LECTURE 13

THEORY OF THE LAST MULTIPLIER (CONT.) APPLICATIONS.

Now consider the most general system of equations (1):

$$\frac{dx}{X} = \frac{dx_1}{X_1} = \ldots = \frac{dx_n}{X_n} , \tag{1}$$

in which $X, X_1, \ldots, X_n$ are given functions of $x, x_1, \ldots, x_n$.

The system is integrated when one knows n first integrals. Suppose that one knows only $(n - 1)$ of them, and let them be:

$$(\gamma) \qquad \left\{ \begin{array}{l} f_2(x, x_1, x_3, \ldots, x_n) = \alpha_2, \\ f_3(x, x_1, x_3, \ldots, x_n) = \alpha_3, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_n(x, x_1, x_3, \ldots, x_n) = \alpha_n. \end{array} \right.$$

From those $(n - 1)$ integrals, we can infer $(n - 1)$ of the variables x_i, for example $x_2, x_3, \ldots, x_n$, as functions of the other two, x and x_1, and of the α. That amounts to saying that the determinant Δ:

$$\Delta = \begin{vmatrix} \dfrac{\partial f_2}{\partial x_2} & \dfrac{\partial f_2}{\partial x_3} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \dfrac{\partial f_3}{\partial x_2} & \dfrac{\partial f_3}{\partial x_3} & \cdots & \dfrac{\partial f_3}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_2} & \dfrac{\partial f_n}{\partial x_3} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{vmatrix}$$

is not identically zero.

More generally, if F denotes an arbitrary function of $x_1, x_2, \ldots, x_n$ then we shall let F' represent what F will become when we replace $x_2, x_3, \ldots, x_n$ with their values that we infer from the system (γ). Having made that convention, it will be clear that x and x_1 satisfy the equation:

$$X_1' \, d\alpha - X' \, dx_1 = 0 ,$$

in which one leaves the α constant.

When one knows a solution M of the equation:

$$(\beta) \qquad X\frac{\partial \log M}{\partial x} + X_1\frac{\partial \log M}{\partial x_1} + \cdots + X_n\frac{\partial \log M}{\partial x_n} + \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = 0\,,$$

one will know an integrating factor to the equation (2). That factor is equal to M' / Δ' ([1]).

In other words:

$$\frac{M'}{\Delta'}(X_1'\,d\alpha - X'\,dx_1) = dF\,(x, x_1, \alpha_2, \alpha_3, \ldots, \alpha_m)\,.$$

The proof differs from the one that was given in the case of two equations only by the complexity of the calculations.

Suppose for the moment that one knows n first integrals $f_1 = \alpha_1$, $f_2 = \alpha_2$, …, $f_n = \alpha_n$. The equations:

$$X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} + \cdots + X_n\frac{\partial f}{\partial x_n} = 0\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$X\frac{\partial f_n}{\partial x} + X_1\frac{\partial f_n}{\partial x_1} + \cdots + X_n\frac{\partial f_n}{\partial x_n} = 0$$

will be equivalent to the following ones:

$$\frac{A}{X} = \frac{A_1}{X_1} = \frac{A_2}{X_2} = \ldots = \frac{A_n}{X_n} = M_1\,,$$

when one sets:

$$R = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x} & \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \\ \alpha & \alpha_1 & \cdots & \alpha_n \end{vmatrix}$$

and

$$A = \frac{\partial R}{\partial \alpha}, \qquad A_1 = \frac{\partial R}{\partial \alpha_1}, \qquad \ldots, \qquad A_n = \frac{\partial R}{\partial \alpha_n}\,,$$

We shall first show that the quantities A_i verify the relation:

$$(4) \qquad \frac{\partial A}{\partial x} + \frac{\partial A_1}{\partial x_1} + \cdots + \frac{\partial A_n}{\partial x_n} \equiv 0\,.$$

([1]) Δ' cannot be zero for any x, x_1, α_2, …, α_n, because otherwise Δ would be identically zero.

Indeed, observe that the left-hand side of (4) is a homogeneous linear function of the second derivatives of $f_1, f_2, \ldots, f_n$.

Now that function cannot contain a term in $\partial^2 f_k / \partial x_i^2$, because the term in question would come from $\partial A_i / \partial x_i$, and $A_i = \partial R / \partial \alpha_i$ contains no derivative of f with respect to the variable x_i .

It can no longer include a term in $\dfrac{\partial^2 f_k}{\partial x_i \, \partial x_j}$, because that term would come from the sum $\dfrac{\partial A_i}{\partial x_i} + \dfrac{\partial A_j}{\partial x_j}$: Set $\beta_i^k = \partial f_k / \partial x_i$, to abbreviate the notation. The term in $\dfrac{\partial^2 f_k}{\partial x_i \, \partial x_j}$ that comes from $\dfrac{\partial A_i}{\partial x_i}$ will have the coefficient $\dfrac{\partial A_i}{\partial \beta_j^k}$ or $\dfrac{\partial^2 R}{\partial \alpha_i \, \partial \beta_j^k}$. The two second-order minors of R that are thus introduced are equal in absolute value, because they are obtained by suppressing the same rows and columns of R. Moreover, they have opposite signs because one can pass from one to the other by permuting two columns in R.

The sum of the terms in the left-hand side of (4) is therefore identically zero, and the relation (4) is then established. If one replaces $A, A_1, \ldots, A_n$ with $M_1 X, M_1 X_1, \ldots, M_1 X_n$ in the relation then one will see that M_1 is an integral of the equation:

$$X \frac{\partial \log M}{\partial x} + X_1 \frac{\partial \log M}{\partial x_1} + \cdots + X_n \frac{\partial \log M}{\partial x_n} + \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = 0 . \tag{3}$$

Having said that, it will be easy to see that M_1' / Δ' is an integrating factor of equation (2). Since one has:

$$A' = M_1' X' , \quad A_1' = M_1' X_1' ,$$

it will suffice to prove that $1 / \Delta'$ is an integrating factor of the equation:

$$A_1' \, dx - A' \, dx_1 = 0 . \tag{2′}$$

To that end, express the variables $x_1, x_2, \ldots, x_n$ in $f_1 (x, x_1, x_2, \ldots, x_n)$ as functions of the $(n - 1)$ variables $\alpha_2, \alpha_3, \ldots, \alpha_n$ and of x and x_1 . We have:

$$f_1 (x, x_1, x_2, \ldots, x_n) = f_1'(x, x_1, \alpha_3, \ldots, \alpha_n) ,$$

and as a result:

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_1'}{\partial x} + \sum_{i=2}^{n} \frac{\partial f_1'}{\partial \alpha_i} \frac{\partial f_i}{\partial x} ,$$

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial f_1'}{\partial x_1} + \sum_{i=2}^{n} \frac{\partial f_1'}{\partial \alpha_i} \frac{\partial f_i}{\partial x_1} ,$$

$$\frac{\partial f_1}{\partial x_2} = \sum_{i=2}^{n} \frac{\partial f_1'}{\partial \alpha_i} \frac{\partial f_i}{\partial x_2},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\frac{\partial f_1}{\partial x_n} = \sum_{i=2}^{n} \frac{\partial f_1'}{\partial \alpha_i} \frac{\partial f_i}{\partial x_n}.$$

Moreover, one knows that:

$$A = \begin{vmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{vmatrix},$$

$$A_1 = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{vmatrix}.$$

If one replaces the $\partial f_1 / \partial x_i$ with the values that are written above then one will see that upon supposing that $x_2, x_3, \dots, x_n$ are expressed as functions of $\alpha_2, \alpha_3, \dots, \alpha_n$, and x, x_1 everywhere, and the expressions A and A_1 reduce to $\frac{\partial f_1'}{\partial x_1}\Delta$ and $-\frac{\partial f_1'}{\partial x_1}\Delta$, respectively, i.e., one will have:

$$A' = \frac{\partial f_1'}{\partial x_1}\Delta, \qquad A_1' = -\frac{\partial f_1'}{\partial x_1}\Delta.$$

As a result:

(5) $$\frac{A_1'\,dx - A'\,dx_1}{\Delta'} = -\left[\frac{\partial f_1'}{\partial x}\,dx + \frac{\partial f_1'}{\partial x_1}\,dx_1\right].$$

If one then gives constant values to $\alpha_2, \alpha_3, \dots, \alpha_n$ then the left-hand side of equation (5) will be the total differential of the function $f_1'(x, x_1, \alpha_3, \dots, \alpha_n)$, which proves that M_1' / Δ' is indeed an integrating factor of equation (2).

Finally, let M be an arbitrary solution to (3). One soon sees that the ratio M / M_1 or N satisfies the equation:

$$X\frac{\partial N}{\partial x} + X_1\frac{\partial N}{\partial x_1} + \cdots + X_n\frac{\partial N}{\partial x_n} = 0,$$

and as a result:

$$N = \varphi(f_1, f_2, \dots, f_n).$$

Hence:

$$M = M_1\ \varphi(f_1, f_2, \ldots, f_n) .$$

If one replaces M_1 as a function of M in equation (5) then it will become:

$$\frac{M'}{\Delta'}(X_1' dx - X' dx_1) = \varphi(f_1, f_2, \ldots, f_n)\, df_1 = dF(x, x_1, \alpha_2, \ldots, \alpha_n) .$$

M' / Δ' is then an integrating factor for equation (2). That is the theorem that we wish to prove.

When one knows an arbitrary solution M of equation (3), it will then suffice to form $(n - 1)$ distinct first integrals of (1) in order to achieve the integration by quadratures. The function M is called a *last multiplier* of the system (1).

In certain cases, one can recognize a last multiplier immediately. That is what happens, in particular, when the expression:

$$\delta = \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = 0$$

is identically zero: One can then set $M = 1$, and the final binomial will admit $1 / \Delta'$ as an integrating factor.

It is fitting to observe, in that regard, that if one replaces $x_1, x_2, \ldots, x_n$ with functions of x that satisfy the system (1) then that equation will become:

$$\frac{dLM}{dx} + \frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right) \equiv 0 .$$

Conversely, if a function $U(x, x_1, \ldots, x_n)$ [when one replaces x_n, x_1 in it with an arbitrary system of in integrals of (1)] verifies the relation:

$$\frac{dU}{dx} = \frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right)$$

then $M = e^{-U}$ will be a last multiplier of (1).

From that, if the expression:

$$\frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right)$$

is a function of only x then a last multiplier will be obtained by a quadrature. More generally, if one has:

$$\frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right) \equiv \psi(x_i)$$

then $M = \exp\left[-\int \psi(x_i)\,dx_i\right]$ will be a multiplier.

Finally, we add that if $x, x_1, \ldots, x_k$ do not enter into any functions X then any multiplier M of (1) that is independent of $x, x_1, \ldots, x_k$ will also be a multiplier of the system:

$$\frac{dx_{(k+1)}}{X_{k+1}} = \frac{dx_{(k+2)}}{X_{(k+2)}} = \ldots = \frac{dx_n}{X_n}.$$

Indeed, $M(x_{k+1}, x_{k+2}, \ldots, x_n)$ verifies the equation:

$$X_{k+1}\frac{\partial \log M}{\partial x_{k+1}} + X_{k+2}\frac{\partial \log M}{\partial x_{k+2}} + \cdots + X_n\frac{\partial \log M}{\partial x_n} = 0.$$

Remark in regard to the case in which one knows only $(n - k)$ **first integrals.** – In the foregoing, we assumed that we had formed $(n - 1)$ first integrals. Knowing a last multiplier would then permit us to find a last integral by quadrature.

We now place ourselves in the case where we know only $(n - k)$ first integrals of equations (1):

(1)
$$\frac{dx}{X} = \frac{dx_1}{X_1} = \ldots = \frac{dx_n}{X_n}.$$

One can infer $(n - k)$ of the variables $x, x_1, \ldots, x_n$ (for example, $x_n, x_{n-1}, \ldots, x_{k+1}$) as functions of the $(k + 1)$ other ones $x, x_1, \ldots, x_k$ and $(n - k)$ constants α from those $(n - k)$ integrals, say, $f_n = \alpha_n$, $f_{(n-1)} = \alpha_{n-1}, \ldots, f_{(k+1)} = \alpha_{k+1}$. Let (F) denote what a function $F(x, x_1, \ldots, x_n)$ will become after that substitution. The variables $x, x_1, \ldots, x_n$ satisfy the equations:

(1′)
$$\frac{dx}{(X)} = \frac{dx_1}{(X_1)} = \ldots = \frac{dx_k}{(X_k)},$$

which must now be integrated. Can knowing a multiplier M of the system (1) serve to integrate the system (1′)? We shall show that it is easy to deduce a multiplier for (1′) from a multiplier of (1).

From the foregoing, an arbitrary multiplier M of (1) satisfies (if we preserve the notations on page 189) the relations:

$$M X = \varphi A, \qquad M X_1 = \varphi A_1, \quad \ldots, \qquad M X_n = \varphi A_n,$$

in which φ denotes an arbitrary function of $f_1, f_2, \ldots, f_n$. Conversely, any function $M(x, x_1, \ldots, x_n)$ that satisfies one of those relations, say, the relation:

$$M X = \varphi(f_1, f_2, \ldots, f_n)\, A,$$

will be a multiplier of (1). Having recalled that, assume that one knows a first integral of (1) (say $f_n = \alpha_n$) and infer one of the variables (for example, x_n) from that integral as a function of the x, x_1, …, x_{n-1}, and α_n. Let F' and $\left(\frac{\partial F}{\partial x_n}\right)$ denote what the functions $F(x, x_1, \ldots, x_n)$ and $\partial F / \partial x_n$ will become when one replaces x_n with that value. One will then have:

$$f_i(x, x_1, \ldots, x_{n-1}, x_n) = f_i'(x, x_1, \ldots, x_{n-1}, x_n),$$

and as a result:

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_i'}{\partial x_j} + \frac{\partial f_i'}{\partial \alpha_n}\frac{\partial f_n}{\partial x_j} \qquad \begin{pmatrix} i = 1, 2, \ldots, (n-1) \\ j = 1, 2, \ldots, (n-1) \end{pmatrix}$$

$$\frac{\partial f_i}{\partial x_n} = \frac{\partial f_i}{\partial \alpha_n}\frac{\partial f_n}{\partial x_n}.$$

If one replaces the $\frac{\partial f_i}{\partial x_j}$, $\frac{\partial f_i}{\partial x_n}$ in the determinant A with those values then one will soon see that A reduces to:

$$\frac{\partial f_n}{\partial x_n}\begin{vmatrix} \frac{\partial f_1'}{\partial x_1} & \frac{\partial f_1'}{\partial x_2} & \cdots & \frac{\partial f_1'}{\partial x_{n-1}} \\ \frac{\partial f_2'}{\partial x_1} & \frac{\partial f_2'}{\partial x_2} & \cdots & \frac{\partial f_2'}{\partial x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{n-1}'}{\partial x_1} & \frac{\partial f_{n-1}'}{\partial x_2} & \cdots & \frac{\partial f_{n-1}'}{\partial x_{n-1}} \end{vmatrix} = \frac{\partial f_n}{\partial x_n}\alpha.$$

One can then write the equality as:

$$M' X' \equiv \varphi(f_1', f_2', \ldots, f_{(n-1)}', \alpha_n)\left(\frac{\partial f_n}{\partial x_n}\right)\alpha.$$

On the other hand, the multipliers μ of the system:

(2) $$\frac{dx}{X'} = \frac{dx_1}{X_1'} = \ldots = \frac{dx_{n-1}}{X_{(n-1)}'}$$

are given by the relation:

$$\mu X' = \psi(f_1', f_2', \ldots, f_{(n-1)}')\,\alpha,$$

in which ψ is an arbitrary function of f_1', f_2', …, $f_{(n-1)}'$. Therefore, the function μ_1 (x, x_1, …, x_{n-1}, α_n) that is defined by the equality:

$$\mu_1 X' = \varphi(f_1', f_2', \ldots, f_{(n-1)}')\,\alpha$$

will be a multiplier of (2). Since one has:

$$\mu_1 = \frac{M'}{\left(\frac{\partial f_n}{\partial x_n}\right)},$$

one will see that knowing a multiplier of (1) will imply that one knows a multiplier of (2).

It is clear that one can reason with the system (2) as one did with the system (1). If one knows a second first integral of (1), $f_{(n-1)} = \alpha_{(n-1)}$ then that integral will correspond to an integral of (2), $f_{(n-1)}' = \alpha_{(n-1)}$. Upon inferring $x_{(n-1)}$ from the latter relation, one will form the system:

$$\frac{d\alpha}{X''} = \frac{d\alpha_1}{X_1''} = \ldots = \frac{d\alpha_{(n-2)}}{X_{(n-2)}''}. \tag{3}$$

The expression $\mu' \Big/ \left(\frac{\partial f_{(n-1)}'}{\partial x_{(n-1)}}\right)$ is a multiplier of (3). μ' and $\left(\frac{\partial f_{(n-1)}'}{\partial x_{(n-1)}}\right)$ denote what μ and $\frac{\partial f_{(n-1)}'}{\partial x_{(n-1)}}$, resp., will become when one expresses x_{n-1} in terms of x, x_1, …, x_{n-2}, α_{n-1}, α_n.

One will arrive at that conclusion by pursuing the argument in the same manner: If one sets:

$$\begin{aligned}
f_n (x, x_1, \ldots, x_n) &= \varphi_n (x, x_1, \ldots, x_n) &&= \alpha_n, \\
f_{(n-1)} (x, x_1, \ldots, x_n) &= \varphi_{(n-1)} (x, x_1, \ldots, x_n) &&= \alpha_{(n-1)}, \\
f_{(n-2)} (x, x_1, \ldots, x_n) &= \varphi_{(n-1)} (x, x_1, \ldots, x_n) &&= \alpha_{(n-2)}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
f_{(k+1)} (x, x_1, \ldots, x_n) &= \varphi_{(k+1)} (x, x_1, \ldots, x_n) &&= \alpha_{(k+1)}
\end{aligned}$$

then the expression:

$$\frac{M}{\frac{\partial \varphi_n}{\partial x_n}\frac{\partial \varphi_{(n-1)}}{\partial x_{(n-1)}}\cdots\frac{\partial \varphi_{(k+1)}}{\partial x_{(k+1)}}},$$

in which one replaces x_n, x_{n-1}, …, x_{k+1} as functions of x, x_1, …, x_k, α_{k+1}, α_{k+2}, …, α_n, will be a last multiplier of the system (1′).

In particular, if one sets $k = 1$ then one will see that the product $\frac{\partial \varphi_n}{\partial x_n} \frac{\partial \varphi_{(n-1)}}{\partial x_{(n-1)}} \cdots \frac{\partial \varphi_{(k+1)}}{\partial x_{(k+1)}}$ coincides with the determinant Δ that was introduced above when one expresses $x_n, \ldots, x_{k+1}$ in terms of $x, \ldots, x_k, \alpha_{k+2}, \ldots, \alpha_n$.

The foregoing makes no assumptions about the values of the constants α and will persist *a fortiori* when one gives particular values to those constants (or at least some of them). From that, let α_n^0 be a certain value of the constant α_n (for example, $\alpha_n^0 = 0$). Assume that one knows a first integral of the system (2) for that value α_n:

$$\varphi_{(n-1)}(x, x_1, \ldots, x_{n-1}) = \alpha_{n-1} .$$

The expression $M / \frac{\partial \varphi_n}{\partial x_n} \frac{\partial \varphi_{(n-1)}}{\partial x_{(n-1)}}$ is once more a multiplier of the system (3) for $\alpha_n = \alpha_n^0$. Similarly, if one knows a first integral of the system (3) for $\alpha_n = \alpha_n^0$, $\alpha_{n-1} = \alpha_{n-1}^0$:

$$\varphi_{n-2}(x, x_2, \ldots, x_{n-2}) = \alpha_{n-2} ,$$

and so on, then the expression (4) will again define a multiplier of (1′) for $\alpha_n = \alpha_n^0$, $\alpha_{n-1} = \alpha_{n-1}^0$, …, $\alpha_{k+2} = \alpha_{k+2}^0$.

We shall soon have occasion to utilize that remark.

We shall now apply the theory of the last multiplier to the equations of dynamics. The first problem that naturally presents itself is the problem of the motion of a free point.

Application to the motion of a free point. – The equations of motion of a free point can be written:

$$dt = \frac{dx}{x'} = \frac{dy}{y'} = \frac{dz}{z'} = \frac{m\,dx'}{X} = \frac{m\,dy'}{Y} = \frac{m\,dz'}{Z}, \tag{1}$$

in which X, Y, Z are given functions of x, y, z, x', y', z', t.

In the general case, one must know six first integrals in order to integrate the system (1).

If X, Y, Z do not depend upon t then it will obviously suffice to know five independent functions of t, since the sixth one can be obtained by quadrature.

If X, Y, Z do not depend upon x', y', z' then it will likewise suffice to know five first integrals (in which t can appear). The expression $\delta = \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}$ that was considered above will be identically zero here. $M = 1$ is then a last multiplier of the system (1), and the sixth integral is obtained by quadrature.

Finally, if X, Y, Z depend upon neither x', y', z' nor t then from a remark that was made above, $M = 1$ will also be a multiplier of the system:

$$(1') \qquad \frac{dx}{x'} = \frac{dy}{y'} = \frac{dz}{z'} = \frac{m\,dx'}{X} = \frac{m\,dy'}{Y} = \frac{m\,dz'}{Z}.$$

It will then suffice to know four first integrals of (1′), i.e., four integrals of (1) that are independent of t. The fifth integral of (1′) will then be given by quadratures, and t is likewise obtained by a quadrature.

Hence, when a free point M is subject to a force that depends upon only the position of the point, it will suffice to know four first integrals of motion (in which time does not figure) in order for the determination of the motion to be achieved by quadratures.

Suppose, for example, that the force is a central force that is a function of only the distance r from the point M to the center O of the force. The theorem of moments provides three first integrals, and the *vis viva* theorem provides a fourth. Since t does not enter into those integrals, the motion can be calculated by quadratures.

We verify that by applying the theory of the last multiplier to that particular case. Since the trajectories are planar, from the theorem of moments, we can take that plane to be the xy-plane. The equations of motion in that plane are (upon setting $m = 1$):

$$(1) \qquad dt = \frac{dx}{x'} = \frac{dy}{y'} = \frac{dx'}{F\frac{x}{r}} = \frac{dy'}{F\frac{y}{r}},$$

in which F is a function of $r = \sqrt{x^2 + y^2}$. Consider the system:

$$(1') \qquad \frac{dx}{x'} = \frac{dy}{y'} = \frac{dx'}{F\frac{x}{r}} = \frac{dy'}{F\frac{y}{r}}.$$

We know two integrals of that system:

$$(a) \qquad f \equiv \tfrac{1}{2}(x'^2 + y'^2) - U = \alpha, \text{ in which } U = \int F(r)\,dr,$$

$$(b) \qquad \varphi \equiv x\,y' - x'\,y = \beta.$$

If one infers x' and y' from (a) and (b) and substitutes them in the equation:

$$x\,y' - x'\,y = 0$$

then that equation will admit:

$$\frac{1}{\dfrac{\partial f}{\partial x'}\dfrac{\partial \varphi}{\partial y'} - \dfrac{\partial f}{\partial y'}\dfrac{\partial \varphi}{\partial x'}} \equiv \frac{1}{x\,x' + y\,y'}$$

for an integrating factor.

In order to confirm that, express $x x' + y y'$ in terms of x and y. Upon multiplying (a) by $2(x^2 + y^2)$ and subtracting the square of (b), one will infer that:

$$(x x' + y y')^2 = 2r^2 (U + \alpha) - \beta^2 = R(r) .$$

One will then have:

$$x x' + y y' = \sqrt{R(r)} ,$$
$$x y' - x' y = \beta .$$

Hence:

$$r^2 x' = -\beta x + x\sqrt{R(r)} ,$$
$$r^2 y' = +\beta y + y\sqrt{R(r)} .$$

One will then have:

$$(c) \qquad \frac{y'\,dx - x'\,dy}{x x' + y y'} = \frac{1}{\sqrt{R(r)}}\frac{1}{r^2}\left[\beta(x\,dx + y\,dy) + (y\,dx - x\,dy)\sqrt{R(r)}\right]$$
$$= \frac{\beta\,dr}{r\sqrt{R(r)}} + \frac{y\,dx - x\,dy}{x^2 + y^2} .$$

If one sets $\theta = \arctan y / x$ then one will see that the left-hand side of (c) is the total differential of the function:

$$\psi = \beta\int_{r_0}^{r} \frac{dr}{r\sqrt{R(r)}} + \theta .$$

That third integral $\psi = \gamma$ permits one to calculate y (and as a result x', y') as a function of x. As for t, it is given by the quadrature:

$$dt = \frac{dx}{x'} .$$

Moreover, one can write:

$$dt = \frac{dx}{x'} = \frac{dy}{y'} = \frac{x\,dx + y\,dy}{x x' + y y'} = \frac{r\,dr}{\sqrt{R(r)}} .$$

Therefore:

$$t = \int_{r_0}^{r} \frac{r\,dr}{\sqrt{R(r)}} .$$

We thus verify the conclusions of the theory of the last multiplier in those particular cases.

Application to the motion of material systems.

I. Motion of a point on a surface. – Let *M* be a material point that moves without friction on the surface:

$$(s) \qquad \varphi(x, y, z, t) = 0 .$$

If X', Y', Z' are the components of the active force then the equations of motion, in the first form that Lagrange gave, can be written (upon setting $m = 1$):

$$(1) \qquad dt = \frac{dx}{x'} = \frac{dy}{y'} = \frac{dz}{z'} = \frac{dx'}{X' + \lambda \frac{\partial \varphi}{\partial x}} = \frac{dy'}{Y' + \lambda \frac{\partial \varphi}{\partial y}} = \frac{dz'}{Z' + \lambda \frac{\partial \varphi}{\partial z}} .$$

We know that λ can be expressed as a function of x, y, z, x', y', z', t. Upon differentiating equation (*s*) twice with respect to *t* and taking (1) into account, it will become:

$$(2) \qquad \frac{\partial \varphi}{\partial x}\left[X' + \lambda \frac{\partial \varphi}{\partial x}\right] + \frac{\partial \varphi}{\partial y}\left[Y' + \lambda \frac{\partial \varphi}{\partial y}\right] + \frac{\partial \varphi}{\partial z}\left[Z' + \lambda \frac{\partial \varphi}{\partial z}\right] + \left(\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x} x' + \frac{\partial \varphi}{\partial y} y' + \frac{\partial \varphi}{\partial z} z'\right) = 0 .$$

If one replaces λ in (1) with its value that is deduced from (2) the equations (1) thus-obtained will admit the integral:

$$\varphi(x, y, z, t) = \alpha\, t + \beta ,$$

or, what amounts to the same thing, the first two integrals:

$$(a) \qquad x' \frac{\partial \varphi}{\partial x} + y' \frac{\partial \varphi}{\partial y} + z' \frac{\partial \varphi}{\partial z} + \frac{\partial \varphi}{\partial t} = \alpha ,$$

$$(b) \qquad \varphi - t\left(x' \frac{\partial \varphi}{\partial x} + y' \frac{\partial \varphi}{\partial y} + z' \frac{\partial \varphi}{\partial z} + \frac{\partial \varphi}{\partial t}\right) = \beta .$$

At least one of the derivatives $\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}$ is not identically zero, say, $\frac{\partial \varphi}{\partial z}$. Infer z' from (*a*) by setting $\alpha = 0$ and then infer *z* from (*b*), which will then reduce to $\varphi = \beta$ by setting $\beta = 0$. Finally, substitute those values of *z* and z' in the equations:

$$(3) \qquad dt = \frac{dx}{x'} = \frac{dy}{y'} = \frac{dx'}{X' + \lambda \frac{\partial \varphi}{\partial x}} = \frac{dy'}{Y' + \lambda \frac{\partial \varphi}{\partial y}} .$$

We thus define four first-order equations for determining x, y, x', y' as functions of t, i.e., in order to determine the motion of the point on the surface $\varphi = 0$. On the other hand, if M (x, y, z, x', y', z', t) is a multiplier of the system (1), in which λ is defined by (2), then we know from a remark that was made above that the expression $M / \left(\frac{\partial \varphi}{\partial z}\right)^2$ will also be a multiplier of (3) after we have replaced z and z' in it with their values that we infer from the equations:

$$(c) \qquad \begin{cases} \varphi(x, y, z, t) = 0 \,, \\ \dfrac{\partial \varphi}{\partial x} x' + \dfrac{\partial \varphi}{\partial y} y' + \dfrac{\partial \varphi}{\partial z} z' + \dfrac{\partial \varphi}{\partial t} = 0 \,. \end{cases}$$

Having said that, I say that the system (1) admits the quantity:

$$R = \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2$$

as a multiplier when the given force (X', Y', Z') does not depend upon velocity.

