{1}{3}M\,l^2\,\theta'^2 + \tfrac{1}{2}M\,R^2\varphi'^2.$$

As for U, one has:

$$U = Mg\,R\cos\varphi.$$

The Lagrange equation that relates to θ is:

$$\frac{d}{dt}\tfrac{2}{3}M\,l^2\,\theta'^2 = 0$$

or

$$\theta' = C,$$
$$\theta = C\,t + C'.$$

That integral results from the theorem of moments when it is applied to the axis Oz. The Lagrange equation that relates to φ is:

$$\frac{d}{dt}MR^2\varphi' = -\,Mg\,R\sin\varphi$$

or

$$\varphi'' = -\frac{g}{R}\sin\varphi.$$

One can replace that equation with the *vis viva* integral. However, that would be pointless since preceding equation is that of a pendular motion. The point O' moves like a pendulum whose rod is fixed at the point O and the bars AB, $A'B'$ turn around the points O and O', respectively, with a uniform rotational velocity.

III. – *Find the motion of a homogeneous sphere that is constrained to slide without friction on a skew helicoid with a horizontal director plane when no active force is exerted on the sphere.*

First of all, the motion of the sphere around its center of gravity G is a uniform rotation around an axis with a fixed direction. Let us then study the motion of the center of gravity.

In order to study that motion, we can ignore the *vis viva* of the sphere around G (which is constant) and consider only the expression:

$$T' = \tfrac{1}{2} M (\xi'^2 + \eta'^2 + \zeta'^2) .$$

Here, we have:

$$\xi = u \cos\psi + \frac{K R \sin\psi}{\sqrt{K^2 + u^2}} ,$$

$$\eta = u \cos\psi - \frac{K R \cos\psi}{\sqrt{K^2 + u^2}} ,$$

$$\zeta = K \psi + \frac{K u}{\sqrt{K^2 + u^2}}$$

(R is the radius of the sphere, K is the pitch of the helicoid). As a result:

$$T = \tfrac{1}{2} M \left[\psi'^2 \left(u^2 + K^2 + \frac{K^2 u^2}{K^2 + u^2} \right) + u'^2 \left(1 + \frac{K^2 u^2}{K^2 + u^2} \right) + 4 \psi' u' \frac{K R}{\sqrt{K^2 + u^2}} \right] .$$

Write down the Lagrange equation that relates to ψ:

$$\frac{d}{dt} \left[M \psi' \left(u^2 + K^2 + \frac{K^2 u^2}{K^2 + u^2} \right) + \frac{2M K R}{\sqrt{K^2 + u^2}} u' \right] = 0 ,$$

so one will have a first integral that one can also form by applying the remark to the helicoidal virtual displacement.

Combine that equation with the *vis viva* integral:

$$(\alpha) \qquad \psi'^2 \left(u^2 + K^2 + \frac{K^2 u^2}{K^2 + u^2} \right) + u'^2 \left(1 + \frac{K^2 R^2}{K^2 + u^2} \right) + 4 \psi' u' \frac{K R}{\sqrt{K^2 + u^2}} = b .$$

It will suffice for one to eliminate ψ' from that equality and the equality:

$$(\beta) \qquad \psi'\left(u^2+K^2+\frac{K^2u^2}{K^2+u^2}\right)+2u'\frac{KR}{\sqrt{K^2+u^2}} = C$$

in order to obtain t as a function of u by an elliptic quadrature:

$$(\alpha') \qquad \pm\, dt = \frac{du[(u^2+K^2)^2-K^2R^2]}{(u^2+K^2)\sqrt{h(u^2+K^2)-C^2(u^2+K^2)+hK^2R^2}} = \frac{du[(u^2+K^2)^2-K^2R^2]}{(u^2+K^2)\sqrt{F(u)}}.$$

As for ψ, it is given as a function of u by an elliptic quadrature and a logarithmic quadrature:

$$(\beta') \qquad d\psi = \frac{du[(u^2+K^2)^2-K^2R^2]}{(u^2+K^2)}\left[\pm\frac{C(u^2+K^2)^2-K^2R^2}{\sqrt{F(u)}}-2KR\sqrt{u^2+K^2}\right].$$

The point G moves on a surface Σ that is parallel to the helicoid like a free point and describes a geodesic of that surface with a constant velocity. One can, moreover, discuss the motion when one supposes that $R < K$.

Discussion. – The derivative du / dt can change sign then only if u attains a value that annuls $F(u)$. Set $u^2+K^2 = V$. The equation:

$$F_1(V) = bV^2 - C^2V + hK^2R^2 = 0$$

has roots V_1 and V_2 that are real or imaginary according to whether the positive number C^2/b is or is not greater than $2KR$. If C^2/b is found between R^2+K^2 and $2KR$ then those roots will be smaller than K^2. If C^2/b is greater than R^2+K^2 then K^2 will be separate V_1 from V_2. From that:

1. Suppose that $C^2/b < R^2+K^2$. $F(u)$ will not be annulled for any value of u. u will always vary in the same sense then, and its absolute value α will increase indefinitely with t. du / dt will tend to $\pm\sqrt{h}$ (according to the sign of u'_0). $d\psi / dt$ will keep a constant sign after a certain time interval and then decrease in absolute value indefinitely:

$$\frac{d\psi}{dt} = \frac{C}{u^2}(1+\varepsilon),$$

in which ε tends to zero when $|u|$ grows without bound. It follows from this that ψ will tend to a limit ψ_1. The point G goes to infinity along a branch of the curve that is asymptotic to the line:

$$\xi = u\cos\psi_1, \quad \eta = u\sin\psi_1, \quad \zeta = K\psi_1 + R.$$

It is appropriate to point out the particular case of $C^2 / b = 2\,K\,R$, in which the two roots r_1 and r_2 of $F_1(v)$ are equal. The double root is then equal to $+ K\,R$, and the equation (α') will give:

$$\sqrt{b}\,t = u + R \arctan \frac{u}{K} + \text{const.}$$

The motion will take place as before, moreover.

2. Suppose that $C^2 / b < R^2 + K^2$. $v_0 = u_0^2 + K^2$ is then greater than the largest root v_1 of $F_1(V)$. One can always suppose that $u_0 > 0$. If u_0' is positive then u will increase constantly and indefinitely as in the first case. If u_0' is negative then then u will decrease down to the value $u_1 = + \sqrt{v_1 - K^2}$ and then grow indefinitely. The derivative $d\psi / du$ is infinite for $u = u_1$. The trajectory will be tangent to the curve $u = u_1$.

3. Suppose that $C^2 / b = R^2 + K^2$. One will then have $v_1 = K^2$, $u_1 = 0$. $F(u)$ will contain u^2 as a factor. If u_0 is positive then u_0' will also be positive, and u will increase indefinitely. If u_0' is negative then u will decrease and tend to zero as t grows without bound. ψ' will tend to $C^2 / (R^2 + K^2)$. The projection of the trajectory onto the xy-plane will admit the circle of radius R whose center is at O as a circular asymptote. If $u_0' = 0$ then one will necessarily find oneself in the second or third case. In the second case, u' will necessarily increase. In the third case, one will have: $u \equiv u(0)$, while ψ will vary in proportion to time.

When R is greater than K, it will be impossible to pursue the discussion if u attains one of the values $\pm u_1 = \pm\sqrt{K(R-K)}$ that annul the denominator of du / dt. Suppose that one has $u = u_1$ for $t = t_1$. du / dt is infinite for $t = t_1$, and one has:

$$\pm (t - t_1) = (u - u_1)^2 \,[\alpha + (u - u_1)\,A]\,,$$

in which α is a positive number. One must take the + sign before $(t - t_1)$, and if one sets $\theta = \pm (t - t_1)^{1/2}$ then one will have:

$$u - u_1 = \theta\,[\alpha + \beta\,\theta + \ldots] = \pm (t - t_1)^{1/2}\,\varphi(t) + (t - t_1)\,\psi(t)\,.$$

Now the initial conditions do not permit one to determine the sign that one must take for $(t - t_1)^{1/2}$. There are two possible trajectories through the point G.

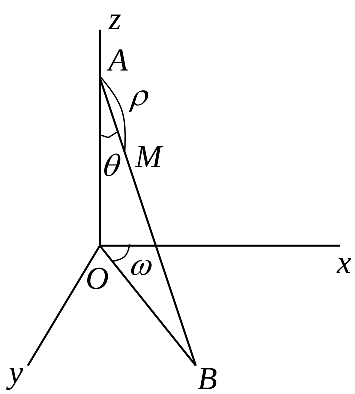
The helix $u = u_1$ of the surface Σ is a locus of singular points of that surface: That singularity in the motion will present itself, moreover, only when C^2 / b is smaller than $2\,K\,R$, i.e., if the roots of $F_1(v)$ are imaginary, because in the opposite case, $F_1(KR)$ would always be negative.

Here is another exercise in which the application of the *vis viva* theorem will encounter some difficulties.

IV. – *One extremity of a bar AB slides without friction on a vertical line OZ, while the other slides on a horizontal plane xOy. That bar is massive and homogeneous, and each of its elements is attracted to the point of intersection O of OZ and that plane in proportion to its mass and in inverse proportion to the square of the distance to that point. Find the motion of the bar.*

The position of *AB* depends upon two parameters, namely, $BOx = \omega$, $BAO = \theta$.

The *via viva* theorem and that of the quantities of motion, when applied to *Oz*, will give two first integrals. However, we shall obtain them by the Lagrange method.



Calculate the vis viva of the bar. – Let *l* be the length of the bar. *M* is an arbitrary point that is defined by $AM = \rho$.

The coordinates are:

$$x = \rho \sin \theta \sin \omega, \text{ so } x' = \rho[\cos\theta\cos\omega\theta' - \sin\theta\sin\omega\omega'],$$
$$y = \rho \sin \theta \sin \omega, \text{ so } y' = \rho[\cos\theta\sin\omega\theta' + \sin\theta\cos\omega\omega'],$$
$$z = (l - \rho) \cos \theta, \text{ so } z' = -(l-\rho)\sin\theta\cdot\theta'.$$

The semi-*vis viva* of the element $d\rho$ that is attained at the point *M* is:

$$\tfrac{1}{2}\{\rho^2(\cos^2\theta\cdot\theta'^2 + \sin^2\theta\cdot\omega'^2) + (l-\rho)^2\sin^2\theta\cdot\theta'^2\}\,d\rho,$$

in which one sets the density equal to unity.

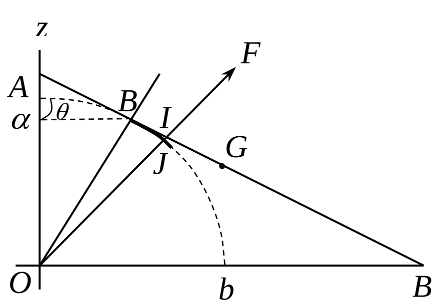
The semi-*vis viva* of the system is then:

$$\tfrac{1}{2}(\cos^2\theta\cdot\theta'^2 + \sin^2\theta\cdot\omega'^2)\int_0^l \rho^2 d\rho + \tfrac{1}{2}\sin^2\theta\cdot\theta'^2\int_0^l (l-\rho)^2\,d\rho\,.$$

Hence:

$$T = \tfrac{1}{6} l^3(\theta'^2 + \omega'^2\sin^2\theta)\,.$$

Calculating the virtual work done by the system. – Let $Q_1\,\delta\theta + Q_2\,\delta\omega$ be the virtual work done under a displacement $(\delta\theta, \delta\omega)$.



First recall a known theorem: The action of the bar *AB* on the point *O* under the stated conditions will be the same as the one that is exerted by an arc of a circle of the same density with its center at *O* that is tangent to *AB* and bounded by *OA* and *OB* when its elements are subject to the same law of attraction to *O*.

It results from this that by reason of symmetry, the actions of the point *O* on the elements of the bar *AB* have a resultant that applies to the point *I* where the bar *AB* meets the bisector of the angle *AOB* and points along *IO*. The point of application in the bar varies at each instant. As for the intensity, an easy calculation will give the value:

$$\frac{f\sqrt{2}}{OP},$$

in which *f* is the coefficient of attraction, or upon setting $f\sqrt{2} = \mu$:

$$\frac{\mu}{OP}.$$

Furthermore:

$$OP = OA \sin\theta = l \sin\theta \cos\theta.$$

Having assumed that, under a virtual displacement $\delta\omega$, the point *I* will displace normally to the attractive force *IO* while remaining fixed in the bar, and the center of gravity *G*, which is the point of application of the weight, will displace horizontally. The virtual work done under the displacement will then be zero, and one will have:

$$Q_2 = 0.$$

In order to evaluate Q_1, consider a displacement $\delta\theta$, and evaluate the virtual work done by weight and the attraction separately.

Weight. – The dimension of the center of gravity is $\frac{1}{2} l \cos\theta$. The virtual work done by weight will then be:

$$-M\,g\,\tfrac{1}{2} l \sin\theta \cdot \delta\theta,$$

i.e.:

$$-\tfrac{1}{2} g\, l^2 \sin\theta \cdot \delta\theta.$$

Attraction. – Calculate the displacement of the point *I*, which is considered to be fixed in the body. Let $AI = \lambda$, while ζ and ξ are the coordinates of *I* with respect to *OZ* and *OB*.

One has:

$$\xi = \lambda \sin\theta, \qquad \zeta = (l - \lambda)\cos\theta, \qquad \xi = \zeta\ (?).$$

Thus:

$$\delta\xi = \lambda \cos\theta \cdot \delta\theta, \qquad \delta\zeta = -(l - \lambda)\sin\theta \cdot \delta\theta.$$

The component $\delta\sigma$ of the displacement along *IO* is:

$$\delta\sigma = -(\delta\xi + \delta\zeta)\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}[(l-\lambda)\sin\theta - \lambda\cos\theta]\,\delta\theta,$$

i.e., upon taking into account the fact that one has:

$$\frac{\lambda}{\cos\theta} = \frac{l-\lambda}{\sin\theta} = \frac{l}{\sin\theta+\cos\theta} \;,$$

one will have:

$$\delta\sigma = \frac{l}{\sqrt{2}}\frac{\sin^2\theta-\cos^2\theta}{\sin\theta+\cos\theta}\,\delta\theta, \qquad\qquad \delta\theta = \frac{l}{\sqrt{2}}(\sin\theta-\cos\theta)\,\delta\theta.$$

The virtual work done by the attraction is:

$$\frac{\mu}{OP}\,d\sigma \;,$$

i.e.:

$$\frac{\mu}{\sqrt{2}}\left(\frac{1}{\cos\theta}-\frac{1}{\sin\theta}\right)\delta\theta\,.$$

Therefore, one finally has:

$$Q_1 = \tfrac{1}{2}\,g\,l^2\sin\theta+\frac{\mu}{\sqrt{2}}\left(\frac{1}{\cos\theta}-\frac{1}{\sin\theta}\right).$$

One remarks that Q_1 and Q_2 are the partial derivatives of the function U :

$$U = -\tfrac{1}{2}\,g\,l^2\cos\theta+\frac{\mu}{\sqrt{2}}\log\left[\frac{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{\tan\frac{\theta}{2}}\right]+\text{const.}$$

Equations of the problem. – One first has the *vis viva* integral:

$$T = U + h\;,$$

i.e.:

$$\tfrac{1}{6}\,l^3\,(\theta'^2+\omega'^2\sin^2\theta) = -\tfrac{1}{2}\,g\,l^2\cos\theta+\frac{\mu}{\sqrt{2}}\log\left[\frac{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{\tan\frac{\theta}{2}}\right]+h\;.$$

The Lagrange equation that relates to ω is:

$$\frac{d}{dt}\left(\tfrac{1}{6}\,l^3\,\omega'\sin^2\theta\right) = 0\;,$$

so:

$$\omega' \sin^2 \theta = \text{const.} = k .$$

That is also the area integral.

If one infers ω from that equation in order to substitute in the *vis viva* equation then that will give:

$$\tfrac{1}{6} l^3 \theta'^2 = -\tfrac{1}{2} g\, l^2 \cos\theta + \frac{\mu}{\sqrt{2}} \log\left[\frac{\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{\tan\frac{\theta}{2}}\right] + h ,$$

or rather:

$$\tfrac{1}{6} l^2 \theta'^2 = \tfrac{1}{6} l^3 \frac{1}{\mathcal{F}(\theta)} .$$

As a result:

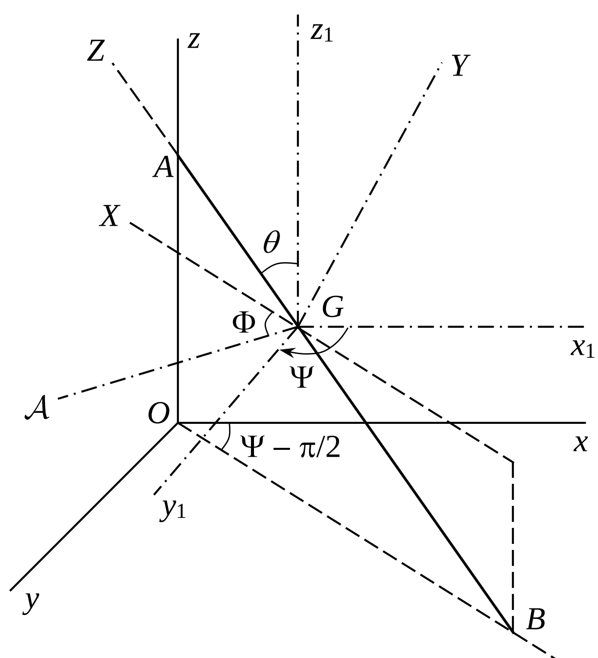
$$t = \int \sqrt{\mathcal{F}(\theta)}\, d\theta ,$$

$$\omega = \int \frac{k\sqrt{\mathcal{F}(\theta)}}{\sin^2\theta}\, d\theta .$$

One is then reduced to quadratures.

We shall now study the motion of a system that depends upon more than two parameters.

V. – *A massive homogeneous solid body of revolution is traversed along its axis by a needle that is fixed in it and one of whose extremities slides without friction along a vertical Oz, while the other slides on a horizontal plane xOy. Study its motion.*



The motion depends upon three parameters: two to fix the position of the axis of revolution and one to fix the position of the body about that axis.

We could get three first integrals by applying:

1. The *vis viva* theorem.

2. The theorem of moments of quantities of motion with respect to *Oz*.

3. The theorem of moments of quantities of motion with respect to the axis of revolution *AB* for the motion around the center of gravity (viz., third Euler equation).

However, we shall appeal to the Lagrange equations.

Calculating the vis viva of the system. – Consider the axes (Ox, Oy, Oz) and the axes that are parallel to them (Gx_1, Gy_1, Gz_1), which are drawn through the center of gravity.

Let $G\mathcal{A}$ be the trace on x_1Gy_1 of the plane that is draw through G perpendicular to AB. Let GX and Gy be two rectangular lines in that plane that are fixed in the solid.

Let:

$$\theta = AGz_1 , \qquad \psi = x_1G\mathcal{A} , \qquad \phi = \mathcal{A}GX .$$

One then concludes that:

$$OAB = \theta , \qquad xOB = \psi - \frac{\pi}{2} .$$

Hence, the angles θ, ψ, ϕ will permit one to determine the position of the body.

Vis viva of the center of gravity. – Let $AG = d$. The coordinates of the point G with respect to (Ox, Oy, Oz) are:

$$\begin{aligned} x &= d \sin\theta \sin\psi , & \text{so:} \quad x' &= d[\sin\theta\cos\psi\cdot\psi' + \cos\theta\sin\psi\cdot\theta'] , \\ y &= d \sin\theta \cos\psi , & y' &= d[\sin\theta\sin\psi\cdot\psi' - \cos\theta\cos\psi\cdot\theta'] , \\ z &= (l - d) \cos\theta , & z' &= -(l-d)\sin\theta\cdot\theta' . \end{aligned}$$

The *vis viva* of the center of gravity, where all of the mass is concentrated, will be:

$$M[d^2(\sin^2\theta\cdot\psi'^2 + \cos^2\theta\cdot\theta'^2) + [(l-d)^2\sin^2\theta\cdot\theta'^2] .$$

Vis viva of the motion around the center of gravity. – Since the body is one of revolution, and θ, ψ, ϕ are the Euler angles, that *vis viva* will be:

$$A(p^2 + q^2) + C r^2 ,$$

with

$$\begin{aligned} p &= \sin\phi\sin\theta\cdot\psi' + \cos\phi\cdot\theta' , \\ q &= \cos\phi\sin\theta\cdot\psi' - \sin\phi\cdot\theta' , \\ r &= \cos\theta\cdot\psi' + \phi' . \end{aligned}$$

It will then be:

$$A(\sin^2\theta\cdot\psi'^2 + \theta'^2) + C(\cos\theta\cdot\psi' + \phi')^2 .$$

As a result, the total *vis viva* of the system will be:

$$2T = (A + M d^2)\sin^2\theta\cdot\psi'^2 + \{A + M[d^2\cos^2\theta + (l-d)^2\sin^2\theta]\}\,\theta'^2 + C(\cos\theta\cdot\psi' + \phi')^2 .$$

Furthermore, there is a force function:

$$U = M\,g\,(l - d)\cos\,\theta.$$

Lagrange equations. – U and T contain neither ϕ nor ψ.