The equation of the last multiplier here is:

$$\frac{d}{dt}(\log M) + \frac{\partial \varphi}{\partial x}\frac{\partial \lambda}{\partial x'} + \frac{\partial \varphi}{\partial y}\frac{\partial \lambda}{\partial y'} + \frac{\partial \varphi}{\partial z}\frac{\partial \lambda}{\partial z'} = 0 \,.$$

Now, if one differentiates the relation (8) with respect to x', y', z', in succession, while observing that X', Y', Z' do not depend upon those variables, then that will give:

$$\frac{\partial \lambda}{\partial x}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right] + 2\left[\frac{\partial^2 \varphi}{\partial x^2} x' + \frac{\partial^2 \varphi}{\partial x\,\partial y} y' + \frac{\partial^2 \varphi}{\partial x\,\partial z} z' + \frac{\partial^2 \varphi}{\partial x\,\partial t}\right] = 0 \,,$$

or rather:

$$R\frac{\partial \lambda}{\partial x'} + 2\frac{\partial}{\partial x}\left(\frac{d\varphi}{dt}\right) = 0 \,,$$

and similarly:

$$R\frac{\partial \lambda}{\partial y'} + 2\frac{\partial}{\partial y}\left(\frac{d\varphi}{dt}\right) = 0 \,,$$

$$R\frac{\partial \lambda}{\partial z'} + 2\frac{\partial}{\partial z}\left(\frac{d\varphi}{dt}\right) = 0 \,.$$

If one observes, as one had verified in the context of the Lagrange equations, that:

$$\frac{\partial}{\partial x}\left(\frac{d\varphi}{dt}\right) = \frac{d}{dt}\left(\frac{\partial\varphi}{\partial x}\right)$$

then one will deduce the following equality from the previous relations:

$$\frac{\partial\varphi}{\partial x}\frac{\partial\lambda}{\partial x'}+\frac{\partial\varphi}{\partial y}\frac{\partial\lambda}{\partial y'}+\frac{\partial\varphi}{\partial z}\frac{\partial\lambda}{\partial z'}+\frac{2}{R}\left[\frac{\partial\varphi}{\partial x}\frac{d}{dt}\left(\frac{\partial\varphi}{\partial x}\right)+\frac{\partial\varphi}{\partial y}\frac{d}{dt}\left(\frac{\partial\varphi}{\partial z}\right)+\frac{\partial\varphi}{\partial y}\frac{d}{dt}\left(\frac{\partial\varphi}{\partial z}\right)\right] = 0 ,$$

or rather:

$$\frac{\partial\varphi}{\partial x}\frac{\partial\lambda}{\partial x'}+\frac{\partial\varphi}{\partial y}\frac{\partial\lambda}{\partial y'}+\frac{\partial\varphi}{\partial z}\frac{\partial\lambda}{\partial z'}+\frac{dR/dt}{R} = 0 .$$

That relation will be true if one supposes that x, y, z, x', y', z' are arbitrary functions of t that satisfy equations (1). Consequently, $M = R$ will be a multiplier of (1).

With that, assume that one knows three first integrals of the motion of a point on the surface. Those integrals can always be put into the form:

$$(d)\qquad \begin{cases} \psi_1(t, x, y, x', y') = \gamma_1, \\ \psi_2(t, x, y, x', y') = \gamma_2, \\ \psi_3(t, x, y, x', y') = \gamma_3. \end{cases}$$

One can infer x', y', y, for example, as functions of x and t from that relation and substitute them in the equation:

$$x'\,dt - dx = 0 .$$

That equation admits the quantity:

$$\frac{R}{\left(\dfrac{\partial\varphi}{\partial z}\right)^2 \Delta}$$

as an integrating factor, in which Δ represents the determinant:

$$\begin{vmatrix} \dfrac{\partial\psi_1}{\partial y} & \dfrac{\partial\psi_1}{\partial x'} & \dfrac{\partial\psi_1}{\partial y'} \\ \dfrac{\partial\psi_2}{\partial y} & \dfrac{\partial\psi_2}{\partial x'} & \dfrac{\partial\psi_2}{\partial y'} \\ \dfrac{\partial\psi_3}{\partial y} & \dfrac{\partial\psi_3}{\partial x'} & \dfrac{\partial\psi_3}{\partial y'} \end{vmatrix} ,$$

and one supposes that x', y', z', y, and z are expressed in terms of x and t, according to (c) and (d).

When the surface and the active force do not depend upon time, t will not appear in either φ or equation (2), or in R and λ, as a result. The multiplier $R / \left(\frac{\partial\varphi}{\partial z}\right)^2$ of the system (3) is also a multiplier of the system:

$$(3') \qquad \frac{dx}{x'} = \frac{dy}{y'} = \frac{dx'}{X' + \lambda \frac{\partial\varphi}{\partial x}} = \frac{dy'}{Y' + \lambda \frac{\partial\varphi}{\partial y}}$$

into which t will not enter. Consequently, in order to solve the problem by quadratures, it will suffice to know two first integrals of (3′).

We then arrive at the following conclusions: When a point moves without friction on a surface and is subject to a given force that does not depend upon the velocity of the point:

1. If the surface varies with time then in order for the motion to be determined by quadratures, it will suffice to know three first integrals of the motion.

2. If neither the surface nor the given force depends upon time then it will suffice to know two first integrals in which t does not figure.

II. Motion of an arbitrary system. – Let Σ be a system of n material points that are subject to p frictionless constraints. The coordinates (x_i, y_i, z_i) of those points are restricted to verify the p equations of constraint:

$$(A) \qquad \begin{cases} \varphi(x_1, y_1, z_1, \ldots, x_i, y_i, z_i, \ldots, x_n, y_n, z_n, t) = 0, \\ \psi(x_1, y_1, z_1, \ldots, x_i, y_i, z_i, \ldots, x_n, y_n, z_n, t) = 0, \\ \chi(x_1, y_1, z_1, \ldots, x_i, y_i, z_i, \ldots, x_n, y_n, z_n, t) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

One can infer p of the $3n$ quantities x_i, y_i, z_i from those p equations, for example, x_n, y_n, z_n, z_{n-1}, … as functions of the $3n - p$ other ones x_1, y_1, z_1, x_2, …, and t. Suppose, for more clarity, that this solution has been performed and write the equations of constraint in the form:

$$(A') \qquad \begin{cases} \varphi \equiv z_n - \varphi_1(x_1, y_1, z_1, x_2, \ldots, t) = 0, \\ \psi \equiv y_n - \psi_1(x_1, y_1, z_1, x_2, \ldots, t) = 0, \\ \chi \equiv x_n - \chi_1(x_1, y_1, z_1, x_2, \ldots, t) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

then the functions φ_1, ψ_1, χ_1, … depend upon only the $(3n - p)$ variables x_1, y_1, z_1, x_2, …, and t.

If X_i, Y_i, Z_i are the components of the active force that is exerted on the point M_1 then the motion of the system will be determined by the equations:

$$(1)\qquad \begin{cases} m_i \dfrac{d^2 x_i}{dt^2} = X_i' + \lambda \dfrac{\partial \varphi}{\partial x_i} + \mu \dfrac{\partial \psi}{\partial x_i} + \nu \dfrac{\partial \chi}{\partial x_i} + \cdots = X_i, \\ m_i \dfrac{d^2 y_i}{dt^2} = Y_i' + \lambda \dfrac{\partial \varphi}{\partial y_i} + \mu \dfrac{\partial \psi}{\partial y_i} + \nu \dfrac{\partial \chi}{\partial y_i} + \cdots = Y_i, \qquad (i = 1, 2, \dots, n) \\ m_i \dfrac{d^2 z_i}{dt^2} = Z_i' + \lambda \dfrac{\partial \varphi}{\partial z_i} + \mu \dfrac{\partial \psi}{\partial z_i} + \nu \dfrac{\partial \chi}{\partial z_i} + \cdots = Z_i. \end{cases}$$

One knows that one can express the coefficients λ, μ, ν, … as functions of x_i, y_i, z_i, x_i', y_i', z_i', and t. It will suffice to differentiate the equations of constraint twice with respect to t. Upon taking equations (1) into account, one will thus obtain p equations such as the following:

$$(B)\qquad \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{1}{m_i}\left[\frac{\partial \varphi}{\partial x_i}\left(X_i' + \lambda \frac{\partial \varphi}{\partial x_i} + \mu \frac{\partial \psi}{\partial x_i} + \cdots\right) + \frac{\partial \varphi}{\partial y_i}\left(Y_i' + \lambda \frac{\partial \varphi}{\partial y_i} + \mu \frac{\partial \psi}{\partial y_i} + \cdots\right) + \frac{\partial \varphi}{\partial z_i}\left(Z_i' + \lambda \frac{\partial \varphi}{\partial z_i} + \mu \frac{\partial \psi}{\partial z_i} + \cdots\right)\right] \\ \qquad\qquad + \left[\dfrac{\partial \varphi}{\partial t} + \displaystyle\sum_{i=1}^{n}\left(\frac{\partial \varphi}{\partial x_i} x_i' + \frac{\partial \varphi}{\partial y_i} y_i' + \frac{\partial \psi}{\partial y_i} z_i'\right)\right]_2 = 0. \end{cases}$$

As we have shown, those p relations are soluble for λ, μ, ν, …, and if one replaces λ, μ, ν, … with those values in equations (1) then the system thus-obtained, which is equivalent to the system:

$$(2)\qquad dt = \frac{dx_i}{x_i'} = \frac{dy_i}{y_i'} = \frac{dz_i}{z_i'} = \frac{m_i\, dx_i'}{X_i} = \frac{m_i\, dy_i'}{Y_i} = \frac{m_i\, dz_i'}{Z_i},$$

will admit the $2p$ first integrals:

$$(a)\qquad \begin{cases} \varphi' = \dfrac{\partial \varphi}{\partial t} + \displaystyle\sum_{i=1}^{n}\left(\frac{\partial \varphi}{\partial x_i} x_i' + \frac{\partial \varphi}{\partial y_i} y_i' + \frac{\partial \varphi}{\partial z_i} z_i'\right) = \alpha', \\ \qquad\qquad\qquad \varphi - t\,\varphi' = \alpha, \\ \psi' = \dfrac{\partial \psi}{\partial t} + \displaystyle\sum_{i=1}^{n}\left(\frac{\partial \psi}{\partial x_i} x_i' + \frac{\partial \psi}{\partial y_i} y_i' + \frac{\partial \psi}{\partial z_i} z_i'\right) = \beta', \\ \qquad\qquad\qquad \psi - t\,\psi' = \beta, \quad \text{etc.} \end{cases}$$

If the equations of constraint have been put into the form (A') then the first integral (a) will contain only z_n', the second one will contain only z_n, the third one will contain only y_n', etc. One then infers z'_n, z_n, y'_n, … from equations (*a*) while setting $\alpha' = \alpha = \beta = \ldots = 0$ and then substitute that in equations (2), in which we suppress the $2p$ equations that include dz'_n, dz_n, dy'_n, etc. We thus form a system of (3) of $(6n - 2p)$ first-order equations between t and the $(6n - 2p)$ variables x_1, x'_1, y_1, y'_1, etc. If M is a multiplier of (2) then the expression $M \,/\, \frac{\partial \varphi'}{\partial z'_n} \frac{\partial \varphi}{\partial z_n} \frac{\partial \psi'}{\partial y'_n} \cdots$ (i.e., M here) will also be a multiplier of the system (3), on the condition that we replace z'_n, z_n, y'_n, … with their values that we infer from (*a*).

It is even clear, moreover, that if the relations (*a*) are not solved with respect to p of the variables then one can again deduce a multiplier of (3) from M.

Finally, if the constraints and the given forces are independent of time then equations (3′) will be obtained by suppressing the first equation (in dt) in the system (3), which does not include t because t does not appear in either the λ, μ, ν, … or the X'_i, Y'_i, Z'_i. Any multiplier of (3) that is independent of t will then be a multiplier (3′).

Having said that, we shall show that we will always know a multiplier of equations (1) when the given forces do not depend upon velocities.

The equation for the multipliers of the system (1) can then be written:

$$(4) \qquad \frac{d}{dt}\log M + \sum_{j=1}^{n}\left(\frac{\partial \lambda}{\partial x'_j}\frac{\partial \varphi}{\partial x_j} + \frac{\partial \lambda}{\partial y'_j}\frac{\partial \varphi}{\partial y_j} + \frac{\partial \lambda}{\partial z'_j}\frac{\partial \varphi}{\partial z_j}\right) + \sum_{j=1}^{n}\left(\frac{\partial \mu}{\partial x'_j}\frac{\partial \psi}{\partial x_j} + \frac{\partial \mu}{\partial y'_j}\frac{\partial \psi}{\partial y_j} + \frac{\partial \mu}{\partial z'_j}\frac{\partial \psi}{\partial z_j}\right) + \cdots = 0 .$$

One can replace $\frac{\partial \lambda}{\partial x'_j}$, $\frac{\partial \lambda}{\partial y'_j}$, … with their values that are deduced from equations (*B*) in the latter equation. Now differentiate those equations with respect to x'_j, y'_j, z'_j upon setting:

$$(\varphi, \varphi) = \sum_{i=1}^{n}\left[\left(\frac{\partial \varphi}{\partial x_i}\right)^2 + \left(\frac{\partial \varphi}{\partial y_i}\right)^2 + \left(\frac{\partial \varphi}{\partial z_i}\right)^2\right],$$

$$(\varphi, \psi) = \sum_{i=1}^{n}\left(\frac{\partial \varphi}{\partial x_i}\frac{\partial \psi}{\partial x_i} + \frac{\partial \varphi}{\partial y_i}\frac{\partial \psi}{\partial y_i} + \frac{\partial \varphi}{\partial z_i}\frac{\partial \psi}{\partial z_i}\right), \qquad \text{etc.,}$$

to abbreviate. That will give:

$$\frac{\partial \lambda}{\partial x'_j}(\varphi, \varphi) + \frac{\partial \lambda}{\partial x'_j}(\varphi, \psi) + \frac{\partial \lambda}{\partial x'_j}(\varphi, \chi) + \cdots + 2\frac{d}{dt}\left(\frac{\partial \varphi}{\partial x_j}\right) = 0 ,$$

or upon observing that $\frac{d}{dt}\left(\frac{\partial \varphi}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{d\varphi}{dt}\right)$ and setting $\frac{d\varphi}{dt} = \varphi'$:

$$(5)\qquad \begin{cases} \dfrac{\partial\lambda}{\partial x'_j}(\varphi,\varphi)+\dfrac{\partial\mu}{\partial x'_j}(\varphi,\psi)+\dfrac{\partial\nu}{\partial x'_j}(\varphi,\chi)+\cdots+2\dfrac{\partial\varphi'}{\partial x_j}=0\ . \\ \text{Similarly}: \\ \dfrac{\partial\lambda}{\partial x'_j}(\psi,\varphi)+\dfrac{\partial\mu}{\partial x'_j}(\psi,\varphi)+\dfrac{\partial\nu}{\partial x'_j}(\psi,\chi)+\cdots+2\dfrac{\partial\psi'}{\partial x_j}=0\ . \end{cases}$$

Consider the determinant:

$$R=\begin{vmatrix} (\varphi,\varphi) & (\varphi,\psi) & (\varphi,\chi) & \cdots & \cdots \\ (\psi,\varphi) & (\psi,\psi) & (\psi,\chi) & \cdots & \cdots \\ (\chi,\varphi) & (\chi,\psi) & (\chi,\chi) & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \\ \vdots & \vdots & \vdots & & \ddots \end{vmatrix}.$$

I say that this determinant satisfies equation (4), i.e., that it is a multiplier of (1). In order to verify that, solve the relation (5). One will find that:

$$\frac{\partial\lambda}{\partial x'_j}=-\frac{2}{R}\left[\frac{\partial\varphi'}{\partial x_j}\cdot\frac{\partial R}{\partial(\varphi,\varphi)}+\frac{\partial\psi'}{\partial x_j}\cdot\frac{\partial R}{\partial(\psi,\varphi)}+\frac{\partial\chi'}{\partial x_j}\cdot\frac{\partial R}{\partial(\chi,\varphi)}+\cdots\right].$$

Similarly:

$$\frac{\partial\lambda}{\partial y'_j}=-\frac{2}{R}\left[\frac{\partial\varphi'}{\partial y_j}\cdot\frac{\partial R}{\partial(\varphi,\varphi)}+\frac{\partial\psi'}{\partial y_j}\cdot\frac{\partial R}{\partial(\psi,\varphi)}+\frac{\partial\chi'}{\partial y_j}\cdot\frac{\partial R}{\partial(\chi,\varphi)}+\cdots\right],$$

$$\frac{\partial\lambda}{\partial z'_j}=-\frac{2}{R}\left[\frac{\partial\varphi'}{\partial z_j}\cdot\frac{\partial R}{\partial(\varphi,\varphi)}+\frac{\partial\psi'}{\partial z_j}\cdot\frac{\partial R}{\partial(\psi,\varphi)}+\frac{\partial\chi'}{\partial z_j}\cdot\frac{\partial R}{\partial(\chi,\varphi)}+\cdots\right].$$

As a result, one will get:

$$\sum_{j=1}^{n}\left(\frac{\partial\lambda}{\partial x'_j}\frac{\partial\varphi}{\partial x_j}+\frac{\partial\lambda}{\partial y'_j}\frac{\partial\varphi}{\partial y_j}+\frac{\partial\lambda}{\partial z'_j}\frac{\partial\varphi}{\partial z_j}\right)+\sum_{j=1}^{n}\left(\frac{\partial\mu}{\partial x'_j}\frac{\partial\psi}{\partial x_j}+\frac{\partial\mu}{\partial y'_j}\frac{\partial\psi}{\partial y_j}+\frac{\partial\mu}{\partial z'_j}\frac{\partial\psi}{\partial z_j}\right)+\cdots$$

$$=-\frac{2}{R}\left[\frac{\partial R}{\partial(\varphi,\varphi)}\sum\left(\frac{\partial\varphi'}{\partial x_j}\frac{\partial\varphi}{\partial x_j}+\frac{\partial\varphi'}{\partial y_j}\frac{\partial\varphi}{\partial y_j}+\frac{\partial\varphi'}{\partial z_j}\frac{\partial\varphi}{\partial z_j}+\right)\right.$$
$$\left.+\frac{\partial R}{\partial(\psi,\varphi)}\sum\left(\frac{\partial\psi'}{\partial x_j}\frac{\partial\varphi}{\partial x_j}+\frac{\partial\psi'}{\partial y_j}\frac{\partial\varphi}{\partial y_j}+\frac{\partial\psi'}{\partial z_j}\frac{\partial\varphi}{\partial z_j}+\right)+\cdots\right],$$

i.e.:

$$= -\frac{1}{R}\left[\frac{\partial R}{\partial(\varphi,\varphi)}\frac{d(\varphi,\varphi)}{dt} + \frac{\partial R}{\partial(\psi,\psi)}\frac{d(\psi,\psi)}{dt} + \cdots + \frac{2\,\partial R}{\partial(\varphi,\psi)}\frac{d(\varphi,\psi)}{dt} + \cdots\right]$$
$$= -\frac{1}{R}\frac{dR}{dt},$$

if one observes that $(\varphi, \psi) = (\psi, \varphi)$.

One can then put equation (4) into the form:

$$\frac{d\,LM}{dt} - \frac{1}{R}\frac{dR}{dt} = 0,$$

i.e., that $M = R$ is a multiplier of (1).

If the constraints do not depend upon time, any more than the given forces, then t will not appear in R, which is also a multiplier of equation (3′).

One then has this theorem:

If the given forces that are exerted on a system of n material points are subject to p frictionless constraints and do not depend upon velocity then in order for the determination of the motion to be achieved by quadrature, it will suffice to know $(6n - 2p - 1)$ *first integrals of the motion.*

Moreover, when the constraints and the given forces are independent of time, it will suffice to consider $(6n - 2p - 2)$ *first integrals into which t does not enter*.

The proof of that theorem will become much quicker if one applies the theory of the last multiplier to the canonical equations. That new proof can be extended to the theorem on continuous systems whose position depends upon a finite number of parameters, in addition, as we shall now prove.

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LECTURE 14

APPLICATION OF THE THEORY OF THE LAST MULTIPLIER TO THE CANONICAL EQUATIONS.

Consider an arbitrary system of canonical equations:

$$(1) \qquad dt = \frac{dq_1}{\dfrac{\partial K}{\partial p_1}} = \frac{dp_1}{\dfrac{\partial K}{\partial q_1} + Q_1} = \frac{dq_2}{\dfrac{\partial K}{\partial p_2}} = \ldots = \frac{dp_k}{\dfrac{\partial K}{\partial q_k} + Q_k} .$$

The equations for the last multiplier M relative to that system can be written as:

$$\frac{d\,LM}{dt} + \frac{\partial Q_1}{\partial p_1} + \frac{\partial Q_2}{\partial p_2} + \cdots + \frac{\partial Q_k}{\partial p_k} = 0 ,$$

since $\dfrac{\partial^2 K}{\partial q_i\,\partial p_j} - \dfrac{\partial^2 K}{\partial p_i\,\partial q_j} \equiv 0$.

If the quantity $\sum \dfrac{\partial Q_i}{\partial p_i}$ is identically zero then $M = 1$ will be a multiplier of equations (1). That will be true when the material system Σ whose motion is determined by equations (1) is frictionless and the given forces do not depend upon velocity.

If one then knows $(2k - 1)$ distinct first integrals of (1) in this case then the last equation can be integrated by quadratures.

When neither the constraints nor the given forces depend upon time, moreover, t will not figure in either K or the Q_i . In order to achieve the integration by quadratures, it will suffice to know $(2k - 2)$ first integrals in which t does not enter.

Let us apply those generalities to the frictionless systems with constraints that are independent of time and whose position is defined by two parameters. If the given forces that are exerted upon such a system depend upon neither velocity nor time then if one is to calculate the motion by quadratures, it will suffice to know two integrals into which t does not enter.

In particular, if the given forces admit a force function $U(q_1, q_2)$ then it will suffice to know a second integral that is distinct from that of *vis viva* and does not include t. An application of the theory of the last multiplier will then lead to some remarkable conclusions in this case that we shall now develop.

Write the canonical equations (while ignoring the first one):

(1′)
$$\frac{dq_1}{\frac{\partial H}{\partial p_1}} = \frac{dp_1}{\frac{\partial H}{\partial q_1}} = \frac{dq_2}{\frac{\partial H}{\partial p_2}} = \frac{dp_2}{\frac{\partial H}{\partial q_2}},$$

in which:

$$H = T - U .$$

Let:

(2)
$$f(q_1, q_2, p_1, p_2) = \alpha$$

be a first integral of (1′) that is distinct from the *vis viva* integral:

(3)
$$H(q_1, q_2, p_1, p_2) = h .$$

Infer p_1 and p_2 as functions of q_1, q_2 from equations (2) and (3):

$$p_1 = \varphi_1(q_1, q_2, \alpha, h) , \qquad p_2 = \varphi_2(q_1, q_2, \alpha, h) .$$

If we replace p_1 and p_2 everywhere with those values then the equation:

$$\frac{\partial H}{\partial p_1}dq_2 - \frac{\partial H}{\partial p_2}dq_1 = 0$$

will admit:

$$\frac{1}{\delta} = \frac{1}{\frac{\partial H}{\partial p_1}\frac{\partial f}{\partial p_2} - \frac{\partial H}{\partial p_2}\frac{\partial f}{\partial p_1}}$$

as an integrating factor. In other words, one has:

(4)
$$\frac{\frac{\partial H}{\partial p_1}dq_2 - \frac{\partial H}{\partial p_2}dq_1}{\delta} = d \cdot F(q_1, q_2, \alpha, h) .$$

I say that the left-hand side of (4) coincides with:

$$\frac{\partial p_1}{\partial \alpha}dq_1 + \frac{\partial p_2}{\partial \alpha}dq_2 .$$

Indeed, from the theory of functional determinants, one knows that:

$$\delta = \frac{1}{\dfrac{\partial p_1}{\partial h}\dfrac{\partial p_2}{\partial \alpha} - \dfrac{\partial p_1}{\partial \alpha}\dfrac{\partial p_2}{\partial h}} .$$

On the other hand, if one replaces p_1, p_2 with φ_1, φ_2 in (2) and (3) then those equations will be verified identically. Upon differentiating (3) with respect to h and α, one will find that:

$$\frac{\partial H}{\partial p_1}\frac{\partial p_1}{\partial h} + \frac{\partial H}{\partial p_2}\frac{\partial p_2}{\partial h} = 1 ,$$

$$\frac{\partial H}{\partial p_1}\frac{\partial p_1}{\partial \alpha} + \frac{\partial H}{\partial p_2}\frac{\partial p_2}{\partial \alpha} = 0 .$$

Hence:

$$\frac{\partial H}{\partial p_1} = \delta \frac{\partial p_2}{\partial \alpha} , \qquad \frac{\partial H}{\partial p_2} = -\delta \frac{\partial p_1}{\partial \alpha} .$$

The equation that remains to be integrated can then be written:

$$\frac{\partial p_1}{\partial \alpha} dq_1 + \frac{\partial p_2}{\partial \alpha} dq_2 = 0 , \tag{4′}$$

and one has assumed that its left-hand side is a total differential $dF (q_1, q_2, \alpha, h)$.

It is quite easy, moreover, to verify the conclusion to which we just arrived by direct calculation by showing that $(p_1\, dq_1 + p_2\, dq_2)$ is an *exact total differential*.

In order for that to be true, it is necessary and sufficient that one should have:

$$\frac{\partial \varphi_1}{\partial q_2} \equiv \frac{\partial \varphi_2}{\partial q_1} .$$

Calculate $\partial p_1 / \partial q_2$ and $\partial p_2 / \partial q_1$ using equations (2) and (3). That will give:

$$\frac{\partial H}{\partial q_1} + \frac{\partial H}{\partial p_1}\frac{\partial p_1}{\partial q_1} + \frac{\partial H}{\partial p_2}\frac{\partial p_2}{\partial q_1} = 0 ,$$

$$\frac{\partial f}{\partial q_1} + \frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial q_1} + \frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial q_1} = 0 .$$

Therefore, one infers that:

$$\left(\frac{\partial H}{\partial q_1}\frac{\partial f}{\partial p_1} - \frac{\partial H}{\partial p_1}\frac{\partial f}{\partial q_1}\right) - \delta \frac{\partial p_2}{\partial q_1} = 0 . \tag{5}$$

One similarly finds that:

(6) $$\left(\frac{\partial H}{\partial q_2}\frac{\partial f}{\partial p_2}-\frac{\partial H}{\partial p_2}\frac{\partial f}{\partial q_2}\right)+\delta\frac{\partial p_1}{\partial q_2} = 0 .$$

Furthermore, the function f satisfies the relation:

(7) $$\frac{\partial H}{\partial q_1}\frac{\partial f}{\partial p_1}-\frac{\partial H}{\partial p_1}\frac{\partial f}{\partial q_1}+\frac{\partial H}{\partial q_2}\frac{\partial f}{\partial p_2}-\frac{\partial H}{\partial p_2}\frac{\partial f}{\partial q_2} = 0$$

identically.

If we the add corresponding sides of (5) and (6) then that will give:

$$\delta\left(\frac{\partial p_2}{\partial q_1}-\frac{\partial p_1}{\partial q_2}\right)\equiv 0 .$$

In all of the foregoing, we have assumed that δ is not identically zero. Under that condition (which is always realized, as we will soon show), we see that the expression $p_1\,dq_1 + p_2\,dq_2$ will be an exact total differential:

$$\int p_1\,dq_1 + p_2\,dq_2 = W(q_1, q_2, \alpha, h) .$$

The integral of equation (4) is given by the equality $\partial W / \partial\alpha = \beta$.

Conversely, if a relation $f = \alpha$, combined with the equation $H = h$, determines functions p_1, p_2 of (q_1, q_2, α, h) such that $p_1\,dq_1 + p_2\,dq_2$ is an exact differential then the function f will verify equation (7) and will be a first integral of (1′), in addition.

From that, consider the partial differential equation:

(8) $$H\left(q_1, q_2, \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}\right) = h$$

and a second relation:

$$f\left(q_1, q_2, \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}\right) = \alpha .$$

The necessary and sufficient condition for those two equations to admit a common integral $W(q_1, q_2, \alpha, h)$ for each value of α and h is that f and H should be coupled by equation (7); in other words, f must be a first integral of (1′).

By definition, if one knows a first integral $f = \alpha$ of (1′) that is distinct from that of *vis viva* $H = h$, and one can infer p_1, p_2 as functions of q_1, q_2, α, h using those two equations then the expression $p_1\,dq_1 + p_2\,dq_2$ will be an exact differential:

$$\int p_1\,dq_1 + p_2\,dq_2 = W(q_1, q_2, \alpha, h) ,$$

and the motion of the system is determined by the equality:

$$\frac{\partial W}{\partial \alpha} = \beta,$$

which defines q_2 as a function of q_1 and the three constants h, α, β.

As for time t, it will be defined as a function of q_1 (for example, with the aid of a quadrature) when one expresses q_2 in terms of q_1 . However, more symmetrically, it should be pointed out that t satisfies the two equalities:

$$dt = \frac{dq_1}{\frac{\partial H}{\partial p_1}} = \frac{dq_2}{\frac{\partial H}{\partial p_2}} .$$

On the other hand, one has:

$$\frac{\partial H}{\partial p_1}\frac{\partial p_1}{\partial h} + \frac{\partial H}{\partial p_2}\frac{\partial p_2}{\partial h} = 1 ,$$

$$\frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial h} + \frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial h} = 0 .$$

Hence:

$$\frac{\partial p_1}{\partial h} = \frac{\partial f / \partial p_2}{\frac{\partial H}{\partial p_1}\frac{\partial f}{\partial p_2} - \frac{\partial H}{\partial p_2}\frac{\partial f}{\partial p_1}}, \qquad \frac{\partial p_2}{\partial h} = \frac{-\partial f / \partial p_1}{\frac{\partial H}{\partial p_1}\frac{\partial f}{\partial p_2} - \frac{\partial H}{\partial p_2}\frac{\partial f}{\partial p_1}} .$$

One can then write:

$$dt = \frac{\frac{\partial f}{\partial p_2}dq_1 - \frac{\partial f}{\partial p_1}dq_2}{\frac{\partial H}{\partial p_1}\frac{\partial f}{\partial p_2} - \frac{\partial H}{\partial p_2}\frac{\partial f}{\partial p_1}} = \frac{\partial p_1}{\partial h}dq_1 + \frac{\partial p_2}{\partial h}dq_2 ,$$

i.e.:

$$t - t_0 = \frac{\partial W}{\partial h} .$$

We will see in the next lecture that this theorem is a particular case of a theorem of Jacobi.

Remark. – All of the foregoing argument supposes essentially that we can solve the two equations $f = \alpha$, $H = h$ for p_1, p_2 . We shall now show that this is always true. In the contrary case, the functional determinant $\delta = \left(\frac{\partial H}{\partial p_1}\frac{\partial f}{\partial p_2} - \frac{\partial H}{\partial p_2}\frac{\partial f}{\partial p_1}\right)$ will be identically zero, and there will exist a relation of the form:

$$\Phi\,(H, f, q_1, q_2) = 0$$

between f and H.

One can infer f from that equation. Otherwise, H would be a simple function of q_1, q_2, or rather q_2 would be expressible as a function of q_1 with no arbitrary constants. We then write the relation thus:

$$f = \psi(H, q_1, q_2) .$$

ψ will define upon at least one of the variables q_1, q_2, since the integral $f = \alpha$ is distinct from that of *vis viva*, by hypothesis.

On the other hand, since $\psi = \alpha$ is an integral, the function ψ must verify the equality:

$$\frac{\partial \psi}{\partial H}(H, H) + \frac{\partial \psi}{\partial q_1}\frac{\partial H}{\partial p_1} + \frac{\partial \psi}{\partial q_2}\frac{\partial H}{\partial p_2} \equiv 0 ,$$

or

$$\frac{\partial \psi}{\partial q_1}\frac{\partial H}{\partial p_1} + \frac{\partial \psi}{\partial q_2}\frac{\partial H}{\partial p_2} \equiv 0 ,$$

or finally:

$$(9) \qquad \frac{\dfrac{\partial \psi}{\partial q_1}}{\dfrac{\partial \psi}{\partial q_2}} \equiv -\frac{\dfrac{\partial H}{\partial p_2}}{\dfrac{\partial H}{\partial p_1}} .$$

$\dfrac{\partial H}{\partial p_1}$ and $\dfrac{\partial H}{\partial p_2}$ are homogeneous linear forms in p_1, p_2 whose determinant is not zero, as one knows. The right-hand side of (9) is therefore a function of p_1 / p_2, say $\chi(p_1 / p_2, q_1, q_2)$. As for the left-hand side, it will coincide with χ only if it depends upon p_1, p_2, and as a result, upon H. One then infers from equation (9) that:

$$H = F\left(\frac{p_1}{p_2}, q_1, q_2\right) ,$$

which is an absurd equality, since H has degree two in p_1, p_2 . We thus arrive at the conclusion that one can always solve the distinct integrals $H = h$, $f = \alpha$ for p_1, p_2 .

We shall now apply the preceding theorems to some particular examples.

I. Motion of a massive point that moves without friction on a paraboloid with a vertical axis. – Let us first apply the theory of the last multiplier to the equations of motion in the first form that Lagrange gave to them.

If the paraboloid is defined by the relation:

$$(A) \qquad \frac{x^2}{\alpha} + \frac{y^2}{\beta} - 2z = 0$$

then the equations of motion will be:

$$(B) \qquad \begin{cases} x'' = \lambda \dfrac{x}{\alpha}, \\ y'' = \lambda \dfrac{y}{\beta}, \\ z'' = -\lambda + g. \end{cases}$$

Differentiate the relation (A) twice. Upon taking (B) into account, that will give:

$$\lambda\left(\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+1\right)+\frac{x'^2}{\alpha}+\frac{y'^2}{\beta}-g = 0 .$$

If one substitutes that value of λ in the first two equations (B) and eliminates z and z' using (A) then the two equations thus-obtained will admit the expression $\left(\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+1\right)$ as a last multiplier. Since z and z' do not figure in λ, the elimination is found to take place because of that fact in its own right, and one immediately obtains:

$$(B') \qquad \begin{cases} \dfrac{x''}{\dfrac{x'^2}{\alpha}+\dfrac{y'^2}{\beta}-g} = -\dfrac{x}{\alpha}\dfrac{1}{\left(\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}+1\right)}, \\ \dfrac{y''}{\dfrac{x'^2}{\alpha}+\dfrac{y'^2}{\beta}-g} = -\dfrac{y}{\beta}\dfrac{1}{\left(\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}+1\right)}. \end{cases}$$

On the other hand, one knows one first integral of those equations, namely, the *vis viva* integral:

$$(a) \qquad 2T \equiv x'^2\left(1+\frac{x^2}{\alpha^2}\right)+y'^2\left(1+\frac{y^2}{\beta^2}\right)+\frac{2x\,y}{\alpha\,\beta}x'\,y' = g\left(\frac{x^2}{\alpha}+\frac{y^2}{\beta}\right)+h .$$

Now multiply the first equation in (B') by x'/α, the second one by y'/β, and add them. Upon integrating, that will give:

$$(b) \qquad f = \left(\frac{x'^2}{\alpha}+\frac{y'^2}{\beta}-g\right)\left(1+\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}\right) = C .$$

Let δ denote the functional determinant of T and f with respect to the variables x', y'. If we infer x', y' from equations (a) and (b) then the expression:

$$(c) \qquad \left(\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+1\right)\frac{y'\,dx - x'\,dy}{\delta}$$

will be an exact total differential $d \cdot F\,(x, y, h, C)$. The motion is defined by the equality F = const.

If we let μ and ν denote the two values of the function:

$$-\left(\alpha+\beta+\frac{x^2}{\alpha}+\frac{y^2}{\beta}\right)\pm\sqrt{\left(\alpha+\beta+\frac{x^2}{\alpha}+\frac{y^2}{\beta}\right)^2-4\left(\frac{\beta\,x^2}{\alpha}+\frac{\alpha\,y^2}{\beta}+\alpha\,\beta\right)}$$

then a laborious calculation will verify that the expression (c) is effectively equal to:

$$\sqrt{\frac{\mu}{(\alpha+\mu)(\beta+\mu)(g\,\mu^2+h\,\mu+C)}}\,d\mu+\sqrt{\frac{\nu}{(\alpha+\nu)(\beta+\nu)(g\,\nu^2+h\nu+C)}}\,d\nu = dF.$$

However, we shall arrive at that result more easily in what follows by appealing to different coordinates.

Thus, here is a case in which Jacobi's theory will show that the motion can certainly be calculated by quadratures, although it would be quite difficult to show that directly, at least with the variables that are employed. Furthermore, one will arrive at the same result by appealing to the canonical equations. If one substitutes the variables p_1 and p_2 that are defined by the equalities:

$$p_1 = x'\left(1+\frac{x^2}{\alpha^2}\right)+y'\,\frac{x\,y}{\alpha\,\beta}\,,$$

$$p_2 = x'\,\frac{x\,y}{\alpha\,\beta}+y'\left(1+\frac{y^2}{\beta^2}\right)$$

for the variables x', y' and in (a) and (b) then one will effortlessly see that the expression $\left(\frac{\partial p_1}{\partial\,?}\,dx+\frac{\partial p_2}{\partial\,?}\,dy\right)$ coincides with the expression (c).

We similarly propose to study the motion of a point M that moves without friction on an ellipsoid and is attracted to the center of the ellipsoid in proportion to its distance from it.

If the ellipsoid is defined by the relation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

then the equations of motion can be written:

$$(B)\qquad \begin{cases} x'' = -k\,x + \lambda\,\dfrac{x}{a^2}, \\[2mm] y'' = -k\,y + \lambda\,\dfrac{y}{b^2}, \\[2mm] z'' = -k\,z + \lambda\,\dfrac{z}{c^2}. \end{cases}$$

In addition to the *vis viva* integral, one defines a second integral in the following manner: Add corresponding sides of equations (B) after multiplying them by $\dfrac{x'}{a^2}$, $\dfrac{y'}{b^2}$, $\dfrac{z'}{c^2}$, respectively, and then by $\dfrac{x}{a^2}$, $\dfrac{y}{b^2}$, $\dfrac{z}{c^2}$, resp. Upon taking the relations:

$$\frac{x'\,x}{a^2} + \frac{y'\,y}{b^2} + \frac{z'\,z}{c^2} = 0\,, \qquad \frac{x''\,x}{a^2} + \frac{y''\,y}{b^2} + \frac{z''\,z}{c^2} + \frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 0$$

into account, one will find that:

$$\frac{x'\,x''}{a^2} + \frac{y'\,y''}{b^2} + \frac{z'\,z''}{c^2} = \lambda\left[\frac{x\,x'}{a^4} + \frac{y\,y'}{b^4} + \frac{z\,z'}{c^4}\right]$$

and

$$-\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2}\right) = -\,k + \lambda\left[\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right].$$

One will then deduce immediately that:

$$\frac{-\dfrac{d}{dt}\left(\dfrac{x'^2}{a^2} + \dfrac{y'^2}{b^2} + \dfrac{z'^2}{c^2} - k\right)}{\dfrac{x'^2}{a^2} + \dfrac{y'^2}{b^2} + \dfrac{z'^2}{c^2} - k} = \frac{\dfrac{d}{dt}\left(\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4}\right)}{\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4}}\,,$$

and finally:

$$\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} - k\right) = C\,.$$

We are then certain that the problem can be achieved by quadratures. However, those quadratures can be carried out comfortably only with the aid of conveniently-chosen coordinates, which are the elliptic coordinates that we shall introduce later on.

II. – Two points M and M_1 are constrained to slide without friction on two helices:

$$x = R\cos\theta,\quad y = R\sin\theta,\quad z = K\theta \quad \text{and} \quad x_1 = R_1\cos\theta_1,\quad y_1 = R_1\sin\theta_1,\quad z_1 = K\theta_1 .$$

The two points repel in proportion to the distance between them. Find the motion of the system.

We know how to form (see page 106) two first integrals of the motion, namely:

$$2T = m(R^2+K^2)\,\theta'^2 + m_1(R_1^2+K^2)\,\theta_1'^2 = -\,2\mu R R_1\cos(\theta-\theta_1) + \mu K^2(\theta_1-\theta)^2 + K,$$

and

$$m(R^2+K^2)\,\theta' + m_1(R_1^2+K^2)\,\theta_1' = C .$$

Replace θ', θ_1' with the canonical variables:

$$p = m(R^2+K^2)\,\theta',\qquad p_1 = m_1(R_1^2+K^2)\,\theta_1' .$$

Upon setting:

$$\alpha = \frac{1}{m(R^2+K^2)} + \frac{1}{m_1(R_1^2+K^2)},\qquad \beta = m m_1(R^2+K^2)(R_1^2+K^2),$$

and

$$U = -\,2\mu R R_1\cos(\theta-\theta_1) + \mu K^2(\theta_1-\theta)^2,$$

we will infer the following values of p and p_1 from the two integrals:

$$\alpha\, p_1 = \frac{C}{m(R^2+K^2)} + \sqrt{\alpha(U+2h) - \frac{C^2}{\beta}},$$

$$\alpha\, p = \frac{C}{m_1(R_1^2+K^2)} - \sqrt{\alpha(U+2h) - \frac{C^2}{\beta}} .$$

The expression $(p\,d\theta + p_1\,d\theta_1)$ is indeed an exact total differential dW. If one introduces the variable $(\theta_1 - \theta) = \varphi$ then one will have:

$$\alpha\, W = C\left[\frac{\theta}{m_1(R_1^2+K^2)} + \frac{\theta_1}{m(R^2+K^2)}\right] + \int_{\varphi_0}^{\varphi}\sqrt{\alpha(U+2h) - \frac{C^2}{\beta}}\,d\varphi .$$

The equalities $\partial W/\partial C = \text{const.}$, $t - t_0 = \partial W/\partial h$, which determine the motion, are:

$$\frac{\theta}{m_1(R_1^2+K^2)}+\frac{\theta_1}{m(R^2+K^2)}+\frac{C}{\beta}\int_{\varphi_0}^{\varphi}\frac{d\varphi}{\sqrt{\alpha(U+2h)-\dfrac{C^2}{\beta}}} = \text{const.},$$

$$t-t_0=\int_{\varphi_0}^{\varphi}\frac{d\varphi}{\sqrt{\alpha(U+2h)-\dfrac{C^2}{\beta}}}$$

here, which are equalities that can also be written:

$$[m(R^2+K^2)+m_1(R_1^2+K^2)]\theta = C\,t - m_1(R_1^2+K^2)\varphi + \text{const.},$$
$$[m(R^2+K^2)+m_1(R_1^2+K^2)]\theta_1 = C\,t + m(R^2+K^2)\varphi + \text{const.}$$

The problem depends upon only the single quadrature that gives *t*. We thus indeed recover the results that were already obtained on page 106.

III. A point *M* is attracted to the origin *O* and released with zero initial velocity in the plane *xOy*. Find the motion of the point when the law of attraction has the form:

$$F = \mu\, r\, x^m\, y^m \qquad \left(r=\sqrt{x^2+y^2}\right).$$

The equations of motion are:

$$x'' = \mu\, x^{m+1}\, y^m,$$
$$y'' = \mu\, x^m\, y^{m+1}$$

here.

Multiply the left-hand side by y', the right-hand side by x', and add them. That will give:

$$d\cdot x'\,y' = \mu\, x^m\, y^m\,[x\,dy+y\,dx]$$

or

$$x'\,y' = \frac{\mu}{m+1}x^{m+1}\,y^{m+1} + a\,.$$

On the other hand, the area integral will give:

$$x\,y'-y\,x' = C\,.$$

One infers from this that:

$$x' = \frac{-C+\sqrt{R}}{2y}, \qquad y = \frac{C+\sqrt{R}}{2x},$$

when one sets:

$$R = C^2 + 4\,x\,y\left[\frac{\mu}{m+1}x^{m+1}y^{m+1} + \alpha\right].$$

From the theory of the last multiplier, the expression:

$$\left(\frac{\partial x'}{\partial C}\frac{\partial y'}{\partial \alpha} - \frac{\partial x'}{\partial \alpha}\frac{\partial y'}{\partial C}\right)(y'\,dx - x'\,dy) \equiv \frac{y'\,dx - x'\,dy}{\sqrt{R}}$$

is an exact total differential. One easily verifies that because one will have:

$$2\,\frac{(y'\,dx - x'\,dy)}{\sqrt{R}} = \frac{dx}{x} - \frac{dy}{y} + \frac{C}{\sqrt{R}}\left(\frac{dx}{x} + \frac{dy}{y}\right)$$

$$= d\cdot L\frac{y}{x} + \frac{C\,du}{u\sqrt{C^2 + 4\cdot u\left(\dfrac{\mu}{m+1}u^{m+1} + \alpha\right)}} = d\,L\frac{y}{\alpha} + dF(u),$$

upon setting $u = x\,y$. The motion is determined by the equality:

$$y = \beta\,x\,e^{-F(xy)},$$

to which one must append the relations:

$$dt = \frac{dx}{x'} = \frac{dy}{y'} = \frac{y\,dx + x\,dy}{x'\,y + y'\,x} = \frac{du}{\sqrt{R(u)}},$$

which is equivalent to the single relation:

$$t = \int \frac{du}{\sqrt{R(u)}} + \text{const.}$$

Thus, here is an application of the theory to a case in which the *vis viva* theorem does not give an integral.

IV. Two massive points M **and** M_1 **of masses** m and m_1 , **resp., are constrained to slide without friction, one of them on the vertical** Oz **and the other on a cylinder of revolution around** Oz. **The two points attract each other according to an arbitrary function of the distance. Motion of the system. –**

Let θ be the angle that the plane MOz makes with the xOz plane. The position of the system depends upon three parameters x, θ, and z. In order to solve the problem by quadratures, it will suffice to know four first integrals in which t does not appear, or rather, five integrals that depend upon t.

The theorem of moments relative to Oz gives:

$$\theta = C\,t + C',$$

which is an equality that is equivalent to two integrals. On the other hand, from the theorem about the motion of the center of gravity, one will have:

$$m\,z + m_1\,z_1 = \tfrac{1}{2}\,g\,t^2 + \alpha\,t + \beta.$$

If one combines those integrals with the *vis viva* integral:

$$m_1\,z_1'^2 + m(R^2\,\theta'^2 + z'^2) = 2g\,(m\,z + m_1\,z_1) + 2U\,(z - z_1) + \text{const.}$$

then one will see that one knows five integrals in which t appears. The motion is then calculated with the aid of quadratures.

Consider separately the four equations:

$$(A) \qquad \frac{dz}{z'} = \frac{dz_1}{z_1'} = \frac{dz'}{m\,g + \dfrac{\partial U}{\partial z}} = \frac{dz_1'}{m_1\,g + \dfrac{\partial U}{\partial z_1}}\,.$$

Those equations are the equations of motion of a system whose *vis viva* is $(m\,z'^2 + m_1\,z_1'^2)$ and is subjected to forces that admit the force function $[g\,(m\,z + m_1\,z_1) + U]$. In addition to the integral:

$$(B) \qquad m\,z'^2 + m_1\,z_1'^2 = 2g\,(m\,z + m_1\,z_1) + 2U\,(z - z_1) + 2h\,,$$

we know an integral of equations (A), namely:

$$(C) \qquad \frac{(m\,z' + m_1\,z_1')^2}{m + m_1} = 2g\,(m\,z + m_1\,z_1) + a\,,$$

which is an integral that is deduced immediately from the foregoing. (It is the *vis viva* integral applied to the motion of the projection onto Oz of the center of gravity of the points M, M_1.)

On the other hand, if we remark that the canonical variables for equations (A) are $p_1 = m\,z'$, $p_2 = m_1\,z_1'$ then the theory that was developed above will show us that the expression:

$$m\,z'\,dz + m_1\,z_1'\,dz_1\,,$$

in which we replace z' and z_1' with the values that we infer from (*R*) and (*C*), must be an exact total differential *dW*. Indeed, a very simple calculation will give:

$$m\,z'\,dz + m_1\,z_1'\,dz_1$$

$$= \frac{1}{\sqrt{m+m_1}}\left[\sqrt{m\,m_1(2U+2h-\alpha)}\,(dz-dz_1)+\sqrt{2g\,(mz+m_1\,z_1)+\alpha}\,(m\,dz+m_1 dz_1)\right]$$

$$= \frac{1}{\sqrt{m+m_1}}\left[\sqrt{m\,m_1[2U(\eta)+2h-\alpha]}\,d\eta+\sqrt{2g\,\xi+\alpha}\,d\xi\right] = dW\,,$$

if one sets $(z - z_1) = \eta$ and $m\,z + m_1\,z_1 = \xi$.

The motion is then determined by the two equations:

$$t = \frac{\partial W}{\partial h} + \text{const.} = \sqrt{\frac{m\,m_1}{m+m_1}}\int_{\eta_1}^{\eta}\frac{d\eta}{\sqrt{2U(\eta)+2h-\alpha}} + \text{const.},$$

and

$$\frac{d\xi}{\sqrt{2\,g\,\xi+\alpha}} = \sqrt{\frac{m\,m_1}{2U(\eta)+2h-\alpha}}\,d\eta\,,$$

or rather

$$\frac{1}{m+m_1}\left(\frac{d\xi}{dt}\right)^2 = 2g\;\xi + \alpha\,.$$

The last equation is nothing but the integral (*C*).

One will arrive at the result more quickly by immediately introducing the variables η and ξ. Indeed, the *vis viva* of the system (M, M_1) is equal to:

$$\frac{1}{m+m_1}(\eta'^2 + m\,m_1\,\xi'^2) + m\,R^2\theta'^2\,.$$

Since one has $\theta' = \theta_0'$ and $\dfrac{\xi'^2}{m+m_1} = 2g\;\xi$, the *vis viva* theorem will imply the following equality:

$$\frac{m\,m_1}{m+m_1}\eta'^2 = 2U + \text{const.}$$

The motion is then determined by the three equalities:

$$\theta = C\,t + C', \qquad \frac{\xi}{m+m_1} = \tfrac{1}{2} g\,t^2 + \alpha\,t + \beta,$$

and

$$\sqrt{\frac{m+m_1}{m\,m_1}}\,dt = \frac{d\eta}{\sqrt{2U(\eta)+2h-\alpha}}\;.$$

It would be easy to apply the theory of the last multiplier to the examples that were treated in the previous chapters. However, this handful of exercises here will suffice to show the utility of that theory: In a great number of cases, it permits one to predict that the problem under study can be solved at one stroke by quadratures, even though the variables employed do not exhibit that fact clearly.

It is therefore appropriate to study the variables that give the simplest form to those quadratures.

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LECTURE 15

JACOBI'S THEOREM. – LIOUVILLE'S THEOREM.

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In the preceding lecture, we showed that if we know a second integral for a two-parameter mechanical problem, in addition to the *vis viva* integral, then the solution to the problem can be achieved by quadratures. That theorem is only a consequence of a general proposition by Jacobi that we shall now develop.

Let S be a material system without friction whose constraints can depend upon time and whose position is defined by the parameters $q_1, q_2, \ldots, q_k$. If the given forces that are exerted on the system admit a force function $U(t, q_1, q_2, \ldots, q_k)$ then the canonical equations of motion of the system can be written:

$$(1)\qquad \left\{\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i}, \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i}, \end{aligned}\right. \qquad (i = 1, 2, \ldots, k),$$

with

$$H = K(t, q_1, \ldots, q_k, p_1, \ldots, p_k) - U(t, q_1, q_2, \ldots, q_k) .$$

Replace the p_i in H with $\partial V / \partial q_i$ and consider the partial differential equation:

$$(2)\qquad \frac{\partial V}{\partial t} + H\left(t, q_1, \ldots, q_k, \frac{\partial V}{\partial q_1}, \ldots, \frac{\partial V}{\partial q_k}\right) = 0 .$$

It is a first-order equation that pertains to the function V of the $(k + 1)$ variables $t, q_1, q_2, \ldots, q_k$, and into which V does not enter explicitly.

Jacobi showed that one can deduce the general integral of the canonical equations (1) from a complete integral of (2).

A complete integral V of (2) is, by definition, an integral $V(t, q_1, q_2, \ldots, q_k, \alpha_1, \ldots, \alpha_k)$ that depends upon k arbitrary constants $\alpha_1, \ldots, \alpha_k$ that permit one to attribute arbitrary values to the $(k + 1)$ derivatives of V for arbitrary values t_0, q_i^0 of t and the q_i, resp., subject to only the condition that they must verify the relation (2).

Here, if the relation (2) is solved for the $\partial V / \partial q_i$ then the integral $V(t, q_1, q_2, \ldots, q_k, \alpha_1, \ldots, \alpha_k)$ will be a complete integral if one can select the $\alpha_1, \ldots, \alpha_k$ in such a manner as to give arbitrary values to the derivatives $\left(\frac{\partial V}{\partial q_1}\right)_0, \ldots, \left(\frac{\partial V}{\partial q_k}\right)_0$. $\left(\frac{\partial V}{\partial t}\right)_0$ will then take the value that is assigned to it by equation (2).

If one considers the equations:

$$(3) \qquad \begin{cases} \dfrac{\partial V}{\partial q_1} = A_1 , \\ \dots\dots\dots\dots \\ \dfrac{\partial V}{\partial q_k} = A_k , \end{cases}$$

in which $A_1, \ldots, A_k$ are arbitrary constants, then one can solve that system (3) for the $\alpha_1, \ldots, \alpha_k$. Analytically, that amounts to saying that the functional determinant of the k functions $\partial V / \partial q_i$ of $\alpha_1, \ldots, \alpha_k$ is not identically zero. That determinant is nothing but:

$$\Delta = \begin{vmatrix} \dfrac{\partial^2 V}{\partial q_1\, \partial \alpha_1} & \dfrac{\partial^2 V}{\partial q_1\, \partial \alpha_2} & \cdots & \dfrac{\partial^2 V}{\partial q_1\, \partial \alpha_k} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 V}{\partial q_k\, \partial \alpha_1} & \dfrac{\partial^2 V}{\partial q_k\, \partial \alpha_2} & \cdots & \dfrac{\partial^2 V}{\partial q_k\, \partial \alpha_k} \end{vmatrix} .$$

Hence, a complete integral is an integral that depends upon k arbitrary constants such that Δ is non-zero. Having recalled that definition, I say that if one knows a complete integral to (2) then the canonical system (1) can be integrated from that fact alone.

Indeed, set:

$$(4) \qquad \begin{cases} \dfrac{\partial V}{\partial \alpha_1} = \beta_1 , \\ \dots\dots\dots\dots \\ \dfrac{\partial V}{\partial \alpha_k} = \beta_k , \end{cases} \qquad (5) \qquad \begin{cases} p_1 = \dfrac{\partial V}{\partial q_1} , \\ \dots\dots\dots\dots \\ p_k = \dfrac{\partial V}{\partial q_k} . \end{cases}$$

The system (4-5), thus-formed, in which one considers the α and the β to be $2k$ arbitrary constants, will define the general integral $p_1(t), \ldots, p_k(t), q_1(t), \ldots, q_k(t)$ of equations (1).

First of all, the relations (4-5) determine the p_i, q_i as functions of t and $2k$ distinct arbitrary constants.

Indeed, on the one hand, equations (4) can be solved for $q_1, q_2, \ldots, q_k$, because the functional determinant of those equations with respect to the $q_1, q_2, \ldots, q_k$ is nothing but Δ. $q_1, q_2, \ldots, q_k$ are thus obtained as functions of $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$. If, on the other hand, one substitutes those values of q_i in (5), then one will obtain $p_1, \ldots, p_k$ as functions of the same quantities.