One will then have:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \phi'}\right) = 0\,, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \psi'}\right) = 0\,,$$

i.e.,

(1) $$\phi' + \psi'\cos\theta = \text{const.} = a\,,$$

$$(A + M\,d^2)\sin^2\theta\cdot\psi' + C\cos\theta(\phi' + \psi'\cos\theta) = \text{const.}$$

If one sets:

$$\alpha = \frac{a\,C}{A + Md^2}$$

then the last equation can be written:

(2) $$\psi'\cdot\sin^2\theta + \alpha\cos\theta = b\,.$$

Finally, upon taking (1) into account and setting:

$$\frac{M\,l\,(l - 2d)}{A + Md^2} = \beta\,, \qquad \frac{2M\,g\,(l - d)}{A + Md^2} = \gamma\,,$$

the *vis viva* equation will give:

(3) $$\psi'^2\sin^2\theta + (1 + \beta\sin^2\theta)\,\theta'^2 = \gamma\cos\,\theta + b\,.$$

If one eliminates ψ' from (2) and (3) then that will give:

(1′) $$\pm\,t = \int\sqrt{\frac{1 + \beta\sin^2\theta}{(\gamma\cos\theta + b)\sin^2\theta - (b - \alpha\cos\theta)^2}}\sin\theta\,d\theta\,, \qquad \text{or} \qquad t = \int F(\theta)\,d\theta\,.$$

As a result, (2) will then give:

(2′) $$\psi = \int\frac{b - \alpha\cos\theta}{\sin^2\theta}\,F(\theta)\,d\theta\,.$$

Finally, (1) gives:

(3′) $$\varphi = \int\left[a - \frac{\cos\theta\,(b - \alpha\cos\theta)}{\sin^2\theta}\right]F(\theta)\,d\theta\,.$$

The problem is then reduced to quadratures.

In order to discuss the motion, one sets cos $\theta = u$. That will give:

$$\pm\, t = \int \sqrt{\frac{1+\beta-\beta u^2}{(\gamma u+b)(1-u^2)-(b-\alpha u)^2}}\, du\,.$$

The numerator of the quantity that is placed under the radical is not annulled when u varies from – 1 to + 1. The denominator admits two roots u_1 and u_2 that are found between – 1 and + 1. θ will then vary between two limits θ_1 and θ_2 . After a certain period T, the motion will recommence, except that ψ and φ will be increased by a certain constant. Observe that sin $\theta = 0$ presents itself as a singular value in equations (2′) and (3′). θ can become equal to 0 only if $b = \alpha$, and equal to p only if $b = -\alpha$. All of the difficulty in introducing the trigonometric lines at an angle of $\theta / 2$ will disappear for those particular initial conditions. For example, if $b = \alpha$ then one will have:

$$\pm\, t = \int \sqrt{\frac{1+\beta\sin^2\theta}{(\gamma\cos\theta+b)\cos^2\dfrac{\theta}{2}-b\sin^2\dfrac{\theta}{2}}}\cos\theta\, d\theta = \int F(\theta)\, d\theta\,,$$

and

$$\psi = \int \frac{b}{2\cos^2\dfrac{\theta}{2}}\, F(\theta)\, d\theta\,,$$

$$\varphi = \int \left[a-\frac{b}{2}+\frac{b}{2}\tan^2\frac{\theta}{2}\right] F(\theta)\, d\theta\,.$$

Those equations are regular in the neighborhood of $\theta = 0$.

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LECTURE 9

APPLICATIONS OF THE LAGRANGE EQUATIONS (CONT.). LIOUVILLE'S THEOREM. RELATIVE MOTION.

Liouville indicated a very extensive case in which the Lagrange equations can be integrated by quadratures.

Consider a frictionless system whose constraints are independent of time and whose *vis viva* is expressed as a function of the parameters $q_1, q_2, \ldots, q_k$ in the form:

(1)
$$2T = \varphi[A_1(q_1)\,q_1'^2 + A_2(q_2)\,q_2'^2 + \cdots + A_k(q_k)\,q_k'^2] ,$$

in which φ is a sum of k functions $\varphi_1, \varphi_2, \ldots, \varphi_k$ of $q_1, q_2, \ldots, q_k$, respectively:

$$\varphi = \varphi_1\,(q_1) + \varphi_2\,(q_2) + \ldots, \varphi_k\,(q_k) .$$

The motion of such a system can always be calculated by quadratures when each active force that is exerted on it, or more generally, when the active forces that are exerted on it, admit a force function $U\,(q_1, q_2, \ldots, q_k)$ of the form:

$$U = \frac{\psi}{\varphi} = \frac{\psi_1(q_1) + \psi_2(q_2) + \cdots + \psi_k(q_k)}{\varphi_1(q_1) + \varphi_2(q_2) + \cdots + \varphi_k(q_k)} .$$

That is Liouville's theorem.

First of all, it will suffice to replace the parameters $q_1, q_2, \ldots, q_k$ with the parameters $u_1, u_2, \ldots, u_k$, which are coupled with latter by the relations:

$$\sqrt{A_i(q_i)}\,dq_i = du_i ,$$

in order to reduce T to the form:

(2)
$$2T = (\varphi_1 + \varphi_2 + \cdots + \varphi_k)(q_1'^2 + q_2'^2 + \cdots + q_k'^2) .$$

Having made that substitution, one writes down the Lagrange equations:

(3)
$$\frac{d}{dt}(\varphi\,q_i') - \frac{1}{2}\frac{\partial\varphi}{\partial q_i}(q_1'^2 + q_2'^2 + \cdots + q_k'^2) = \frac{\partial U}{\partial q_i} .$$

The equality of the *vis viva* :

(4) $$T = \frac{\varphi}{2}(q_1'^2 + \cdots + q_k'^2) = U + h$$

is a consequence of those equations. Now multiply the two sides of equation (3) by $2\varphi q_i'$. That will give:

$$\frac{d}{dt}(\varphi^2 q_i'^2) - q_i' \frac{\partial \varphi}{\partial q_i} \varphi(q_1'^2 + \cdots + q_k'^2) = 2q_i' \varphi \frac{\partial U}{\partial q_i},$$

or rather, upon taking (4) into account:

$$\frac{d}{dt}(\varphi^2 q_i'^2) - 2q_i' \frac{\partial \varphi}{\partial q_i}(U + h) = 2q_i' \varphi \frac{\partial U}{\partial q_i},$$

or finally:

$$\frac{d}{dt}(\varphi^2 q_i'^2) = 2q_i' \frac{\partial}{\partial q_i} \varphi(U + h) ,$$

but:

$$\varphi (U + h) = \psi_1 (q_1) + \ldots + \psi_k (q_k) + b [\varphi_1 (q_1) + \ldots, \varphi_k (q_k)] .$$

Therefore:

$$\frac{d}{dt}(\varphi^2 q_i'^2) = 2q_i' \frac{d}{dq_i}(\psi_i + b\varphi_i) = 2\frac{d}{dt}(\psi_i + b\varphi_i),$$

or finally:

$$\tfrac{1}{2}\varphi^2 q_i'^2 = \psi_i + b\, \varphi_i + \alpha_i .$$

The α_i and b are arbitrary constants that are subject to the single condition that equation (4) must be verified:

$$T = U + b .$$

Now the equalities:

$$\tfrac{1}{2}\varphi q_i'^2 = \frac{\psi_i + b\varphi_i + \alpha_i}{\varphi}$$

imply that:

$$T = \frac{\varphi}{2}(q_1'^2 + q_2'^2 + \cdots + q_k'^2) = \frac{\psi}{\varphi} + b + \alpha_1 + \alpha_2 + \cdots + \alpha_k$$
$$= U + b + \sum_i \alpha_i .$$

It is then necessary and sufficient that one should have:

$$\sum_i \alpha_i = 0 .$$

By definition, the Lagrange equations in this case are equivalent to the equalities:

$$\varphi^2\, q_i'^2 = 2\,(\psi_i + b\,\varphi_i + \alpha_i) = 2\,F_i\,(q_i) \qquad (i = 1, 2, \ldots, k)\,,$$

in which h and α denote $(k + 1)$ arbitrary constants that are subject to the single condition that:

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = 0\,.$$

One deduces from this that:

$$\frac{\sqrt{2}\,dt}{\varphi} = \frac{dq_1}{\sqrt{F_1(q_1)}} = \frac{dq_2}{\sqrt{F_2(q_2)}} = \ldots = \frac{dq_k}{\sqrt{F_k(q_k)}}\,.$$

The last $(k - 1)$ equalities give the relations between $q_1, q_2, \ldots, q_k$ and $k - 1$ arbitrary constants by quadratures. t is given by one last quadrature when one has expressed, for example, $q_2, q_3, \ldots, q_k$ as functions of q_1. One further points out that one has:

$$\frac{\sqrt{2}\,dt}{\varphi} = \frac{1}{\varphi_1 + \varphi_2 + \cdots + \varphi_k}\left[\frac{\varphi_1\,dq_1}{\sqrt{F_1}} + \frac{\varphi_2\,dq_2}{\sqrt{F_2}} + \cdots + \frac{\varphi_k\,dq_k}{\sqrt{F_k}}\right],$$

and as a result:

$$\sqrt{2}\,dt = \frac{\varphi_1\,dq_1}{\sqrt{F_1}} + \frac{\varphi_2\,dq_2}{\sqrt{F_2}} + \cdots + \frac{\varphi_k\,dq_k}{\sqrt{F_k}}\,,$$

or

$$\sqrt{2} + \text{const.} = \int\frac{\varphi_1\,dq_1}{\sqrt{F_1}} + \int\frac{\varphi_2\,dq_2}{\sqrt{F_2}} + \cdots + \int\frac{\varphi_k\,dq_k}{\sqrt{F_k}}\,.$$

We have supposed that the *vis viva* T reduces to the form:

$$T = \frac{\varphi}{2}[q_1'^2 + q_2'^2 + \cdots + q_k'^2]\,.$$

If we now revert to the more general form:

$$2T = [\varphi_1(q_1) + \varphi_2(q_2) + \cdots + \varphi_k(q_k)]\,[A_1(q_1)\,q_1'^2 + A_2(q_2)\,q_2'^2 + \cdots + A_k(q_k)\,q_k'^2]$$

then it will result immediately from the foregoing that in the case where U has the form:

$$U = \frac{\psi}{\varphi} = \frac{\psi_1(q_1) + \psi_2(q_2) + \cdots + \psi_k(q_k)}{\varphi_1(q_1) + \varphi_2(q_2) + \cdots + \varphi_k(q_k)}\,,$$

the motion will be determined by the equalities:

$$\frac{\sqrt{2}\,dt}{\varphi_1+\varphi_2+\cdots+\varphi_k} = \frac{dq_1\sqrt{A_1}}{\sqrt{F_1}} = \frac{dq_2\sqrt{A_2}}{\sqrt{F_2}} = \ldots = \frac{dq_k\sqrt{A_k}}{\sqrt{F_k}} ,$$

in which $F_i = \psi_i\,(q_i) + b\ \varphi_i\,(q_i) + \alpha_i$. b and the α_i are constants that are subject to the conditions:

$$\sum \alpha_i = 0 .$$

Those equalities are equivalent to the following ones:

$$\int\sqrt{\frac{A_1}{F_1}}\,dq_1+\beta_1 = \int\sqrt{\frac{A_2}{F_2}}\,dq_2+\beta_2 = \ldots = \int\sqrt{\frac{A_k}{F_k}}\,dq_k+\beta_k ,$$

and

$$\sqrt{2}\,t+\text{const.} = \int\varphi_1\sqrt{\frac{A_1}{F_1}}\,dq_1+\int\varphi_2\sqrt{\frac{A_2}{F_2}}\,dq_2+\cdots+\int\varphi_k\sqrt{\frac{A_k}{F_k}}\,dq_k .$$

That is Liouville's theorem in its most general form.

We shall now indicate some applications of that theorem.

First of all, when no active force is exerted on the system, it will suffice that the *vis viva* should have the form:

$$T = \tfrac{1}{2}(\varphi_1+\varphi_2+\cdots+\varphi_k)(A_1\,q_1'^2+A_2\,q_2'^2+\cdots+A_k\,q_k'^2)$$

if one is to integrate the motion by the preceding method. The functions F_i that were introduced above then reduce to $b\ \varphi_i + \alpha_i$, respectively.

For example, if the ds^2 of a surface can be put into the form:

$$ds^2 = (\varphi_1+\varphi_2)(A_1\,q_1'^2+A_2\,q_2'^2)$$

then the geodesics of that surface will be given by the equation:

$$\int\frac{dq_1\sqrt{A_1}}{\sqrt{b\,\varphi_1+\alpha_1}} = \int\frac{dq_2\sqrt{A_2}}{\sqrt{b\,\varphi_2-\alpha_1}} = \text{const.}$$

(One can set $b = 1$ in that equation.)

That is what happens with the second-degree surfaces when one introduces elliptic coordinates. However, that is a point to which we shall return in what follows.

In particular, when the surface is a plane (or applicable to a plane), one can put the ds^2 into the form in question in an infinitude of ways. Thus, let x and y be the Cartesian coordinates of a point in the plane. One has:

$$ds^2 = dx^2 + dy^2 .$$

Liouville's theorem applies if U has the form:

$$U = \psi_1(x) + \psi_2(y) .$$

Similarly, employ polar coordinates:

$$ds^2 = dr^2 + r^2\, d\theta^2 = r^2\left[\frac{dr^2}{r^2} + d\theta^2\right] .$$

Here, $\varphi = r^2$, $A_1 = 1/r^2$, $A_2 = 1$. Liouville's theorem will apply if one has:

$$U = \frac{1}{r^2}\left[\psi_1(r) + \psi_2(\theta)\right].$$

In order to do that, it is necessary and sufficient that the active force that is exerted on the point M should admit a force function, and that its component F_N that is normal to the radius vector OM should vary along each radius vector in inverse proportion to the square of OM : $F_N = K(\theta)/r^2$. The relation between r and t is the same as if F_N were zero.

When the surface is one of revolution, one can give ds^2 the expression:

$$ds^2 = dr^2[1+\varphi'^2(r)] + r^2\, d\theta^2 = r^2\left\{\frac{dr^2\,[1+\varphi'^2(r)]}{r^2} + d\theta^2\right\}.$$

Liouville's theorem applies if U has the form:

$$U = \frac{1}{r^2}\left[\psi_1(r) + \psi_2(\theta)\right].$$

One can deduce from that, for example, that the motion of a point on a skew helicoid with a director plane is determined by quadrature when the active force that is exerted on the point admits a force function and that along each rectilinear generator, the component of that force along the tangent to the helix that passes through that point will vary in inverse proportion to the square of the distance from the point to the axis of the surface.

As a more complicated example, let us treat the following classical problem:

I. – *A free material point is attracted to two fixed points F and* F' *in inverse proportion to the square of the distance between them and to the point O that is the midpoint of* FF' *in proportion to the distance to that point. Find the motion of the point.*

First suppose that the point is released with no initial velocity or with an initial velocity that is found in the plane $F\,M\,F'$. The point *M* constantly moves in that plane. Take the *x*-axis to be the line *OF* and the *y*-axis to be the perpendicular *Oy*. Among the conics that have *F* and F' for their foci and whose equations can be written:

$$\frac{x^2}{u} + \frac{y^2}{u - c^2} = 1 , \tag{1}$$

there exist two of them that pass through the point *M* at a given instant. For given *x* and *y*, equation (1), which is in terms of *u*, will indeed admit two real roots and positions, one of which λ is greater than c^2 and corresponds to an ellipse, while the other one μ is smaller than c^2 and corresponds to a hyperbola. One can determine the position of the point *M* in the plane with the aid of the two parameters λ and μ. One has:

$$\frac{x^2}{\lambda} + \frac{y^2}{\lambda - c^2} = 1 ,$$

$$(0 < \mu < c^2 < \lambda)$$

$$\frac{x^2}{\mu} + \frac{y^2}{\mu - c^2} = 1 .$$

One deduces from this that:

$$c^2\, x^2 = \lambda\, \mu , \qquad c^2\, y^2 = (\lambda - c^2)(c^2 - \mu) ,$$

and as a result:

$$2\frac{dx}{x} = \frac{d\lambda}{\lambda} + \frac{d\mu}{\mu} ,$$

$$2\frac{dy}{y} = \frac{d\lambda}{\lambda - c^2} + \frac{d\mu}{c^2 - \mu} ,$$

so

$$4\,ds^2 = 4\,(dx^2 + dy^2) = (\lambda - \mu)\left[\frac{d\lambda^2}{\lambda\,(\lambda - c^2)} + \frac{d\mu^2}{\mu\,(c^2 - \mu)}\right] .$$

The ds^2 is then expressed as a function of λ and μ in the Liouville form. We can write:

$$2T = m\left(\frac{ds}{dt}\right)^2 = (\lambda - \mu)\left[\frac{m}{4}\frac{\lambda'^2}{\lambda\,(\lambda - c^2)} + \frac{m}{4}\frac{\mu'^2}{\mu\,(c^2 - \mu)}\right] .$$

Here:

$$\varphi_1 = \lambda\,, \qquad \varphi_2 = -\,\mu\,, \qquad A_1 = \frac{m}{4\lambda\,(\lambda - c^2)}\,, \qquad A_2 = \frac{m}{4\mu\,(c^2 - \mu)}\,.$$

In order for Liouville's theorem to apply, it is necessary and sufficient that one should have:

$$U = \frac{\psi_1(\lambda) + \psi_2(\mu)}{\lambda - \mu}\,.$$

Now set: $r = FM$, $r' = F'M$, $\rho = OM$. The latter equation will become:

$$U = \frac{K}{r} + \frac{K'}{r'} + \alpha\,\rho^2\,,$$

in which K, K', α are given coefficients. However, since $\sqrt{\lambda}$ is the major axis of the ellipse $u = \lambda$, one will have:

$$r + r' = 2\sqrt{\lambda}\,,$$

and similarly, upon considering the hyperbola $u = \mu$:

$$r - r' = 2\sqrt{\mu}\,.$$

Hence:

$$r = \sqrt{\lambda} + \sqrt{\mu}\,, \qquad r' = \sqrt{\lambda} - \sqrt{\mu}\,,$$

$$\frac{K}{r} = \frac{K}{\sqrt{\lambda} + \sqrt{\mu}} = \frac{K(\sqrt{\lambda} - \sqrt{\mu})}{\lambda - \mu}\,,$$

$$\frac{K'}{r'} = \frac{K'}{\sqrt{\lambda} - \sqrt{\mu}} = \frac{K'(\sqrt{\lambda} + \sqrt{\mu})}{\lambda - \mu}\,.$$

Finally:

$$\rho^2 = x^2 + y^2 = \lambda + \mu + \text{const.},$$

$$\alpha\,\rho^2 = \frac{\alpha\,(\lambda^2 - \mu^2)}{\lambda - \mu} + \text{const.}$$

As a result:

$$U = \frac{\sqrt{\lambda}\,(K + K') + \alpha\,\lambda^2 + \sqrt{\mu}\,(K - K') - \alpha\,\mu^2}{\lambda - \mu} = \frac{\psi_1(\lambda) + \psi_2(\mu)}{\lambda - \mu}\,.$$

That result further persists when one supposes that the point M is attracted (or repelled) by the x-axis and the y-axis in inverse proportion to the cubes of its distances to those axes. In order to take those forces into account, it is necessary to add a term U_1 to U that has the form:

$$U_1 = \frac{\beta}{x^2} + \frac{\gamma}{y^2} ,$$

in which β and γ are given coefficients.

However, one knows that:

$$\frac{1}{x^2} = \frac{c^2}{\lambda\,\mu} = \frac{c^2}{\lambda-\mu}\left(\frac{1}{\mu}-\frac{1}{\lambda}\right),$$

$$\frac{1}{y^2} = \frac{c^2}{(\lambda-c^2)(c^2-\mu)} = \frac{c^2}{\lambda-\mu}\left(\frac{\lambda}{\lambda-c^2}+\frac{\mu}{c^2-\mu}\right).$$

Therefore:

$$U_1 = \frac{1}{\lambda-\mu}\left(\frac{-\beta\,c^2}{\lambda}+\frac{\gamma\,\lambda}{\lambda-c^2}+\frac{\beta\,c^2}{\mu}+\frac{\gamma\,\mu}{c^2-\mu}\right).$$

Thus, the sum $U + U_1$ still has the form:

$$\frac{\psi_1(\lambda)+\psi_2(\mu)}{\lambda-\mu} .$$

The motion of the point is determined by the equalities:

$$(2) \qquad \sqrt{\frac{8}{m}}\,\frac{dt}{\lambda-\mu} = \frac{d\lambda}{\sqrt{\lambda(\lambda-c^2)(\psi_1+\beta\,\lambda+\alpha_1)}} = \frac{d\mu}{\sqrt{\mu(c^2-\mu)(\psi_2-\beta\,\mu-\alpha_1)}},$$

with

$$\psi_1(\lambda) = (K+K')\sqrt{\lambda}-\frac{\beta\,c^2}{\lambda}+\frac{\gamma\,\lambda}{\lambda-c^2} ,$$

$$\psi_2(\mu) = (K-K')\sqrt{\mu}-\alpha\,\mu^2+\frac{\beta\,c^2}{\mu}+\frac{\gamma\,\lambda}{c^2-\mu} .$$

Those equalities imply the following:

$$\sqrt{\frac{8}{m}}\,t + \text{const.} = \int\frac{\lambda\,d\lambda}{\sqrt{(\lambda-c^2)(\psi_1+\beta\,\lambda+\alpha_1)}}-\int\frac{\mu\,d\mu}{\sqrt{(c^2-\mu)(\psi_2-\beta\,\mu-\alpha_1)}} .$$

However, we know that in this case the motion of the point is uniform and rectilinear:

$$x = a\,t + b\,, \qquad y = a'\,t + b'\,, \qquad y = a\,x + p\,.$$

If one replaces x and y as functions of λ, μ then one will see that the equations (1) can be integrated algebraically.

The integral of the equation:

$$\frac{d\lambda}{\sqrt{\lambda(\lambda - c^2)(\lambda + \alpha_1)}} = \frac{d\mu}{\sqrt{\mu(\mu - c^2)(\mu + \alpha_1)}} \tag{4}$$

is then algebraic and will have the form:

$$A\sqrt{\lambda\,\mu} + A'\sqrt{(\lambda - c^2)(c^2 - \mu)} + A'' = 0\,.$$

A, A', A'' are constants whose values one can easily find by expressing x_0, y_0, y_0' as functions of the initial values $\mu_0 = -\alpha_1$, $\lambda_0 = C_1\left(\frac{du}{d\lambda}\right)_0 = \rho$:

$$A = \sqrt{C\,(c^2 + \alpha_1)}\,, \qquad A' = \sqrt{-\alpha_1\,(C - c^2)}\,, \qquad A'' = c^2\sqrt{-\alpha_1\,(c^2 + \alpha_1)}\,.$$

C is an arbitrary constant in those equalities.

If one likewise supposes that the only force that is exerted on the point M is an attraction to the point O then equations (2) will become:

$$\sqrt{\frac{8}{m}}\,\frac{dt}{\lambda - \mu} = \frac{d\lambda}{\sqrt{\lambda(\lambda - c^2)(\alpha\,\lambda^2 + \beta\,\lambda + \alpha_1)}} = \frac{d\mu}{\sqrt{\mu(\mu - c^2)(\alpha\,\mu^2 + \beta\,\mu + \alpha_1)}}\,.$$

One knows that x and y can then be expressed as functions of t with the aid of exponentials and that the relation between x and y is algebraic and of degree two. Equations (5) are then integrated with the aid of exponentials, and the integral of the equation:

$$\frac{d\lambda}{\sqrt{\lambda(\lambda - c^2)(\alpha\,\lambda^2 + \beta\,\lambda + \alpha_1)}} = \frac{d\mu}{\sqrt{\mu(\mu - c^2)(\alpha\,\mu^2 + \beta\,\mu + \alpha_1)}}$$

is algebraic. One knows that those equations are introduced into the theory of elliptic functions.

In the foregoing, we assumed that the point was launched in the plane $F\,M\,F'$. Let us now suppose that the initial conditions are arbitrary.

The forces that are exerted on the point M (viz., attractions to the points F, F', and O) all meet the axis Ox. Apply the theorem of moments to that axis. Upon letting R denote the distance from the point M to the axis and letting θ denote the angle between the plane MOx and the fixed plane xOy, that will give:

$$R^2 \frac{d\theta}{dt} = \text{const.} = \gamma.$$

Having said that, let us study the relative motion of the point M in the plane MOx. In other words, let us study the motion of M with respect to the tri-rectangular trihedron Oxy_1z_1, where Oy_1 is the perpendicular that is drawn to Ox in the plane MOx at each instant. The point M constantly remains in the plane xOy_1 with respect to that trihedron. Let us evaluate the projection of the force that is exerted on M relative to the axes Oxy_1z_1 onto that plane. The composite centrifugal force is normal to the plane MOx. The guiding force of (F_c) is situated in that plane and points along the perpendicular PM to Ox. That force will have the expression:

$$m\left(\frac{d\theta}{dt}\right)^2 y_1$$

when measured along Oy_1. On the other hand:

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{\gamma^2}{R_4} = \frac{\gamma^2}{y_1^4};$$

thus:

$$F_c = \frac{m\gamma^2}{y_1^3}.$$

It follows from this that the point M moves in the plane xOy_1 as if the point were attracted to the point F and F' in inverse proportion to the square of the distance, to the point O in direct proportion to the distance, and is repelled by the axis Ox in inverse proportion to the cube of the distance with a coefficient of repulsion that is equal to $m\gamma^2$. The motion of the point will then be determined by the equalities that were written above. Observe that this result will persist when the axis Ox attracts (or repels) the point M in inverse proportion to the cube of its distance to the point.

Moreover, one will arrive at the same conclusions by studying the absolute motion of M with the aid of the Lagrange equations. Indeed, determine the position of the point M with the aid of elliptic coordinates λ, μ in the plane MOx, and the angle θ that the latter plane makes with the xOy-plane. One immediately finds that:

$$T = \frac{m}{a}\left\{\frac{(\lambda-\mu)}{4}\left[\frac{\lambda'^2}{\lambda(\lambda-c^2)} + \frac{\mu'^2}{\mu(c^2-\mu)}\right] + y_1^2\,\theta'^2\right\} = T_1 + \tfrac{1}{2} m\, y_1^2\,\theta'^2.$$

The Lagrange equations can be written:

$$\frac{d}{dt}\cdot m\, y_1^2\,\theta' = 0 \qquad \text{or} \qquad y_1^2\,\theta' = \gamma,$$

(*i*)

$$\frac{d}{dt}\frac{\partial T_1}{\partial \lambda'} - \frac{\partial T_1}{\partial \lambda} - \tfrac{1}{2}\, m\,\theta'^2\, y_1\,\frac{\partial y_1}{\partial \lambda} = \frac{\partial U}{\partial \lambda}.$$

Now, upon taking (*i*) into account:

(*j*)
$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial \lambda'}\right) - \frac{\partial T_1}{\partial \lambda} = \frac{\partial U}{\partial \lambda} + \frac{m\gamma^2}{y_1^3}\frac{\partial y_1}{\partial \lambda} = \frac{\partial U_1}{\partial \lambda},$$

if one sets:

$$U_1 = U + \frac{m\gamma^2}{2y_1^2} = \frac{K}{r} + \frac{K'}{r'} + \alpha\,\rho - \frac{m\gamma^2}{2y_1^2}.$$

Similarly:

(*k*)
$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial \mu'}\right) - \frac{\partial T_1}{\partial \mu} = \frac{\partial U_1}{\partial \mu}.$$

Those results coincide quite well with the ones that we just obtained.

Here is another application to the motion of a point on a surface:

II. – *A point M moves without friction on a second-degree cone. It is attracted to the summit O of the cone in inverse proportion to the square of the distance OM and to the internal axis of the cone in inverse proportion to the cube of its distance MP to that axis. Find the motion of the point.*

Define the axes *Oxyz* to be the three principal axes of the cone, where *Oz* is the internal axis. The equation of the cone is:

$$z^2 = \alpha^2 x^2 + \beta^2 y^2.$$

Define the position of a point on the surface with the aid of the two parameters ρ and u :

$$\rho^2 = x^2 + y^2 + z^2, \qquad u = \frac{y}{x}.$$

The curvilinear coordinates $u = u_0$ and $\rho = \rho_0$ are defined by the rectilinear generators and their orthogonal trajectories.

One immediately finds that:

$$x^2 = \frac{\rho^2}{1+\alpha^2+u^2(1+\beta^2)},$$

$$y^2 = \frac{u^2\rho^2}{1+\alpha^2+u^2(1+\beta^2)},$$
$$z^2 = \frac{(\alpha^2+u^2\beta^2)\,\rho^2}{1+\alpha^2+u^2(1+\beta^2)}.$$

If one sets $D = 1+\alpha^2+u^2(1+\beta^2)$ then one will deduce these following three equalities:

$$\frac{dx}{x} = \frac{d\rho}{\rho} - \frac{u\,(1+\beta^2)\,du}{D},$$
$$\frac{dy}{y} = \frac{d\rho}{\rho} + \frac{(1+\alpha^2)\,du}{u\,D},$$
$$\frac{dz}{z} = \frac{d\rho}{\rho} + \frac{(\beta^2-\alpha^2)\,u\,du}{(\alpha^2+u^2\beta^2)D},$$

so upon replacing x^2, y^2, z^2 as functions of *r* and *u*, the sum $x^2+y^2+z^2$ will have the value:

$$ds^2 = d\rho^2 + \frac{\rho^2[\alpha^2(1+\alpha^2)+u^2\beta^2(1+\beta^2)]}{(\alpha^2+u^2\beta^2)\,D^2}\,du^2.$$

As a result, if one sets:

$$A\,(u) = \frac{\alpha^2(1+\alpha^2)+u^2\beta^2(1+\beta^2)}{(\alpha^2+u^2\beta^2)[1+\alpha^2u^2(1+\beta^2)]^2}$$

then one will have:

$$T = \tfrac{1}{2}m\rho^2\left[\frac{\rho'^2}{\rho^2}+A(u)\,u'^2\right].$$

Upon substituting the parameter θ for the parameter *u*, one will have:

$$\rho = \int\sqrt{A(u)}\,du = \int\sqrt{\frac{\alpha^2(1+\alpha^2)+u^2\beta^2(1+\beta^2)}{\alpha^2+u^2\beta^2}}\,\frac{du}{1+\alpha^2+u^2(1+\beta^2)},$$

and ds^2 will reduce to the form:

$$ds^2 = d\rho^2+\rho^2d\theta^2,$$

which is an expression for ds^2 in the plane in terms of polar coordinates. That result is easy to predict geometrically, moreover.

Let us now evaluate *U*. Let *r* = *MP*:

$$U = \frac{K}{\rho} + \frac{K'}{r^2}.$$

Moreover:

$$r^2 = x^2 + y^2 = \rho^2 - z^2 = \frac{\rho^2 (1+u^2)}{D},$$

so:

$$U = \frac{1}{\rho^2}\left(K\rho + \frac{K' D}{1+u^2}\right) = \frac{1}{\rho^2}\,[\psi_1(\rho) + \psi_2(u)]\ .$$

Liouville's theorem thus applies here. If one further desires that the problem comes down to the problem of motion of a point in a plane when it is subject to a force that admits a force function and whose component normal to the radius vector OM varies along OM in inverse proportion to OM^2.

The equations that define the motion are:

$$\sqrt{\frac{2}{m}}\,\frac{dt}{\rho^2} = \frac{dt}{\rho\sqrt{F_1}} = du\,\sqrt{\frac{A(u)}{F_2}},$$

with:

$$F_1 = K\rho + b\rho^2 + C,$$

$$F_2 = \frac{K'(1+u^2)}{(1+\alpha^2) + u^2(1+\beta^2)} - C.$$

Thus:

$$\sqrt{\frac{2}{m}}\,dt = \frac{\rho\, d\rho}{\sqrt{K\rho + b\rho^2 + C}}$$

and

$$\frac{d\rho}{\rho\sqrt{K\rho + b\rho^2 + C}} = du\,\sqrt{\frac{\alpha^2(1+\alpha^2) + u^2\,\beta^2\,(1+\beta^2)}{(\alpha^2 + u^2\,\beta^2)[1+\alpha^2 + u^2\,(1+\beta^2)]\{K'(1+u^2) - C[1+\alpha^2 + u^2\,(1+\beta^2)]\}}}\ .$$

The relation between ρ and t is independent of the coefficient of attraction to the axis Oz.

In conclusion, we add that one must often apply Liouville's theorem in the simple case where φ is a constant, i.e., the case in which T has:

$$T = A_1(q_1)\,q_1'^2 + A_2(q_2)\,q_2'^2 + \cdots + A_k(q_k)\,q_k'^2\ .$$

It is necessary and sufficient that one must have:

$$U = \psi_1(q_1) + \psi_2(q_2) + \ldots + \psi_k(q_k)\ .$$

Let us recall, for example, the problem that was treated before (see page 37).

III. – *Two points A and B of a massive solid body slide without friction along two fixed parallel lines. Find the motion of the system.*

While preserving the notation on pages 37 and 38, one will find that:

$$2T_1 = \psi'^2(a + b\cos^2\psi) + M\,\zeta'^2\,,$$
$$U = \lambda\cos\,\psi + \mu\sin\,\psi + r\,\zeta\,,$$

in which a, b, M, λ, μ, r are given coefficients. The motion is then determined by the equations:

$$\sqrt{2}\,dt = d\psi\sqrt{\frac{a + b\cos^2\psi}{\lambda\cos\psi + \mu\sin\psi + \beta}} = d\zeta\sqrt{\frac{M}{\nu\,\zeta + \alpha}}\,,$$

in which α and β are constants.

We will recover Liouville's theorem along a different path and return to its applications.

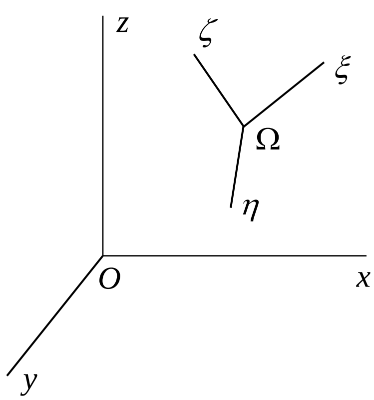
LECTURE 10

APPLICATION OF THE LAGRANGE EQUATIONS TO THE STUDY OF THE RELATIVE MOTION OF SYSTEMS

Let two systems of coordinate axes (Ox, Oy, Oz) and ($\Omega\xi$, $\Omega\eta$, $\Omega\zeta$) be animated with respect to each other by an arbitrary given motion.

Let	a, b, c	denote	the coordinates of	Ω	with respect to	$Oxyz$
	α, β, γ	″	the direction cosines of	$\Omega\xi$	″	″
	α', β', γ'	″	″ ″	$\Omega\eta$	″	″
	$\alpha'', \beta'', \gamma''$	″	″ ″	$\Omega\zeta$	″	″

Those twelve quantities are given functions of time t.



Consider a free material point and suppose that one knows the form of the force (F) relative to the axes $Oxyz$ that is exerted on that point. One proposes to study the motion of a point M with respect to the axes $\Omega\xi\eta\zeta$.

It is clear that when one knows the motion of M with respect to the first set of axes, from the formulas for changing axes, one will know its motion with respect to the second one, but that calculation is generally inconvenient.

A better process consists of using Coriolis's theorem. One calculates the force relative to the axes $\Omega\xi\eta\zeta$ that is exerted on M by the formula:

$$(F_r) = (F_a) - m\,(\gamma_r) - m\,(\gamma_c)$$

and studies the motion of a point that is subject to the force F_r .

In place of that, one can calculate the *vis viva* $2T$ of the point M in its motion with respect to $Oxyz$ as a function of the parameters ξ, η, ζ, and write the three Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \xi'}\right) - \frac{\partial T}{\partial \xi} = \Xi\,, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \eta'}\right) - \frac{\partial T}{\partial \eta} = \mathrm{H}\,, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \zeta'}\right) - \frac{\partial T}{\partial \zeta} = \mathrm{Z}\,.$$

If the force (F) admits a force function U then the three right-hand sides are equal to $\frac{\partial U}{\partial \xi}$, $\frac{\partial U}{\partial \eta}$, $\frac{\partial U}{\partial \zeta}$, respectively.

More generally, let M be a material point that is or is not free. Let (F) be the active force that is exerted on it relative to the axes $Oxyz$. Express its coordinates ξ, η, ζ as functions of the most convenient parameters q_i (and also t, necessarily). In order to study the motion of the point M with respect to the axes $\Omega\xi\eta\zeta$, one can pursue the following methods:

Method 1. – One calculates the active force (ϕ) relative to $\Omega\xi\eta\zeta$ using the Coriolis formula:

$$(\phi) = (F) - m\,(\gamma_e) - m\,(\gamma_c)$$

and the *vis viva* $2T$ relative to $\Omega\xi\eta\zeta$:

$$T_1 = \tfrac{1}{2} m(\xi'^2 + \eta'^2 + \zeta'^2) \ .$$

One then writes the Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right) - \frac{\partial T}{\partial q} = Q \ ,$$

where $Q\ \delta q$ denotes the virtual work done by ϕ during the displacement δq. The equations thus-obtained will define the parameter q, and as a result, ξ, η, ζ, as a functions of t.

Method 2. – One calculates the *vis viva* $2T_1$ with respect to Ox, Oy, Oz, instead. One can calculate the *vis viva* $2T_2$ with respect to Ox, Oy, Oz. One only has to write down the equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right) - \frac{\partial T}{\partial q} = Q$$

then, in which:

$$\sum Q\,\delta q = \sum (X\,\delta x + Y\,\delta y + Z\,\delta z) \ ,$$

if X, Y, Z are the components of (F) along (Ox, Oy, Oz).

The ultimate equations to which one arrives are the same as before, but the calculation is more convenient.

In order to calculate T_2, it is not necessary to form the expressions for x, y, z as functions of q and t. It is easy to calculate the velocity V of M with respect to Ox, Oy, Oz as functions of ξ, η, ζ and $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, $\frac{d\zeta}{dt}$. Indeed, the velocity of the point M at each instant is the same as if the point had been animated with a translatory motion whose velocity is the same as that of the origin Ω and a rotational motion ω around a certain axis Ω*I*.

Let	u, v, w	denote the components of	the velocity of Ω	along	$\Omega\xi, \Omega\eta, \Omega\zeta$
"	p, q, r	" "	the rotation Ω*I*	"	"

The velocity of the point *M* with respect to *Ox*, *Oy*, *Oz* has the components along $\Omega\xi, \Omega\eta, \Omega\zeta$:

$$V_\xi = \frac{d\xi}{dt} + u + q\zeta - r\eta ,$$
$$V_\eta = \frac{d\eta}{dt} + v + r\xi - p\zeta ,$$
$$V_\zeta = \frac{d\zeta}{dt} + w + p\eta - q\xi .$$

Thus:

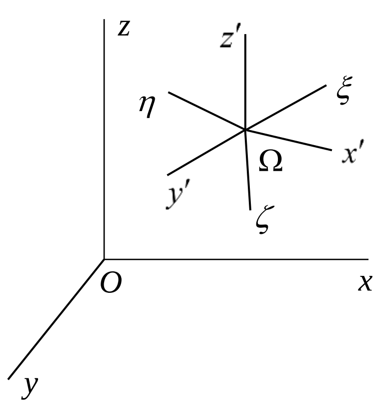
$$T_2 = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left[\left(\frac{d\xi}{dt} + u + q\zeta - r\eta\right)^2 + \left(\frac{d\eta}{dt} + v + r\xi - p\zeta\right)^2 + \left(\frac{d\zeta}{dt} + w + p\eta - q\xi\right)^2\right].$$

It is easy to develop the expression for that *vis viva*.

We remark that there is a term that one can neglect in the calculation. It is $\frac{1}{2}m(u^2 + v^2 + w^2)$. Indeed, it contains only *t* and will contribute nothing to the derivatives with respect to q_i and q_i'.

Method 3. – One sometimes presents the preceding calculation in a somewhat-different form ([9]):

One considers a system of intermediate axes $(\Omega x', \Omega y', \Omega z')$ that are parallel to (*Ox*, *Oy*, *Oz*). One calculates the *vis viva* and the active force with respect to those axes and applies the Lagrange equations.



The active force with respect to those axes is:

$$(F') = (F) - m\,(\gamma) ,$$

in which *F* is the active force with respect to (*Ox*, *Oy*, *Oz*) and γ is the acceleration of the point Ω with respect to (*Ox*, *Oy*, *Oz*).

γ is calculated with no difficulty, and as a result, *F* will be obtained quite easily.

As for the *vis viva* T_3 relative to $(\Omega x', \Omega y', \Omega z')$, if one keeps the preceding notations then it will be obvious that:

([9]) **Gilbert** – Annales de la Société Scientifique de Bruxelles **6** (1882), pp. 270, *et seq*.

$$T_3 = \tfrac{1}{2} m \left[\left(\frac{d\xi}{dt} + q\zeta - r\eta \right)^2 + \left(\frac{d\eta}{dt} + r\xi - p\zeta \right)^2 + \left(\frac{d\zeta}{dt} + p\eta - q\xi \right)^2 \right].$$

One finally writes out the Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T_3}{\partial q'} \right) - \frac{\partial T_3}{\partial q} = Q ,$$

in which $\sum Q\,\delta q$ is the virtual work done by the force F'.