On the other hand, those $2k$ constants α_i, β_i are distinct. In other words, one can select them in such a way that one gives arbitrary values q_i^0, p_i^0 to q_i, p_i, resp., at $t = t_0$. Indeed, if one sets $t = t_0$, $q_i = q_i^0$ in equations (5) then one can select the α in such a fashion that one gives arbitrary values p_i^0 to the p_i. When the α are determined in that way, say $\alpha_i = \alpha_i^0$, the values of β will be obtained immediately upon setting $t = t_0$, $q_i = q_i^0$, $\alpha_i = \alpha_i^0$ in (4).

From that, if one can prove that any system of functions $p_i(t)$, $q_i(t)$ that verifies the relations (4-5) also verifies the equations (1), it will be clear that the relations (4-5) define the *general integral of the canonical system*.

In order to prove that this is true, calculate the derivatives $\frac{dp_i}{dt}$, $\frac{dq_i}{dt}$ using (4-5). That will give:

$$
(4') \qquad \left\{ \begin{array}{l} \dfrac{\partial^2 V}{\partial\alpha_1\,\partial t} + \dfrac{\partial^2 V}{\partial\alpha_1\,\partial q_1}\dfrac{dq_1}{dt} + \dfrac{\partial^2 V}{\partial\alpha_1\,\partial q_2}\dfrac{dq_2}{dt} + \cdots + \dfrac{\partial^2 V}{\partial\alpha_1\,\partial q_k}\dfrac{dq_k}{dt} = 0, \\ \dotfill \\ \dotfill \end{array} \right.
$$

$$
(5') \qquad \left\{ \begin{array}{l} \dfrac{dp_1}{dt} = \dfrac{\partial^2 V}{\partial q_1\,\partial t} + \dfrac{\partial^2 V}{\partial q_1^2}\dfrac{dq_1}{dt} + \dfrac{\partial^2 V}{\partial q_1\,\partial q_2}\dfrac{dq_2}{dt} + \cdots + \dfrac{\partial^2 V}{\partial q_1\,\partial q_k}\dfrac{dq_k}{dt} = 0, \\ \dotfill \\ \dotfill \end{array} \right.
$$

One can infer dq_i / dt from (4′) and dp_i / dt from (5′). One must substitute those values in (1) and confirm that the relations thus-obtained are verified for any system $p_i(t)$, $q_i(t)$ that satisfies the relations (4-5) [if V is a complete integral of (2)]. Instead of doing that, it is legitimate for one to replace the dq_i / dt, dp_i / dt in (4′) and (5′) with their values that are inferred from (1) and see if the conditions that are calculated in that way are consequences of equations (4-5).

First, make that substitution in (4′). One will find that:

$$
(6) \qquad \left\{ \begin{array}{l} \dfrac{\partial^2 V}{\partial\alpha_1\,\partial t} + \dfrac{\partial^2 V}{\partial\alpha_1\,\partial q_1}\dfrac{\partial H}{\partial p_1} + \dfrac{\partial^2 V}{\partial\alpha_1\,\partial q_2}\dfrac{\partial H}{\partial p_2} + \cdots + \dfrac{\partial^2 V}{\partial\alpha_1\,\partial q_k}\dfrac{\partial H}{\partial q_k} = 0, \\ \dotfill \end{array} \right.
$$

The equation that we just wrote down must be a consequence of the relations (4-5). By hypothesis, V will satisfy equation (2) for any $t, q_1, \ldots, q_k, \alpha_1, \ldots, \alpha_k$. V will then identically satisfy the equation that is obtained by differentiating (2) with respect to α_1, i.e., the equation:

$$\frac{\partial^2 V}{\partial t\,\partial \alpha_1} + \frac{\partial H}{\partial\left(\dfrac{\partial V}{\partial q_1}\right)}\frac{\partial^2 V}{\partial q_1\,\partial \alpha_1} + \frac{\partial H}{\partial\left(\dfrac{\partial V}{\partial q_2}\right)}\frac{\partial^2 V}{\partial q_2\,\partial \alpha_1} + \cdots + \frac{\partial H}{\partial\left(\dfrac{\partial V}{\partial q_k}\right)}\frac{\partial^2 V}{\partial q_k\,\partial \alpha_1} \equiv 0\,.$$

On the other hand, if one replaces the p_i in the first equation in (6) with their values that one infers from (5) then it will coincide with the preceding identity. It will then be indeed a consequence of the relations (4-5). One will likewise verify the other equations (6).

That shows us that the first group of canonical equations:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad\qquad (i = 1, 2, \ldots, k)$$

is verified by the solutions $p_i(t)$, $q_i(t)$ to (4-5). If we now replace $\dfrac{dp_i}{dt}$ with $-\dfrac{\partial H}{\partial q_i}$, and $\dfrac{dq_i}{dt}$ with $\dfrac{\partial H}{\partial p_i}$ in equations (5′) then that will give:

$$(7)\qquad \begin{cases} \dfrac{\partial H}{\partial q_1} + \dfrac{\partial^2 V}{\partial q_1\,\partial t} + \dfrac{\partial^2 V}{\partial q_1^2}\dfrac{\partial H}{\partial p_1} + \dfrac{\partial^2 V}{\partial q_1\,\partial q_2}\dfrac{\partial H}{\partial p_2} + \cdots + \dfrac{\partial^2 V}{\partial q_1\,\partial q_k}\dfrac{\partial H}{\partial p_k} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

On the other hand, V will satisfy the relation that is obtained by differentiating (2) with respect to q_1, which is a relation that can be written:

$$\frac{\partial^2 V}{\partial t\,\partial q_1} + \frac{\partial H}{\partial q_1} + \frac{\partial H}{\partial\left(\dfrac{\partial V}{\partial q_1}\right)}\frac{\partial^2 V}{\partial q_1^2} + \frac{\partial H}{\partial\left(\dfrac{\partial V}{\partial q_2}\right)}\frac{\partial^2 V}{\partial q_2\,\partial q_1} + \cdots + \frac{\partial H}{\partial\left(\dfrac{\partial V}{\partial q_k}\right)}\frac{\partial^2 V}{\partial q_k\,\partial q_1} = 0\,.$$

If one takes equations (5) into account then the first equation in (7) will be no different from that identity. One verifies the other equations in (7) similarly.

The proof is then complete, and one arrives at the following conclusion:

The integration of the system (1) *can be achieved when one knows a complete integral of equation* (2), *and the equations that resolve the problem of mechanics are:*

$$\frac{\partial V}{\partial \alpha_1} = \beta_1\,,$$

$$\ldots\ldots\ldots\ldots,$$

$$\frac{\partial V}{\partial \alpha_k} = \beta_k .$$

They determine $q_1, q_2, \ldots, q_k$ *as functions of* t *and the* $2k$ *arbitrary constants* α_i , β_i .

We shall indicate a certain number of cases in which the Jacobi equation can be replaced with a simpler equation.

Case in which H **does not depend upon time.** – Suppose that the constraints do not depend upon time, any more than the force function U. t will not enter into H then.

There will then exist complete integrals of equation (2) that have the form:

$$V = - h\, t + W\,(q_1, q_2, \ldots, q_k, \alpha_1, \ldots, \alpha_{k-1}, h) ,$$

in which h is a constant.

Indeed, replace V with $- h\, t + W\,(q_1, \ldots, q_k)$ in equation (2). W must then satisfy the equation:

$$H\left(q_1, \ldots, q_k, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_k}\right) = h . \tag{2′}$$

If $W\,(q_1, q_2, \ldots, q_k, \alpha_1, \ldots, \alpha_{k-1}, h)$ is a complete integral of (2) for each value of h then one can choose the $\alpha_1, \ldots, \alpha_{k-1}$ in such a manner that one can give arbitrary values to $\frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_k}$, for example, for arbitrary h. It will follow from this that the function $V = - h\, t + W$ is a complete integral of equation (2). Indeed, if one wishes that $\frac{\partial V}{\partial q_1}, \ldots, \frac{\partial V}{\partial q_k}$ should take arbitrary values $p_1^0, \ldots, p_k^0$, resp., for $t_0, q_1^0, \ldots, q_k^0$ then one can begin by setting $h = H_0$, where H_0 is the value of H that one will get when one replaces q_i, $\frac{\partial W}{\partial q_i}$ with q_i^0, p_i^0, resp., in it. One then chooses $\alpha_1, \ldots, \alpha_{k-1}$ in such a fashion that one will give the values $p_1^0, \ldots, p_{(k-1)}^0$ to $\frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_{(k-1)}}$, resp.

In this case, the motion is defined by the equalities:

$$\frac{\partial V}{\partial h} = - t + \frac{\partial W}{\partial h} = \beta, \qquad \text{or} \qquad t = \frac{\partial W}{\partial h} + \beta ,$$

and

$$\frac{\partial W}{\partial \alpha_i} = \beta_i \qquad\qquad (i = 1, 2, \ldots, k-1).$$

In particular, if $k = 2$ then we will recover the theorem that was proved in the previous chapter. Indeed, suppose that one knows a complete integral $V = -h\,t + W(q_1, q_2, \alpha, h)$ of equation (2). The two equations:

$$\frac{\partial W}{\partial q_1} = A_1\,, \qquad \frac{\partial W}{\partial q_2} = A_2$$

can be solved for h and α, so W will satisfy the two equations:

$$H\left(q_1, q_2, \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}\right) = h\,, \qquad f\left(q_1, q_2, \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}\right) = \alpha\,. \tag{8}$$

In order for equations (8) to be compatible, as we know, it is necessary and sufficient that $f = \alpha$ should be an integral of the canonical system (1). Therefore, any complete integral $W(q_1, q_2, \alpha, h)$ of the equation $H = h$ can be regarded as a common integral to the two equations (8), in which $f = \alpha$ is a certain first integral of (1). On the other hand, if the function W satisfies the two equations then we have proved that the motion of the system is determined by the equalities:

$$\frac{\partial W}{\partial \alpha} = \beta\,, \qquad t = \frac{\partial W}{\partial h} + \beta'\,.$$

Those are precisely the equalities to which Jacobi's theorem will lead.

Knowing a first integral $f(q_1, q_2, p_1, p_2) = \alpha$ of (1) will then permit one to determine a complete integral of the Jacobi equation in this particular case with the aid of a simple quadrature of a total differential. In the next lecture, we will see what the analogue of that theorem is for equations (8) when the number of variables k is arbitrary.

Case in which several parameters q do not enter into H. – Suppose that H has the form:

$$H = H(t, q_{i+1}, q_{i+2}, \ldots, q_k, p_1, p_2, \ldots, p_k)\,.$$

One can look for an integral of the Jacobi equation that has the form:

$$V = \alpha_1 q_1 + \alpha_2 q_2 + \ldots + \alpha_i q_i + W(t, q_{i+1}, \ldots, q_k)\,.$$

The function W must satisfy the equation:

$$\frac{\partial W}{\partial t} + H\left(t, q_{i+1}, \ldots, q_k, \alpha_1, \ldots, \alpha_i, \frac{\partial W}{\partial q_{i+1}}, \ldots, \frac{\partial W}{\partial q_k}\right) = 0\,.$$

If one knows a complete integral $(t, q_{i+1}, q_{i+2}, \ldots, q_k, \alpha_1, \alpha_2, \ldots, \alpha_i, \alpha_{i+1}, \ldots \alpha_k)$ to the latter equation when one gives arbitrary values to the constants $\alpha_1, \ldots \alpha_k$ then the function $V = \alpha_1 q_1 + \ldots + \alpha_i q_i + W$ will be a complete integral of the Jacobi equation, as one will see quite easily. The motion of the system is determined by the equations:

$$q_1 + \frac{\partial W}{\partial \alpha_1} = \beta_1 , \qquad \ldots, \qquad q_i + \frac{\partial W}{\partial \alpha_i} = \beta_i ,$$
$$\frac{\partial W}{\partial \alpha_{i+1}} = \beta_{i+1} , \qquad \ldots, \qquad \frac{\partial W}{\partial \alpha_k} = \beta_k .$$

When H is independent of t at the same time as $q_1, q_2, \ldots, q_k$, one sets:

$$V = -h\,t + \alpha_1 q_1 + \alpha_2 q_2 + \ldots + \alpha_i q_i + W(q_{i+1}, \ldots, q_k) ,$$

and one seeks a complete integral W of the equation:

$$H\left(q_{i+1}, \ldots, q_k, \alpha_1, \ldots, \alpha_i, \frac{\partial W}{\partial q_{i+1}}, \ldots, \frac{\partial W}{\partial q_k}\right) = h .$$

The equalities that define the motion are then:

$$t = \frac{\partial W}{\partial h} + \gamma, \quad q_1 = \beta_1 - \frac{\partial W}{\partial \alpha_1}, \qquad \ldots, \qquad q_i = \beta_i - \frac{\partial W}{\partial \alpha_i},$$

$$\frac{\partial W}{\partial \alpha_{i+1}} = \beta_{i+1} , \ \ldots, \qquad \frac{\partial W}{\partial \alpha_{k-1}} = \beta_{k-1} .$$

In particular, if $i = k - 1$ then W (which is a function of only the variable q_k) will be defined by a quadrature:

$$H\left(q_k, \alpha_1, \ldots, \alpha_{k-1}, \frac{\partial W}{\partial q_k}\right) = h .$$

That case often presents itself in problems with two parameters: t and q_1 will then be given as functions of q_2 by the equations:

$$t = \gamma + \frac{\partial W}{\partial h}(q_2, \alpha, h), \qquad q_1 = \beta - \frac{\partial W}{\partial \alpha}(q_2, \alpha, h).$$

In this particular case, the canonical equations admit the integral $p_1 = \alpha$, and when that is combined with the equality $H = h$, that will determine the function:

$$\int p_1\, dq_1 + p_2\, dq_2 \equiv \alpha\, q_1 + W(q_2, \alpha, h)$$

precisely.

Applications. – We shall now indicate some examples to which those remarks apply. We shall first study the motion of a material point M under the action of a central force that is a function of only the distance $r = OM$ from the point M to the center O of the force.

If we refer the point to polar coordinates r and θ in the plane of the trajectory then we will have:

$$T = \tfrac{1}{2} m[r'^2 + r^2\theta'^2] = \frac{1}{2m}\left[p_1^2 + \frac{p_2^2}{r^2}\right],$$

$$H = \frac{1}{2m}\left[p_1^2 + \frac{p_2^2}{r^2}\right] - U(r).$$

The Jacobi equation admits an integral of the form $V = -h\,t + \alpha\,\theta + W(r, \alpha, h)$. W is given by the quadrature:

$$W = \int \frac{\sqrt{2mr^2(U+h) - \alpha^2}}{r}\, dr,$$

while t and θ are given by the equalities:

$$t = \int \frac{mr\, dr}{\sqrt{2mr^2(U+h) - \alpha^2}} + \text{const.}, \quad \theta = \int \frac{\alpha\, dr}{r\sqrt{2mr^2(U+h) - \alpha^2}} + \text{const.}$$

Those quadratures coincide with the ones that we obtained before using other methods. However, it would be appropriate to remark that those two quadratures are found to be performed by that fact in its own right if we know how to perform the single quadrature that gives W. Jacobi's method exhibits that fact quite clearly.

Similarly, let us calculate the motion of a point M on a surface of revolution when the given force admits a force function $U(r)$, where r denotes the distance from the point M to the axis of revolution, and θ denotes the angle between the two planes zOM and xOz. If $z = \varphi(r)$ is the equation of the surface then we will have:

$$T = \tfrac{1}{2} m[r'^2(1+\varphi'^2) + r^2\,\theta'^2] = \frac{1}{2m}\left[\frac{p_1^2}{1+\varphi'^2} + \frac{p_2^2}{r^2}\right],$$

$$H = \frac{1}{2m}\left[\frac{p_1^2}{1+\varphi'^2} + \frac{p_2^2}{r^2}\right] - U(r).$$

The functions $W(r, \alpha, h)$ and $t(r)$, $\theta(r)$ are determined here by the equalities:

$$W = \int \frac{\sqrt{(1+\varphi'^2)[2mr^2(U+h)-\alpha^2]}}{r}\,dr\,,$$

$$t = \int \frac{mr\sqrt{1+\varphi'^2}\,dr}{\sqrt{2mr^2(U+h)-\alpha^2}}\,, \qquad \theta = \int \frac{\alpha\sqrt{1+\varphi'^2}}{r\sqrt{2mr^2(U+h)-\alpha^2}}\,dr\,.$$

It would be appropriate to make the same remark as before on the subject of those two integrals. Furthermore, that remark can be repeated in all of the analogous cases.

It would be easy to apply Jacobi's method to all of the examples that were treated before in which H could be reduced to something that depended upon only q_k and the variables p_i by a convenient choice of variables. We shall confine ourselves here to recalling the example that was treated on page 116.

A massive, homogeneous, solid body of revolution is traversed along its axis by a needle to which it is subject and one of whose extremities slides without friction on one vertical Oz, while the other slides on the horizontal plane xOy. Let us study its motion.

The *vis viva* $2T$ of the system (see page 117) is equal to:

$$(A+Md^2)\sin^2\theta\,\psi'^2 + \{A+M[d^2\cos^2\theta+(l-d)^2\sin^2\theta]\}\,\theta'^2 + [C\cos\theta\,\psi'+\varphi']^2\,,$$

and the force function U is $M\,g\,(l-d)\cos\theta$. If one sets:

$$p_1 = \frac{\partial T}{\partial\theta'}\,, \qquad p_2 = \frac{\partial T}{\partial\varphi'}\,, \qquad p_3 = \frac{\partial T}{\partial\psi'}$$

then that will give:

$$H = \frac{(p_3 - C\cos\theta\,p_2)^2}{2(A+Md^2)\sin^2\theta} + \frac{p_1^2}{2\{A+M[d^2\cos^2\theta+(l-d)^2\sin^2\theta]\}} + \frac{p_2^2}{2} - Mg\,(l-d)\cos\theta\,.$$

One can take the function W to be the function that is defined by:

$$W = \alpha\,\varphi + \alpha'\,\psi + W_1(\theta)\,,$$

with

$$W_1 = \int d\theta \sqrt{\left\{A+M[d^2\cos^2\theta+(l-d)^2\sin^2\theta]\right\}\left\{2Mg\,(l-d)\cos\theta+2h-d^2-\frac{\alpha'-(\alpha\cos\theta)^2}{(A+Md^2)\sin^2\theta}\right\}}\,,$$

and the motion will then be determined by the equalities:

$$t = \frac{\partial W_1}{\partial h} + \text{const.}, \qquad \varphi = \beta - \frac{\partial W_1}{\partial \alpha}, \qquad \psi = \beta' - \frac{\partial W_1}{\partial \alpha'},$$

in which α, β, α', β', h are arbitrary constants. One thus recovers the equations on page 118. The three quadratures that give t, φ, and ψ can be performed once one has performed the one that gives W_1.

Theorems of Liouville and Staeckel.

Now suppose that the *vis viva* of the system has the form:

$$T = \frac{\varphi_1(q_1) + \varphi_2(q_2) + \cdots + \varphi_k(q_k)}{2}\,[A_1(q_1)\,q_1'^2 + A_2(q_2)\,q_2'^2 + \cdots + A_k(q_k)\,q_k'^2],$$

and that the forces are derived from the potential:

$$U = \frac{\psi_1(q_1) + \psi_2(q_2) + \cdots + \psi_k(q_k)}{\varphi_1(q_1) + \varphi_2(q_2) + \cdots + \varphi_k(q_k)}.$$

One has the expression for H:

$$H = \frac{1}{2(\varphi_1 + \varphi_2 + \cdots + \varphi_k)}\left[\frac{p_1^2}{A_1} + \frac{p_2^2}{A_2} + \cdots + \frac{p_k^2}{A_k}\right] - \frac{\psi_1 + \psi_2 + \cdots + \psi_k}{\varphi_1 + \varphi_2 + \cdots + \varphi_k}.$$

Conversely, if H has that form then T and U will have the indicated form (see pp. 165).

The partial differential equation in W, namely:

$$\sum_{i=1}^{k} \frac{1}{A_i}\left(\frac{dW}{dq_i}\right)^2 = 2\sum_{i=1}^{k} (\psi_i + h\,\varphi_i), \tag{2'}$$

will then admit a complete integral of the form:

$$W = W_1(q_1) + W_1(q_1) + \ldots + W_k(q_k).$$

Indeed, it suffices to take $W_i(q_i)$ to be the function that is given by the equality:

$$W_i(q_i) = \int dq_i \sqrt{2A_i\,(\psi_i + h\,\varphi_i + \alpha_i)},$$

in which the α_i are constants that are constrained by the single condition that:

$$\alpha_1 + \alpha_2 + \ldots + \alpha_k = 0 .$$

The function $W = \sum W_i$ will then satisfy equation (2′) identically.

If one sets $F_i = \sqrt{\psi_i + h\,\varphi_i + \alpha_i}$ then the motion will be defined by the equalities:

$$\frac{\partial W_i}{\partial \alpha_i} - \frac{\partial W_k}{\partial \alpha_k} = \text{const.} \quad [i = 1, 2, \ldots, (k-1), \text{ since } \alpha_k = -(\alpha_1 + \ldots + \alpha_{k-1})]$$

and

$$t + \text{const.} = \frac{\partial W}{\partial h},$$

which are equalities that can be written:

$$\int \sqrt{\frac{A_1}{F_1}}\, dq_1 + \beta_1 = \int \sqrt{\frac{A_2}{F_2}}\, dq_2 + \beta_2 = \ldots = \int \sqrt{\frac{A_k}{F_k}}\, dq_k + \beta_k ,$$

and

$$t\sqrt{2} + \text{const.} = \int \varphi_1 \sqrt{\frac{A_1}{F_1}}\, dq_1 + \varphi_2 \sqrt{\frac{A_2}{F_2}}\, dq_2 + \cdots + \varphi_k \sqrt{\frac{A_k}{F_k}}\, dq_k ,$$

resp. One thus recovers the same equalities that one would get by starting from Lagrange's equations (see pages 120-123). However, on the one hand, those equalities will follow more directly from Jacobi's theorem. On the other hand, one sees that the $2k$ integrals that define the motion can be performed if one knows how to perform the quadratures that define the W_i .

In another lecture, we will give some new applications of Liouville's theorem. I will now indicate a generalization of that theorem. That generalization is due to P. Staeckel, who extended it to an arbitrary number of parameters. In order to not complicate the notation, we shall adopt the case of three parameters.

Let Δ be the determinant:

$$\Delta = \begin{vmatrix} \varphi_1(q_1) & \varphi_2(q_2) & \varphi_3(q_3) \\ \psi_1(q_1) & \psi_2(q_2) & \psi_3(q_3) \\ \chi_1(q_1) & \chi_2(q_2) & \chi_3(q_3) \end{vmatrix},$$

and let Φ_1, Φ_2, Φ_3 be the minors of Δ with respect to the elements of the first row, while $\Psi_1, \ldots$ and $\Xi_1, \ldots$ are the minors relative to $\psi_1, \ldots, \chi_1, \ldots$, resp.

Suppose that the *vis viva* $2T$ of a system and the force function U have the forms:

$$(A) \qquad 2T = \Delta\left(\frac{q_1'^2}{\phi_1} + \frac{q_2'^2}{\phi_2} + \frac{q_3'^2}{\phi_3}\right), \qquad U = \frac{f_1(q_1)\,\phi_1 + f_2(q_2)\,\phi_2 + f_3(q_3)\,\phi_3}{\Delta} .$$

The Jacobi method will permit one to determine the motion by quadratures.

Indeed, we have:

$$H = \frac{1}{2\Delta}[p_1^2\,\Phi_1 + p_2^2\,\Phi_2 + p_3^2\,\Phi_3] - \frac{f_1\,\Phi_1 + f_2\,\Phi_2 + f_3\,\Phi_3}{\Delta}$$

here, and equation (2′) for W will be written:

$$(2') \qquad \left(\frac{\partial W}{\partial q_1}\right)^2 \Phi_1 + \left(\frac{\partial W}{\partial q_2}\right)^2 \Phi_2 + \left(\frac{\partial W}{\partial q_3}\right)^2 \Phi_3 = 2\,(f_1\,\Phi_1 + f_2\,\Phi_2 + f_3\,\Phi_3 + h\,\Delta)\,.$$

Let us see whether there exists a complete integral of the form:

$$W = W_1\,(q_1) + W_2\,(q_2) + W_3\,(q_3)\,.$$

If we observe that we have:

$$\sum_{1,2,3} \varphi\,\Phi \equiv \Delta\,, \quad \sum_{1,2,3} \psi\,\Phi \equiv 0\,, \quad \sum_{1,2,3} \chi\,\Phi \equiv 0$$

then when we set:

$$\left(\frac{\partial W}{\partial q_1}\right)^2 = 2\,[f_1 + h\,\varphi_1 + \alpha\,\psi_1 + \alpha'\,\chi_1] = F_1\,(q_1)\,,$$

$$\left(\frac{\partial W}{\partial q_2}\right)^2 = 2\,[f_2 + h\,\varphi_2 + \alpha\,\psi_2 + \alpha'\,\chi_2] = F_2\,(q_2)\,,$$

$$\left(\frac{\partial W}{\partial q_3}\right)^2 = 2\,[f_3 + h\,\varphi_3 + \alpha\,\psi_3 + \alpha'\,\chi_3] = F_3\,(q_3)\,,$$

equation (2′) will be verified identically: h, α, α' denote constants.

The motion is defined by the equalities:

$$\beta = \int \frac{\psi_1\,dq_1}{\sqrt{F_1(q_1)}} + \int \frac{\psi_2\,dq_2}{\sqrt{F_2(q_2)}} + \int \frac{\psi_3\,dq_3}{\sqrt{F_3(q_3)}}\,,$$

$$\gamma = \int \frac{\chi_1\,dq_1}{\sqrt{F_1(q_1)}} + \int \frac{\chi_2\,dq_2}{\sqrt{F_2(q_2)}} + \int \frac{\chi_3\,dq_3}{\sqrt{F_3(q_3)}}\,,$$

and

$$\sqrt{2}\,t + \text{cost.} = \int \frac{\varphi_1\,dq_1}{\sqrt{F_1(q_1)}} + \int \frac{\varphi_2\,dq_2}{\sqrt{F_2(q_2)}} + \int \frac{\varphi_3\,dq_3}{\sqrt{F_3(q_3)}}\,.$$

The preceding theorem includes Liouville's theorem as a special case, because if T and U have the form that Liouville's theorem requires then one will see directly that one can always put it into the form (A).

LECTURE 16

STUDY OF REAL TRAJECTORIES. EQUATIONS OF THE TRAJECTORIES WHEN THE FORCES ARE ZERO OR DERIVED FROM A POTENTIAL.

Number of arbitrary constants that the trajectories depend upon. – Consider a system S in which neither the constraints nor the forces depend upon time. The motion of that system is determined by the Lagrange equations:

$$\text{(1)} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = Q_i\,(q_1, q_2, \ldots, q_k, q_1', q_2', \ldots, q_k'), \qquad \frac{dq_i}{dt} = q_i' \qquad (i = 1, 2, \ldots, k),$$

in which:

$$2T \equiv \sum_{i,j} q_i'\, q_j'\, A_{ij}\,(q_1, q_2, \ldots, q_k) \qquad (A_{ij} \equiv A_{ji})\,.$$

If t_0, $q_2^0, \ldots, q_k^0, q_1'^0, \ldots, q_k'^0$ are the values of t, $q_2, \ldots, q_k$, $q_1', \ldots, q_k'$ for a given value of q_1 (say $q_1 = 0$) then those equations will define $q_1, q_2, \ldots, q_k$ as functions of $t - t_0$ and $(2k - 1)$ arbitrary constants $q_2^0, \ldots, q_k^0, q_1'^0, \ldots, q_k'^0$. As a result, $q_2, \ldots, q_k$ will be functions of q_1 that depend upon at most $2k - 1$ arbitrary constants ([2]). On the other hand, they depend upon at least $(2k - 2)$ constants, because $q_2, \ldots, q_k$, and $\dfrac{dq_2}{dq_1}\left(\text{or } \dfrac{q_2'}{q_1'}\right), \ldots, \dfrac{dq_k}{dq_1}\left(\text{or } \dfrac{q_k'}{q_1'}\right)$ can take arbitrary values for a given q_1. We shall see that in general the number ν of constants that the trajectories of the system depend upon is indeed equal to $2k - 1$ and find the conditions under which that number will reduce to $2k - 2$.