We remark that:

$$T_3 = \mathcal{T} + G + \mathcal{V},$$

upon setting:

$$\mathcal{T} = \tfrac{1}{2} m \left[\left(\frac{d\xi}{dt} \right)^2 + \left(\frac{d\eta}{dt} \right)^2 + \left(\frac{d\zeta}{dt} \right)^2 \right],$$

$$G = \tfrac{1}{2} m \left[(q\zeta - r\eta)^2 + (r\xi - p\zeta)^2 + (p\eta - q\xi)^2 \right] ,$$

$$\mathcal{V} = m \left[\frac{d\xi}{dt}(q\zeta - r\eta) + \frac{d\eta}{dt}(r\xi - p\zeta) + \frac{d\zeta}{dt}(p\eta - q\xi) \right] .$$

Those three terms can be interpreted as follows:

$\mathcal{T}$ is the *vis viva* of the motion of the point with respect to $\Omega\xi, \Omega\eta, \Omega\zeta$. G is the *vis viva* of the point that is due to the instantaneous rotation about the axis ΩI. If r is the distance from M to ΩI then we will have:

$$G = \tfrac{1}{2}\,(mr^2)\,\omega^2 .$$

Finally, the expression for $\mathcal{V}$ is written in the form:

$$\mathcal{V} = p \cdot m \left(\eta \frac{d\zeta}{dt} - \zeta \frac{d\eta}{dt} \right) + \cdots + \cdots$$

If we let P denote the moment with respect to Ω of the quantity of motion of the point M during its motion with respect to $(\Omega\xi, \Omega\eta, \Omega\zeta)$ and let P_ξ, P_η, P_ζ be its projections onto those axes then we will have:

$$\mathcal{V} = p\,P_\xi + q\,P_\eta + r\,P_\zeta$$

or

$$\mathcal{V} = \omega \cdot P \cos(\omega, P) .$$

Therefore, $\mathcal{V}$ is the geometric product of the instantaneous rotation ΩI with the moment with respect to Ω of the quantity of motion of the point due to its motion with respect to the reference axes $\Omega\xi$, $\Omega\eta$, $\Omega\zeta$.

Those geometric interpretations facilitate the calculation of the *vis viva* in certain cases.

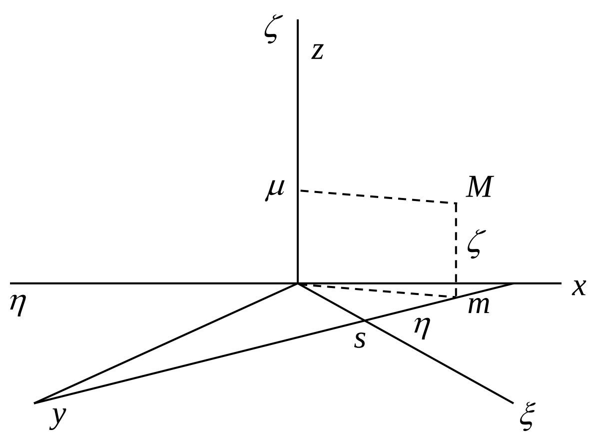
Remark. – It is important to observe that the three indicated methods will lead to the identical differential equations for defining the parameters q that determine the position of point.

Example. – *Motion of a massive point that moves without friction in a vertical plane that is animated with a rotational motion around a vertical axis.*

Let Oz be the rotational axis, while Ox, Oy are two fixed horizontal axes.

Take the reference axes to be ($O\xi$, $O\eta$, $O\zeta$), in which $O\xi$ is perpendicular to the given plane, while $O\zeta$ coincides with Oz.

Suppose that the mass of the moving point is equal to unity. Define the position of that point by its coordinates: η, ζ, while ξ is zero. Finally, let ω be the velocity of rotation.



First method. – One has:

$$2T_4 = \zeta'^2 + \eta'^2 .$$

Calculating the virtual work. – In this particular case, one should not be preoccupied with the composite centrifugal force, which is normal to the plane that describes the moving point, and consequently, to any virtual displacement.

The guiding force is $\omega^2 \cdot M\mu$, in which $M\mu$ is the distance from M to Oz; it points from M to Oz. The given force is gravity. One immediately finds that:

$$Q_\zeta = -g ,$$
$$Q_\eta = \omega^2\eta .$$

The corresponding Lagrange equations are:

$$\begin{cases} \zeta'' = -g , \\ \eta'' = \omega^2\eta , \end{cases}$$

whose first integrals are:

$$\zeta' = -g\,t + A , \qquad \eta'^2 = \omega^2\,\eta^2 + B ,$$

and the complete integral is:

$$\zeta = -\tfrac{1}{2} g t^2 + At + A', \qquad \eta = C e^{\omega t} + D e^{-\omega t}.$$

Those integrals can be obtained by applying the *vis viva* theorem to the projections into *Oz*.

Second method. – One has:

$$x = l \cos \omega t + \eta \sin \omega t,$$
$$y = l \sin \omega t - \eta \cos \omega t,$$
$$z = \zeta,$$

in which l denotes the distance from O to the given plane.

One then concludes that:

$$2T_2 = l^2\omega^2 + \eta^2\omega^2 + \eta'^2 + 2l\,\omega\eta' + \zeta'^2.$$

There is a force function:

$$U = -g\,\zeta.$$

The Lagrange equations are:

$$\begin{cases} \zeta'' = -g, \\ \dfrac{d}{dt}(\eta' + l\,\omega) - \eta\,\omega^2 = 0 \quad \text{or} \quad \eta'' - \omega^2\eta = 0. \end{cases}$$

Third method. – The intermediate axes coincide with *Ox*, *Oy*, *Oz*. As a result, T_3 will be identical to T_2 . One confirms that with the aid of Gilbert's formulas. One has:

$$2\,\mathcal{T} = \zeta'^2 + \eta'^2,$$
$$2\,\mathcal{G} = (l^2 + \eta^2)\,\omega^2,$$
$$2\,\mathcal{V} = 2\omega \cdot l\,\eta \qquad\qquad (\omega, P) = 0.$$

As a result:

$$T_3 = T_2.$$

The origin of the instantaneous axis is fixed. One will then have only gravity as the given force, and one will recover the preceding equations.

One easily treats the same problem by supposing that, in addition, the point is attracted by each element of the *Oz* axis according to a function of distance, and similarly by supposing that the point moves on a vertical cylinder that turns uniformly around a vertical axis.

Extending the preceding consideration to the relative motion of systems.

When one has to study the relative motion of a system of points with respect to axes $\Omega\xi$, $\Omega\eta$, $\Omega\zeta$ that are animated with a known motion relative to the fixed axes *Ox*, *Oy*, *Oz* and when one knows the active force that is exerted on each point of the system with respect to *Ox*, *Oy*, *Oz*, one can again employ the three methods that were indicated in the case of a single point.

One refers the position of the points of the system to the parameters *q* that are most convenient for defining those positions relative to the moving axes $\Omega\xi$, $\Omega\eta$, $\Omega\zeta$, and one then calculates the *vis viva* of the system as a function of the *q*, the q', and *t* :

– with respect to the moving axes $\Omega\xi$, $\Omega\eta$, $\Omega\zeta$,

– with respect to the fixed axes *Ox*, *Oy*, *Oz* ,

– with respect to the intermediate axes $\Omega x'$, $\Omega y'$, $\Omega z'$.

In order to apply the Lagrange equations, it remains to determine the quantities *Q*.

In the first case, one calculates the total virtual work done by the given forces with respect to *Ox*, *Oy*, *Oz*, and the Coriolis forces with the sign changed.

In the second case, one calculates the total virtual work done by the given forces with respect to *Ox*, *Oy*, *Oz*.

In third case, one calculates the total virtual work done by the given forces and the guiding forces m (γ_e), where γ_e is the acceleration of the point Ω.

Gilbert gave an interesting form to the equations that one obtains by the third method, and which will result immediately from the foregoing.

Let us place ourselves in the case where the given forces admit a force function *U*.

Let γ_ξ, γ_η, γ_ζ be the components along $\Omega\xi$, $\Omega\eta$, $\Omega\zeta$ of the acceleration γ_e at the point Ω. Set:

$$N = -\sum m(\xi\gamma_\xi + \eta\gamma_\eta + \zeta\gamma_\zeta).$$

We will then has:

$$Q = \frac{\partial(U+N)}{\partial q}.$$

Furthermore, the semi-*vis viva* T_3 is the sum of three terms:

One of them $\mathcal{T}$ is the semi-*vis viva* of the relative motion of the system with respect to $\Omega\xi$, $\Omega\eta$, $\Omega\zeta$.

The second one $\mathcal{G}$ is one-half the product of the moment of inertia of the system at the instant in question with respect to the instantaneous axis Ω of the system $\Omega\xi\eta\zeta$ in its motion around Ω with the square of the instantaneous rotation ω.

The third one $\mathcal{V}$ is the product of the instantaneous rotation ω with the moment of the quantity of motion of the system with respect to Ω (in its relative motion with respect to $\Omega\xi$, $\Omega\eta$, $\Omega\zeta$) and the cosine of the angle between those two quantities.

Thus:

$$T_3 = \mathcal{T} + \mathcal{G} + \mathcal{V}.$$

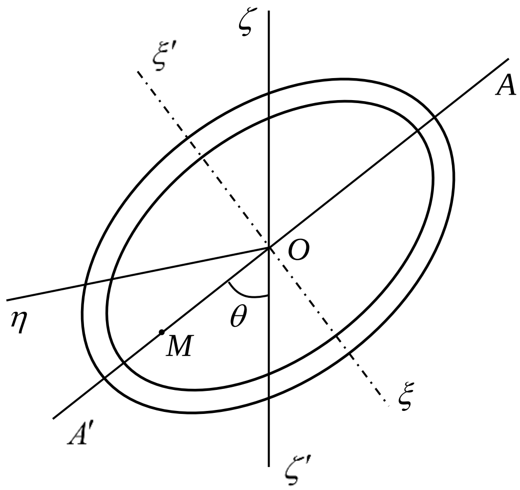
The equations of motion will then take the form:

$$\frac{d}{dt}\left(\frac{\partial T_3}{\partial q'}\right) = \frac{\partial (U+N)}{\partial q}.$$

Applications:

I. – *A massive homogeneous torus T moves around a horizontal equatorial diameter* $\xi O \xi'$. *That diameter is animated with a uniform rotation* ω *around the vertical to the center O. An additional weight m is placed at a point M on the equatorial diameter that is perpendicular to* $\xi O \xi'$. *Find the motion of the torus around the diameter* $\xi O \xi'$ ([10]).

(*Agrégation*, 1874)



Let $OM = d$, while A and C are the moments of inertia of the torus with respect to an equatorial diameter and the axis of revolution, resp.

The motion will be defined by the angle θ between the radius OM and the descending vertical.

One has:

$$2\mathcal{T} = (A + m\,d^2)\,\theta'^2,$$

$$2\mathcal{G} = \omega^2[A\cos^2\theta + C\sin^2\theta] + \omega^2 m\,d^2\sin^2\theta,$$

$$2\mathcal{V} = 0, \qquad \text{because} \qquad (\omega, P) = \pi/2.$$

Therefore:

$$2\,T_3 = (A + m\,d^2)\,\theta'^2 + \omega^2(C - A + m\,d^2)\sin^2\theta + \omega^2 A.$$

The origin of the axes is fixed. Therefore $N = 0$.

The active force reduces to the additional weight because O is the center of gravity of the torus.

The force function is then:

$$U = -\,m\,g\,d\cos\theta.$$

The Lagrange equation that solves the problem is:

$$(A + m\,d^2)\,\theta'^2 - \omega^2(C - A + m\,d^2)\sin\theta\cos\theta = -\,m\,g\,d\sin\theta.$$

All that remains is to integrate that second-order differential equation.

([10]) **Gilbert** – Treatise on the application of the Lagrange method to problems of relative motion, *loc. cit.*, pp. 276.

II. – *A solid body of revolution is fixed to its center of gravity. The axis of revolution Gx is subject to remain in a fixed plane, moreover. Find the motion of the solid with respect to terrestrial objects while taking into account the motion of the Earth (Foucault gyroscope).*

Let us first treat the problem using the first method by applying the Coriolis theorem.

Let $Gxyz$ be the principal axes of inertia of the solid, while $G\,x_1\,y_1\,z_1$ are axes that are fixed with respect to the Earth. We choose the $x_1\,y_1$-plane to be the plane in which Gz moves and the Gx-axis to be the projection into that plane of the segment of the terrestrial rotation (ω) or (GS): GS is the North-South direction of the Earth's axis. We take Gz_1 to point on the same side as GS with respect to the plane $x_1G\,y_1$.

The position of the solid depends upon two parameters: for example, the two Euler angles φ and ψ, while $\theta = \pi / 2$.

With respect to axes that are fixed stellar directions that pass through the center of the Earth, the active force that is exerted on each point M of the solid is a quantity $m\,(\gamma)$ that is proportional to the mass of M and the fixed direction relative to $G\,x_1\,y_1\,z_1$. The active force relative to the terrestrial axes that is exerted on that point is equal to:

$$m\,(\gamma) - m\,(\gamma_e) - m\,(\gamma_c)\ .$$

The quantity (γ_e) is reasonably invariable (relative to the axes $G\,x_1\,y_1\,z_1$) for the various points of the solid and no matter what their positions might be. We can then set:

$$m\,(\gamma) - m\,(\gamma_c) = m\,(g)\ .$$

The quantity (g) does not vary with respect to the terrestrial axes.

Let us take the latter approximation into account. In order to do that, we consider axes of fixed stellar directions that pass through G. The active force relative to those axes that is exerted on M is:

$$m\,(\gamma) - m\,(F_c)\ ,$$

in which (F_c) denotes the acceleration of G due to the motion of the Earth around its center. One can set $m\,(\gamma) - m\,(F_c) = m\,(g)$, in which the quantity (g) is invariable with respect to $G\,x_1\,y_1\,z_1$.

Furthermore, the active force relative to the latter axes that is exerted on M is equal to:

$$m\,(g) - m(\gamma'_e) - m\,(\gamma_c)\ .$$

The quantity (γ_c) is the same as above. As for the quantity (γ'_e), it can be defined as follows: Let Q be the foot of the perpendicular to GS that is based at M. (γ'_e) then points along MQ and is equal to $\omega^2 M\,Q$. We therefore neglect a term of order ω^2 .

Let us now evaluate $-\,(F_c) = -\,m\,(\gamma_e)$. If we construct a quantity GM that is equipollent to the velocity (V_r) of the point M (relative to the axes $G\,x_1\,y_1\,z_1$) then we know that the quantity $(\gamma_e / 2)$

is equal to the velocity that the point M will have while turning around GS with the angular velocity ω. Therefore, let X_c, Y_c, Z_c denote the components of – (F_c) along the axes $Gxyz$, let a, b, c and v_x, v_y, v_z denote the components of (ω) and (V_r), resp., along the same axes, and finally let p, q, r denote the components of the instantaneous rotation of the solid. That will give:

$$X_c = -2\,m\,(b\,v_z - c\,v_y)\,,$$
$$Y_c = -2\,m\,(c\,v_x - a\,v_z)\,,$$
$$Z_c = -2\,m\,(a\,v_y - b\,v_x)\,,$$

with

$$v_x = q\,z - r\,y\,, \qquad v_y = r\,x - p\,z\,, \qquad v_z = p\,y - q\,x\,.$$

Therefore:

$$X_c = -2\,m\,[b\,(p\,y - q\,x) - c\,(r\,x - p\,z)]\,,$$
$$Y_c = -2\,m\,[c\,(q\,z - r\,y) - a\,(p\,y - q\,x)]\,,$$
$$Z_c = -2\,m\,[a\,(r\,x - p\,z) - b\,(q\,z - r\,y)]\,.$$

Having done that, we know that the work done by the composite centrifugal force is zero. The same thing is true for the work done by the forces (g), which admits a resultant that passes through G. The *vis viva* is then constant during the motion.

(1) $$A(p^2 + q^2) + C\,r^2 = h\,.$$

On the other hand, the reactions of the plane $x_1\,G\,y_1$ meet all of the axis Gz. We then write the third Euler equation:

$$C\,\frac{dr}{dt} = N\,.$$

Here:

$$N = \sum (x\,Y_c - y\,X_c)\,,$$

and if one takes into account the fact that the axes $Gxyz$ are principal axes of inertia then one will find that:

$$N = -2\,a\,q \sum m\,x^2 + 2\,b\,p \sum m\,y^2 = G\,(b\,p - a\,q)\,.$$

Thus:

(2) $$\frac{dr}{dt} = b\,p - a\,q\,.$$

Now introduce the variables φ and ψ. If λ denotes the angle xGS then one will have:

$$a = \omega \sin \lambda \sin \varphi + \omega \cos \lambda \cos \psi \cos \varphi\,,$$
$$b = \omega \sin \lambda \cos \varphi - \omega \cos \lambda \cos \psi \sin \varphi\,,$$
$$c = \omega \sin \lambda\,.$$

Furthermore, since θ is equal to $\pi / 2$, $d\theta / dt$ and cos θ will be equal to zero, while sin $\theta = 1$. It then results that:

$$p = \psi' \sin\varphi, \quad q = \psi' \cos\varphi, \quad r = \varphi'.$$

Equation (2) will then become:

$$\frac{dr}{dt} = -\ \omega \cos\lambda \cos\psi \frac{d\psi}{dt},$$

or rather:

$$r = \frac{d\varphi}{dt} = -\ \omega \cos\lambda \sin\psi + K,$$

in which K denotes a constant. As for equation (1), it will give:

$$A\psi'^3 + C\varphi'^3 = h.$$

The problem is then solved by quadratures. One will have:

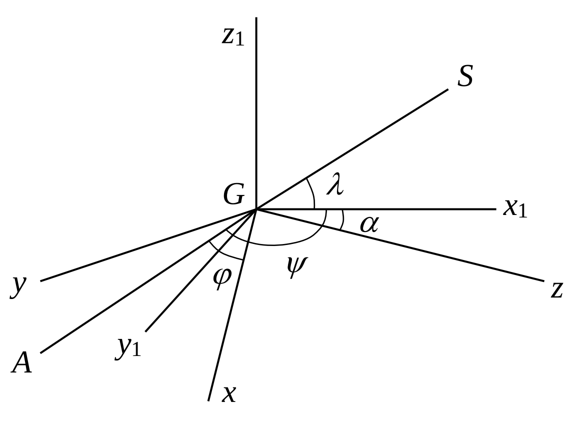
$$A\psi'^3 + C(K - \omega\cos\lambda\sin\psi)^2 = h,$$

or rather:

$$dt = \frac{A\,d\psi}{\sqrt{h - C(K - \omega\cos\lambda\sin\psi)^2}},$$

and similarly:

$$d\varphi = \frac{(\omega\cos\lambda\sin\psi + K)\,d\psi}{\sqrt{h - C(K - \omega\cos\lambda\sin\psi)^2}}.$$



If one sets sin $\psi = u$ then t and φ will be expressed as functions of u by elliptic quadratures. sin ψ is a doubly-periodic function of t.

The discussion is completed with no difficulty. If one supposes that at the beginning of the motion, $d\varphi / dt$ is positive and very large, while K is positive and very large, and one neglects the term in ω^2 then:

$$\left(\frac{d\psi}{dt}\right)^2 = L^2 \sin\psi + \text{const.},$$

with:

$$L^2 = \frac{2KC\omega\cos\lambda}{A}.$$

On the other hand, if α denotes the angle x_1Gz then one will have $\alpha = \psi - \pi / 2$. Thus:

$$\left(\frac{d\alpha}{dt}\right)^2 = L^2 \cos\alpha + \text{const.},$$

i.e., the axis Gz is animated with pendular motion around Gx_1 . In reality, as a result of air resistance and friction, the axis Gz will stop at Gx_1 after a certain amount of time, i.e., along the projection of the Earth's axis onto the fixed plane x_1Gy_1 .

Now employ Gilbert's method. Calculate the semi-*vis viva* T_3 of the motion of the system with respect to axes Σ of fixed stellar directions that pass through G. We have:

$$T_3 = T + \mathcal{G} + \mathcal{V}.$$

Here, T is the semi-*vis viva* of the solid with respect to the axes $G\,x_1\,y_1\,z_1$:

$$2T = A(p^2 + q^2) + C\,r^2 = A\psi'^2 + C\,\varphi'^2.$$

$\mathcal{G}$ is one-half the product of ω^2 with the moment of inertia of the solid with respect to GS. The equation of the ellipsoid of inertia is:

$$A(x^2 + y^2) + C\,z^2 = 1 .$$

If u denotes the angle between GS and GZ and ρ^2 is the inverse of the desired moment of inertia then one will have:

$$A\sin^2 u + B\cos^2 u = \frac{1}{\rho^2}.$$

On the other hand, one immediately finds that:

$$\cos u = \sin\,\psi \cos\,\lambda .$$

Therefore:

$$\frac{1}{\rho^2} = A(1 - \cos^2\lambda \sin^2\psi) + C\cos^2\lambda \sin^2\psi,$$

and as a result:

$$2\mathcal{G} = \omega^2 (C - A)\cos^2\lambda \sin^2\psi + \omega^2 A\ .$$

Finally, let us calculate $\mathcal{V}$. If (M) denotes the moment of the relative quantities of motion of the solid with respect to G, and β denotes the angle between (M) and GS then we will have:

$$\mathcal{V} = \omega\,M \cos\,\beta .$$

However, the components (*M*) along *Gx*, *Gy*, *Gz* are *Ap*, *Aq*, *Cr*. Let us calculate the direction cosines ξ, η, ζ of *GS* with respect to *Gx*, *Gy*, *Gz*, resp. We will find (upon taking into account the fact that $\theta = \pi / 2$) that:

$$\begin{aligned}\xi &= \quad \cos\lambda \cos\psi \cos\varphi + \sin\lambda \sin\varphi ,\\ \eta &= -\cos\lambda \cos\psi \sin\varphi + \sin\lambda \cos\varphi ,\\ \zeta &= \quad \cos\lambda \sin\psi .\end{aligned}$$

Moreover:

$$\mathcal{V} = A\,(p\,\xi + q\,\eta) + C\,r\,\zeta ,$$

so upon replacing p with $\psi' \sin\varphi$, q with $\psi' \cos\varphi$, and r with φ', we will have:

$$\mathcal{V} = A\psi' \sin\lambda + C\varphi' \cos\lambda \sin\psi .$$

By definition:

$$\begin{aligned}2T_3 &= A\psi'^2 + C\varphi'^2 + 2\,\omega(A\psi' \sin\lambda + C\varphi' \cos\lambda \sin\psi) + \omega^2 (C - A)\cos^2\lambda \sin^2\psi + u^2 A\\ &= 2\,(T_2 + T_1 + T_0) .\end{aligned}$$

Moreover, the active force (relative to the axes Σ) that is exerted on each point *M* of solid is $m(g)$. $U + \mathcal{N}$ is therefore zero here.

The equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial T_3}{\partial \varphi'}\right) = \frac{\partial T_3}{\partial \varphi} ,$$

$$\frac{d}{dt}\left(\frac{\partial T_3}{\partial \psi'}\right) = \frac{\partial T_3}{\partial \psi} .$$

The first one gives:

$$\frac{d}{dt}\,(C\varphi' + \omega C \cos\lambda \sin\psi) = 0 ,$$

or rather:

$$\varphi' = -\,\omega \cos\lambda \sin\psi + K .$$

Furthermore, we can substitute the generalized equation of *vis viva* (see page 94) for the second equation:

$$\frac{d}{dt}(T_2 - T_0) = Q_1\,q_1' + Q_2\,q_2' + \cdots + Q_k\,q_k' - \frac{\partial T}{\partial t} ,$$

which will give:

$$T_3 - T_0 = A\psi'^2 + C\varphi'^2 + \omega^2 (C - A)\cos^2\lambda \sin^2\psi + \omega^2 A = h$$

here.

If one neglects the term $\omega^2 (C - A)\cos^2\lambda \sin^2\psi$ [which amounts to neglecting the forces $m(\gamma_e')$ that were considered above] then one will indeed find the equation:

$$A\psi'^2 + C\,\varphi'^2 = \text{const.}$$

When one keeps the aforementioned term, t and φ will be given as functions of ψ by two elliptic quadratures.

LECTURE 11

APPLYING THE LAGRANGE EQUATIONS TO THE STUDY OF SMALL MOTIONS.

When the constraints on a frictionless system are independent of time and the active forces that are exerted on it admit a force function U, one knows that the necessary and sufficient conditions for equilibrium are:

$$\frac{\partial U}{\partial q_1} = 0 , \qquad \frac{\partial U}{\partial q_2} = 0 , \qquad \ldots, \qquad \frac{\partial U}{\partial q_k} = 0 .$$

(q_1, q_2, …, q_k are the K independent parameters that the position of the system depends upon.)

Those equalities are necessary conditions for U to present a maximum or a minimum. We shall show that if the force function is a maximum then the equilibrium is stable. That theorem is due to Lejeune-Dirichlet.

We can always suppose that the values of the parameters that correspond to the equilibrium position are $q_1 = 0$, $q_2 = 0$, …, $q_k = 0$, and that U is zero for those values. By definition, the equilibrium will be stable if the system deviates from its equilibrium position as little as one desires for initial conditions that are sufficiently close to equilibrium conditions. More precisely, let q_i^0, $q_i'^0$ be the initial conditions under which one releases the system, and let ε be a number that is given in advance and is as small as one pleases. One can find a number η that is small enough that the q_i^0, $q_i'^0$ are less than η in absolute value such that the $| q_i |$ will remain less than ε under the entire duration of motion.

Having recalled that definition, assume that U is zero and a maximum for the values $q_1 = 0$, $q_2 = 0$, …, $q_k = 0$. I say that the equilibrium is *stable*.

Indeed, since U is a maximum, one can find a number ε that is small enough that U is negative when the $| q_i |$ do not exceed ε and are not all zero. Furthermore, all of the numbers less than ε enjoy the same property:

$$\begin{aligned} &U < 0 , && \text{if} \quad | q_i | \leq \varepsilon , \\ &U = 0 , && \text{if} \quad q_1 = q_2 = \ldots = q_k = 0 . \end{aligned}$$

In particular, give the values of $+ \varepsilon$ and $- \varepsilon$ to q_i, give all of the other parameters q_j values whose modulus is less than or equal to ε, and let A_i be the greatest value of U for those values of q. A_i is an essentially-negative number. Moreover, let A the greatest of the numbers A_i. A is negative and non-zero. From that, one will necessarily have:

$$U \leq A$$

when one of the parameters q_i attains one of the values $\pm\,\varepsilon$, while the other parameters have moduli that are not greater than ε.

On the other hand, write the *vis viva* equation:

$$T = U + \sum \tfrac{1}{2} m v_0^2 - U_0 = U + h\,.$$

One can always find a number η that is small enough that the $|\,q_i^0\,|$ and the $|\,q_i'^0\,|$ do not exceed η, so the constant $h = \left(\sum \tfrac{1}{2} m v_0^2 - U_0\right)$ will be less than $-A$, i.e., such that one has $h + A < 0$. Under those conditions, the $|\,q_i\,|$ will remain less than ε under the entire duration of the motion. Otherwise, one would have at least one parameter q_i that attains one of the values $\pm\,\varepsilon$ at the instant t, while the moduli of the other parameters would not exceed ε, and one would have:

$$U + h \le A + h < 0\,,$$

so

$$T < 0\,,$$

which would be impossible.

The equilibrium is therefore stable. Q.E.D.

I should add that the *vis viva* of the system itself remains less than h, which can be chosen to be as small as one pleases.

Study of small oscillations of a system. – The Lagrange equations are then convenient to the study of small motions of the system around its stable equilibrium. The method that we shall indicate applies to the case in which the equilibrium is unstable, as well, but only for a very short length of time, unless one knows in advance that the system deviates only slightly from its equilibrium position for the given initial conditions.

Therefore, place the system under initial conditions that are very close to stable equilibrium conditions. The parameters q_i vary only slightly under the motion, so one can regard the coefficients a_{ij} in the *vis viva:*

$$T = \tfrac{1}{2} \sum_{i,j} a_{ij}\, q_i'\, q_j' \qquad (a_{ij} = a_{ji})$$

as constants α_{ij} : α_{ij} denotes the value of the function a_{ij} for $q_1 = q_2 = \ldots = q_k = 0$. Similarly, develop the force function in powers of q_i :

$$U = U_0 + U_1 + U_2 + \ldots,$$

$U_0 = 0$, U_1, U_2, … are homogeneous functions of degree one, two, …, resp., then U_1 will be identically zero, by hypothesis (since $\partial U / \partial q_i = 0$ for $q_1 = q_2 = \ldots = q_k = 0$). We will then reduce U to the term:

$$U_2 = \tfrac{1}{2} \sum_{i,j} \beta_{ij}\, q_i'\, q_j' \qquad (\alpha_{ij} = \alpha_{ji})$$

Those approximations amount to considering the q_i and q_i' to be infinitely-small of first order and neglecting the quantities of the form $q_i\ q_j$, $q_i' q_j$, $q_i' q_j'$, and higher-order quantities, in comparison to the latter quantities. Indeed:

$$\frac{\partial T}{\partial q_i'} = \sum_{j=1}^{k} a_{ij}\, q_j' ,$$

$$\frac{d}{dt}\frac{\partial T}{\partial q_i'} = \sum_{j=1}^{k} \alpha_{ij}\, q_j'' + \varepsilon , \quad \frac{\partial T}{\partial q_i} = \varepsilon' ,$$

$$\frac{\partial T}{\partial q_i} = \sum_{j=1}^{k} \beta_i\, q_j + \eta ,$$

in which ε, ε', and η, are negligible compared to the q_j, from the preceding convention. We can then replace the Lagrange equation:

(1) $$\frac{d}{dt}\frac{\partial T}{\partial q_i'} - \frac{\partial T}{\partial q_i} = \frac{\partial U}{\partial q_i}$$

with the approximate equality:

(2) $$\sum_{j=1}^{k} \alpha_{ij}\, q_j'' = \sum_{j=1}^{k} \beta_i\, q_j .$$

In other words, one can consider the a_{ij} in the Lagrange equations (viz., the coefficients in T) to be constants α_{ij}, and reduce U to U_2 . The method will break down when the determinant δ of the α_{ij} is zero. That will then amount to changing the parameters q_j .

Having said that, equation (2) can be written:

$$\frac{d^2}{dt^2}\sum_{j=1}^{k} \alpha_{ij}\, q_j = \sum_{j=1}^{k} \beta_i\, q_j .$$

Introduce the function:

$$s = \tfrac{1}{2}\sum_{i,j} \alpha_{ij}\, q_i\, q_j .$$

We will have:

$$\sum_{j=1}^{k} \alpha_{ij}\, q_j = \frac{\partial s}{\partial q_i} .$$

Therefore, equation (2) is equivalent to the following one:

(3) $$\frac{d^2}{dt^2}\frac{\partial s}{\partial q_i} = \frac{\partial U_2}{\partial q_i} ,$$

in which s and U_2 are two homogeneous functions of degree two with respect to the q_i .

Equations (3) form a system of n linear equations with constant coefficients. One will then have:

$$q_i = A_1^i e^{\alpha_1 t} + A_2^i e^{\alpha_2 t} + \cdots,$$

in which the exponents are real or imaginary.

It is appropriate to point out that even if the characteristic equation of the system (3) has multiple roots, the integrals will contain time only in the exponentials.

Indeed, if one is given two quadratic forms s and U_2, at least one of which includes nothing but squares, then one can always make the product terms in the two forms disappear by the same linear substitution. Now, s includes nothing but squares, since otherwise upon annulling all of the q' except one, one would annul s, and the *vis viva* T would be zero (for $q_1 = q_2 = \ldots = q_k = 0$), even though not all of the velocities are zero.

One can then reduce s and U to the forms:

$$s = \tfrac{1}{2}\sum_i a_i q_i^2 ,$$
$$U_2 = \tfrac{1}{2}\sum_i b_i q_i^2$$

by the same linear substitution, and equations (3) will become:

$$\frac{d^2 q_i}{dt^2} = \beta_i q_i .$$

If one supposes that the equilibrium is stable then the q_i can enter into only trigonometric symbols. Indeed, U_2 is negative for small values of q_i . Hence, the β_i are negatives $\beta_i = -\gamma_i^2$, and one will have:

$$q_i = A_i \sin \gamma_i t + B_i \cos \gamma_i t .$$

Characteristic equation of the system (3). – Let us look for the characteristic equation of the system:

(3) $$\frac{d^2}{dt^2}\frac{\partial s}{\partial q_i} = \frac{\partial U_2}{\partial q_i},$$

directly.

In order to do that, I multiply each side of equation (3) by a constant λ_i and add them. That will give:

$$\frac{d^2}{dt^2}\sum \lambda_i \frac{\partial s}{\partial q_i} = \sum \lambda_i \frac{\partial U_2}{\partial q_i} .$$

Now, one has:

$$\sum \lambda_i \frac{\partial s}{\partial q_i} = \sum q_i \frac{\partial s}{\partial q_i} .$$

Similarly:

$$\sum \lambda_i \frac{\partial U_2}{\partial q_i} = \sum q_i \frac{\partial U_2}{\partial \lambda_i} ,$$

in which $\frac{\partial s}{\partial \lambda_i}$, $\frac{\partial U_2}{\partial \lambda_i}$ are the derivatives of s and U_2, resp., after one has replaced the q_i with the λ_i. One can then write:

$$\frac{d^2}{dt^2} \sum_i \frac{\partial s}{\partial \lambda_i} q_i = \sum_i \frac{\partial U_2}{\partial \lambda_i} q_i .$$

If one sets:

$$\sum_i \frac{\partial s}{\partial \lambda_i} q_i = \frac{1}{\mu} \sum_i \frac{\partial U_2}{\partial \lambda_i} q_i = Q$$

then the equation to be integrated will then become:

$$\frac{\partial^2 Q}{\partial t^2} = \mu\, Q .$$

If I can find k distinct values of μ (which correspond to k systems of values λ_i) then I will know $2k$ distinct integrals of the system (3), which is then found to have been integrated.

Now, the indeterminates λ_i, μ must satisfy the condition:

$$\sum \left(\mu \frac{\partial s}{\partial \lambda_i} - \frac{\partial U_2}{\partial \lambda_i} \right) q_i = 0 ,$$

i.e., the k equations:

$$\mu \frac{\partial s}{\partial \lambda_i} - \frac{\partial U_2}{\partial \lambda_i} = 0 \qquad (i = 1, 2, \ldots, k).$$

However:

$$s = \tfrac{1}{2} \sum_{i,j} a_{ij}\, \lambda_i\, \lambda_j ,$$

$$U_2 = \tfrac{1}{2} \sum_{i,j} b_{ij}\, \lambda_i\, \lambda_j .$$

One will then have:

$$\mu \sum_j a_{ij}\, \lambda_j - \sum_j b_{ij}\, \lambda_j = 0 ,$$

or rather:

$$(\mu\, a_{i1} - b_{i1})\, \lambda_1 + (\mu\, a_{i2} - b_{i2})\, \lambda_2 + \ldots + (\mu\, a_{ik} - b_{ik})\, \lambda_k = 0 \qquad (i = 1, 2, \ldots, k).$$

In order for those k equations, which are linear and homogeneous with respect to the λ_i, to admit solutions for which all of the λ_i are non-zero, it is necessary and sufficient that the determinant Δ of those equations should be zero. One thus forms an equation in μ of degree k, which is the desired characteristic equation. If its k roots are distinct then one will get k distinct functions Q :

$$Q = \sum \lambda_i \frac{\partial s}{\partial q_i},$$

and the system (3) is integrated. If the characteristic equation admits a double root μ_1 then the system of equations that defines the ratios of the λ will become indeterminate, and one can make that double root correspond to two distinct functions Q, say, Q_1 and Q_1', and one will have:

$$\frac{d^2 Q_1}{dt^2} = \mu\, Q_1 ,$$
$$\frac{d^2 Q_1'}{dt^2} = \mu\, Q_1' .$$

More generally, if μ_1 is a multiple root of order i then all of the minors of order $i - 1$ of the determinant Δ are zero for $\mu = \mu_1$, and one can make i distinct functions Q correspond to that root μ_1. The system (3) is then found to be integrated in all cases.

In practice, one can begin by making just the products in s disappear. s will then reduce to the form:

$$s = \tfrac{1}{2} \sum a_i\, q_i^2 ,$$

so one will have the equations:

$$\frac{d^2}{dt^2} \sum_i \lambda_i\, q_i = \sum_i \frac{\partial U_2}{\partial \lambda_i} q_i ,$$

and if one sets:

$$\sum_i \lambda_i\, q_i = \frac{1}{\mu} \sum_i \frac{\partial U_2}{\partial \lambda_i} = Q$$

then one will have the conditions:

$$\frac{1}{\lambda_1} \frac{\partial U_2}{\partial \lambda_1} = \frac{1}{\lambda_2} \frac{\partial U_2}{\partial \lambda_2} = \ldots = \frac{1}{\lambda_k} \frac{\partial U_2}{\partial \lambda_k} = \mu .$$

For example, if k is equal to 3 then the preceding equations will define the axes of the quadric:

$$U(\lambda_1, \lambda_2, \lambda_3) = 1 .$$

If the equation in μ admits a double root then the surface will be one of revolution. There will be an infinitude of axes, so an infinitude of values λ_2 / λ_1 , λ_3 / λ_1 that correspond to that value of μ. If the equation in μ is a triple root then one can take λ_1, λ_2, λ_3 arbitrarily.

Applications.

I. *Study of the motion of a spherical pendulum in the neighborhood of its equilibrium position*. – We can define the position of the point M on the sphere by its longitude φ and its latitude θ. Indeed, φ is a function of the point x, y, z of the sphere that becomes indeterminate at the highest and lowest points on the sphere. If one forms the *vis viva:*

$$T = \tfrac{1}{2} m R^2 (\theta'^2 + \sin^2\theta\,\varphi'^2)$$

then the determinant δ of the a_{ij} (see page 150) will be:

$$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

here (for $\theta = 0$ and φ arbitrary), and the method cannot be applied.

We take the origin of the axes to be the center O of the sphere, while the z-axis points in the direction of gravity, and the parameters are the x and y coordinates. We have:

$$x = x\,, \quad y = y\,, \quad z = \sqrt{R^2 - x^2 - y^2}\,,$$

so we find immediately that:

$$T = \tfrac{1}{2} m \left(x'^2 + y'^2 + \frac{(x\,x' + y\,y')^2}{R^2 - x^2 - y^2} \right);$$

hence:

$$s = \tfrac{1}{2} m \left(x'^2 + y'^2 \right).$$

On the other hand:

$$U = m\,g\,(z - R) = U_2 + \varepsilon\,.$$

A very simple calculation will show that for $x = 0$, $y = 0$ ($z = + R$), one has:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2} = \frac{-1}{R}\,, \qquad \text{and} \qquad \frac{\partial^2 z}{\partial x\,\partial y} = 0\,.$$

Therefore:

$$U_2 = -\,\frac{mg}{2R}\,(x^2 + y^2)\,.$$

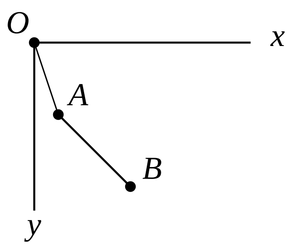
Hence, the two equations:

$$x'' = -\frac{g}{R}x, \quad y'' = -\frac{g}{R}y$$

will determine x and y:

$$x = A\cos t\sqrt{\frac{g}{R}} + B\sin t\sqrt{\frac{g}{R}} \quad \text{and} \quad y = A'\cos t\sqrt{\frac{g}{R}} + B'\sin t\sqrt{\frac{g}{R}}\,.$$

II. *Let two massive homogeneous bars OA, AB have densities ρ and ρ' and lengths l and l', resp. The first one is fixed at its extremity O and articulates with AB at A. Find the motion of the system when released with no initial velocity in a vertical plane in the neighborhood of its stable equilibrium position.* – The system moves in the initial plane xOy, where Oy is the direction of gravity. Let θ denote the angle between OA and Oy and let φ denote the angle between AB and Oy. The force function U is:



$$U = g\left[\tfrac{1}{2}\rho l^2\cos\theta + \rho' l'(l\cos\theta + \tfrac{1}{2}l'\cos\varphi)\right] + \text{const.}$$

here.

The stable equilibrium position corresponds to the values $\theta = 0$, $\varphi = 0$ of the two parameters.