Solve equations (1) for the q_i''. If Δ denotes the discriminant of the quadratic form T, and α_i denotes what Δ will become when one replaces the elements of the i^{th} column with $Q_1, Q_2, \ldots, Q_k$, and finally if P_i represents quadratic form with respect to the q_i' then one can write:

$$\text{(2)} \qquad q_i'' = P_i + \frac{\alpha_i}{\Delta} = P_i + \beta_i \qquad (i = 1, 2, \ldots, k).$$

On the other hand, we have:

$$q_i'' = \frac{d^2 q_i}{dq_1^2}\, q_1'^2 + \frac{dq_i}{dq_1}\, q_1''\,,$$

([2]) In what follows, I will say that any system of such functions $q_2, q_3, \ldots, q_k$ of q_1 define a trajectories of S.

so upon replacing q_1'' and q_i'' with their values that we infer from (2):

$$(3)\qquad \frac{d^2 q_i}{dq_1^2} = \frac{\left[P_i - \frac{dq_i}{dq_1} P_1\right] + \left[\beta_i - \frac{dq_i}{dq_1}\beta_1\right]}{q_1'^2} \qquad (i = 2, 3, \dots, k).$$

In what follows, let $q'_{(i)}$, $q''_{(i)}$, … represent the derivatives $\frac{dq_i}{dq_1}$, $\frac{d^2 q_i}{dq_1^2}$, …, and let π_i denote what P_i will become when one replaces q_1' with 1, q_2' with $q'_{(2)}$, …, q_k' with $q'_{(k)}$.

Equation (3) will then become:

$$(4)\qquad q''_{(i)} = \pi_i - q'_{(i)}\,\pi_1 + \frac{\beta_i - q'_{(i)}\,\beta_1}{q_1'^2} \qquad (i = 2, 3, \dots, k).$$

Those β are functions of $q_1, q_2, \dots, q_k, q_1', q'_{(2)}, \dots, q'_{(k)}$. There are two cases to be distinguished: If the right-hand side of equations (4) is independent of q_1' then the system (4) will form a system of $(k - 1)$ second-order equations that involve the k variables $q_1, q_2, \dots, q_k$, and those equations will define $q_2, q_3, \dots, q_k$ as functions of q_1 and $(2k - 2)$ arbitrary constants $q_2^0, \dots, q_k^0, q'^0_{(2)}, \dots, q'^0_{(k)}$. If, on the contrary, q_1' appears in at least one of equations (4) (say, the equation for $i = 2$) then one can take:

$$q_2, \dots, q_k, q_1', q'_{(2)}, \dots, q'_{(k)}$$

arbitrarily for a given q_1 and choose $q_1'^0$ in order to give $q''^0_{(2)}$ an arbitrary value, so the functions $q_2, q_3, \dots, q_k$ of q_1 will then depend upon $(2k - 2)$ distinct constants.

In order for the right-hand sides of all of equations (4) to be independent of q_1', it is necessary and sufficient that the $(k - 1)$ expressions $\frac{\beta_i - (q_i'/q_1')\,\beta_1}{q_1'^2}$ should be homogeneous of degree zero with respect to the q_i', in such a way that the result to which we will arrive can be stated thus:

The number ν *of distinct constants upon which the trajectories of the system depend is equal to* $(2k - 1)$ *or* $(2k - 2)$. *In order for it to reduce to* $(2k - 2)$, *it is necessary and sufficient that the* $(k - 1)$ expressions $(\beta_i\, q_1' - \beta_1\, q_i')$ *should be homogeneous of degree three with respect to the* q_i'.

If one lets a_{ij} denote the minor of Δ relative to A_{ij} then one will have:

$$(5)\qquad \beta_i = \frac{1}{\Delta}[a_{i1}\,Q_1 + a_{i2}\,Q_2 + \cdots + a_{ik}\,Q_k].$$

The ν will then be equal to $(2k - 2)$, in particular, whenever the forces Q_i are homogeneous of degree two with respect to q_1', q_2', …, q_k'.

When the Q_i do not depend upon the velocities, the same thing will be true for the β_i. In order for ν to be equal to $(2k - 2)$, it is necessary and sufficient that the expressions $\beta_i\, q_1' - \beta_1\, q_i'$ should be identically zero; in other words, that one should have:

(6) $$\frac{\beta_1}{q_1'} = \frac{\beta_2}{q_2'} = \ldots = \frac{\beta_k}{q_k'} .$$

It is possible for the β to not depend upon the q_i' only if one has:

$$\beta_1 \equiv \beta_2 \equiv \ldots \equiv \beta_k \equiv 0 ,$$

and as a result ([3]):

$$Q_1 \equiv Q_2 \equiv \ldots \equiv Q_k \equiv 0 .$$

The trajectories of a system for which the Q_i are zero thus depend upon $(2k - 2)$ parameters. We shall give the name of *geodesics* for the ds^2 of T to those trajectories upon setting:

$$ds^2 = \sum A_{ij}\, dq_i\, dq_j .$$

That is the case for a system without friction that is not subject to any given forces. When the forces Q_i, which are functions of only q_1, q_2, …, q_k, are not all zero, the trajectories will always depend upon $(2k - 1)$ parameters.

A remarkable case is the one in which the forces depend upon velocities so the trajectories of the system will coincide with the geodesics of ds^2. In order for that to be true, it is necessary and sufficient that equations (6) should be verified; in other words, that one should have:

$$\beta_1 = \lambda\, q_1', \qquad \beta_2 = \lambda\, q_2', \qquad \ldots, \qquad \beta_k = \lambda\, q_k',$$

and as a result, from (5):

(7) $$\lambda\, q_i' = \frac{1}{\Delta}\left[a_{i1}\, Q_1 + a_{i2}\, Q_2 + \cdots + a_{ik}\, Q_k\right] \qquad (i = 1, 2, \ldots, k) .$$

However, on the other hand, let $p_i = \dfrac{\partial T}{\partial q_i'}$, so one knows that one has:

$$q_i' = \frac{1}{\Delta}\left[a_{i1}\, p_1 + a_{i2}\, p_2 + \cdots + a_{ik}\, p_k\right].$$

([3]) Equations (5), in which one annuls the β_i, have no other solution than $Q_1 \equiv Q_2 \equiv \ldots \equiv Q_k \equiv 0$, because their determinant is a power of Δ, and as a result, it will not be zero.

Equations (7) will then be equivalent to the following ones:

$$(8) \qquad Q_1 = \lambda \frac{\partial T}{\partial q_1'}, \qquad Q_2 = \lambda \frac{\partial T}{\partial q_2'}, \quad \ldots, \qquad Q_k = \lambda \frac{\partial T}{\partial q_k'} .$$

In order for the trajectories of (1) to be the same when all of the forces (Q_i) are zero, it will therefore be necessary and sufficient that $Q_1, Q_2, \ldots, Q_k$ should be proportional to $\frac{\partial T}{\partial q_1'}$, $\frac{\partial T}{\partial q_2'}$, …, $\frac{\partial T}{\partial q_k'}$.

Examples. – Let us interpret those conditions (8) in certain special cases. First of all, let T be the *vis viva* of a free material point, so one has : $T = \frac{1}{2}m[x'^2 + y'^2 + z'^2]$, and the conditions (8) express the idea that X, Y, Z are proportional to x', y', z', i.e., the force that is exerted on that point will have the velocity for the line of action. When a free point is subject to a force that constantly points in the direction of its velocity (or in the opposite sense), it will describe a line, no matter what the initial conditions.

Now, if the system is composed of free material points M_i then one will have: $T = \sum \frac{1}{2} m_i [x_i'^2 + y_i'^2 + z_i'^2]$, and the conditions (8) express the idea that the force (F_i) that is exerted upon each point M_i is directed along the velocity of M_i (in the same or opposite sense), and that one should have $\frac{F_i}{m_i v_i} = \frac{F_1}{m_1 v_1}$ ($i = 2, 3, \ldots, k$), moreover.

Finally, let us treat the case of a point M that moves on a surface S. The conditions (8) then express the idea that the force that acts upon it is constantly in the normal plane to the surface and tangent to the trajectory. Indeed, let $x(q_1, q_2)$, $y(q_1, q_2)$, $z(q_1, q_2)$ be the Cartesian coordinates of a point on the surface. The total force that is exerted on M is, as one knows, the resultant of the normal reaction and the given force (F') [viz., the active force and the force of friction, if it exists]. Let (F_1) or (X_1, Y_1, Z_1) be the projection of the given force (F') onto the tangent plane to S. One has the following relations between Q_1, Q_2, X_1, Y_1, Z_1 :

$$(9) \qquad Q_1 = X_1 \frac{\partial x}{\partial q_1} + Y_1 \frac{\partial y}{\partial q_1} + Z_1 \frac{\partial z}{\partial q_1}, \qquad Q_2 = X_1 \frac{\partial x}{\partial q_2} + Y_1 \frac{\partial y}{\partial q_2} + Z_1 \frac{\partial z}{\partial q_2}.$$

Conversely, if (F_1) denotes a segment that is tangent to S whose projections satisfy equations (9) then it will denote the component of (F') that is tangent to S. In order for that segment (F_1) to be tangent to the trajectory, it is necessary that one must have:

$$\frac{X_1}{\frac{\partial x}{\partial q_1}q_1' + \frac{\partial x}{\partial q_2}q_2'} = \frac{Y_1}{\frac{\partial y}{\partial q_1}q_1' + \frac{\partial y}{\partial q_2}q_2'} = \frac{Z_1}{\frac{\partial z}{\partial q_1}q_1' + \frac{\partial z}{\partial q_2}q_2'},$$

and as a result, upon calling the common value of those ratios μ:

$$Q_1 = \mu\left\{q_1'\left[\left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2\right] + q_1'\left(\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2} + \frac{\partial y}{\partial q_1}\frac{\partial y}{\partial q_2} + \frac{\partial z}{\partial q_1}\frac{\partial z}{\partial q_2}\right)\right\} = \frac{\mu}{m}\frac{\partial T}{\partial q_1'},$$

and similarly, $Q_2 = \dfrac{\mu}{m}\dfrac{\partial T}{\partial q_2'}$.

Conversely, if Q_1 and Q_2 have that form then the segment:

$$X_1 = \mu\left(\frac{\partial x}{\partial q_1}q_1' + \frac{\partial x}{\partial q_2}q_2'\right), \qquad Y_1 = \mu\left(\frac{\partial y}{\partial q_1}q_1' + \frac{\partial y}{\partial q_2}q_2'\right), \qquad Z_1 = \mu\left(\frac{\partial z}{\partial q_1}q_1' + \frac{\partial z}{\partial q_2}q_2'\right)$$

will be a tangent segment to the trajectory, so to the surface that satisfies equations (9) and, as a result, it will represent the component of (F') that is tangent to S. The conditions (8) then express the fact that (F') is projected onto the tangent plane to S along tangent to the trajectory. That will be the case, for example, with a point that moves without friction on a surface in a resisting medium with no other force acting upon it.

We shall now study the systems that are subject to forces that are independent of velocity exclusively.

Study of the trajectories in the case where the forces do not depend upon velocity. – From the foregoing, it would be suitable to subdivide this case into two other ones according to whether all of the forces are zero or not.

I. *All of the coefficients Q_i are zero*. – The trajectories (viz., geodesics of ds^2) will then depend upon $(2k – 2)$ constants and are defined [see page 235] by $(k – 1)$ equations of the form:

$$q''_{(i)} = \pi_i - q'_{(i)}\,\pi_i \qquad (i = 2, 3, \ldots, k) . \tag{4′}$$

Once those equations are integrated, the motion of the system will be given by the equality:

$$dt = \frac{ds}{\sqrt{h}} = \frac{dq_1}{\sqrt{h}}\cdot\sqrt{\sum A_{ij}\, q'_{(i)}\, q'_{(j)}} ,$$

in which $q'_{(i)} = dq_i / dq_1$, $q'_1 = 1$, and h represents an arbitrary constant for each geodesic. Therefore, if one expresses $q_2, q_3, \ldots, q_k$ as functions of q_1 and $(2k - 2)$ arbitrary constants $c_1, c_2, \ldots, c_{2k-2}$ then one will have:

$$t = c\int f[q_1, c_1, c_2, \ldots, c_{2k-2}]\, dq_1 + c',$$

in which c represents any one of the constants. More generally, let:

$$\varphi[q_1, q_2, \ldots, q_k, q'_{(2)}, \ldots, q'_{(k)}, c] = \alpha$$

be a first integral of equations (4′) that depends upon one arbitrary parameter c. One can write:

$$dt = \varphi\sqrt{\sum A_{ij}\, q'_{(i)}\, q'_{(j)}}\, dq_1,$$

and that equality, when combined with equations (4′), define the same motion as (1). Conversely, if the equality:

$$(a) \qquad dt = \psi[q_1, q_2, \ldots, q_k, q'_{(2)}, \ldots, q'_{(k)}, c]\sqrt{\sum A_{ij}\, q'_{(i)}\, q'_{(j)}}\, dq_1 ,$$

in which ψ is an arbitrary function, is compatible with equations (1), in other words, if one combines it with equations (4′) then it will define a motion on S. ψ is a first integral of the geodesics.

Indeed, consider an arbitrary geodesic. From (1), one will have $\frac{dt}{ds} = \frac{1}{\sqrt{h}}$ all along that geodesic, and as a result, from (a), one must also have: $\psi = 1/\sqrt{h}$ = const. Therefore, ψ is a first integral of (4′).

We will soon give an explicit form to equations (4′).

For the moment, I shall insist upon only the fact that the same trajectory can correspond to an infinitude of distinct motions, and I intend the word "motion" to mean the same positions, but different velocities. Moreover, those motions are deduced from just one of them by multiplying all velocities by the same numerical constant.

II. *Not all of the coefficients* Q_i ($q_1, q_2, \ldots, q_k$) *are zero*. *Equations of the trajectories*. – We know that the trajectories depend upon $(2k - 1)$ parameters. In order to construct the differential equations that define those trajectories without the intermediary of t, observe that from (4) [see page 2], we first have:

$$(10) \qquad \frac{q''_{(i)} + q'_{(i)}\,\pi_1 - \pi_i}{\beta_i - q'_{(i)}\,\beta_1} = \frac{q''_{(2)} + q'_{(2)}\,\pi_1 - \pi_2}{\beta_2 - q'_{(2)}\,\beta_1} \qquad (i = 3, 4, \ldots, k),$$

in which the common value of those ratios is $\left(\frac{dt}{dq_1}\right)^2$.

On the other hand, if one differentiates the equality:

$$q_1'^2 = \frac{\beta_2 - q'_{(2)}\,\beta_1}{q''_{(2)} + q'_{(2)}\,\pi_1 - \pi_2} \equiv \frac{\psi_2}{\chi_2}$$

with respect to q_1 and points out that $\frac{d}{dq_1}\cdot q_1'^2 = 2\,q_1'\,q_1''\frac{dt}{dq_1} = 2\,q_1''$, and replaces q_1'' with $(\pi_1\,q_1'^2 + \beta_1) \equiv \pi_1\,(\psi_2 / \chi_2) + \beta_1$ then one will have:

$$2\,\pi_1\frac{\psi_2}{\chi_2} + 2\beta_1 = \frac{d}{dq_1}\frac{\psi_2}{\chi_2},$$

or rather:

(11) $$\frac{\dfrac{d}{dq_1}[q''_{(2)} + q'_{(2)}\,\pi_1 - \pi_2]}{q''_{(2)} + q'_{(2)}\,\pi_1 - \pi_2} + \pi_1 = \frac{\dfrac{d}{dq_1}(\beta_2 - q'_{(2)}\,\beta_1)}{\beta_2 - q'_{(2)}\,\beta_1}\cdot\frac{2\beta_1\,[q''_{(2)} + q'_{(2)}\,\pi_1 - \pi_2]}{\beta_2 - q'_{(2)}\,\beta_1},$$

which is an equation of the form:

$$q'''_{(2)} = \frac{-3\,q''^{\,2}_{(2)} + q''_2\,M_3 + M_5}{M_0 - q'_{(2)}},$$

in which M_3 and M_5 denote polynomials of degrees three and five, resp., in $q'_{(2)}, q'_{(3)}, \ldots, q'_{(k)}$, and M_0 is a function of the q_i .

Since equations (10) and (11) define the trajectories, it would be easy to give them a more symmetric form, but that is hardly important in the context of our objective.

Observe that in equations (10) and (11), the expressions $\pi_1, \ldots, \pi_k$ depend upon only the *vis viva* T. Only the coefficients β_i vary with the forces Q_i . Furthermore, observe that the geodesics of T are obtained by equating all of the numerators χ_i of the ratios (10) to zero.

A first consequence of those remarks is that the geodesics of T belong to the trajectories no matter what the forces $Q_i\,(q_1, q_2, \ldots, q_k)$. Indeed, since those geodesics satisfy the equations $\chi_i = 0$, they will satisfy equations (10). On the other hand, equation (11) can be written:

$$\frac{d\chi_2}{dq_1} = \chi_2\,A,$$

and since one has $\chi_2 = 0$, $d\chi_2 / dq_1 = 0$ for an arbitrary geodesic, it will also be verified. The geodesics then define a congruence of trajectories with $(2k - 2)$ parameters.

Determination of time. – When the system (10), (11) has been integrated, one will know q_2, q_3, …, q_k as functions of q_1 and $(2k - 1)$ arbitrary constants, namely, the initial values of q_2'', …, q_k'', $q_{(2)}'^{\,0}$, $q_{(3)}'^{\,0}$, …, $q_{(k)}'^{\,0}$, $q_{(2)}''^{\,0}$ for q_1^0. One and only one trajectory will correspond to those initial values, when taken at random. All along that trajectory, the motion will be defined by any one of the equalities:

$$dt = dq_1\sqrt{\frac{\chi_i}{\psi_i}} \equiv dq_1\sqrt{\frac{q_{(i)}'' + q_{(i)}'\,\pi_1 - \pi_i}{\beta_i - q_{(i)}'\,\beta_1}}\ .$$

One sees that t is calculated by a quadrature and that for a given trajectory, one will have:

$$t = \pm f(q_1) + \text{const.},$$

in which f denotes a well-defined function of q_1 (which depends upon the trajectory).

The values q_1^0, q_2^0, …, q_k^0, $q_1'^{\,0}$, …, $q_k'^{\,0}$ correspond to one and only one system of values q_1^0, q_2^0, …, q_k^0, $q_{(2)}'^{\,0}$, …, $q_{(k)}'^{\,0}$, $q_{(2)}''^{\,0}$, and that system will not change when one changes the signs of all the $q_i'^{\,0}$. It follows from this that if each point of the material system occupies the same position with equal, but directly opposite, velocities at the instants t_1 and t_2 then one can pass from the first motion to the second one by changing t into $t_1 + t_2 - t$.

Indeed, one has:

$$t - t_1 = + f(q_1) - f(q_1^0)$$

for the first motion, and:

$$t - t_2 = - f(q_2) - f(q_1^0)$$

for the second one, if q_1^0 is the common value of q_1 at the two instants t_1 and t_2.

One sees that a given trajectory can be traversed in only two distinct manners. For each position of the system along that trajectory, the velocities q_1', q_2', …, q_k' will be determined, up to sign, by the equalities:

$$q_1' = \pm\sqrt{\frac{\psi_i}{\chi_i}}\,, \quad q_2' = q_{(2)}'\,q_1',\ \dots,\ q_k' = q_{(k)}'\,q_1'.$$

In particular, if that trajectory is a geodesic then one will always have $q_1' = \infty$, or if one prefers, $dt / dq_1 = 0$. Conversely, if one takes arbitrary values for q_1^0, …, q_k^0, $q_{(2)}'^{\,0}$, …, $q_{(k)}'^{\,0}$, $q_{(2)}''^{\,0}$ that annul χ_2 then the trajectory that is defined by those values will be unique, and as a result, it will coincide with the geodesic that satisfies those initial conditions, since that geodesic is also a trajectory. All along that trajectory, dt / dq_1 will then be zero. One can further say that the congruence of geodesics is the $(2k - 2)$-parameter congruence that is obtained by subjecting the $(2k - 1)$ parameters of the trajectory to the condition that $1 / T_0 = 0$. In particular, if there exists a force function then each value h of the constant of the *vis viva* integral will correspond to $(2k - 1)$ parameters, so the congruence of geodesics will correspond to the value $h = \infty$.

Remarkable trajectories. – Nevertheless, there can exist exceptional trajectories that we call *remarkable* and for which the preceding conclusions break down: They are the trajectories that give the form 0/0 to the ratios χ_i / ψ_i; in other words, that they satisfy both of the equalities:

$$q''_{(i)} + \pi_1 q'_{(i)} - \pi_i = 0 ; \qquad \beta_i - q'_{(i)} \beta_1 = 0 \qquad (i = 2, 3, \ldots, k) .$$

Those trajectories are then the geodesics that simultaneously satisfy all of the equations:

$$(12) \qquad q'_{(2)} = \frac{\beta_2}{\beta_1}, \ldots, \; q'_{(k)} = \frac{\beta_k}{\beta_1} .$$

In general, the system (12) and the equations of the geodesics have no common integral: In all cases, from (12), those common integrals cannot depend upon more than $(k - 1)$ arbitrary constants.

There is an infinitude of possible motions on one of those remarkable trajectories. Indeed, we can replace the equations of motion with equations (4):

$$(4) \qquad 0 = q''_{(i)} + q'_{(i)} \pi_1 - \pi_i + \frac{\beta_1 q'_{(i)} - \beta_i}{q_1'^2} \equiv \chi_i - \frac{\psi_i}{q_1'^2} ,$$

combined with one of the Lagrange equations, or if one prefers, with the *vis viva* equality:

$$T = \int [Q_1 + Q_2 \, q'_{(2)} + \cdots + Q_k \, q'_{(k)}] \, dq_1 .$$

By hypothesis, the trajectory considered will satisfy the equations $\chi_i = 0$, $\psi_i = 0$, so it will satisfy the equations (4). The motion on that trajectory will then be defined by the single equality:

$$(13) \qquad \tfrac{1}{2} q_1'^2 \sum A_{ij} \, q'_{(i)} \, q'_{(j)} = \int [Q_1 + Q_2 \, q'_{(2)} + \cdots + Q_k \, q'_{(k)}] \, dq_1 ,$$

or rather:

$$dt = dq_1 \sqrt{\frac{\varphi(q_1)}{2[f(q_1) + h]}} ,$$

in which φ and f are two functions of q_1 that are determined by the trajectory considered.

Observe that one can always assume that the coefficient φ of $\frac{1}{2} q_1'^2$ in (13) will be non-zero if the trajectory corresponds to a real motion of the system. Indeed, one has $2T = \sum m_j (x_j'^2 + y_j'^2 + z_j'^2)$, and it is always legitimate to take $(q_1, q_2, \ldots, q_k)$ to be k of the coordinates x_j, y_j, z_j that are independent. Under those conditions, $2T$ will be annulled under a real motion of the system only if all of the q' are zero ([4]). Since one has $\varphi = 2T[q_1, q_2, \ldots, q_k, 1, q'_{(2)}, \ldots, q'_{(k)}]$, φ cannot be annulled for a real position of the system and real values of $q'_{(2)}$, $q'_{(3)}$, …, $q'_{(k)}$. Observe, moreover, that any real motion of S will correspond to a real trajectory, but the converse is not necessarily true. It can happen that q_1, q_2, …, q_k are real, but certain coordinates x_j, y_j, z_j of the points M_j of S are imaginary.

The motion of the system on a remarkable trajectory is that of a system with complete constraints. The material points of such a system will traverse only one trajectory, but they can traverse it in an infinitude of ways.

Suppose, for example, that one is dealing with a free material point that is referred to the rectangular coordinates x, y, z and is subjected to the force (F) or X, Y, Z. The remarkable trajectories are the lines D (if they exist) such that all along D, the force has D for its line of action. Notably, if F is a force that issues from the origin O then the remarkable trajectories will be composed of all of the lines that pass through O, which is a congruence with $(k - 1) = 2$ parameters.

Similarly, if M is a point that moves on a surface Σ then the exceptional trajectories are the geodesics of Σ that are tangent at each point to the projection of the force onto the plane tangent to Σ.

Study of real trajectories.

Consider a real trajectory that is not a remarkable trajectory. The motion along that trajectory is defined by any one of the equalities $\frac{dt}{dq_1} = \sqrt{\frac{\chi_i}{\psi_i}}$. It will then be real if χ_i / ψ_i is positive and imaginary if χ_i / ψ_i is negative. The real trajectories (C) are then subdivided into two categories: The trajectories (C') along which the motion is real and the trajectories (C'') along which it is imaginary. However, that raises the question: Can part of the same analytic trajectory belong to the class (C'), while part of it belongs to the class (C'')? In order to answer that question concisely, it is necessary to discuss certain properties of motion in detail.

Exhibiting some properties of motion:

In what follows, we shall suppose that the parameters q_i have been chosen in such a fashion that any real position of the system S corresponds to real values of q_i, and that Δ, the discriminant of T, is not annulled for any real position of S. For any system of real values of q_i that corresponds

([4]) In regard to this, we add that if the q_i have been chosen in that way then the discriminant Δ of T will not be annulled for any real position of the system and for real values of the q_i. The expression $\sum m_j (x_j'^2 + y_j'^2 + z_j'^2)$ will then be annulled for some real values of x_j', y_j', z_j' that are not all zero.

to real positions of S, the functions A_{ij}, Q_i can have several real determinations ([5]). However, we assume that those determinations define just as many continuous functions of the real variables q_i that possess continuous first and second derivatives, except perhaps for certain exceptional values that we shall call *singular* values of the values of q_i that correspond to *singular* positions of the system S.

Now consider a system of real values of the q_i, namely, $q_1 = \alpha_1$, $q_2 = \alpha_2$, …, $q_k = \alpha_k$, in the neighborhood of which the determination that is made for each of the functions A_{ij}, Q_i is regular, by which I mean that they remain well-defined and continuous, along with their first and second derivatives. If we solve the Lagrange equations (1) that define the motion of S for the q_i'' then we will get the equations:

$$\frac{dq_i}{dt} = q_i', \qquad \frac{dq_i'}{dt} = P_i + \beta_i, \tag{2}$$

whose right-hand sides are well-defined and continuous, along with their first derivatives, while the variables q_i' will remain finite and the q_i will remain close to a_1, a_2, …, a_k. There will then exist one and only one system of integrals of (2), namely, $q_1(t)$, …, $q_k(t)$, $q_1'(t)$, …, $q_k'(t)$, such that for $t = t_0$, the q_i will take the values a_i, and the q_i' will take the values $q_i'^0$ that are given in advance.

Let us make that theorem more precise: One can find a number δ such that the functions Q_i are regular for the values of q_i that satisfy the equalities:

$$|q_i - a_i| \le \delta \qquad (i = 1, 2, \ldots, k).$$

If L is an arbitrary number that we take to be greater than δ then the right-hand sides of equations (2) will remain less than a certain limit M in absolute value, while the inequalities:

$$|q_i - a_i| \le \delta, \qquad |q_i'| \le L \qquad (i = 1, 2, \ldots, k)$$

will be verified. From that, let q_i^0, $q_i'^0$ be a system of values that satisfies the conditions:

$$|q_i^0 - \alpha_i| \le \frac{\delta}{2}, \qquad |q_i'^0| \le \frac{L}{2} \qquad (i = 1, 2, \ldots, k).$$

For any system q_i, …, q_i' such that one has:

([5]) For the given initial conditions of the system S, the *vis viva* T and the forces are well-defined. However, if the same system of values of the q_i corresponds to several positions of S then the A_{ij} and Q_i will have several determinations. For a position of S that is taken at random, there will be no ambiguity about which of those determinations that one must choose.

$$| q_i - q_i^0 | \le \frac{\delta}{2}, \qquad | q_i' - q_i'^0 | \le \frac{\delta}{2},$$

the right-hand sides of (2) will remain less than M in modulus. From the fundamental theorem on differential equations ([6]), the system of integrals $q_i(t)$, $q_i'(t)$ that takes the values q_i^0, $q_i'^0$ for $t = t_0$ will be a system of functions of t that is well-defined and continuous, at least in the interval from $t_0 - \delta / 2M$ to $t_0 + \delta / 2M$.