Let us calculate the *vis viva*. We will have for OA:

$$\sum mv^2 = \int_0^l \rho r^2\,\theta'^2 dr = \tfrac{1}{3}\rho l^3\theta'^2,$$

and for AB:

$$\sum mv^2 = \int_0^{l'} \rho'\left[r^2\,\theta'^2 + 2\,r\,l\cos(\theta-\varphi)\theta'\varphi' + \tfrac{1}{3}l^3\varphi'^2\right]dr'$$
$$= \rho' l^2 l'\theta'^2 + \rho' l' l^2\cos(\theta-\varphi)\theta'\varphi' + \tfrac{1}{3}\rho' l'^3\varphi'^2.$$

As a result:

$$2T = l^2(\tfrac{1}{3}\rho l + \rho' l')\theta'^2 + \rho' l' l^2\cos(\theta-\varphi)\theta'\varphi' + \tfrac{1}{3}\rho' l'^3\varphi'^2.$$

The expressions for s and U_2 here are then:

$$2s = l^2(\tfrac{1}{3}\rho l + \rho' l')\theta^2 + \rho' l\, l'^2\,\theta\varphi + \tfrac{1}{3}\rho' l'^3\varphi^2,$$

and

$$U_2 = -\frac{g}{2}[l(\tfrac{1}{2}\rho l + \rho' l')\theta^2 + \tfrac{1}{2}\rho' l'^2\varphi^2].$$

The approximate motion will be determined from the equations:

$$(\alpha) \qquad \begin{cases} \dfrac{l}{g}\left[1-\dfrac{\rho l}{3\rho l+6\rho' l'}\right]\theta''+\dfrac{\rho' l'^2}{g\,(\rho l+2\rho' l')}\,\varphi''=-\theta, \\ \dfrac{l}{g}\,\theta''+\dfrac{2l'}{3g}\,\varphi''=-\varphi, \end{cases}$$

which is a system of the form:

$$A\theta''+B\varphi'' = -\,\theta,$$
$$B\theta''+C\varphi'' = -\,\varphi.$$

Let us look for its characteristic equation. Multiply the first one by λ, the second one by λ', and determine λ and λ' in such a fashion that one has:

$$\frac{A\lambda+B'\lambda'}{\lambda} = \frac{B\lambda+C\lambda'}{\lambda'} = \mu\,.$$

The value of μ is given by the equality:

$$(A-\mu)(C-\mu) - BB' = 0\,,$$

in which B and B' have the same sign. The two values μ_1 and μ_2 of μ are real and outside of the two positive numbers A and C, and positive, moreover. Indeed, $AC-BB'$ is positive, since otherwise the *vis viva* T would be annulled without all of the velocities being zero. A direct calculation will further show that one has:

$$AC-BB' = \frac{l\,l'}{3g^2(\rho l+2\rho' l')}\left(\tfrac{4}{3}\rho l+\rho' l'\right).$$

We can take the corresponding multipliers λ and λ' to be the numbers:

$$\lambda=\mu-C=\mu-\frac{2l'}{3g},\quad \lambda'=B=\frac{\rho' l'^2}{g\,(\rho l+2\rho' l')},$$

and if we take:

$$Q_1 = (\mu_1 - C)\;\theta + B\;\varphi,$$
$$Q_2 = (\mu_2 - C)\;\theta + B\;\varphi,$$

or rather (upon introducing the variable $\psi = B\,\varphi - C\,\theta$):

$$Q_1 = \mu_1\;\theta + \psi, \qquad Q_2 = \mu_2\;\theta + \psi,$$

then we will know that Q_1 and Q_2 have the form:

$$Q_1 = a_1 \cos \omega_1 t + b_1 \sin \omega_1 t ,$$
$$Q_2 = a_2 \cos \omega_2 t + b_2 \sin \omega_2 t .$$

a_1, b_1, a_2, b_2 are arbitrary constants in those equalities, while $\omega_1 = \sqrt{\mu_1}$, $\omega_2 = \sqrt{\mu_2}$.

The resulting expressions for θ and ψ [after dividing all of the constants by $(\mu_1 - \mu_2)$] will then be:

$$\theta = a_1 \cos \omega_1 t + b_1 \sin \omega_1 t - a_2 \cos \omega_2 t - b_2 \sin \omega_2 t ,$$
$$\psi = \omega_1^2 (a_2 \cos \omega_2 t - b_2 \sin \omega_2 t) - \omega_2^2 (a_1 \cos \omega_1 t + b_1 \sin \omega_1 t) .$$

If the system is released with no initial velocity at time $t = 0$ then the constants b_1 and b_2 must satisfy the two conditions:

$$\omega_1 b_1 - \omega_2 b_2 = 0 ,$$
$$\omega_1^2 \omega_2 b_2 - \omega_2^2 \omega_1 b_1 = 0 ,$$

and since ω_1^2 is not equal to ω_2^2, b_1 and b_2 will be zero, one will then have:

$$\theta = a_1 \cos \omega_1 t - a_2 \cos \omega_2 t ,$$
$$\psi = \omega_1^2 a_2 \cos \omega_2 t - \omega_2^2 a_1 \cos \omega_1 t ,$$

or rather, upon letting θ_0 and ψ_0 denote the initial values of θ and ψ:

$$(\beta) \qquad \begin{cases} \theta = \dfrac{1}{\omega_1^2 - \omega_2^2} \left[(\theta_0 \, \omega_1^2 + \psi_0) \cos \omega_1 t - (\theta_0 \, \omega_2^2 + \psi_0) \cos \omega_2 t \right], \\[2ex] \psi = \dfrac{1}{\omega_1^2 - \omega_2^2} \left[(\theta_0 \, \omega_1^2 + \psi_0) \cos \omega_1 t - (\theta_0 \, \omega_2^2 + \psi_0) \cos \omega_2 t \right]. \end{cases}$$

Since θ_0 and ψ_0 are very small, θ and ψ will remain very small. If the ratio ω_1 / ω_2 is commensurable then the motion will be periodic.

If one supposes that $\rho = \rho' = 1$ and $l = l'$ then equations (α) will become:

$$\tfrac{8}{3} \theta'' + \varphi'' = - \frac{3g}{l} \theta ,$$
$$\theta' + \frac{2\varphi''}{3} = - \frac{g}{l} \varphi ,$$

and the numbers μ_1, μ_2 will be the roots of the equation:

$$\mu^2 - \frac{14 l}{9g} \mu + \frac{7 l}{27 g^2} = 0 ,$$

so

$$\mu_1 = \frac{l}{9g}(7+2\sqrt{7}) , \qquad \mu_2 = \frac{l}{9g}(7-2\sqrt{7}) .$$

In order to get θ and ψ, it will suffice to replace ω_1 and ω_2 with $\sqrt{\mu_1}$ and $\sqrt{\mu_2}$, resp., in the equalities (β). In this particular case, $\psi = (l' / 3g)(\psi - 2\theta)$.

LECTURE 12

CANONICAL EQUATIONS. THEORY OF THE LAST MULTIPLER.

The motion of a system that depends upon k parameters is determined by the k Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = Q_i \qquad (i = 1, 2, \ldots, k).$$

Those equations, which are linear in $q_1'', q_2'', \ldots, q_k''$ can be solved for those variables. We proved that by establishing that the principal determinant of the expressions $\partial T_2 / \partial q_i'$, which are linear and homogeneous in $q_1', q_2', \ldots, q_k'$, is not zero (T_2 is the second-degree homogeneous part of $T = T_2 + T_1 + T_0$).

The Lagrange equations define a system of k second-order equations for the k functions $q_1, q_2, \ldots, q_k$ of t. However, that system is equivalent to the system of $2k$ equations:

(1) $$\frac{dq_i}{dt} = q_i', \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = Q_i \qquad (i = 1, 2, \ldots, k),$$

which are first-order equations in which the $2k$ functions q_i, q_i' of t figure. When those equations (2) are solved for the dq_i' / dt, they can be written:

$$\frac{dq_i}{dt} = q_i', \qquad \frac{dq_i'}{dt} = f_i(t, q_1, q_2, \ldots, q_k, q_1', q_2', \ldots, q_k') \qquad (i = 1, 2, \ldots, k).$$

However, one must solve them in each case.

Poisson indicated a change of variables under which one replaces equation (1) with a system of $2k$ first-order equations that are found to solved for the derivatives that appear in them.

The method consists of replacing the variables $q_1', q_2', \ldots, q_k'$ with new variables $p_1, p_2, \ldots, p_k$ that are coupled with the preceding ones by the k relations:

(2) $$p_i = \frac{\partial T}{\partial q_i'} \qquad (i = 1, 2, \ldots, k).$$

Equations (2), which are linear in q'_1, q'_2, …, q'_k, can be solved for those variables, since the determinant of the expressions $\partial T_2 / \partial q'_i$ is non-zero. One then infers the q'_i as functions of the p_i from equations (2) in the form:

(2′) $$q'_i = B^i(t, q_1, q_2, \ldots, q_k) + \sum_{j=1}^{k} p_j A^i_j(t, q_1, q_2, \ldots, q_k) \;.$$

Upon substituting those values for q'_i in equations (1), one will obtain $2k$ first-order equations in t, p_i, q_i .

Those $2k$ equations are found to be soluble for the dq_i / dt, dp_i / dt . Indeed, they can be written:

(3) $$\frac{dq_i}{dt} = q'_i \,, \qquad \frac{dp_i}{dt} = \frac{\partial T}{\partial q_i} + Q_i \qquad (i = 1, 2, \ldots, k) \;.$$

In those equations, one must replace q'_1, q'_2, …, q'_k with their values that one derived from (2) everywhere. However, the remarkable fact is that after that substitution has been made, the right-hand sides of equations (3) can be expressed with the aid of the Q_i and the partial derivatives of the same function K (t, q_1, q_2, …, q_k, p_1, p_2, …, p_k).

Indeed, set:

$$K = p_1\, q'_1 + p_2\, q'_2 + \cdots + p_k\, q'_k - T \;.$$

The quantity K can be regarded as a function of the variables t, q_1, q_2, …, q_k, q'_1, q'_2, …, q'_k, p_1, p_2, …, p_k that are coupled by equations (2).

Let t be constant in K ($t = t_0$) and give variations δq_i, $\delta q'_i$, δp_i to q_i, q'_i, p_i that are compatible with equations (2) (for that value of t). One will have:

$$\delta K = \sum_i (p_i\, \delta q'_i + q'_i\, \delta p_i) - \sum_i \left(\frac{\partial T}{\partial q_i} \delta q_i + \frac{\partial T}{\partial q'_i} \delta q'_i \right) ,$$

i.e., upon taking (2) into account:

$$dK = \sum_i q'_i\, \delta p_i - \sum_i \frac{\partial T}{\partial q_i} \delta q_i \;.$$

On the other hand, let (K) denote what the function K will become when one replaces the q' in it with functions of p, q, and t. Leave t constant ($t = t_0$) and give variations δp, δq to p, q. That will give:

$$\delta (K) = \sum_i \frac{\partial (K)}{\partial p_i} \delta p_i + \sum_i \frac{\partial (K)}{\partial q_i} \delta q_i \;.$$

Now, $\delta(K)$ is identical to δK for arbitrary values of δp, δq. That demands that one must have:

$$q'_i = \frac{\partial(K)}{\partial p_i},$$
$$-\frac{\partial T}{\partial q_i} = \frac{\partial(K)}{\partial q_i}.$$

The latter equations are then consequences of equations (3).

That implies the following conclusion: Equations (3) can be written:

$$(4) \qquad \begin{cases} \dfrac{dp_i}{dt} = -\dfrac{\partial K}{\partial q_i} + Q_i \\[2mm] \dfrac{dq_i}{dt} = \dfrac{\partial K}{\partial p_i}, \end{cases}$$

upon setting:

$$K = \sum p_i \, q'_i - T$$

and assuming that one replaces the q' in K with functions of the p, q, and t using the relations:

$$p_i = \frac{\partial T}{\partial q'_i}.$$

Equations (4) are called the *canonical equations.*

Case in which the constraints are independent of time. – In this case, we have:

$$\sum p_i \, q'_i = \sum \frac{\partial T}{\partial q'_i} q'_i = 2T,$$

because T is homogeneous of degree two in $q'_1, q'_2, \ldots, q'_k$. Therefore:

$$K = 2T - T = T.$$

Let (T) denote the function T when one has replaced the q' as functions of the p, q. The canonical equations are written:

$$\frac{dp_i}{dt} = -\frac{\partial(T)}{\partial q_i} + Q_i,$$

$$\frac{dq_i}{dt} = \frac{\partial(T)}{\partial p_i} \; .$$

The p_i are linear homogeneous functions of the q_i', and conversely. $(T) = K$ is a homogeneous function of degree two in the p_i . One has both:

$$p_i = \frac{\partial T}{\partial q_i'}, \qquad q_i' = \frac{\partial(T)}{\partial p_i} \qquad \text{and} \qquad \frac{\partial T}{\partial q_i} = -\frac{\partial(T)}{\partial q_i} \; .$$

Conversely, if (K) is a homogeneous function of degree two in $p_1, p_2, \dots, p_k$ then the constraints are independent of time. Indeed, the equalities:

$$q_i' = \frac{\partial(K)}{\partial p_i}$$

show us that the p_i are linear homogeneous functions of the q_i'. As a result, the function:

$$T = \sum p_i \, q_i' - K$$

(in which the p are expressed as functions of the q', q, and t) is a homogeneous function of the q_i'. That will demand that T_1 and T_0 are zero, or that T_0 is equal to:

$$\sum m \left[\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2 \right],$$

and that sum can be zero only if the $\partial x / \partial t$, $\partial y / \partial t$, $\partial z / \partial t$ are zero identically, i.e., if the constraints are independent of time.

Case in which the constraints depend upon time. – Decompose T into the sum of three terms:

$$T = T_2 + T_1 + T_0 \, ,$$

as we did before.

We can write:

$$K = \sum \left(\frac{\partial T_2}{\partial q_i'} + \frac{\partial T_1}{\partial q_i'} \right) q_i' - T_2 - T_1 - T_0 \; .$$

Now:

$$\sum_i \frac{\partial T_2}{\partial q_i'} q_i' = 2T_2 \, , \qquad \sum_i \frac{\partial T_1}{\partial q_i'} q_i' = T_1 \, ,$$

Therefore:

$$K = T_2 - T_0 .$$

T_2 is a quadratic form in the q', and T_0 depends upon only the q and t. One replaces the q' as functions of the p in T_2, and since the q' are linear, but no longer homogeneous, with respect to the p, T_2 will become a function (T_2) of degree two in $p_1, p_2, \ldots, p_k$, but it will no longer be homogeneous.

Here, one has:

$$p_i = \frac{\partial T_2}{\partial q_i'} + \frac{\partial T_1}{\partial q_i'} , \qquad q_i' = \frac{\partial (T_2)}{\partial p_i} , \quad - \frac{\partial (T_2)}{\partial q_i} = \frac{\partial T_2}{\partial q_i} + \frac{\partial T_1}{\partial q_i} .$$

One sees, by definition, that $(K) = (T_2) - T_0$ is a polynomial of degree two with respect to the p whose coefficients depend upon t and the q in an arbitrary fashion.

Observe that the canonical equations can be written:

$$\frac{dq_i}{dt} = \frac{\partial (T_2)}{\partial p_i} ,$$
$$\frac{dp_i}{dt} = -\frac{\partial (T_2)}{\partial p_i} - \frac{\partial T_0}{\partial q_i} + Q_i ,$$

In the particular case where T_0 is simply a function of t, it is legitimate to replace (K) with (T_2).

Case in which there exists a force function. – Suppose that $Q_1, Q_2, \ldots, Q_k$ are derivatives with respect to $q_1, q_2, \ldots, q_k$ of the same function U, which can depend upon time, but not the velocities, moreover:

$$P_i = \frac{\partial U}{\partial q_i}(q_1, q_2, \ldots, q_k, t) .$$

After introducing the variables p_i, the function of U, which is independent of the q_i', will be independent of the p_i, and if one sets:

$$H = K - U$$

then the canonical equations can be written:

$$(5) \qquad \begin{cases} \dfrac{dp_i}{dx} = -\dfrac{\partial H}{\partial q_i} , \\[2ex] \dfrac{dq_i}{dx} = \dfrac{\partial H}{\partial p_i} , \end{cases}$$

because $\dfrac{\partial H}{\partial p_i} = \dfrac{\partial K}{\partial p_i} - \dfrac{\partial U}{\partial p_i} = \dfrac{\partial K}{\partial p_i}$. The right-hand sides are then expressed with the aid of only the function H. When the constraints depend upon time, U will generally depend upon time, as well as H. When the constraints are independent of time, $K = T$, and as a result:

$$H = (T) - U .$$

(T) is then a quadratic form with respect to the p_i that does not depend upon t. If, on the other hand, t does not appear in U then H will be a simple function of p_i, q_i :

$$H = \sum_{i,j} p_i\, p_j\, A_{ij}(q_1, q_2, \ldots, q_k) - U(q_1, q_2, \ldots, q_k) .$$

In the latter case, the *vis viva* theorem will provide an integral of motion, namely, the integral:

$$T - U = h ,$$

or rather:

$$H = h .$$

That equality must be a consequence of the canonical equations that define the motion. In other words, if one replaces the p_i, q_i in H with an arbitrary system of integrals $p_i(t)$, $q_i(t)$ of equations (5) then the function $H_1(t)$ thus-obtained must reduce to a constant. Now calculate dH_1 / dt :

$$\frac{dH_1}{dt} = \sum \left(\frac{\partial H}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial H}{\partial p_i} \frac{dp_i}{dt} \right) ,$$

or rather, since the $p_i(t)$, $q_i(t)$ satisfy equations (5):

$$\frac{dH_1}{dt} = \sum \left(\frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \equiv 0 ,$$

which is what we wished to verify.

Remark. – The Poisson transformation is particularly simple in the case where T has the form:

$$2T = \sum_i A_i(q_1, q_2, \ldots, q_k)\, q_i'^2 .$$

One then has:

$$p_i = A_i\, q_i'$$

and

$$2\,(T) = 2\,(K) = \sum_i \frac{p_i^2}{A_i}.$$

Conversely, if (K) has the form:

$$(K) = \tfrac{1}{2}\sum_i B_i\, p_i^2$$

then one will have:

$$q_i' = B_i\, p_i$$

and

$$2T = \sum_i \frac{q_i'^2}{B_i},$$

moreover. The constraints will then be independent of time, and as a result, the B_i will not include t.

That is what happens, for example, when one studies the motion of a free point with respect to orthogonal curvilinear coordinates. If those coordinates are Cartesian coordinates then:

$$2T = m\,(x'^2 + y'^2 + z'^2)\,,$$

so

$$p_1 = m\,x'\,, \qquad p_2 = m\,y'\,, \qquad p_3 = m\,z'\,,$$

and

$$2T = \frac{1}{m}\,[p_1^2 + p_2^2 + p_3^2]\ .$$

If those coordinates are polar coordinates in space r, θ, φ then one knows that:

$$2T = m\,(r'^2 + r^2\,\theta'^2 + r^2 \sin^2\theta\,\varphi'^2)\ .$$

Thus:

$$p_1 = m\,r'\,, \qquad p_2 = m\,r^2\theta'\,, \quad p_3 = m\,r^2 \sin^2\theta\,\varphi'\,,$$

and

$$2\,(T) = \frac{1}{m}\left[p_1^2 + \frac{p_2^2}{r^2} + \frac{p_3^2}{r^2\sin^2\theta}\right].$$

That is what happens when one studies the motion of a point on a surface when it is referred to the orthogonal coordinates on the surface. Hence, let a point M be referred to polar coordinates $r = OM$, $\theta = xOM$ in a plane xOy, and let it be attracted to the point O according to a function of distance. One will have:

$$2T = m\,(r'^2 + r^2\,\theta'^2)\,, \qquad p_1 = m\,r'\,, \qquad p_2 = \frac{m\,\theta'}{r^2}\,,$$

$$2\,(T) = \frac{1}{m}\left[p_1^2 + \frac{p_2^2}{r^2}\right],$$

$$H = (T) - U\,(r)\,.$$

The canonical equations are:

$$(\alpha) \qquad \left\{\begin{aligned} \frac{dr}{dt} &= \frac{p_1'}{m}, \\ \frac{dp_1'}{dt} &= \frac{p_2'}{m\,r^3} + \frac{\partial U}{\partial r}, \end{aligned}\right. \qquad \left\{\begin{aligned} \frac{d\theta}{dt} &= \frac{p_2}{m\,r^2}, \\ \frac{dp_2'}{dt} &= 0. \end{aligned}\right.$$

The latter equality then gives p_2' = const. (viz., the area integral). On the other hand, $H = h$ is an integral of the system (α). It would be easy to achieve the integration of that system directly. However, that integration will be presented in what follows as a consequence of some very general theorems that we shall now prove.

Meanwhile, I shall make one last remark in regard to the relative motion of systems.

Canonical equations of relative motion. – Let a system S be subject to given forces with respect to a system $Oxyz$.

One wishes to study its motion relative to the axes $O_1\,x_1\,y_1\,z_1$, which are animated with respect to the first ones by a given motion.

As we said, we refer the system to parameters $q_1, q_2, \ldots, q_k$ that are most convenient for determining the position relative to the axes $O_1\,x_1\,y_1\,z_1$. In order to form the canonical system that determines the motion, we can apply the three methods that we indicated in the context of the Lagrange equations (see Lecture 10).

1. Form the *vis viva* $2T_1$ with respect to $O_1\,x_1\,y_1\,z_1$ and take the canonical variables to be the variables $p_i = \partial T_1' / \partial q_i'$. (One should take care to add the two Coriolis forces to the given forces that are exerted on each point of the system.)

2. Form the *vis viva* $2T_2$ with respect to the axis $Oxyz$ and take the conjugate variables to be the variables $p_i = \partial T_2 / \partial q_i'$.

3. Form the *vis viva* $2T_3$ with respect to intermediate axes $O_1\xi\eta\zeta$ that are parallel to the axes $Oxyz$ and take the conjugate variables to be the variables $p_i = \partial T_2 / \partial q_i'$. [One must take care to add the force $m\,(\gamma_e)$ to the given forces that are exerted on the point M of mass m, where (γ_e) is the acceleration of O_1 with respect to $Oxyz$.]

The three methods lead to distinct canonical systems, because the conjugate variables are different. On the contrary, the three methods lead to the same Lagrange equations. Furthermore, if one refers to what was said in Lecture 10 about the calculation of T_1, T_2, T_3 then one will see that forming those canonical systems will present no difficulties.

There is no reason to stress that one can make the same variables $q_1, q_2, \ldots, q_k$ correspond to different conjugate variables, and as a result, to different canonical systems. Indeed, the Lagrange equations can be written:

$$\frac{d}{dt}\frac{\partial T}{\partial q_i'} - \frac{\partial T}{\partial q_i} = Q_i ,$$

and the variables $p_i = \partial T / \partial q_i'$ form a first system of conjugate variables. But then set:

$$T = T' + \Omega ,$$

in which Ω denotes a linear function of the q_i' whose coefficients depend upon the q_i and t:

$$\Omega = A_0(q_1, q_2, \ldots, q_k, t) + q_1' A_1(q_1, q_2, \ldots, q_k, t) + \cdots + q_k' A_k(q_1, q_2, \ldots, q_k, t) \;.$$

That will make:

$$\frac{d}{dt}\frac{\partial T}{\partial q_i'} - \frac{\partial T}{\partial q_i} = Q_i - \frac{d}{dt}\frac{\partial \Omega}{\partial q_i'} + \frac{\partial \Omega}{\partial q_i} = Q_i' .$$

Q_i' depends upon t, the q_i, and the q_i'. If we introduce the variables $p_i' = \partial T' / \partial q_i'$ then the system of equations between the q_i, p_i', and t will be canonical. It sometimes happens that for a convenient choice of the function Ω, T' will once more represent the semi-*vis viva* of the system with respect to certain axes: That is what happens for the functions T_1, T_2, T_3 considered above.

We shall not dwell upon that question any further but move on to a study of the most important properties of canonical systems. The canonical form is, in fact, the form of the equations of motion that applies to most of the applications in the theory of systems of first-order differential equations. Before developing the applications, we shall establish the general theorems to which we shall have recourse.

Generalities on differential equations. – Let:

$$dt = \frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n} \tag{1}$$

be a system of first-order differential equations between the functions $x_1, x_2, \ldots, x_n$ of t, where X_1, $X_2, \ldots, X_n$ denote given functions of $x_1, x_2, \ldots, x_n$, and t.

The general integral of such a system depends upon n arbitrary constants: One can give arbitrary values x_1^0, x_2^0, …, x_n^0 to x_1, x_2, …, x_n for $t = t_0$. That integral can then be put into the form:

$$x_1 = \varphi_1(t, (t_0), x_1^0, x_2^0, \ldots, x_n^0) ,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_n = \varphi_n(t, (t_0), x_1^0, x_2^0, \ldots, x_n^0) .$$

(t_0) represents an arbitrary number (0 or 1, for example), and x_1^0, x_2^0, …, x_n^0 are arbitrary constants.

One can suppose that those relations have been solved for the x_1^0, x_2^0, …, x_n^0, and one will obviously have:

$$x_1^0 = \varphi_1(t_0, (t), x_1, x_2, \ldots, x_n) ,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$x_n^0 = \varphi_1(t_0, (t), x_1, x_2, \ldots, x_n) .$$

One gives the name of *first integral* of the system (1) to any relation of the form:

$$f(t, x_1, x_2, \ldots, x_n) = \text{const.}$$

that is verified for an arbitrary system $x_1(t)$, $x_2(t)$, …, $x_n(t)$ of integrals of equations (1). In other words, if one replaces the variables x_1, x_2, …, x_n in f with functions of t that satisfy equations (1) then f will reduce to a constant.

If $f = \alpha$ is a first integral then $F(f) = \beta$ will also be a first integral. More generally, let:

$$f_1 = \alpha_1 , \qquad f_2 = \alpha_2 , \qquad \ldots, \qquad f_m = \alpha_m$$

be m first integrals. $F(f_1, f_2, \ldots, f_m) = \beta$ will again be a first integral. One says that the m integrals $f_1 = \alpha_1$,…, $f_m = \alpha_m$ are distinct if none of the functions f_i can be expressed as a function of the $(m - 1)$ other ones in the form:

$$f_i = F(f_1, f_2, \ldots, f_{i-1}, f_{i+1}, f_m) .$$

In order for the system (1) to be integrable, it will suffice that one knows n distinct first integrals:

$$f_1 = \alpha_1 , f_2 = \alpha_2 , \ldots, f_m = \alpha_m . \tag{2}$$

Indeed, those n equations can be solved for x_1, x_2, …, x_n. Otherwise, there would exist a relation of the form:

$$F(f_1, f_2, \ldots, f_m, t) = 0 .$$

One could infer t from that relation, since no relation that is independent of t could exist between the f_i. t = const. would then be a first integral of the system (1), which is absurd.

On the other hand, one can arrange the α in such a fashion as to give arbitrary values to x_1, x_2, …, x_n for $t = t_0$, as the equalities:

$$\alpha_1^0 = f_1(t_0, x_1^0, x_2^0, \ldots, x_n^0), \qquad \text{etc.},$$

show immediately.

The system of n functions x_1, x_2, …, x_n of t that is defined by equations (2) then represents the general integral of equations (1).

Arbitrary integrals. – One says, in a general manner, that the relation:

$$F(t, x_1, x_2, \ldots, x_n, \alpha_1, \alpha_2, \ldots, \alpha_n) = 0$$

is an *integral* of the system (1) if it is verified identically when one replaces the x_1, x_2, …, x_n in it with an arbitrary solution (1) for convenient values of the constants α.

When $F = 0$ depends upon only one constant α_1, one can solve for that constant and write:

$$\alpha_1 = f(t, x_1, x_2, \ldots, x_n).$$

A first integral is then an integral that depends upon only one constant.

Theorem:

When an integral $F = 0$ depends upon m distinct constants, one can deduce m first integrals from that integral.

We shall clarify what we mean when we say that $F = 0$ depends upon m distinct constants.

In order to do that, suppose that one has replaced the variables x_1, x_2, …, x_n in F with the most general functions of t that satisfy equations (1), namely:

$$x_1(t, (t_0), x_1^0, x_2^0, \ldots, x_n^0),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_n(t, (t_0), x_1^0, x_2^0, \ldots, x_n^0).$$

F will become:

$$\mathcal{F}(t, (t_0), x_1^0, x_2^0, \ldots, x_n^0, \alpha_1, \alpha_2, \ldots, \alpha_m).$$

If one expresses the idea that the function $\mathcal{F}(t)$ is zero for any t then one will get some relations between the constants x^0 and α. In order for $F = 0$ to be an integral of (1), it is necessary and sufficient that the relations between the α should be compatible for arbitrary x^0.

One says that the integral $F = 0$ depends upon m distinct constant when those relations that couple the m constants α to x_1^0, x_2^0, …, x_n^0, (t_0) satisfy the following conditions:

1. The set of all of them forms a system of m distinct relations.

(We remark that there can no longer exist m distinct relations. Otherwise, one could infer α_1, α_2, …, α_n from m of them, and upon substituting those values in the $(m+1)^{\text{th}}$ one, one would obtain a condition:

$$\psi(x_1^0, x_2^0, \dots, x_n^0, (t_0)) = 0 ,$$

which would not be verified identically. x_1^0, x_2^0, …, x_n^0 would not be arbitrary for $t = t_0$ then.)

2. Those m relations, which one can suppose have been solved for the α :

$$\alpha_i = X_i(x_1^0, x_2^0, \dots, x_n^0, (t_0)) , \tag{3}$$

permit one to dispose of the x^0 in such a manner as to give arbitrary values to the α. In other words, for arbitrarily-given values of the α, the relations (3) between the x^0 will be compatible, or rather, there will exist no relation between the α that is independent of the x^0.

When those conditions are not fulfilled, one can write the integral $F = 0$ in a form that includes less than m constants.

First of all, if the number of distinct relations between the α (and the x^0) is equal to $m - k$ then one can annul k of the constants α while leaving the other ones arbitrary.

Now if the number of distinct relations is equal to m then there will exist relations between the α, say $\alpha_m = h(\alpha_1, \alpha_2, \dots, \alpha_{m-1})$. Upon replacing α_m with that expression, one will reduce the integral to one that depends upon less than m constants.

Conversely, when the stated conditions are verified, it will be impossible to reduce the number of constants in the integral $F = 0$ by a change of constants:

$$\alpha_1 = \theta_1(\beta_1, \beta_2, \dots, \beta_{m-1}) , \qquad \dots, \qquad \alpha_m = \theta_m(\beta_1, \beta_2, \dots, \beta_{m-1}) \qquad (p < m) .$$

Indeed, if that is true then the relation $F = 0$ will be verified for conveniently-chosen values of the constants β_1, β_2, …, β_{m-1} when one replaces x_1, x_2, …, x_n with an arbitrary system of integrals of (1) in it. Now for arbitrary values of the β, the α will not be arbitrary but will satisfy certain relations. The relation $\mathcal{F}(t) \equiv 0$ will then be verified (the x^0 being arbitrary) for non-arbitrary values of the α, which is contrary to hypothesis.

Having said that, it is easy to see that if $F = 0$ depends upon m distinct constants then one can deduce m distinct first integrals.

That is certainly possible, since one has:

$$\alpha_i = X_i(x_1^0, x_2^0, \ldots, x_n^0, (t_0)) \qquad (i = 1, 2, \ldots, m) .$$

If one gives an arbitrary value t to t_0 and lets $x_1, x_2, \ldots, x_n$ denote the values of the integral functions $x_i(t)$ for that value of t then one will once more have:

$$\alpha_i = X_i(x_1, x_2, \ldots, x_n, t) .$$

In other words, $X_i = \alpha_i$ is a first integral. The m first integrals thus-obtained will be distinct, moreover, since the X_i are not coupled by any relation.

In order to form those m first integrals explicitly, one proceeds as follows: Solve $F = 0$ for α_m; that will give:

$$\alpha_m = y(t, x_1, x_2, \ldots, x_n, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}) .$$

Differentiate that with respect to t :

$$(3) \qquad 0 = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \psi}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial \psi}{\partial x_n}\frac{dx_n}{dt} .$$

$x_1, x_2, \ldots, x_n$ are functions of t that satisfy equations (1), i.e., one has:

$$\frac{dx_1}{dt} = X_1 , \qquad \frac{dx_2}{dt} = X_2 , \qquad \ldots, \qquad \frac{dx_n}{dt} = X_n .$$

Therefore:

$$0 = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x_1} X_1 + \frac{\partial \psi}{\partial x_2} X_2 + \cdots + \frac{\partial \psi}{\partial x_n} X_n = F'$$

is an integral of the system (1). $F = 0$ depends effectively upon $(m - 1)$ constants $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$. Otherwise, α_{m-1}, for example, would not appear and the relation $F' = 0$ would be verified as a result of the relation (3) for convenient values of the $\alpha_1, \alpha_2, \ldots, \alpha_{m-2}$, since the constant α_{m-1} would be arbitrary. Upon integrating the relation (3), one will see that the relation $\alpha_m = \psi$ will be verified, since α_{m-1} is arbitrary (for example, equal to zero), and the m constants α will not be distinct.

Furthermore, the $(m - 1)$ constants that enter into the integral $F' = 0$ are distinct, since $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ can be expressed as functions of the x^0, $\alpha_i = X_i$, and there exists no relation between the X_i .

If one reasons with the integral $F' = 0$ as one did with the first one, and so on, then one will arrive at an equality of the form:

$$\alpha_1 = X_1(x_1, x_2, \ldots, x_n, t) ,$$

which will be a first integral.

Upon substituting that value of α_1 in the preceding integral and solving it for α_2, one will have a second integral, and so on.

We remark that when the method of calculation that we just presented is applied to an arbitrary relation $F(t, x_1, x_2, \ldots, x_n, \alpha_1, \alpha_2, \ldots, \alpha_m) = 0$, that will permit us to recognize:

1. Whether that relation is an integral of (1).

2. How many distinct constants that integral depends upon.

More precisely: The question is found to come down to that of recognizing whether a relation $\alpha = X(x_1, x_2, \ldots, x_n, t)$ is a first integral. That is a question that shall address in a moment.

Before we do that, in order for the system (1) to be integrable, it will suffice to find an integral $F = 0$ that depends upon n distinct constants.

The simplest problem that one can propose (and which, as a result, has the best chance of being solved) is therefore the search for first integrals.

LECTURE 12 (cont.)

Properties of first integrals. – We said that if we know n first integrals of the system (1) then the system will be found to be completely integrable. One can infer $x_1, x_2, \ldots, x_n$ as functions of t and n constants using those n integrals.

More generally, if one knows $m = (n - k)$ distinct first integrals of (1) then one can use those integrals to infer m of the variables, conveniently chosen, as functions of the other k and of t. Indeed, a first integral includes at least one of the variables x_i, say x_1 (otherwise t = const. would be a first integral.) Solve it for x_1 and replace x_1 with the value thus-obtained in the other $(m - 1)$ integrals. Since the m integrals are distinct, the right-hand sides of the new integrals:

$$\alpha_1 = f_2'(\alpha_1, t, x_2, \ldots, x_n), \qquad \text{etc.}$$

will not reduce to functions of α_1, and f_2' will include at least one of the variables $x_2, \ldots, x_n$, say x_2. Infer x_2 from the second integral and substitute the value thus-obtained in the following ones. After $(m - 1)$ analogous operations, one then forms the expression for x_{n-k}, for example, as a function of $x_{(n-k+1)}, \ldots, x_n, t$, and some constants. If one proceeds in the opposite direction then one will see that $x_{(n-k+1)}, \ldots, x_1$ are found to be expressed as functions of the same quantities.

If we replace $x_1, x_2, \ldots, x_{n-k}$ with those values in the last k equations (1):

$$dt = \frac{dx_{(n-k+1)}}{X_{(n-k+1)}} = \ldots = \frac{dx_n}{X_n}$$

then we will have more to integrate than a system of k first-order equations between $(k + 1)$ variables. It will suffice to find k distinct first integrals of that system, for example, the k integrals:

$$x^0_{(n-k+1)} = \varphi_{(n-k+1)}(t, x_{(n-k+1)}, \dots, x_n) \qquad (i = 1, 2, \dots, k)\,.$$

The φ depend upon the $\alpha_1, \dots, \alpha_{n-k}$: If we replace the constants with $f_1, f_2, \dots, f_{n-k}$ then we will get k first integrals of (1):

$$x^0_{(n-k+i)} = f_{(n-k+i)}(t, x_1, x_2, \dots, x_n)\,.$$

Those integrals, when combined with the preceding $(n - k)$, will define a system of n distinct first integrals, because one can infer $x_1, x_2, \dots, x_n$ as functions of t and n constants.

Analytic condition for $(n - k)$ **first integrals to be distinct.** – From the foregoing, in order for n first integrals to be distinct, it is necessary and sufficient that the determinant:

$$\begin{vmatrix} \dfrac{df_1}{dx_1} & \dfrac{df_1}{dx_2} & \cdots & \dfrac{df_1}{dx_n} \\ \dfrac{df_2}{dx_1} & \dfrac{df_2}{dx_2} & \cdots & \dfrac{df_2}{dx_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{df_n}{dx_1} & \dfrac{df_n}{dx_2} & \cdots & \dfrac{df_n}{dx_n} \end{vmatrix}$$

should not be identically zero.

More generally, in order for $(n - k)$ first integrals to be distinct, it is necessary and sufficient that at least one of the determinants that are obtained by suppressing k columns from the rectangular matrix:

$$\begin{vmatrix} \dfrac{df_1}{dx_1} & \dfrac{df_1}{dx_2} & \cdots & \dfrac{df_1}{dx_n} \\ \dfrac{df_2}{dx_1} & \dfrac{df_2}{dx_2} & \cdots & \dfrac{df_2}{dx_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{df_{(n-k)}}{dx_1} & \dfrac{df_{(n-k)}}{dx_2} & \cdots & \dfrac{df_{(n-k)}}{dx_n} \end{vmatrix}$$

must not be identically zero.

Observe that there cannot exist more than n distinct first integrals, because if there existed $(n + 1)$ then one could infer $x_1, x_2, \dots, x_n$ from n of them and substitute them in the $(n + 1)^{\text{th}}$ one, which would give the relation:

$$\alpha_{n+1} = \varphi\,(t, \alpha_1, \alpha_2, \dots, \alpha_n)\,.$$

Since the $(n + 1)$ integrals are distinct, t would have to figure effectively in φ. t = const. would then be a first integral, which is impossible.

From that, when one knows n distinct first integrals of the system (1), say $\alpha_i = f_i$, any other first integral $\beta = \varphi(t, x_1, \ldots, x_n)$ will be expressed as a function of those n integrals:

$$\beta = F(\alpha_1, \alpha_2, \ldots, \alpha_n), \qquad \varphi = F(f_1, f_2, \ldots, f_n) .$$

That shows that an arbitrary integral will not depend upon more than n distinct constants.

The search for first integrals. – As we said, the search for a first integral presents the simplest problem that one can propose. We shall show, moreover, that such an integral can be characterized by a very simple condition that is deduced directly from equations (1). On the contrary, in order to recognize whether a relation $F = 0$ that depends upon several constants is an integral, one must eliminate the constants in each particular case.

Let:

$$\alpha = f(t, x_1, \ldots, x_n)$$

be a first integral. Any system of integrals $x_1(t), x_2(t), \ldots, x_n(t)$ of (1) must verify the relation:

$$0 = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_k}\frac{dx_k}{dt},$$

i.e., that relation must be a consequence of equations (1).

There will then exist functions $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $t, x_1, \ldots, x_n$ such that one has:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt} = \lambda_1\left(\frac{dx_1}{dt} - X_1\right) + \cdots + \lambda_n\left(\frac{dx_n}{dt} - X_n\right),$$

or rather:

$$(A) \qquad \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}x_1' + \frac{\partial f}{\partial x_2}x_2' + \cdots + \frac{\partial f}{\partial x_n}x_n' = \lambda_1(x_1' - X_1) + \cdots + \lambda_n(x_m' - X_n) .$$

That identity implies the relations:

$$-\frac{\partial f}{\partial t} = \lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_n X_n ,$$

$$\lambda_1 = \frac{\partial f}{\partial x_1}, \qquad \lambda_2 = \frac{\partial f}{\partial x_2}, \qquad \ldots, \qquad \lambda_n = \frac{\partial f}{\partial x_n}.$$

The multipliers λ must then satisfy the conditions:

$$(B) \qquad \begin{cases} \dfrac{\partial \lambda_i}{\partial x_j} = \dfrac{\partial \lambda_j}{\partial x_i}, \\[2ex] \dfrac{\partial(\lambda_1 X_1)}{\partial x_i} + \dfrac{\partial(\lambda_2 X_2)}{\partial x_i} + \cdots + \dfrac{\partial(\lambda_n X_n)}{\partial x_i} = -\dfrac{\partial \lambda_i}{\partial t}. \end{cases}$$

Conversely, if a system of functions $\lambda\,(t, x_1, \ldots, x_n)$ verifies equations (B) then the function:

$$f = \int \lambda_1\, dx_1 + \lambda_2\, dx_2 + \cdots + \lambda_n\, dx_n + (\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n)\, dt$$

will be equal to a constant that defines a first integral.

n distinct first integrals correspond to n linearly-distinct systems of multipliers λ, and conversely.

The direct search for those multipliers λ is very complicated, in general. However, one can find one or more systems of such multipliers immediately in certain cases. For example, that is what happens for the *vis viva* integral and most of the usual integrals of dynamics. Therefore, suppose that one has the following equations:

$$m\frac{dx_1}{dt} = X, \qquad \frac{dx}{dt} = x_1,$$
$$m\frac{dy_1}{dt} = Y, \qquad \frac{dy}{dt} = y_1,$$
$$m\frac{dz_1}{dt} = Z, \qquad \frac{dz}{dt} = z_1,$$

in which one has:

$$X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt} = \frac{dU}{dt}.$$

Form the expression:

$$x_1\left(m\,\frac{dx_1}{dt} - X\right) + y_1\left(m\,\frac{dy_1}{dt} - Y\right) + z_1\left(m\,\frac{dz_1}{dt} - Z\right) - X\left(\frac{dx}{dt} - x_1\right) - Y\left(\frac{dy}{dt} - y_1\right) - Z\left(\frac{dz}{dt} - z_1\right).$$

It reduces to:

$$\frac{d}{dt}\left[\frac{m}{2}(x_1^2 + y_1^2 + z_1^2) - U\right] = \frac{df}{dt}.$$

The multipliers λ are x_1, y_1, z_1, X, Y, Z here.