Let us study what happens for initial values $q_i'^0$ that are very large in absolute value. Take one of the parameters q to be an independent variable, which is chosen in such a way that its derivative (dq / dt) is not less than any of the other derivatives $q_i'^0$ in absolute value: Let q_1 be that parameter. If we set:

$$\frac{1}{q_1'} \equiv \frac{dt}{dq_1} = r_1$$

then equations (2) will become:

$$(2') \qquad \begin{cases} \dfrac{dt}{dq_1} = r_1, & \dfrac{dr_1}{dq_1} = -r_1\,\Pi_1 - r_1^2\,\beta_1, \\[2ex] \dfrac{dr_1}{dq_1} = q_{(i)}', & \dfrac{dq_{(i)}'}{dq_1} = \Pi_1 - q_{(i)}'\,\Pi_1 + (\beta_i - q_{(i)}'\beta_1)\,r_1^2. \end{cases}$$

Let M_1 be the maximum modulus of the right-hand sides of equations (2′) when one has, simultaneously:

$$| q_i - a_i | \le \delta, \quad r_1 \le L_2, \qquad | q_{(i)}' | \le 1 + \frac{\delta}{2} \qquad (i = 1, 2, \ldots, k)$$

(with the condition $L_1 > \delta$). On the other hand, let the values q_i^0, r_1^0, $q_{(i)}'^0$ satisfy the conditions:

$$| q_i^0 - \alpha_i | \le \frac{\delta}{2}, \qquad r_1^0 \le \frac{L_1}{2}, \qquad | q_i'^0 | \le 1 .$$

There exists one and only one system of integrals of (2′) $t(q_1)$, $r_1(q_1)$, $r_i(q_1)$, $q_{(i)}'(q_1)$ that takes the values t_0, q_i^0, r_1^0, $q_{(i)}'^0$ for $q_1 = q_1^0$, and those integrals will be continuous, at least in the interval from $q_1^0 - \varepsilon_1$ to $q_1^0 + \varepsilon_1$, when ε_1 denotes the quantity $\delta / 2M_1$.

In particular, if $r_1^0 = 0$ then that system will have the form:

$$t \equiv t_0, \qquad r_1 \equiv 0, \qquad q_i = \varphi_i(q_1), \quad q_{(i)}' = \varphi_i'(q_1),$$

([6]) See **Picard**, *Traité d'analyse*, Tome II, Chapter XI, page 308.

as equations (2′) will show immediately ([7]).

That shows that r_1 will never be annulled in the interval $q_1^0 - \varepsilon_1$ to $q_1^0 + \varepsilon_1$, or at least it will not be identically zero. t will then vary constantly in the same sense when q_1 grows from $q_1^0 - \varepsilon_1$ to $q_1^0 + \varepsilon_1$ and will pass from the value $t_0 - \eta$ to the value $t_0 + \eta'$ or from the value $t_0 + \eta'$ to the value $t_0 - \eta$. It follows from this that $q_1(t)$, $q_2(t)$, …, $q_k(t)$ are functions of t that are continuous, along with their first derivatives, in the interval from $t_0 - \eta$ to $t_0 + \eta'$, and that q_1 passes from one of the values $q_1^0 \pm \varepsilon_1$ to the other when t varies in that interval.

The same argument can be repeated when the independent variable is a different parameter q_j. Equations (2) will correspond to analogous equations: Let M_j be the maximum modulus of the right-hand sides of those equations when one has, at the same time:

$$| q_i - a_i | \leq \delta, \quad \left| \frac{dt}{dq_j} \right| \leq L_1, \quad \left| \frac{dq_i}{dq_j} \right| \leq 1 + \frac{\delta}{2} \qquad (i = 1, 2, \ldots, k).$$

The quantity $\varepsilon_1 = \delta / 2M$ will correspond to the quantity $\varepsilon_j = \delta / 2 M_j$. I shall let ε denote the smallest of the quantities ε_j.

Those propositions allow us to prove an important property of motion. Consider the system of integrals $q_i(t)$, $q_i'(t)$ of equations (2) such that for $t = t_0$, $q_i = q_i^0$, $q_i' = q_i'^0$, the values of q_i^0 are not singular values of A_{ij}, Q_i. When one makes t increase when starting from t_0, several situations can present themselves: The motion might remain regular for any value of t. (By that, I mean that it remains finite and continuous, and the system S does not pass through any singular position, moreover.) One or more of the parameters q_i might become infinite or indeterminate when t tends to a certain value t_1. Finally, the system S might tend to a singular position. However, can it happen that when t tends to t_1, the system S will tend to a non-singular position and the velocities q_i' will become indeterminate or infinite? We shall see that this can never happen. More precisely, assume that when t tends to t_1, the parameters q_1, q_2, …, q_k tend to the values a_1, a_2, …, a_k, respectively, in whose neighborhood the determinations that were taken for the A_{ij}, Q_i remain regular. Under those conditions, the q_i' will tend to finite limits, respectively, and the motion will remain regular outside of the instant t_1.

Indeed, there are two possibilities: Either the q_i' all tend to zero when t tends to t_1 (which would prove the theorem) or the modulus of at least one of the q_i' is greater than a certain limit λ for certain values of t that are as close to t_1 as one desires. Then consider the number ε that was introduced above and corresponds to the conditions:

$$| q_i - a_i | \leq \delta, \quad \left| \frac{dt}{dq_j} \right| \leq \frac{1}{\lambda}, \quad \left| \frac{dq_j}{dq_i} \right| \leq 1 + \frac{\delta}{2} \qquad (i = 1, 2, \ldots, k).$$

([7]) The trajectory $q_i = \varphi_i(q_1)$ is a geodesic.

($1/\lambda$ replaces L_1). By hypothesis, we can find an instant t' that is sufficiently close to t_1 that when t varies from t' to t_1, each variable q_i will remain between $a_i - \alpha$ and $a_i + \alpha$, where α denotes an arbitrary number that is less than $\delta / 2$ and $\varepsilon / 2$. Now let t_0 be a value of t that is found between t' and t_1 and for which the greatest of the moduli $|q_i'|$, namely, $|q_1'|$, exceeds λ. One has, for $t = t_0$:

$$|q_i^0 - a_i| < \frac{\delta}{2}, \qquad \left|\frac{dt}{dq_1}\right|_0 < \frac{1}{\lambda}, \quad \left|\frac{dq_i}{dq_1}\right|_0 \leq 1 \qquad (i = 1, 2, \ldots, k).$$

When q_1 varies from $q_i^0 - \varepsilon$ to $q_i^0 + \varepsilon$, t will vary from $t_0 - \eta$ to $t_0 + \eta'$ (or from $t_0 + \eta'$ to $t_0 - \eta$). I say that t_1 is found between t_0 and $t_0 + \eta'$. In other words, when t varies from t_0 to $t_0 + \eta'$, and as a result between t' and t_1, q_1 will vary between q_1^0 and $q_1^0 \pm \varepsilon$. However, between t' and t_1, one has: $|q_1 - a_i| \leq \alpha \leq \varepsilon / 2$, so there will be two values of q_1 in that interval that cannot differ by ε. Therefore, the instant t_1 is found between t_0 and $t_0 + \eta'$, and since the functions $q_i(t)$, $q_i'(t)$ are continuous in that interval, the motion will remain regular at the instant t_1 and beyond. Q.E.D.

We can then state the following theorem:

Theorem:

When the system S tends to a non-singular position as t tends to t_1, its velocities will tend to a limit, and the motion can be continued regularly beyond t_1.

If all of the q_i' are non-zero for $t = t_1$ (say $q_1' \neq 0$) then the ratios q_i' / q_1' will have well-defined values. The same thing will be true when the q_i''s are all annulled for $t = t_1$. Indeed, one has:

$$q_i'' = \frac{(t-t_1)^2}{1\cdot 2}[\beta_i(a_1, a_2, \ldots, a_k) + \delta_i] = \frac{(t-t_1)^2}{2}(\beta_i^0 + \delta_i) \qquad (i = 1, 2, \ldots, k)$$

in that case, in which the δ_i tend to zero with $t - t_1$. Furthermore, none of the β_i^0 are zero. In other words, the unique system of integrals that satisfies the initial conditions $q_i = a_i$, $q_i' = 0$ (for $t = t_1$) will be the system $q_i(t) \equiv a_i$, $q_i'(t) \equiv 0$ ([8]). Therefore, let $\beta_1^0 \neq 0$. The ratios $\dfrac{q_i'}{q_1'} = \dfrac{dq_i}{dq_1}$ take the values $\dfrac{\beta_i^0}{\beta_1^0}$ for $t = t_1$. That shows that as t tends to t_1, the system S cannot tend to a (regular) equilibrium position with a *vis viva* that tends to zero.

([8]) The equilibrium conditions of the system S are obviously $\beta_i = 0$, and the equalities $q_i = a_i$ define that equilibrium.

I must add that the integrals $q_i\,(t)$ are even functions of $t - t_1$ then. In other words, if one sets $t - t_1 = \tau = \theta^2$ then one will have:

$$q_i = \alpha_i + \beta_i\,\theta + c_i\,\theta^2 + \ldots \qquad (i = 1, 2, \ldots, k)\,.$$

Indeed, if one changes τ into $-\tau$ in the integrals $q_i\,(\tau)$ then one will again get a system of integrals. When the first system satisfies the initial conditions q_i^0, $q_i'^0$ for $t = t_1$, the second one will satisfy the conditions $q_i^0, -q_i'^0$. Since the $q_i'^0$ are zero here, the initial conditions will remain the same, and the two systems of integrals will then coincide: $q_i\,(\tau) \equiv q_i\,(-\tau)$. When t goes beyond the instant t_1, the system S will reverse: At the instant $t_1 + \tau$, it will pass through the same position that it passed through at the instant $t_1 - \tau$, but the velocities will have changed sense.

Now suppose that the system S tends to a non-singular position $(a_1, a_2, \ldots, a_k)$ as t increases indefinitely. I say that all of the velocities necessarily tend to zero. Indeed, assume that this is not true and repeat the argument that was made above while keeping the same notation: By hypothesis, for any value of t that is greater than a certain limit t', one will have: $|\,q_i - a_i\,| \leq \alpha \leq \varepsilon / 2$ $(i = 1, 2, \ldots, k)$. However, on the other hand, there exist values t_0 of t that are greater than t', and are such that at least one of the parameters, say q_1, varies by ε when t varies from t_0 to $t_0 + \eta'$. There is then a contradiction. The q_i' tend to zero with $1 / t$.

Moreover, the position $(a_1, a_2, \ldots, a_k)$ is a position of equilibrium of the system S. Indeed, make the change of variables $t = 1 / \theta$. One will have:

$$\frac{dq_i}{dt} = q_i' = -\theta^2\,\frac{dq_i}{d\theta}\,, \qquad \frac{dq_i'}{dt} = -\theta^2\,\frac{dq_i'}{d\theta}\,,$$

and as a result:

$$\frac{dq_i}{d\theta} = -\frac{q_i'}{\theta^2}\,, \qquad \frac{dq_i'}{d\theta} = \frac{-\beta_i + \Pi_i}{\theta^2}\,.$$

By hypothesis, when θ tends to zero, the $q_1, \ldots, q_k$ will tend to $a_1, \ldots, a_k$, and the q_i' will tend to zero.

It follows from this that the $\beta_i\,(a_1, a_2, \ldots, a_k)$ must be zero because if $\beta_i\,(a_1, a_2, \ldots, a_k) = \beta_i^0$ then one will have:

$$\frac{dq_i'}{d\theta} = \frac{-\beta_i^0 + \delta_i'}{\theta^2}\,,$$

δ_i' will tend to zero with θ, and q_i' will increase indefinitely when θ tends to zero. It is then necessary that $\beta_1^0, \beta_2^0, \ldots, \beta_k^0$ must be zero.

We know, moreover, that conversely when t increases, the system S cannot tend to a (regular) equilibrium position with a *vis viva* that is annulled without t increasing beyond any limit.

It would be fitting to observe here that the q_i' tend to zero without the ratios $\dfrac{q_i'}{q_1'} = \dfrac{dq_i}{dq_1}$ necessarily having a limit. In order to convince ourselves of that, consider the motion in a plane of a point (x, y) that is subject to the force $X = 2y$, $Y = -2x$. The equations of motion:

$$x'' = 2y, \quad y'' = -2x$$

admit the family of integrals:

$$x = e^{-t}\,[\alpha \cos t - \beta \sin t]\,, \qquad y = e^{-t}\,[\alpha \sin t + \beta \cos t]\,,$$

in which α, β are two arbitrary real constants. When t increases indefinitely, x and y will tend to zero, as well as x', y', but the ratio $\dfrac{dy}{dx} = \dfrac{\alpha \tan t + \beta}{\alpha - \beta \tan t}$ will not tend to any limit.

In the foregoing, we supposed that t is increasing. However, all of the conclusions will obviously persist if one makes t decrease, since it is legitimate to change t into $-t$.

Properties of real trajectories. Return to the study of trajectories. – To abbreviate the language, let us agree to regard $q_1, q_2, \ldots, q_k$ as the k rectangular coordinates of a point M in the k-dimensional space E_k. The trajectories $q_i = \varphi_i(q_1)$ $[i = 2, \ldots, k]$ will be curves C in that space, and the differentials $dq_1, dq_2, \ldots, dq_k$ will define the direction of the tangent at a point on one such curve.

Finally, set:

$$d\sigma = \sqrt{dq_1^2 + dq_2^2 + \cdots + dq_k^2}\,,$$

in which the arc-length σ denotes the length of the segment of the curve C that is found between two points M and M', and extend the integral $\int \sqrt{dq_1^2 + dq_2^2 + \cdots + dq_k^2}$ along the curve MM' (all of the elements being positive). When a curve (C) is regular (by that, I mean that it admits a continuous tangent at each point), one can suppose that $q_1, q_2, \ldots, q_k$ are expressed as functions of arc-length σ, which is measured by starting from a fixed point M_0 and proceeding positively in one sense and negatively in the other. Each value of σ will then correspond to a well-defined point $(q_1, q_2, \ldots, q_k)$ on (C).

We shall consider only the domain in the real space E_k in which the points $(q_1, q_2, \ldots, q_k)$ correspond to real positions of the system, and we shall study the trajectories that belong to that domain exclusively. (What we say will apply to other real trajectories, moreover.) We say *singular points* in the space E_k to mean the points at which the functions A_{ij}, Q_i cease to be regular. Under the most unfavorable hypothesis, those points will form a $(k-1)$-dimensional surface in E_k, namely, the surface $\psi(q_1, q_2, \ldots, q_k) = 0$. In particular, that surface constitutes the boundary of the domain E_k when that space is not considered to be enveloped by all of space.

Finally, we shall say *equilibrium points N* to mean the points $q_1, q_2, \ldots, q_k$ that correspond to an equilibrium position of *S*, i.e., where $\beta_1, \beta_2, \ldots, \beta_k$ are annulled. In the most general case, those points will be isolated points.

True trajectories. Conjugate trajectories. Mixed trajectories. – If one changes *t* into *it* in the equations of motion (1) then those equations will not be altered, except for the fact that the Q_i will all change sign. The trajectories that are defined by the system (1) and the system that is obtained by changing Q_i into $-Q_i$ will then coincide.

That new motion will be called the motion that is *conjugate* to the true motion. That motion is the motion of the system *S* when one changes the senses of all the given forces without changing their directions or magnitudes. If the true motion is imaginary along a real trajectory C'' then the conjugate motion along that same trajectory will be real, since $\left(\dfrac{dt}{dq_1}\right)^2$, which is negative under the first motion, will change sign when one changes *t* into *it*. We give the name of *true trajectories* to the real arcs (C') of the trajectory along which the true motion is real and the name of *conjugate trajectories* to the arcs (C'') along which the conjugate motion is real. When one changes *t* into *it*, the two classes of trajectories (C') and (C'') will permute.

Having said that, give the system *S* the real initial conditions $q_1^0, q_2^0, \ldots, q_k^0, q_1'^0, q_2'^0, \ldots, q_k'^0$ at the instant t_0 and measure the arc-length σ of the trajectory *C* by starting from the initial point M_0 and proceeding in the sense that makes σ begin by increasing with *t*. σ will continue to increase with *t* as long as *t* does not attain a value t_1 for which the motion ceases to be regular or a value t_1 for which $d\sigma / dt$, and as a result, all of q_i', are annulled.

We adopt the first hypothesis to begin with: When *t* tends to t_1, either the point $(q_1, q_2, \ldots, q_k)$ does not tend to any point *M* at a finite distance in the space E_k (in which case, σ would increase indefinitely) or the point $(q_1, q_2, \ldots, q_k)$ tends to a singular point *N* in E_k. From the foregoing, no other case would be possible.

Under the second hypothesis, in which all of the q_i' are annulled when *t* tends to t_1, σ will increase up to a certain limit σ_1, then decrease and take on the same value at $t_1 + \alpha$ that it had at $t_1 - \alpha$. The point $(q_1, q_2, \ldots, q_k)$ moves backward in its trajectory.

Indeed, the point $(a_1, a_2, \ldots, a_k)$, or M_1, which corresponds to the value σ_1 of σ, cannot be an equilibrium point (otherwise, t_1 would be infinite), and the trajectory will be defined by equations of the form:

$$q_i = a_i + b_i\,\theta + c_i\,\theta^2 + \cdots \qquad (i = 1, 2, \ldots, k)$$

in the neighborhood of M_1, in which $\theta = (t - t_1)^2$, and none of the b_i are zero. That shows us that (*C*) can be extended up to the point M_1 while always admitting a continuous tangent. Furthermore, since one has:

$$\frac{d\sigma}{dt} = (t - t_1)\left[-\sqrt{b_1^2 + b_2^2 + \cdots + b_k^2} - \varepsilon\right] = (t - t_1)\,(B + \varepsilon)$$

(in which ε tends to zero with $t - t_1$), σ will pass through a maximum σ_1 for $t = t_1$. The point M_1 will be called a *point of regression* (*point d'arret*) of the trajectory (C). The conjugate motion will be real along the segment M_1M of (C), which follows the segment M_0M_1 . Finally, if the trajectory (C) is not a remarkable trajectory then $\left(\frac{d\sigma}{dt}\right)^2$ will have a well-defined value $\varphi(\sigma)$ at each point of M_0M : The equality $(\sigma - \sigma_1) = (t-t_1)^2(B/2+\varepsilon')$ and its consequence $\left(\frac{d\sigma}{dt}\right)^2 = (\sigma-\sigma_1)(2B+\varepsilon')$ prove that $\left(\frac{d\sigma}{dt}\right)^2$ will remain a continuous function of σ (but with its sign changed) when one crosses a point of regression while varying M from M_0 to M along (C).

We give the name of *mixed* trajectories to those trajectories Γ that possess at least one point of regression M_1 . They define a family that depends upon k arbitrary constants, for example, the coordinates $a_1, a_2, \ldots, a_k$ of a point of regression. Indeed, take one of the derivatives q_i' (say, q_1') to be the independent variable and study $q_1, q_2, \ldots, q_k, q_2', \ldots, q_k'$ as functions of q_1'. If one considers all of the trajectories then one can give the values $q_1^0, q_2^0, \ldots, q_k^0, q_2'^0, \ldots, q_k'^0$, for $q_1'^0 = 0$ arbitrarily. In order to get the mixed trajectories, one sets $q_2'^0 = 0, \ldots, q_k'^0 = 0$. The congruence of trajectories Γ then depends upon k arbitrary constants: An infinitude of trajectories that depend upon one parameter then passes through a given point ($q_1, q_2, \ldots, q_k$). It can nonetheless happen that those k parameters $a_1, a_2, \ldots, a_k$ are not distinct: In order for that to happen, it is necessary that an arbitrary mixed trajectory Γ should correspond to an infinitude of values of the constants $a_1, a_2, \ldots, a_k$ such that one can take at least one of them arbitrarily.

It will then be necessary that all of the points of a segment of Γ must be points of regression, and a result, that an infinitude of motions will be possible on Γ. In other words, the mixed trajectories must be remarkable trajectories. On the other hand, since at least one mixed trajectory will pass through an arbitrary point M_0 or $(q_1^0, q_2^0, \ldots, q_k^0)$ (namely, the one that admits M_0 as a point of regression), the congruence Γ will depend upon at least $(k - 1)$ distinct constants: We then arrive at the following conclusion: The congruence of mixed trajectories (Γ) depends upon k distinct constants, except in the case where there exists a $(k - 1)$-parameter congruence of remarkable trajectories ([9]), in which case, that congruence will coincide with the congruence (Γ).

Moreover, a remarkable trajectory (γ) must be regarded as a mixed trajectory, in the sense that an arbitrary point ($a_1, a_2, \ldots, a_k$) of the trajectory γ must be a point of regression for one of the motions of S along γ. Indeed, all of those motions are defined by an equality of the form (see page 242):

$$(a) \qquad T = f(q_1) + k ,$$

([9]) We saw (page 242) that the remarkable trajectories depend upon at most $(k - 1)$ parameters.

in which h is an arbitrary constant. Let $q_1 = a_1$ for $t = t_0$, and let $h = -f(a_1)$: At the point $(a_1, a_2, \ldots, a_k)$ of (γ), the *vis viva*, and as a result $q_1', q_2', \ldots, q_k'$, will be annulled, and the system S will reverse along (γ) when t passes from $t_0 - \varepsilon$ to $t_0 + \varepsilon$ for the motion in question. However, there can exist motions that always take place in the same sense along (γ): In other words, it can happen that $f(q_1) + h$ is never annulled for other values of h, as we will soon verify in an example.

Now suppose that as t increases indefinitely, the motion remains regular and the *vis viva* is not annulled: There two possible cases: Either the point $(q_1, q_2, \ldots, q_k)$ does not tend to any point at a finite distance in the space E_k (in which case, σ would increase indefinitely with t) or $(q_1, q_2, \ldots, q_k)$ will tend to a point $(a_1, a_2, \ldots, a_k)$ that will then be an equilibrium point N'. The trajectory does not necessarily have a tangent at that point and cannot be continued analytically beyond it.

From the foregoing, we can state the following conclusions:

Let M_0M be a continuous fragment of the same real trajectory (C) that does not pass through either a singular point N of E_k or an equilibrium point N': The curve (C) admits a continuous tangent along the arc M_0M, and the total length of that axis is a certain finite number σ.

Furthermore, if (C) is not a mixed trajectory (which is the general case in which the mixed trajectories depend upon only k constants) then the arc M_0M will always be traversed in the same sense during a finite time, whether the motion is a true or a conjugate one. That will also be true when the trajectory (C) is mixed if it possesses no point of regression between M_0 and M. The entire arc MM' will then belong to either the class (C') or the class (C'').

If (C) is a mixed trajectory (without being a remarkable trajectory) then, in general, it can possess only one point of regression M_1. When that point M_1 belongs to the arc M_0M, that arc will decompose into two parts M_0M_1 and M_1M, both of which are traversed twice in opposite senses (in a finite length of time), one of which will be the true motion, and the other of which will be the conjugate one. However, it can happen that there exist several points of regression $M_1, M_2, \ldots$ between M_0 and M ([10]), but there is always just a finite number of them. Indeed, suppose that the function $\left(\frac{d\sigma}{dt}\right)^2 = \varphi(\sigma)$ admits an infinitude of zeroes between M_0 and M that correspond to values (increasing, for example) $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ of σ. σ_n remains less than σ' (viz., the length of M_0M), while tending to σ' as n increases indefinitely. The function $\varphi(\sigma)$ is continuous, so it will be annulled when $\sigma = \sigma'$, and one will have (when the corresponding point M' is not an equilibrium point):

$$\varphi(\sigma) = (\sigma - \sigma')(2B + \varepsilon) \qquad (B \neq 0),$$

which shows that $\varphi(\sigma)$ will admit no other zeroes besides σ' in the neighborhood of σ'. The hypothesis is therefore absurd. From that, one can always decompose the arc $M_0 M$ into a finite number of segments that either belong to the class (C') or to the class (C'') entirely.

([10]) It is appropriate to observe that the arc of the trajectory M_1M_2 will correspond to a real periodic motion (either true or conjugate).

If (C) is a remarkable trajectory then each point of $M_0\,M$ is a point of regression for a one of the corresponding motions, but one can always take h in the equality:

$$(\alpha) \qquad T = f(q_1) + h = F(\sigma) + h$$

to be sufficiently large that $M_0\,M$ is traversed in the same sense in its entirety. Furthermore, none of the motions that take place along C can admit two points of regression between M_0 and M. Indeed, assume that T is annulled at M_1 and M_2 : From the equality (α), $F'(\sigma)$ (which is continuous between M_0 and M) will be annulled between M_1 and M_2, and therefore between M_0 and M. Let σ' be the first zero of $F'(\sigma)$ that one encounters upon starting from a point μ of M_0M where $F'(\sigma)$ is not zero and proceeding towards M (or towards M_0). I say that the point M' or σ' of (C) is an equilibrium point N'. In order to see that, it will suffice to consider the motion along (C) that is defined by the equality:

$$T = F(\sigma) - F(\sigma') = (\sigma - \sigma')^2 F_1(\sigma) \ ,$$

since $F_1(\sigma)$ is annulled at most once between μ and M' (say, at M_1), and the sign of $F_1(\sigma)$ is constant along a finite arc M_1M'. Upon supposing that it is positive (which is legitimate, since otherwise one could change t into it), one will have:

$$dt = \frac{d\sigma}{\sigma - \sigma'}\, G(\sigma) \ ,$$

in which G remains greater than a certain positive number when σ varies between $\sigma' - \alpha$ and σ'. t will then increase indefinitely when σ tends to σ', or rather σ will tend to σ' when t increases indefinitely. That will be possible only if M' is an equilibrium point ([11]). It will follow from this

([11]) It is easy to verify that conclusion as follows: One has:

$$F'(\sigma) = Q_1 \frac{dq_1}{d\sigma} + \cdots + Q_k \frac{dq_k}{d\sigma} ,$$

but on the other hand, since the trajectory (C) is remarkable, one knows that $dq_1 / \beta_1 = dq_i / \beta_i$. Therefore:

$$F'(\sigma) = \frac{Q_1\,\beta_1 + \cdots + Q_k\,\beta_k}{\sqrt{\beta_1^2 + \cdots + \beta_k^2}} ,$$

or rather [since $Q_i = \sum_{j=1}^{k} A_{ij}\, B_j$, from equations (6) on page 236]:

$$F'(\sigma) = \frac{T(q_1, q_2, \ldots, q_k, \beta_1, \beta_2, \ldots, \beta_k)}{\sqrt{\beta_1^2 + \cdots + \beta_k^2}} ,$$

which is an expression that be zero only if all of the β_i are zero. When all of the points of (C) are equilibrium points, the geodesic (C) will be traversed with a constant *vis viva* (that is arbitrary, moreover), since there are no given forces.

that for h sufficiently large ($h > h_1$), the segment M_0M will be traversed in its entirety under the corresponding motion. For h less than a certain limit h_2, the same segment will be traversed in its entirety under the conjugate motion. For h between h_1 and h_2, the segment will decompose into two continuous segments, one of which is traversed under the true motion, while the other one is traversed under the conjugate motion.

Branches of singular trajectories. – Now assume that the segment considered $M_0\,M$ includes equilibrium points N' (but not singular points). Let N' be the first equilibrium point that one encounters along $M_0\,M$ upon starting from M_0. Now study the segment M_0N', while first supposing that (C) is not a remarkable trajectory.

Under that hypothesis, $\left(\frac{d\sigma}{dt}\right)^2$ will be a function of σ. $\varphi(\sigma)$ will be continuous as long as M remains between M_0 and N', but what will happen that when one makes M tend to N'? I say that $\varphi(\sigma)$ will tend to a limit. First of all, if $\varphi(\sigma)$ tends to zero then the proposition will have been proved. If that is not true then one can find points M along (C) that are close as one wishes to N' ([12]) and are such that $f(\sigma)$ (and as a result, at least one of the $q_1'^2$) has an absolute value at M that is greater than a certain fixed number λ^2. Upon repeating the argument on page 246) identically (either with the true motion or the conjugate motion), one will see that the system S must reach the position N' in a finite length of time with a well-defined, finite *vis viva*, and as a result, the motion, like the trajectory (C), can be regularly extended beyond N'. In summary, $f(\sigma)$ will tend to a finite limit $f(N')$ when one makes M tend to N' along the arc M_0N', and if that limit $f(N')$ is not zero then the point N' will be an ordinary point of (C).