That is the first form that one can give to the condition that characterizes a first integral. However, the following form is more convenient in applications. The relation:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt} = 0$$

is verified by any system of integrals $x_1(t), \ldots, x_n(t)$. The same thing is true for the relation:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1} X_1 + \cdots + \frac{\partial f}{\partial x_n} X_n = 0 , \tag{3}$$

which is no different from the first one when one takes equations (1) into account. On the other hand, since one disposes of the integration constants in such a fashion that $x_1, x_2, \ldots, x_n$ will take arbitrary values $x_1^0, x_2^0, \ldots, x_n^0$ for $t = t_0$, equation (3) must be verified when one gives arbitrary values to $t, x_1, x_2, \ldots, x_n$. It is therefore an identity. Hence, one has the following theorem: If $f = \alpha$ is a first integral of (1) then f will verify equation (3) identically.

Conversely, any solution to equation (3) will define a first integral of the system (1). Indeed, replace the variables $x_1, x_2, \ldots, x_n$ in f with an arbitrary system of integrals of (1) and calculate df / dt; that will give:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1} X_1 + \cdots + \frac{\partial f}{\partial x_n} X_n = 0 .$$

Therefore, f = const. is a first integral of (1).

Therefore, in order for $f = \alpha$ to be a first integral of (1), it is necessary and sufficient that f should be a solution to the linear partial differential equation:

$$\frac{\partial f}{\partial t} + X_1 \frac{\partial f}{\partial x_1} + X_2 \frac{\partial f}{\partial x_2} + \cdots + X_n \frac{\partial f}{\partial x_n} = 0 . \tag{3}$$

Observe that, from the foregoing, if $f_1, f_2, \ldots, f_n$ are n distinct integrals of equation (3) then the general integral of that equations will be $F(f_1, f_2, \ldots, f_n)$. That is a well-known proposition that is easy to prove directly. Let the equation be:

$$X \frac{\partial f}{\partial x} + X_1 \frac{\partial f}{\partial x_1} + \cdots + X_n \frac{\partial f}{\partial x_n} = 0 ,$$

for more symmetry, and let $f_1, f_2, \ldots, f_{n+1}$ be $(n + 1)$ integrals of that equation. The $(n + 1)$ relations:

$$X \frac{\partial f_i}{\partial x} + X_1 \frac{\partial f_i}{\partial x_1} + \cdots + X_n \frac{\partial f_i}{\partial x_n} = 0 ,$$

will be compatible for values of the X_j that are not all zero only if the determinant of the $\partial f_i / \partial x_j$ is identically zero. There will then exist at least one relation between the f_i .

Let us apply those results to a system of canonical equations:

$$\frac{dq_i}{dt} = \frac{\partial K}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial K}{\partial q_i} + Q_i \ .$$

Equation (3) takes the form:

$$\frac{\partial f}{\partial t} + \sum \left(\frac{\partial f}{\partial q_i}\frac{\partial K}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial K}{\partial q_i} \right) + \sum \frac{\partial f}{\partial p_i} Q_i = 0 \ .$$

In the case where the Q_i are partial derivatives of a function U ($q_1, \ldots, q_k, t$), one has:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \ ,$$

and equation (3) will become:

$$\frac{\partial f}{\partial t} + \sum \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right) = 0 \ .$$

In a general manner, if f and H are two arbitrary functions of the p_i, q_i then Poisson used the symbol (f, H) to denote the expression $\sum \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right)$. With that notation, equation (3) is written:

$$\frac{\partial f}{\partial t} + (f, H) = 0 \ .$$

When H does not depend upon t, one can then write the canonical equations as:

$$\frac{dq_1}{\dfrac{\partial H}{\partial p_1}} = \ldots = \frac{dq_k}{\dfrac{\partial H}{\partial p_k}} = \frac{dp_1}{-\dfrac{\partial H}{\partial q_1}} = \ldots = \frac{dp_k}{-\dfrac{\partial H}{\partial q_k}} = dt$$

and ignore t by neglecting the last equation. If one seeks a first integral that is independent of t:

$$f(q_1, q_2, .., q_k, p_1, p_2, \ldots, p_k) = \alpha$$

then the function f must satisfy the equation:

$$(f, H) = 0 \ .$$

We shall now begin the theory of the last multiplier (which is due to Jacobi), which allows us to achieve the integration of a system (1) by quadratures in a large number of cases when we know a certain number of first integrals ($\nu < n$).

First observe that when t does not appear explicitly in the equations of motion of a system, it will suffice to know $(2k - 1)$ first integrals that are independent of t. Indeed, one can infer $(2k - 1)$ of the quantities p_i, q_i as functions of another one (say, q_1) from those $(2k - 1)$ integrals, and upon substituting those values in the first canonical equation, one will see that t is given as a function of q_1 by a quadrature:

$$dt = A(q_1)\,dq_1 .$$

The same remark obviously applies if one of the variables q_i does not appear explicitly in the canonical equations.

The theory of the last multiplier leads to the following proposition:

When a material system without friction whose constraints are independent of time is subject to given forces that depend upon neither velocity nor time, it will suffice to know $(2k - 2)$ distinct first integrals of motion (in which t do not appear) in order for the determination of the motion to be achieved by quadratures. (k is the number of parameters upon which the position of the system depends.)

When the constraints or the given forces depend upon time, it suffices to know $(2k - 1)$ distinct first integrals (into which t does not enter).

Theory of the last multiplier. – The integration of a first-order equation:

$$\frac{dx}{X} = \frac{dy}{Y} \tag{1}$$

amounts to the determination of a first integral of this equation:

$$f(x, y) = \alpha ,$$

i.e., a function f that satisfies the equation:

$$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} = 0 .$$

The search for f is equivalent to the search for a multiplier M such that the expression:

$$M(Y\,dx - X\,dy)$$

is an exact total differential. If one knows such a multiplier then the function:

$$f = \int M\,(Y\,dx - X\,dy)$$

will define a first integral. In order for M to be an integrating factor of equation (1), it is necessary and sufficient that one should have:

$$\frac{\partial(MY)}{\partial y} + \frac{\partial(MX)}{\partial x} = 0\,,$$

or rather:

$$X\frac{\partial M}{\partial x} + Y\frac{\partial M}{\partial y} + M\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = 0\,,$$

or finally:

(2) $$X\frac{\partial \log M}{\partial x} + Y\frac{\partial \log M}{\partial y} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0\,.$$

That is the equation that M or log M must satisfy.

In certain cases, one can find a multiplier immediately. (For example, if $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \equiv 0$ then $M = 1$ will be a multiplier.) In order to do that, it is convenient to write equation (2) in a slightly-different form. Suppose that one has replaced y with an arbitrary integral y (x) of equation (1) in that equation, so one can write:

$$\frac{\partial \log M}{\partial x} + \frac{\partial \log M}{\partial y}\,y' + \frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = 0$$

or rather:

(2′) $$\frac{d(\log M)}{dx} + \frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = 0\,,$$

Conversely, if a function M (x, y) satisfies equation (2) when one replaces y with an arbitrary integral of (1) then M will be an integrating factor. Indeed, M will verify equation (2) when one replaces y with an arbitrary integral of (1) in it, i.e., for arbitrary values of x, y.

From that, if one knows a function U (x, y) such that one has:

$$\frac{\partial U}{\partial x} = \frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)$$

when y is an integral of (1) then $M = e^{-U}$ will be a multiplier.

Suppose, for example, that:

$$\frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = \varphi(x)\,.$$

A multiplier *M* will be given by the equality:

$$\log M = -\int \varphi(x)\,dx$$

Similarly, if:

$$\frac{1}{Y}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = \psi(x)$$

then a multiplier will be given by the equality:

$$\log M = -\int \psi(x)\,dx .$$

That is what happens for the first-order linear equation:

$$y' = A\,y + B ,$$

or

$$\frac{dx}{1} = \frac{dy}{A\,y + B} .$$

Here, the equation of the last multiplier is:

$$\frac{d}{dx}\log M + A(x) = 0 ,$$

so

$$M = e^{-\int A(x)\,dx} .$$

We shall study the theory of the integrating factor in the case of a systems of two first-order differential equations and then the case of an arbitrary system.

Case of two differential equations. – Let the system be:

(1) $$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} ,$$

in which X, Y, Z are given functions of x, y, z.

In order to integrate those equations, it will suffice to know two first integrals:

$$f(x, y, z) = \alpha ,$$
$$\varphi(x, y, z) = \beta .$$

Suppose that one knows one of them, say, $\varphi(x, y, z) = \beta$. One can infer z (for example) as a function of x, y, and β from that integral and substitute that in X and Y. Let X', Y' denote what X, Y become after that substitution. x and y will then be coupled by the equation:

(2) $$Y'\,dx - X'\,dy = 0 .$$

If one knows an integrating factor $\lambda(x, y, \beta)$ of equation (2) then that equation can be integrated by a quadrature. We shall show that if $M(x, y, z)$ satisfies the relations:

$$X\frac{\partial \log M}{\partial x} + Y\frac{\partial \log M}{\partial y} + Z\frac{\partial \log M}{\partial z} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \equiv 0$$

then the expression M / φ'_z (in which z is replaced with a function of x, y, and b) will be an integrating factor of equation (2).

For the moment, add a second integral $f = \alpha$ of the system (1) to the integral $\varphi = \beta$. We will have:

$$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z} \equiv 0 ,$$

$$X\frac{\partial \varphi}{\partial x} + Y\frac{\partial \varphi}{\partial y} + Z\frac{\partial \varphi}{\partial z} \equiv 0 ,$$

i.e.:

$$\frac{A}{X} = \frac{B}{Y} = \frac{C}{Z} ,$$

when we set:

$$A = \frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial z} - \frac{\partial \varphi}{\partial y}\frac{\partial f}{\partial z} ,$$

$$B = \frac{\partial f}{\partial z}\frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial z}\frac{\partial f}{\partial x} ,$$

$$C = \frac{\partial f}{\partial x}\frac{\partial \varphi}{\partial y} - \frac{\partial \varphi}{\partial x}\frac{\partial f}{\partial y} .$$

Let M be the common value of the ratios A / X, B / Y, C / Z. I shall first say that M_1 / φ'_x, in which z is expressed in terms of x, y, and β, is an integrating factor for equation (2), resp.

Indeed, let A', B', M_1', and $(\partial\varphi / \partial z)$ denote what A, B, M_1, and $(\partial\varphi / \partial z)$, become when one replaces z as a function of x, y, and β. One will obviously have $A' = M_1' X'$, $B' = M_1' Y'$. It will suffice for me to show that $1/\left(\dfrac{\partial \varphi}{\partial z}\right)$ is an integrating factor of the equation:

$$B'\,dx - A'\,dy = 0\,.$$

Now, replace the variable z in $f(x, y, z)$ with the variable β that is defined by the relation $\varphi = \beta$, and let $f'(x, y, \beta) \equiv f(x, y, z)$. Calculate A and B as functions of the derivatives of f'; that will give:

$$\frac{\partial f}{\partial x} = \frac{\partial f'}{\partial x} + \frac{\partial f'}{\partial \beta}\frac{\partial \varphi}{\partial x}\,,$$

$$\frac{\partial f}{\partial y} = \frac{\partial f'}{\partial y} + \frac{\partial f'}{\partial \beta}\frac{\partial \varphi}{\partial y}\,,$$

$$\frac{\partial f}{\partial z} = \qquad \frac{\partial f'}{\partial \beta}\frac{\partial \varphi}{\partial z}\,.$$

One deduces from this that:

$$A = \frac{\partial f'}{\partial y}\frac{\partial \varphi}{\partial z}\,,$$

$$B = \frac{\partial f'}{\partial x}\frac{\partial \varphi}{\partial z}\,,$$

in which z is expressed in terms of x, y, and β everywhere, or rather:

$$A' = \frac{\partial f'}{\partial y}\left(\frac{\partial \varphi}{\partial z}\right),$$

$$B' = \frac{\partial f'}{\partial x}\left(\frac{\partial \varphi}{\partial z}\right),$$

and as a result:

$$\frac{B'\,dx - A'\,dy}{\left(\dfrac{\partial \varphi}{\partial z}\right)} \equiv -\left(\frac{\partial f'}{\partial x}dx + \frac{\partial f'}{\partial y}dy\right).$$

If one gives a constant value to β then one will see that $1/\left(\dfrac{\partial \varphi}{\partial z}\right)$ is an integrating factor for the equation $B'\,dx - A'\,dy = 0$. The corresponding integral of that equation will be $f'(x, y, \beta) \equiv$ const.

Having established that lemma, observe that the functions A, B, C verify the equality:

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0$$

identically.

One will then have:

$$\frac{\partial(M_1X)}{\partial x}+\frac{\partial(M_1Y)}{\partial y}+\frac{\partial(M_1Z)}{\partial z}=0 .$$

In other words, M_1 is an integral of the partial differential equation:

$$X\frac{\partial\log M}{\partial x}+Y\frac{\partial\log M}{\partial y}+Z\frac{\partial\log M}{\partial z}+\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0 . \tag{3}$$

Now compare M_1 to an arbitrary integral M of equation (3). Upon subtracting corresponding sides of the two identities (3) that relate to M and M_1, one will get the relation:

$$X\frac{\partial\log(M/M_1)}{\partial x}+Y\frac{\partial\log(M/M_1)}{\partial y}+Z\frac{\partial\log(M/M_1)}{\partial z}=0 .$$

Consequently, $M / M_1 = N$ will verify the equation:

$$X\frac{\partial N}{\partial x}+Y\frac{\partial N}{\partial y}+Z\frac{\partial N}{\partial z}=0 ,$$

i.e., N will be a function of f and φ. Therefore:

$$M = M_1\,\chi(f, \varphi) .$$

The proposition that we have in mind is proved with that. Indeed, replace M_1 with a function of M in the equality:

$$\frac{M_1}{\left(\dfrac{\partial\varphi}{\partial z}\right)}(Y'\,dx-X'\,dy)=d\cdot f'(x,y,\beta)$$

(in which β is constant). If M' denotes what M will become when one expresses z in terms of x, y, and β then one will obviously have:

$$M'=M_1'\,\chi(f',\beta) ,$$

and as a result:

$$\frac{M'}{\left(\dfrac{\partial\varphi}{\partial z}\right)}(Y'\,dx-X'\,dy)=\chi(f',\beta)\,df'=dF(x,y,\beta) .$$

Therefore, if *M* is a solution of equation (3) then $M'/\left(\frac{\partial\varphi}{\partial z}\right)$ will be an integrating factor of equation (2). Q. E. D.

The following theorem results from this:

When one knows an integral M of the equation:

$$X\frac{\partial\log M}{\partial x}+Y\frac{\partial\log M}{\partial y}+Z\frac{\partial\log M}{\partial z}+\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0 \tag{3}$$

in order to achieve the integral of that system by quadratures, it will suffice to know a first integral of the system (1). *One gives the name of* ***last multiplier*** *to M.*

A very simple case in which one finds a multiplier *M* immediately is the one in which:

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0 .$$

$M = 1$ is then the solution of (3), and $1/\left(\frac{\partial\varphi}{\partial z}\right)$ is an integrating factor of equation (2).

More generally, one can give a slightly-different form to equation (3) by regarding *y* and *z* as functions of *x* that satisfy the system (1). Equation (3) will then be written:

$$\frac{d}{dx}(\log M)+\frac{1}{X}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)=0 .$$

Find a function *U* (*x*, *y*, *z*) such that one will have:

$$\frac{dU}{dx}=\frac{1}{X}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)$$

when *y* and *z* are arbitrary integrals *y* (*x*), *z*(*x*) of the system (1), so $M = \exp\int U(x,y,z)\,dx$ will be a multiplier. (One sees that by arguing as one did in the case of only one equation.)

That is what happens especially when the quantity $\frac{1}{X}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)$ is a function of only *x*, and similarly, if the quantities $\frac{1}{Y}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)$ or $\frac{1}{Z}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)$ reduce to functions of *y* or *z*.

Applications. – As an application, consider the general second-order equation:

$$\frac{d^2 y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right) .$$

It is equivalent to the system:

$$\frac{dx}{1} = \frac{dy}{y'} = \frac{dy'}{F(x, y, y')} .$$

Here:

$$X = 1 , \qquad Y = 1 , \qquad Z = F(x, y, y') .$$

As a result:

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = \frac{\partial F}{\partial y'} .$$

In order for that quantity to be zero, it is necessary and sufficient that F should not depend upon y'.

Thus, if one is given the equation:

$$\frac{d^2 y}{dx^2} = F(x, y)$$

then in order for the integration to be achieved by a quadrature, it will suffice to know a first integral $\varphi(x, y, y') = \beta$. One infers y' as a function of x, y, and β from that integral, namely, $y' = \psi(x, y, b)$, and the equation:

$$\psi\, dx - dy = 0$$

will admit $1/\left(\frac{\partial \varphi}{\partial y}\right)$ for an integrating factor.

In the case where:

$$F(x, y, y') = \alpha(x)\, y' + \beta(x, y) ,$$

one determines a multiplier M by a simple quadrature:

$$\frac{d}{dx} \log M + \alpha(x) = 0 .$$

From that, if one studies the motion of a point that moves without friction on a fixed curve subject to a given force F that does not depend upon the velocity of the point then in order to achieve the solution to the problem by quadratures, it will suffice to know a first integral of motion.

The same thing will be true if the given force depends upon the velocity V in such a fashion that the tangential component F_T has the form:

$$F_T = \alpha(s)\,V + \beta(s, t)\,,$$

in which s denotes the arc of the fixed curve.

One knows that the motion of the point is, in fact, determined by the equation:

$$m\,\frac{d^2 s}{dt^2} = F_T\,,$$

in that case.

More generally, the Lagrange equation that determines the motion of a system with complete constraints is written:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right) - \frac{\partial T}{\partial q} = Q\,.$$

If Q does not depend upon q' then it will suffice to know a first integral of motion if one is to determine the motion by quadratures.

Indeed, the Lagrange equation is equivalent to the equations:

$$\frac{dt}{1} = \frac{dq}{q'} = \frac{dq'}{\left(\dfrac{u}{\dfrac{\partial^2 T}{\partial q'^2}}\right)}\,,$$

in which one has set:

$$u = \frac{\partial T}{\partial q} + Q - \frac{\partial^2 T}{\partial q'\,\partial t} - \frac{\partial^2 T}{\partial q'\,\partial q}\,q'\,.$$

The equation (3) that corresponds to the system (1) is:

$$\frac{\partial M}{\partial t} + \frac{\partial M}{\partial q}\,q' + \frac{\partial M}{\partial q'}\,\frac{u}{\dfrac{\partial^2 T}{\partial q'^2}} + M\,\frac{\partial}{\partial q'}\left(\frac{u}{\dfrac{\partial^2 T}{\partial q'^2}}\right) = 0\,.$$

I say that the function $M = \dfrac{\partial^2 T}{\partial q'^2}$ is a multiplier: In order to see that, it will suffice to replace M with $\dfrac{\partial^2 T}{\partial q'^2}$ in the preceding equation. It will become:

$$0 \equiv \frac{\partial^3 T}{\partial q'^2\,\partial t} + \frac{\partial^3 T}{\partial q'^2\,\partial q} q' + \frac{\partial^3 T}{\partial q'^3} \frac{u}{\dfrac{\partial^2 T}{\partial q'^2}} + \frac{\partial^2 T}{\partial q'^2} \left[-\frac{\partial^3 T}{\partial q'^3} \frac{u}{\left(\dfrac{\partial^2 T}{\partial q'^2}\right)^2} + \frac{1}{\dfrac{\partial^2 T}{\partial q'^2}} \left(\frac{\partial^2 T}{\partial q\,\partial q'} - \frac{\partial^3 T}{\partial q'^2\,\partial t} - q' \frac{\partial^3 T}{\partial q'^2\,\partial q} - \frac{\partial^2 T}{\partial q'\,\partial q} \right) \right].$$

The theorem is thus proved. One can arrive at it more easily by appealing to the canonical equations:

(1′)
$$\frac{dt}{1} = \frac{dq}{\dfrac{\partial K}{\partial p}} = \frac{dp}{-\dfrac{\partial K}{\partial p} + Q} .$$

The expression $\dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y} + \dfrac{\partial Z}{\partial z}$ is equal to $\left(\dfrac{\partial^2 K}{\partial p\,\partial q} - \dfrac{\partial^2 K}{\partial p\,\partial q} + \dfrac{\partial Q}{\partial p} \right)$ here, i.e., to zero, since Q does not depend upon p. $M = 1$ will then be a multiplier for the canonical system (1′).

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