If, on the contrary, $f(N') = 0$ then we say that the arc M_0N' is a *singular* branch: There are two cases to be distinguished according to whether $f(\sigma)$ is or is not annulled an infinitude of times between M_0 and N'. We first consider the latter case.

1. There exist only a finite number of zeroes of $f(\sigma)$ between M_0 and N'. It will then suffice to consider the segment M_1N' that is adjacent to N' and in which $f(\sigma)$ keeps a constant sign, and it is legitimate to suppose that it is positive. When t increases and the system is placed between M_1 and M with a positive value of $d\sigma/dt$, it will tend to N', and cannot attain the equilibrium position in a finite length of time. The system will then tend to N' along (C) when t increases indefinitely. It can happen that the real curve (C) has no tangent at the point N' and that it cannot be prolonged beyond N'. As for the arc-length σ, or M_0N, it can tend to a finite limit σ' or increase indefinitely when M tends to N'. The equations $x'' = 2y$, $y'' = -2x$ that were cited above (page 249) offer us an example of this first case.

([12]) I intend that to mean that the coordinates $q_1, q_2, \ldots, q_k$ of M differ as little as one desires from the coordinates $a_1, a_2, \ldots, a_k$ of N'.

2. $f(\sigma)$ admits an infinitude of zeroes $M_1, M_2, \ldots, M_n, \ldots M_0$ and N'. The points of regression $M_1, M_2, \ldots, M_n, \ldots$ tend to N' when n increases indefinitely. The segment M_0M will then decompose into an infinitude of segments $M_1M_2, M_2M_3, \ldots$ that tend to N' and correspond to as many periodic motions (which alternate between true and conjugate ones) whose amplitudes will tend to zero. As an example of that case (which is clearly an exception), we cite the equations:

$$(A) \qquad \begin{cases} x'' = \frac{1}{2}\,x\,y\left[\frac{7}{4}(x^2+y^2)^2 - 3\right] + \frac{1}{4}(x^2+y^2)[5y^2 - x^2], \\ y'' = \frac{7}{8}\,y^2(x^2+y^2)^2 - \frac{3}{2}\,x\,y\,(x^2+y^2) + \frac{1}{2}x^2 - y^2. \end{cases}$$

One of the trajectories that are defined by (A) is the following one: $x = \dfrac{\cos\theta}{\theta^{1/2}}$, $y = \dfrac{\sin\theta}{\theta^{1/2}}$, in which once more $\theta = 1/r^2$, in terms of polar coordinates. That trajectory passes through the origin (an asymptotic point), which is an equilibrium point, and the corresponding motion will be defined by the equality:

$$dt = \frac{d\theta\,\theta^{1/4}}{\sqrt{\sin\theta}} = \frac{-2}{t^3}\frac{dr}{\sqrt{r\sin 1/r^2}}\;.$$

Along each arc of the curve $2n\,\pi < \theta < (2n+1)\,\pi$, the motion will be real and periodic. Along the arcs $(2n+1)\,\pi < \theta < 2\,(n+1)\,\pi$, it will be the conjugate motion that is periodic.

When M tends to N', σ will tend to a limit σ' or increase indefinitely, so the real curve (C) might or might not be analytically continued beyond N' according the situation. Finally, it can happen that it includes an infinitude of equilibrium positions that form a sequence.

If one considers the equality $dt = d\sigma / \sqrt{\varphi(\sigma)}$ then $\varphi(\sigma)$ will change sign at each of the zeroes $\sigma_1, \sigma_2, \ldots$, of $\varphi(\sigma)$, and when M tends to N', t will satisfy the relation:

$$t - t_0 = \int_{\sigma_1}^{\sigma_1} \frac{d\sigma}{\sqrt{\varphi(\sigma)}} + i\int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{\sqrt{-\varphi(\sigma)}} + \int_{\sigma_2}^{\sigma_3} \frac{d\sigma}{\sqrt{\varphi(\sigma)}} + \cdots$$

It is always legitimate to choose the positive value of the radical in each case, so the value of $t - t_0$ will then have the form:

$$t - t_0 = (\alpha_1 + \alpha_3 + \alpha_5 + \ldots) + i\,(\alpha_2 + \alpha_4 + \ldots)\,,$$

in which all of the α are positive. It will then follow from this that $|\,t - t_0\,|$ will increase indefinitely when M tends to N'. In other words, t will tend to a limit $A + i\,A'$, and if one replaces t with $(A + i\,A') + t$ then one will see that when t tends to zero (for imaginary values, but that is unimportant), the system will tend to a regular equilibrium position, while the *vis viva* tends to zero, which is impossible.

One will then obtain all of the *singular* branches of the trajectories that pass through an equilibrium point N' (or $\alpha_1, \alpha_2, \ldots, \alpha_k$) by searching for all of the integrals $q_i(\theta)$, $q_i'(\theta)$ of the system:

$$\frac{dq_i}{d\theta} = \frac{-q_i'}{\theta^2}, \quad \frac{dq_i'}{d\theta} = \frac{-\beta_i + \Pi_i}{\theta^2},$$

and for real values, the q_i will tend to α_i and the q_i' will tend to zero when θ tends to zero according to a certain law.

It remains for us to discuss the case that we have left aside in which the trajectory M_0M' is remarkable. First observe that any point M_1 of a remarkable trajectory $M_0\,M_1$ or (γ) is a *regular* point on that trajectory. Indeed, we can place the point M at a point of $M_0\,M_1$ that is as close to M_1 as we desire and has a velocity tangent to (γ) that is as large as we desire. The argument on page 246 then shows that M will go past M_1 for a regular motion. The trajectory (γ) will then continue regularly beyond M_1.

Having said that, let N' be an equilibrium point that is situated on (γ) and consider the equality:

$$T = F(\sigma) + h .$$

$F(\sigma')$ will have a finite value for $\sigma = \sigma'$, say $F(\sigma') = 0$. If h is zero then the point M will tend to N' when t increases indefinitely (under either the true motion or the conjugate motion). For the other values of h, N' will be an ordinary point of the motion, which can then present two points of regression that include the point N'. If σ' is only a double zero of $F(\sigma)$ then one will have:

$$T = h + h + (\sigma - \sigma')^2 [A + \varepsilon]$$

in the neighborhood of σ'. When the number A is negative, the motions that correspond to small positive values of h will be periodic around N'. If A is positive then the same remark will apply to the conjugate motion.

Furthermore, the trajectory (γ) can include an infinite number of equilibrium points N' that form a sequence. That can be seen in the example of the two equations:

$$2x'' = x^3\left[5x\sin\frac{1}{x} - \cos\frac{1}{x}\right], \qquad y'' = 0,$$

which admit the *remarkable* trajectories $y = y_0$, along which, the motion will be defined by the relation:

$$\left(\frac{dx}{dt}\right)^2 = x^5 \sin\frac{1}{x} + h .$$

All of the roots x_i of the equality tan $1 / x = x / 5$ correspond to equilibrium points N', and the roots x_i have $x = 0$ for a limit.

In that example, the force X is continuous, along with its first derivatives $\partial X / \partial x$, $\partial X / \partial y \equiv 0$, in the neighborhood of $x = 0$. However, it is appropriate to remark that the singularity in question will not present itself when the coefficients A_{ij}, Q_i are holomorphic functions of $q_1, q_2, \ldots, q_k$ in the domain of $a_1, a_2, \ldots, a_k$ or N'. Indeed, the point N' will then be a regular analytic point of (γ), and the variables $q_2, \ldots, q_k$ will be holomorphic functions of σ when σ is in the neighborhood of σ', and the point $F'(\sigma) = Q_1 \dfrac{dq_1}{d\sigma} + Q_2 \dfrac{dq_2}{d\sigma} + \cdots + Q_k \dfrac{dq_k}{d\sigma}$ will then be holomorphic in the neighborhood of σ', and the point will necessarily be an isolated zero of $F'(\sigma)$.

Finally, observe that the β_i can be zero all along (γ). The *vis viva* of motion along (γ) will then be constant. In order for that to be true, it is necessary and sufficient that a *geodesic* of T should be a locus of equilibrium points N'.

Remark concerning the case in which the forces Q_i are derived from a potential $U(q_1, q_2, \ldots, q_k)$. – If there exists a force function U, and if the equilibrium points N' are isolated, moreover, then the trajectories (C) that pass through an equilibrium point will be necessarily exceptional.

First of all, the trajectories (C) for which N' is an ordinary point ($T \neq 0$ at N') depend upon only k parameters. As for the ones for which N is a singular point ($T = 0$ at N'), they satisfy the condition: $U(a_1, a_2, \ldots, a_k) + h = \sigma$. Those trajectories then correspond to particular values of the *vis viva* constant, and as a result, they cannot depend upon more than $2k - 2$ parameters.

Conclusion: All of the preceding discussion can then be summarized by:

I. – When the forces $Q_i(q_1, q_2, \ldots, q_k)$ are not all zero, any arbitrary trajectory (C) can be traversed in only two distinct manners ([13]), and the second motion is deduced from the first one by changing the sign of t.

There can nonetheless be exceptions for certain trajectories (γ) that are called *remarkable*, which correspond to an infinitude of motions (which are pairwise in the opposite sense) that depend upon an arbitrary constant. Such trajectories (which do not exist, in general) are necessarily geodesics of T and depend upon more than $(k - 1)$ parameters.

II. – Every continuous arc of a real trajectory (C) that does not pass through either a singular point N or an equilibrium point N' of E_k is *regular* (i.e., it admits a continuous tangent and curvature at each point).

The only points that can be singular points or *extremities* of a trajectory are the points N and N' then. Any continuous arc (C) of a real trajectory that does not pass through either a point N or

([13]) One does not regard two motions to be distinct when one of them is deduced from the other by t in t + const.

a point N' will then be traversed in the same sense in its entirety, whether the motion is a true real motion or its conjugate. In the former case, we say that (C) is a *true* trajectory (C'), and in the latter case that it is a *conjugate* trajectory (C'').

There nonetheless exists an exceptional class of trajectories (Γ) that we call *mixed* trajectories, and on them there will be finite segments that include no points like N or N', so they will be composed, in part, of arcs of type (C') and in part of arcs of type (C''). Any point M that separates two adjacent arcs (C') and (C'') along (Γ) is a *point of regression*, where the real motion changes sense. The *remarkable* trajectories must be regarded as *mixed* trajectories. The congruence of mixed trajectories will then depend upon exactly k parameters, except in the particular case where the congruence of remarkable trajectories attains its maximum number (k – 1) of parameters, in which case, the two congruences will coincide.

III. – If an arbitrary point M in the space E_k is neither a singular point N nor an equilibrium point N' then an infinitude of *regular* trajectories will pass through it in a neighborhood of M that will depend upon k parameters, which are the values of q'_1, q'_2, …, q'_k at M, and no other trajectories will pass through M. Among those trajectories, there are ones of type (C') and ones of type (C'') in the neighborhood of M. Nevertheless, there is one and only one of them that admits the point M as a *point of regression*, namely, the *mixed* trajectory (Γ_1), which corresponds to a zero *vis viva* at M. That trajectory (Γ_1) belongs to the one-parameter congruence of mixed trajectories (Γ) that pass through M. Nonetheless, in the case where the congruence of remarkable trajectories depends upon (k – 1) parameters, and as a result agrees with the congruence of trajectories (Γ), the trajectory (Γ_1) will be the *only* mixed trajectory that passes through M.

If the point M in question is an equilibrium point N' then once more an infinitude of trajectories that depend upon k parameters will pass through that point that are regular in the neighborhood N'. The trajectory Γ_1 will then reduce to the point N' ($q_1 \equiv a_1$, …, $q_k \equiv a_k$). However, the most important fact is that other trajectories (C) can exist besides those trajectories that also pass through N' and have arbitrary singularities at N', and are such that the segment MN' (which is sufficiently small) is never traversed by a real motion (whether true or conjugate) in a finite time.

All of the trajectories (C) are obtained by searching for the integrals $q_i(\theta)$, $q'_i(\theta)$ of the system:

$$\frac{dq_i}{d\theta} = -\frac{q'_i}{\theta^2}, \quad \frac{dq'_i}{d\theta} = -\frac{\beta_i + \Pi_i}{\theta^2},$$

such that the q_i tend to a_1, …, a_k, and the q'_i tend to zero for real values when one makes θ tend to zero according to a certain law.

When a trajectory (C) that is not a remarkable trajectory passes through the point N' without coinciding with one of the *singular* branches (C), the point N' will be an ordinary point of (C), and it will be crossed by a regular motion (whether true or conjugate). If the trajectory (C)

coincides with one those singular branches $N'M$ then either the system will tend to N' along MN' (with a *vis viva* that tends to zero) when t increases indefinitely or MN' will decompose into an infinitude of arcs (which will tend to N') that correspond to just as many periodic motions (true or conjugate). When a remarkable trajectory (γ) passes through N', N' will always be an ordinary point of (γ), and there will always exist an infinitude of periodic motions on (γ) for which M oscillates around N' (which are true or conjugate motions) and motions for which M tends to either N' or some other equilibrium point N'' that is as close to N' as one desires as t increases indefinitely.

Finally, when the forces are derived from a potential and there exist only *isolated* equilibrium positions, a trajectory that is taken at random will not include any singular arcs (C). It will then follow from this that (except for certain exceptional trajectories) any continuous arc of the real trajectory that includes no singular points N of E_k will be traversed completely in the same sense under either a true motion or its conjugate.

Remark. – In all of the foregoing discussion, we have overlooked the case in which the trajectory passes through a singular point N of E_k . When the system S tends to a singular position N (as t tends to t_1), it might happen that the q' do not tend to any limit. In general, it can also happen that even when the q' do have a limit, knowing the velocities at the position N will be insufficient for one to determine the ultimate motion. There would then be no reason to pursue the *analytical* study of motion any further.

By definition, when t starts from t_0 and increases, it can happen that the system S goes to infinity when t tends to t_1, or it might not tend to any limit point, or finally, it might tend to a singular position N. However, as long as that is not the case, we have seen that the velocities will remain well-defined for each value of t, and the preceding discussion will be valid.

Before applying the preceding considerations to some examples, we shall make a few more observations on the subject of similitude in mechanics.

On similitude in mechanics. Conjugate motions.

When one changes T into CT in a system:

$$(1) \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right)-\frac{\partial T}{\partial q_i} = Q_i\,(q_1, q_2, \ldots, q_k)\,, \qquad \frac{dq_i'}{dt} = q_i' \qquad (i = 1, 2, \ldots, k),$$

and changes the Q_i into $c\,Q_i$, in which C and c denote two constants, if one wishes to pass from the first motion to the second then it will suffice to change t into $\sqrt{\dfrac{c}{C}}\,t_1$.

Indeed, write the new equations:

(1′)
$$\frac{d}{dt_1}\sum_j A_{ij}\frac{dq_j}{dt_1} - \tfrac{1}{2}\sum_{j,l}\frac{\partial A_{jl}}{\partial q_i}\frac{dq_j}{dt_1}\frac{dq_l}{dt_1} = \frac{c}{C}Q_i \ .$$

If one sets $t_1 = \sqrt{\frac{C}{c}}\,t$ then one will have $\frac{dq_i}{dt_1} = \sqrt{\frac{c}{C}}\frac{dq_i}{dt}$, $dt_1 = \sqrt{\frac{C}{c}}\,dt$, and equations (1′) will be transformed into the system (1).

It follows from this that equations (1) and (1′) will define the same trajectories. However, the motion on those trajectories will be the same only if C / c is equal to unity. If c / C is positive then the true real trajectories will be the same for the two systems. If c / C is negative then the true trajectories of the same system will be the conjugate trajectories of the second one, and conversely. In particular, if $C = 1$ and $c = -1$ then one can pass from the first motion to the second one by changing t into $i\,t$. One will recover the result that was pointed out before, namely, that the motion that is conjugate to the true motion will be the motion of the system when one changes the sign of Q_i, i.e., when one changes the senses of all given forces without changing their direction or magnitudes.

From that, consider a system S of material points x_i, y_i, z_i that are subject to certain constraints and a homothetic system Σ or $\xi_i = \lambda\, x_i$, $\eta_i = \lambda\, y_i$, $\zeta_i = \lambda\, z_i$ that is subject to corresponding constraints. The *vis viva* of the new system will be $\lambda^2 T$, if T is that of the first one. On the other hand, subject the new system to some given forces (Φ_i) or Ξ_i, H_i, Z_i that are homothetic to the forces (F_i) or X_i, Y_i, Z_i that are exerted on the first one: $\Xi_i = \mu\, X_i$, $\mathrm{H}_i = \mu\, Y_i$, $\mathrm{Z}_i = \mu\, Z_i$. If Q'_j denotes the coefficient Q_j relative to the second system then one will have:

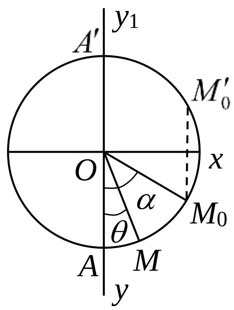
$$Q'_j = \sum \Xi\frac{\partial\xi}{\partial q_j} + \mathrm{H}\frac{\partial\eta}{\partial q_j} + \mathrm{Z}\frac{\partial\zeta}{\partial q_j} = \lambda\,\mu\sum X\frac{\partial x}{\partial q_j} + Y\frac{\partial y}{\partial q_j} + Z\frac{\partial z}{\partial q_j} = \lambda\,\mu\,Q_j \ .$$

It follows from this that the relations between the q_i will be the same for the two motions and that one passes from the first motion to the second one by changing t into $\sqrt{\frac{\mu}{\lambda}}\,t$. The trajectories of second system Σ will then be homothetic to the trajectories of S and will be deduced from the formulas $\xi_i = \lambda\, x_i$, $\eta_i = \lambda\, y_i$, $\zeta_i = \lambda\, z_i$. The motion of Σ is deduced from the motion of S with the aid of the preceding formula and changing t into $\sqrt{\frac{\mu}{\lambda}}\,t$, moreover. If the homothetic correspondence between S and Σ and the correspondence between the (F) and the (Φ) have the same (viz., $\lambda\,\mu > 0$) then the true trajectories of Σ will be the transforms of the true trajectories of S. Otherwise ($\lambda\,\mu < 0$), they will be the transforms of the conjugate trajectories of S. If $\mu = \lambda$, i.e., if one transforms the points S and the forces F together, then the new motion can be deduced from the first one by the formulas $\xi_i = \lambda\, x_i$, $\eta_i = \lambda\, y_i$, $\zeta_i = \lambda\, z_i$. If $\mu = -\lambda$ then the new motion can be deduced from the motion that is conjugate to the first one by using the same formulas.

One sees that, by definition, *if two similar systems are subject to similar forces then the trajectories will also be similar.* That principle of similitude in mechanics was introduced for the first time by Bertrand.

We have said that, on the one hand, the equations of motion will not change when one changes t into $-t$. In other words, when a system S without friction that has constraints that are independent of time is subject to forces that depend upon neither the velocities nor time, the motions of that system will be *reversible*.

I shall add that Appell has inferred an interpretation of imaginary time in mechanics from a consideration of conjugate motions that applies to many interesting special cases. We shall confine ourselves to citing the case of the *simple pendulum* as one example.



Let O be a circle in the vertical plane xOy, and let Oy be the direction of gravity. The position M of a point that moves without friction on the circle is determined by the angle $\theta = MOy$. If one releases the point M with zero velocity from M_0 ($\theta = \alpha$) then the motion will be defined by the equality:

$$\sqrt{\frac{g}{t}}\,dt = \frac{d\theta}{2\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} \, ,$$

or rather, upon setting $\sin\theta/2 = u \sin\alpha/2$:

$$(\alpha) \qquad \sqrt{\frac{g}{l}}\,t = \int_1^u \frac{du}{-\sqrt{(1-u^2)(1-k^2u^2)}} \qquad \left(k^2 = \sin^2\frac{\alpha}{2}\right),$$

i.e., $u = \operatorname{sn}\left(t\sqrt{\frac{g}{l}} + \text{const.}\right)$. The function sn admits periods that are expressed by $4k$ and $2ik'$ with:

$$k = \int_0^1 \frac{du}{+\sqrt{(1-u^2)(1-k^2u^2)}} \, , \qquad k' = \int_1^{1/k} \frac{du}{+\sqrt{(1-u^2)(1-k^2u^2)}} \, .$$

The time that it takes for M to go from M_0 to A is equal to $\sqrt{\frac{l}{g}}\,k$. If one now changes the sense of gravity (without changing its direction or magnitude) then the equation of the new motion will be obtained by changing t into it, and one will have:

$$\sqrt{\frac{g}{l}}\,t = \int_1^u \frac{du}{+\sqrt{(1-u^2)(k^2u^2-1)}} \, .$$

θ will then increase from α to π, and the time that it takes for M to go from M_0 (or $u = 1$) to the point A' (or $u = \dfrac{1}{\sin\alpha/2} = \dfrac{1}{k}$) will be equal to $\sqrt{\dfrac{l}{g}}\,k'$.

One can further say that if $4k$ and $2ik'$ are periods of the function sn that correspond to the modulus $k^2 = \sin^2 \alpha / 2$ then the time that it takes for the moving point that is released without velocity at M_0 ($\theta = \alpha$) or at M_0' ($\theta = \pi - \alpha$) to arrive at A will be equal to $\sqrt{\dfrac{l}{g}}\,k$ under the first hypothesis and $\sqrt{\dfrac{l}{g}}\,k'$ under the second one.

Remark. – We just saw that if we replace the forces Q_i with the forces $Q_i' = c\,Q_i$ (c being a constant) then the trajectories will not be modified. It is appropriate to point out that those forces $c\,Q_i$ are the only forces $Q_i'(q_1, q_2, \ldots, q_k)$ that will generate the same trajectories when they are substituted for the forces Q_i in the system (1).

Indeed, write the differential equations of the trajectories (see page 239) as:

$$\text{(10)} \qquad \frac{q''_{(i)} + q'_{(i)}\,\Pi_1 - \Pi_i}{\beta_i - q'_{(i)}\,\beta_1} = \frac{q''_{(2)} + q'_{(2)}\,\Pi_1 - \Pi_i}{\beta_2 - q'_{(2)}\,\beta_1} \qquad (i = 3, 4, \ldots, k)$$

and

$$\text{(11)} \qquad \frac{\dfrac{d}{dq_1}[q''_{(2)} + q'_{(2)}\,\Pi_1 - \Pi_2]}{q''_{(2)} + q'_{(2)}\,\Pi_1 - \Pi_2} + 2\Pi_1 = \frac{\dfrac{d}{dq_1}[\beta_2 - q'_{(2)}\,\beta_1] - 2\beta_1\,[q''_{(2)} + q'_{(2)}\,\Pi_1 - \Pi_2]}{\beta_2 - q'_{(2)}\,\beta_1}\,.$$

The differential system will have the form:

$$\text{(10}'\text{)} \qquad \frac{d^2 q_i}{dq_1^2} = \frac{d^2 q_2}{dq_i^2}\,\frac{\beta_i - \beta_1 \dfrac{dq_i}{dq_1}}{\beta_2 - \beta_1 \dfrac{dq_2}{dq_1}} + L_i \qquad (i = 3, 4, \ldots, k)$$

and

$$\text{(11}'\text{)} \qquad \frac{d^3 q_i}{dq_1^3} = \chi_2\,\frac{d}{dq_1}\log\beta_1 + L_2'\,,$$

in which L_i includes only the first derivatives, and L_2' is defined with the aid of the coefficients of T and the ratios β_i / β_1 .

Having said that, assume that one replaces the forces Q_i with some other forces $Q_i'(q_1, q_1, \ldots, q_k)$. The coefficients β_i (q_1, q_2, …, q_k) will become $\beta_i'(q_1, q_2, \ldots, q_k)$. In order for equations (10) and (11) remain unaltered by that substitution, it is first necessary that one must have:

$$\frac{\beta_i - \beta_1 \dfrac{dq_i}{dq_1}}{\beta_2 - \beta_1 \dfrac{dq_2}{dq_1}} \equiv \frac{\beta_i' - \beta_1' \dfrac{dq_i}{dq_1}}{\beta_2' - \beta_1' \dfrac{dq_2}{dq_1}}$$

identically, and as a result:

(12) $$\frac{\beta_1'}{\beta_1} \equiv \frac{\beta_2'}{\beta_2} \equiv \ldots \equiv \frac{\beta_k'}{\beta_k} .$$

It will then follow [from (11′)] that:

$$\frac{d}{dq_1}\log \beta_1 \equiv \frac{d}{dq_1}\log \beta_1' ,$$

or rather that:

$$\frac{\partial}{\partial q_1}\log \beta_1 \equiv \frac{\partial}{\partial q_1}\log \beta_1' , \qquad \ldots, \qquad \frac{\partial}{\partial q_k}\log \beta_1 \equiv \frac{\partial}{\partial q_k}\log \beta_1' ,$$

which will imply the consequence:

$$\beta_1' \equiv c\, \beta_1 ,$$

so, from (12):

$$\beta_1' = c\, \beta_1 , \qquad \beta_2' = c\, \beta_2 , \qquad \ldots, \qquad \beta_k' = c\, \beta_k ,$$

and one immediately deduces the equalities:

$$Q_1' = c\, Q_1 , \qquad Q_2' = c\, Q_2 , \qquad \ldots, \qquad Q_k' = c\, Q_k .$$

Q.E.D.

More generally, if one replaces T with $C\,T$ and the Q_i with $Q_i'(q_1, q_1, \ldots, q_k)$, where C denotes a constant, then in order for the trajectories to the remain the same, is necessary and sufficient for one to have $Q_i' \equiv C\, Q_i$.

Applications of the preceding generalities to some examples.

Consider a free material point (x, y, z) that is subjected to a force F whose projections X, Y, Z are analytic functions of x, y, z that are holomorphic for all real values of those variables.

The *geodesics* here are the lines in space: The *remarkable* trajectories (if they exist) will then be lines D, and if they form a congruence then it will depend upon at most two parameters. In what case will those trajectory lines depend upon precisely two parameters? In order to account for that case, it is sufficient to observe that at each point (x, y, z) of the line D, the line of action of (F) must coincide with D. If one passes a line (D) through any point (x, y, z) then the set of all lines of action of the force (F) must coincide with the congruence of lines D. Conversely, if the lines of action of (F), which generally form a *complex*, form a congruence then every line of that congruence will be a remarkable trajectory. The only case in which the remarkable trajectories depend upon two parameters is then the case in which the complex of forces (F) reduced to a congruence. In particular, when there exists a force function U (x, y, z), the forces (F) will form a complex, unless the level surfaces U = const. are *parallel*. The normals to those surfaces will then form a congruence.

As for the *singular* trajectories, if (a, b, c) is an equilibrium point then they will be obtained in any case by searching for all of the integrals of the system:

$$\frac{dx}{x'} = \frac{dy}{y'} = \frac{dz}{z'} = \frac{dx'}{X} = \frac{dy'}{Y} = \frac{dz'}{Z}$$

that satisfy the initial conditions: $x = a$, $y = b$, $z = c$, $x' = y' = z' = 0$.

Any continuous arc of the trajectory is necessarily regular, except perhaps at an equilibrium point (a, b, c), and can be extended indefinitely in a regular fashion ([14]), as long as one does not encounter an equilibrium point. However, the singular branches that pass through an equilibrium point (a, b, c) can present arbitrary singularities at that point, and in particular, they can terminate there.

Now assume that the forces are derived from a potential U and that the equilibrium points are isolated, moreover, i.e., that the three derivatives of U are annulled simultaneously only at isolated points. A trajectory that is taken at random will not include any singular branch. Any continuous arc of such a thing will be traversed in its entirety in the same sense by a regular motion, whether true or conjugate. There will be an exception only for some special trajectories, namely, the *mixed* trajectories and the singular trajectories. The former depend upon *three* parameters, unless the level surfaces are *parallel*, in which case the mixed trajectories will coincide with the congruence of remarkable trajectories. As far as singular trajectories such as MN' are concerned (N' being an equilibrium point), we know only that the arc MN' is never traversed by the moving point in a finite time by a real motion (whether true or conjugate): The moving point tends to N' when t increases indefinitely, or rather, MN' decomposes into an infinitude of arcs that correspond to periodic motions.

When the level surfaces are *parallel*, any equilibrium point N' will be a multiple point of the level surface that contains it, and an infinitude of normals will pass through that point that form a cone, which are just as many remarkable trajectories D. One can define initial conditions for the moving point such that it will tend to N' on D when t increases indefinitely (for a true or conjugate motion, as the case may be).

([14]) I intend that to mean that x, y, z remain holomorphic functions of the arc-length σ.

I shall add some observations in regard to the periodic motions. For an arbitrary true motion to be periodic, it is necessary and sufficient that an arbitrary true trajectory should be a closed curve. Indeed, take a random trajectory: At any point M of that trajectory (that is not a double point), the velocity of the moving point will have a well-defined value. On the other hand, the trajectory will be traversed in its entirety in the same sense by a regular motion in a finite time interval t_1, and after the time t_1, the moving point will return to the starting point M with the same velocity and point in the same sense, and the same motion will begin again.

Similarly, in order for any true motion whose initial conditions are subject to a certain inequality to be periodic, it is necessary and sufficient that all of the true trajectories (whose $2k - 1$ parameters satisfy a certain inequality) should be closed curves.

If one desires that the particular motions should be periodic then there are several cases to be distinguished.

In order for a *true* trajectory that is neither mixed nor remarkable to correspond to a periodic motion, it is necessary and sufficient that it should be closed and contain no singular branch.

In order for a *mixed* trajectory to correspond to a periodic motion, it is necessary and sufficient that it should present at least two points of regression that do not include any singular branch, and between which the true motion will be real (and not the conjugate motion).

In order for a *remarkable* trajectory (which is then a line D) to correspond to a periodic motion, it is necessary and sufficient that the line D should pass through an equilibrium point N', where the value of the function U along the line possesses a maximum. In that case, there exists an infinitude of periodic motions in which the moving point will oscillate about N' along D with a constant amplitude of oscillation that is as small as one desires.

Examples:

1. As one particular application, let us study the motion of a gravitating point.

The forces form a congruence here, namely, the congruence of vertical lines. Those lines will be both remarkable trajectories and mixed trajectories for the point. There is no equilibrium position, so there will be no singular branches.

If one defines the direction and sense of gravity then the trajectories will be parabolas:

$$y = \lambda x + \mu ,$$
$$z = \alpha x^2 + \beta x + \gamma ,$$

in which $\alpha, \beta, \gamma, \lambda, \mu$ denote arbitrary constants. Those trajectories are true for $\alpha > 0$ and conjugate for $\alpha < 0$. The two classes will permute when one changes the sense of gravity. Any arc of the parabola will be traversed in the same sense in a finite time under either the true motion or the conjugate motion.

As for the motion along a vertical, it will necessarily present one and only one point of regression (the culmination point) when t varies from $-\infty$ to $+\infty$.

There are no periodic motions.

2. Now consider a material point (of mass 1) that is attracted to the origin in proportion to the distance: $F = -\ k^2\, r$.

The forces once more form a congruence, namely, the congruence of lines that issue from the origin. Those lines D are both remarkable and mixed trajectories of the moving point. The origin O is an equilibrium point, and it is the only one.

The real trajectories are the real conics that have the origin for their centers. The true trajectories are the ellipses, and the conjugate trajectories are hyperbolas.

All of the true motions are periodic. Along a line D, the function U has a maximum at the origin O. All of the motions along D will present two points of regression that are equidistant from O. As one knows, the oscillations are *tautochronous*.

If one considers a point that is *repelled* by the origin in proportion to the distance then the real trajectories will not change, the true trajectories will be hyperbolas, and the conjugate trajectories will be ellipses. Along a line D, the motion will present one and only one point of regression between $t = -\infty$ and $t = +\infty$ or none of them, according to whether the absolute value V_0 of the initial velocity is less than or greater than $+ k\, r_0$, respectively, where r_0 is the initial distance from the origin. If $V_0 = k\, r_0$ then the moving point will tend to O when t tends to $+\infty$ (or to $-\infty$). No motion is periodic.

In the latter case, x, y, z are rational functions of e^{kt}, and as a result, they will admit the imaginary period $2i\pi / k$. That period corresponds to the real period $2\pi / k$ of the conjugate motion, and indeed when it is attractive, x, y, z will be rational functions of tan $k\, t / 2$

3. Finally, let us study the motion of a material point of mass 1 that is subject to the force $X = 2y$, $Y = -2x$, $Z = 0$.

The forces form a complex here. The z-axis is a locus of equilibrium points, and since it is also a geodesic, it will be a trajectory that the moving point can traverse with a constant and arbitrary velocity.

Do there exist other remarkable trajectories? The trajectories must be lines and verify the equations:

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{0} .$$

The line $x = 0$, $y = 0$ is the only line that satisfies those conditions.

If one observes that the equations of motion can be written:

$$\frac{d^2(x+iy)}{dt^2} + 2i\,(x + i\,y) = 0 , \qquad\qquad z'' = 0$$

then one will see that the motion is defined by the equalities:

$$x + i\,y = (\alpha + i\,\beta)\,e^{(1-i)t} + (\gamma + i\,\delta)\,e^{(-1+i)t}, \qquad z = \lambda\,t + \mu,$$

or rather:

$$(C) \qquad \begin{cases} x = e^{t}(\alpha\cos t + \beta\sin t) + e^{-t}(\gamma\cos t - \delta\sin t), \\ y = e^{t}(\beta\cos t - \alpha\sin t) + e^{-t}(\delta\cos t + \gamma\sin t), \\ z = \lambda\,t + \mu, \end{cases}$$

in which α, β, γ, δ, λ, μ are arbitrary constants. Since x is always annulled a certain number of times (along with y) when t varies from $-\infty$ to $+\infty$, one can suppose that x is equal to zero for $t = 0$, which amounts to setting $\alpha = -\gamma$ in the expression for x, y.

The *true* real trajectories are obtained by giving real values to α, β, γ, δ, λ, and t, while the *conjugate* real trajectories are obtained by changing t into $i\,t'$ in the expression for $x + i\,y$. x, y, z will then be expressed in the following form:

$$(C') \qquad \begin{cases} x = e^{t'}(\alpha'\cos t' - \beta'\sin t') + e^{-t'}(\gamma'\cos t' + \delta'\sin t'), \\ y = e^{t'}(\beta'\cos t' + \alpha'\sin t') + e^{-t'}(\delta'\cos t' - \gamma'\sin t'), \\ z = \lambda'\,t' + \mu', \end{cases}$$

and one will see that if one sets $\alpha' = \alpha$, $\beta' = -\beta$, $\gamma' = \gamma$, $\delta' = -\delta$, and if one changes y into $-y$ then one will recover equations (C). The conjugate trajectories are then symmetric to the true trajectories with respect to the zx-plane (as well as with respect to the zy-plane). Furthermore, the symmetric curve of a true trajectory with respect to an arbitrary point of Oz is again a true trajectory. The set of true trajectories also admits symmetry planes that are easy to see.

Among the real trajectories (which depend upon five constants), the mixed trajectories form a congruence that depends upon three constants. In order for a trajectory to mixed, it is necessary and sufficient that it should present a point (that is not an equilibrium point) where x', y', z' are simultaneously annulled. It will first be necessary then that λ should be zero, so as a result, that $x' + i\,y'$ will be annulled, for a real value of t that one can always suppose to be zero. In order for $x' + i\,y'$ to be annulled with t, it is necessary and sufficient that $\alpha + i\,\beta = \gamma + i\,\delta$, or that $\alpha = \gamma$, $\beta = \delta$. The mixed trajectories will then be given by the equalities:

$$(C) \qquad \begin{cases} x = \alpha\cos t\,(e^{t} + e^{-t}) + \beta\sin t\,(e^{t} - e^{-t}), \\ y = \alpha\sin t\,(e^{-t} - e^{t}) + \beta\cos t\,(e^{t} + e^{-t}), \\ z = \mu. \end{cases}$$

They present one and only one point of regression, $x = \alpha$, $y = \beta$, $z = \mu$. x, y, z will take those same values again for values of t that are equal and of opposite sign. However, if one sets $t = i\,t'$ then one will get the other real portion of the mixed trajectory:

$$(C') \qquad \begin{cases} x = \alpha \cos t' (e^{t'} + e^{-t'}) - \beta \sin t' (e^{t''} - e^{-t'}), \\ y = \alpha \sin t' (e^{-t'} - e^{t'}) + \beta \cos t' (e^{t'} + e^{-t'}), \\ z = \mu. \end{cases}$$

If one sets $\theta = t^2 = - t'^2$ then the functions $x(\theta)$, $y(\theta)$, and $z = \mu$ that are defined by (C) or (C') will be the same analytic functions of θ, and when θ varies from $-\infty$ to $+\infty$, the point (x, y, z) will traverse all of the real mixed trajectory. That trajectory decomposes into two parts, one of which is true and the other of which is conjugate, that are separated by the point $\theta = 0$. The curve that is symmetric to a mixed trajectory with respect to the zy-plane (or the zx-plane) is also a mixed trajectory, but the symmetric part of a true segment will be a conjugate segment, and conversely.

Let us now consider the *singular* trajectories. Those trajectories cannot include an infinite number of points of regression, since any one of the trajectories will possess at most one point of regression. The true singular branches are then obtained by increasing (or decreasing) t indefinitely through real values and determining whether the moving point will tend to a limiting position N'. That point will necessarily be an equilibrium point, and therefore, a point on Oz. In order for x and y to tend to zero as t decreases indefinitely, it is necessary and sufficient that γ and δ should be zero. In order for $|z|$ to not increase indefinitely, λ must be zero. Since, on the other hand, one can suppose that $\alpha = -\gamma$, so one will have $\alpha = 0$ here, one will have, by definition, the trajectories:

$$x = \beta e^t \sin t, \quad y = \beta e^t \cos t, \quad z = \mu,$$

which depend upon two arbitrary constants. One will get the same trajectories by making t increase indefinitely (it will suffice to change t into $-t$). The conjugate singular branches are obtained by changing t into $i t'$, which will give:

$$x' = -\beta' e^{t'} \sin t', \qquad y' = \beta' e^{t'} \cos t', \qquad z' = \mu.$$

These are the symmetric images with respect to the yz-plane of the true singular branches. If one eliminates t then one will find that $r = C e^{-\theta}$, $z = \mu$ for true branches and $r = C e^{\theta}$, $z = \mu$ for the conjugate ones, in which r and θ denote the polar coordinates of a point in the xy-plane. Those curves admit the origin as an asymptotic point. When the moving point describes one of those singular trajectories, if it is launched, for example, in the sense of the point N' then it will tend to that point without ever reaching it when t increases indefinitely ([15]).

Any true trajectory that does not belong to either the mixed trajectories (a three-parameter congruence) or the singular trajectories (a two-parameter congruence) will be traversed in its entirety in the same sense when t increases from $-\infty$ to $+\infty$.

([15]) We gave (page 255) an example in which a singular branch decomposes into an infinitude of segments that tend to N' and correspond to just as many periodic motions (which are alternately true or conjugate).

Equations of the trajectories when the forces are zero or derive from a potential.

When the forces are zero, we saw (pp. 238) that the trajectories will depend upon $(2k - 2)$ constants, and we have indicated the means of forming the differential equations of those trajectories. We shall now give an explicit form for those differential equations

Let:

$$2T \equiv \sum A_{ij}\, q_i' \, q_j' \qquad (A_{ij} = A_{ji})$$

be the *vis viva* of the system, and consider the Lagrange equations:

(1) $$\frac{d}{dt}\frac{\partial T}{\partial q_i'} - \frac{\partial T}{\partial q_i} = 0 \qquad (i = 1, 2, \ldots, k),$$

which define the motion in the absence of forces.

As one knows, those equations will imply the consequence: $T = h$. If one lets T_1 denote what T will become when one replaces q_1' with 1, q_2' with $dq_2 / dq_1 = q_{(2)}'$, …, and q_k' with $dq_k / dq_1 = q_{(k)}'$ in it then the first integral can be written:

$$q_1'^2 \, T_1 = h ,$$

or rather:

(2) $$dt = dq_1 \sqrt{\frac{T_1}{h}} \; .$$

It is legitimate to replace equations (1) with the last $(k - 1)$ of those equations:

(1′) $$\frac{d}{dt}\frac{\partial T}{\partial q_i'} - \frac{\partial T}{\partial q_i} = 0 \qquad (i = 2, 3, \ldots, k),$$

combined with equation (2). If one now replaces dt with $dq_1 \sqrt{\frac{T_1}{h}}$ everywhere in (1) then one will define the differential equations of the trajectories. Upon remarking that $\frac{\partial T}{\partial q_i'}$ is linear with respect to q_i', and is equal to $q_1' \frac{\partial T_1}{\partial q_{(i)}'}$ or $\frac{\sqrt{h}}{T_1}\frac{\partial T_1}{\partial q_{(i)}'}$ as a result, and that $\frac{\partial T}{\partial q_i}$ is likewise equal to $\frac{h}{T_1}\frac{\partial T_1}{\partial q_i}$, one will see that equations (1), thus-transformed, will become:

$$\frac{\sqrt{h}}{T_1}\frac{d}{dq_1}\left(\frac{\sqrt{h}}{T_1}\frac{\partial T_1}{\partial q_{(i)}'}\right) - \frac{h}{T_1}\frac{\partial T_1}{\partial q_i} = 0 \qquad (i = 2, 3, \ldots, k),$$

or rather:

(3)
$$\frac{d}{dq_1}\left(\frac{1}{T_1}\frac{\partial T_1}{\partial q'_{(i)}}\right)-\frac{1}{\sqrt{T_1}}\frac{\partial T_1}{\partial q_i}=0 \qquad (i = 2, 3, \ldots, k).$$

If we now set:

$$f=\sqrt{T_1}=\sqrt{A_{11}+2A_{12}\,q'_{(2)}+\cdots+2A_{1k}\,q'_{(k)}+\sum_{i>1,\,j>1}A_{ij}\,q'_{(i)}\,q'_{(j)}}$$

then the system (3) will be written:

(4)
$$\frac{d}{dq_1}\frac{\partial f}{\partial q'_{(i)}}-\frac{\partial f}{\partial q_i}=0 \qquad (i = 2, 3, \ldots, k).$$

For example, if $k = 2$ then the geodesics of the surface whose ds^2 is $A_{11}\,dq_1^2+2A_{12}\,dq_1\,dq_2+A_{22}\,dq_2^2$ will be given by the second-order equation:

$$\frac{d}{dq_1}\frac{\partial f}{\partial q'_{(2)}}-\frac{\partial f}{\partial q_2}=0,$$

in which $f=\sqrt{A_{11}+2A_{12}\,q'_{(2)}+A_{22}\,q'^{\,2}_{(2)}}$ and $q'_{(2)}=dq_2/dq_1$.

Equations (4) are the ones that one finds by annulling the variation of the integral $\int f\,dq_1$, when it is taken along an arbitrary curve $q_i = \varphi_i(q_1)$ whose extremities are fixed.

One indeed verifies in that manner that the geodesics depend upon only $(2k - 2)$ constants, and one forms the $(k - 1)$ second-order differential equations that define them explicitly. It would be easy to prove directly that those equations, which are linear with respect to the $q''_{(i)}$, can be solved for those variables, but that would result from what was said at the beginning of this chapter.

Principle of least action. – The procedure that we just employed applies just as well to the case in which the forces are not zero, but are derived from a *potential U*. Indeed, under that hypothesis, write down the Lagrange equations:

$$\frac{d}{dt}\frac{\partial T}{\partial q'_i}-\frac{\partial T}{\partial q_i}=\frac{\partial U}{\partial q_i} \qquad (i = 1, 2, \ldots, k).$$

Those equations imply the consequence that $T - U = h$, so it will be legitimate for us to replace them with $(k - 1)$ of them (the last ones, for example) with:

(1′)
$$\frac{d}{dt}\frac{\partial T}{\partial q'_i}-\frac{\partial T}{\partial q_i}=\frac{\partial U}{\partial q_i} \qquad (i = 2, 3, \ldots, k),$$

combined with the equation:

$$(2) \qquad dt = dq_1 \sqrt{\frac{T_1}{U+h}}\,,$$

in which the notation is the same as it was recently.

If one expresses dt in terms of dq_1 as in (2) everywhere in equations (1′) then since one has:

$$\frac{\partial T}{\partial q_i'} = \sqrt{\frac{U+h}{T_1}}\frac{\partial T_1}{\partial q_{(i)}'}\,, \qquad \frac{\partial T}{\partial q_i} = \frac{U+h}{T_1}\frac{\partial T_1}{\partial q_i}$$

here, it will become:

$$(3) \qquad \sqrt{\frac{U+h}{T_1}}\frac{d}{dq_1}\left(\sqrt{\frac{U+h}{T_1}}\frac{\partial T_1}{\partial q_{(i)}'}\right) - \frac{U+h}{T_1}\frac{\partial T_1}{\partial q_i} = \frac{\partial U}{\partial q_i} \qquad (i = 2, 3, \ldots, k),$$

and those equations are the differential equations of the trajectories that correspond to the value h of the constant of the *vis viva* integral. Those trajectories define a $(2k - 2)$-parameter congruence.

Equations (3) can also be written:

$$(4) \qquad \frac{d}{dq_1}\left(\sqrt{\frac{U+h}{T_1}}\frac{\partial T_1}{\partial q_{(i)}'}\right) - \sqrt{\frac{U+h}{T_1}}\frac{\partial T_1}{\partial q_i} - \sqrt{\frac{T_1}{U+h}}\frac{\partial U}{\partial q_i} = 0 \qquad (i = 2, 3, \ldots, k).$$

In that form, one sees that if one sets:

$$f = \sqrt{(U+h)T_1}$$

then they will coincide with the equations:

$$(5) \qquad \frac{d}{dq_1}\frac{\partial f}{\partial q_{(i)}'} - \frac{\partial f}{\partial q_i} = 0\,, \qquad q_{(i)}' = \frac{dq_i}{dq_1} \qquad (i = 2, 3, \ldots, k).$$

Let us examine this last equation in more detail. Consider an arbitrary curve C in k-dimensional space: $q_2 = \varphi_2(q_1), \ldots, q_k = \varphi_k(q_1)$ that is constrained to have both of its extremities fixed at $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$. If one expresses the idea that the integral $\int_C f\,dq_1$ has zero variation when one passes from one particular curve C_1 to a neighboring curve C then one will find that the curve C_1 must satisfy the differential equations (5) precisely. Moreover, one proves that if the fixed extremities $(a_1, a_2, \ldots, a_k)$, or M_1, and $(b_1, b_2, \ldots, b_k)$, or M_2, are sufficiently close then the integral that is taken along C will be a minimum ([16]). One can then say that the integral $\int_{M_1}^{M_2} f\,dq_1$ is minimal for the natural trajectory of the system $(q_1, q_2, \ldots, q_k)$ that corresponds to the value h of the *vis viva*

([16]) On the subject of higher geometry, see Darboux, (Tome II, Chapters VI, VII, and VIII).

constant and passes through the points M_1 and M_2 . That is what the principle of least action consists of.

Hamilton's principle. – When the forces are derived from a potential and one leaves the Lagrange equations in their original form:

(1) $$\frac{d}{dt}\frac{\partial T}{\partial q_i'} - \frac{\partial (T+U)}{\partial q_i} = 0 \qquad (i = 1, 2, \ldots, k),$$

one can attach them to the calculus of variations in another way.

Indeed, let $q_1, q_2, \ldots, q_k$ be k arbitrary functions of t that are subject to the condition that $q_1, q_2, \ldots, q_k$ have fixed the values $(a_1, a_2, \ldots, a_k)$ for $t = t_0$ and $(b_1, b_2, \ldots, b_k)$ for $t = t_1$. Now consider the integral $\int_{t_0}^{t_1}(T+U)\,dt$. If one expresses the idea that the variation of that integral is zero when one passes from a particular system of functions $q_i = \varphi_i(t)$ to a neighboring system then one will find that those functions $q_i = \varphi_i(t)$ must verify equations (1) precisely. Those equations then express the idea that the variation of the integral $\int_{t_0}^{t_1}(T+U)\,dt$ must be zero for the motion $q_i = \varphi_i(t)$ of the material system, which is a motion that is defined by the conditions that the system must occupy two well-defined positions $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ for $t = t_0$ and $t = t_1$. That is what Hamilton's principle consists of.

Remarks on the principle of least action. – When the forces are derived from a potential, the principle of least action reduces the search for trajectories to the search for geodesics of an ds^2 that depends upon the arbitrary constant h. Indeed, if one sets:

$$T' = (U + h)\,T \qquad \text{or} \qquad ds'^2 = (U+h)\,ds^2 = (U+h)\sum A_{ij}\,dq_i\,dq_j$$

then the geodesics of ds'^2 will be given by the equations:

(5) $$\frac{d}{dq_1}\frac{\partial f}{\partial q'_{(i)}} - \frac{\partial f}{\partial q_i} = 0\,, \qquad \frac{dq_i}{dq_1} = q'_{(i)} \qquad (i = 2, 3, \ldots, k),$$

in which $f = ds' / dq_1 = \sqrt{(U+h)T_1}$.

If one agrees to call any $(2k - 2)$-parameter congruence of trajectories that corresponds to a well-defined value a of the constant h of the *vis viva* integral a *natural congruence* then one will see that the natural congruence $h = a$ will coincide with the geodesics of $(U+a)\,ds^2$, or if one prefers, with the geodesics of $(\lambda U + \mu)\, ds^2$, where $\mu / \lambda = a$. The geodesics of ds^2 define a natural congruence that will then correspond to $a = \infty$, as we know already.

The study of the trajectories when there exists a force function then comes down to integrating the system (5), i.e., to an explicit system of $(k - 1)$ second-order equations that depend upon one arbitrary parameter h. It is important to remark that those equations are of the same type that the calculus of variations provides, and as a result, as we will soon see, they will possess some important properties that facilitate their integration. In particular, we know a last multiplier.

Once the trajectories are known, the motion will be defined with the help of just one quadrature. What sort of relations exist between the motions that are defined, on the one hand, by the first system of Lagrange equations:

$$(1) \qquad \frac{d}{dt}\frac{\partial T}{\partial q_i'} - \frac{\partial T}{\partial q_i} = \frac{\partial U}{\partial q_i}, \qquad \frac{dq_i}{dt} = q_i' \qquad (i = 1, 2, \ldots, k),$$

and on the other hand, by the system with no forces:

$$(1') \qquad \frac{d}{dt}\frac{\partial T'}{\partial q_i'} - \frac{\partial T'}{\partial q_i} = 0, \qquad \frac{dq_i}{dt} = q_i' \qquad (i = 1, 2, \ldots, k),$$

in which $T' = (U + h)\, T$?

We know that the trajectories are the same for the two systems. Let us consider one of those trajectories. The motion along that trajectory, according to (1), is defined by the equality:

$$dt^2 = \frac{ds^2}{U + h},$$

and from (1′), by:

$$dt_1^2 = \alpha\,(U + h)\, ds^2,$$

in which α denotes an arbitrary constant. One then passes from the first motion to the second one by changing dt^2 into $\dfrac{dt_1^2}{\alpha\,(U + h)^2}$, in which α is an arbitrary constant.

Darboux's transformation.

We just said that when the forces are derived from a potential, the trajectories of a system will coincide for each value of h with the geodesics of $(U + h)\, ds^2$. From that, consider a system whose ds^2 is equal to $(\alpha U + \beta)\, ds^2$, and is subject to forces whose potential is $\dfrac{\gamma U + \delta}{\alpha U + \beta}$. The trajectories of that new system will coincide with the geodesics of $d\sigma^2$, when we set:

$$d\sigma^2 = [\gamma U + \delta + h_1(\alpha U + \beta)]\, ds^2 ,$$

in which h_1 denotes the constant of the new *vis viva* integral. If one establishes the following relation between h and h_1:

$$h = \frac{\delta + \beta\, h_1}{\gamma + \alpha\, h_1}$$

then the geodesics of ds^2 and $(U + h)\, ds^2$ (which differ by only a constant factor) will coincide. Hence, one concludes that:

If one replaces T with $(\alpha\, U + \beta)\, T$ and U with $U' = \dfrac{\gamma U + \delta}{\alpha U + \beta}$ in a system $[T, U]$ of Lagrange equations then the trajectories will not change. Every natural congruence $h = a$ that is composed of the former trajectories is a natural congruence $h_1 = a_1$ that is composed of the new trajectories. The value of h_1 that corresponds to one value of h is given by the equality:

$$h_1 = \frac{\delta - \gamma\, h_1}{\alpha\, h - \beta} .$$

That transformation was pointed out for the first time by Darboux.

The motions that are defined by each of the two systems will not be the same on the same trajectory. Indeed, if we represent time by t in the former motion and by t_1 in the latter one then we will have the two equalities:

$$ds^2 = (U + h)\, dt^2 = \left(U + \frac{\delta + \beta\, h_1}{\gamma + \alpha\, h_1}\right) dt^2$$

and

$$(\alpha U + \beta)\, ds^2 = \left(\frac{\gamma U + \delta}{\alpha U + \beta} + h_1\right) dt_1^2 .$$

Hence, upon eliminating h_1, we will infer that:

(6) $$(\alpha\,\delta - \beta\,\gamma)\, dt_1^2 = (\alpha U + \beta)^2 [\alpha\, ds^2 - (\alpha U + \beta)\, dt^2] .$$

That is the relation that exists between dt and dt_1 .

Since it is legitimate to add a constant to the force function, one can always suppose that U' has the form $\delta / \alpha\, U$ (in which α is non-zero). The relation (6) will then become (with $\beta = 0$):

$$\left(\frac{dt_1}{dt}\right)^2 = \frac{\alpha^2}{\delta}\, U^2 \left[\frac{ds^2}{dt^2} - U\right] = \frac{\alpha}{\delta}\, U^2\, h ,$$

or rather:

$$\left(\frac{dt}{dt_1}\right)^2 = \frac{1}{\alpha U^2}\left[\alpha U \frac{ds^2}{dt_1^2} - \frac{\delta}{\alpha U}\right] = \frac{h_1}{\alpha U^2} .$$

Those equalities show us that the expressions $\frac{1}{U}\frac{dt_1}{dt}$ and $U\frac{dt}{dt_1}$ are the first integrals of the two respective systems, namely, the two *vis viva* integrals.

Any first integral of the first system corresponds to an integral of the second one that is obtained by replacing dt as a function of dt_1 using (6). The two systems are reciprocal, moreover, i.e., conversely, the first one is deduced from the second one by a Darboux transformation. All of the transforms of the transformed system coincide with the transforms of the original system.

Properties of differential equations that are produced by the calculus of variations.

Let $x_1, x_2, \ldots, x_n$ be n arbitrary functions of one variable x, and let $x_1', x_2', \ldots, x_n'$ be their derivatives. Consider the integral $\int_a^b f(x, x_1, \ldots, x_n, x_1', \ldots, x_n')\,dx$, in which the x_i are subject to the single condition that they must take given values at $x = a$ and $x = b$. f represents a given arbitrary function on which one makes the single hypothesis that its Hessian Δ relative to the n variables x_i' is not identically zero. If one annuls the variation of the integral in question then one will find, as we recalled, that the functions $x_i(x)$ must verify the differential system:

$$(A) \qquad \frac{d}{dx}\frac{\partial f}{\partial x_i'} - \frac{\partial f}{\partial x_i} = 0, \qquad \frac{dx_i}{dx} = x_i' \qquad (i = 1, 2, \ldots, n).$$

It is a system of second-order equations that can be solved for the x_i'' that enter into them linearly because the determinant of the coefficients of the x_i'' is nothing but the Hessian Δ of f. Such a system (A) enjoys some properties that are analogous to those of a Lagrange system for which U exists. More generally, any system:

$$(B) \qquad \frac{d}{dx}\frac{\partial f}{\partial x_i'} - \frac{\partial f}{\partial x_i} = X_i(x_1, x_2, \ldots, x_n), \qquad \frac{dx_i}{dx} = x_i' \qquad (i = 1, 2, \ldots, n)$$

enjoys properties that are analogous to those of a Lagrange system in which forces do not depend upon velocities.

First of all, one can reduce an arbitrary system (B) to the canonical form. It suffices to set:

$$p_i = \frac{\partial f}{\partial x_i'} \qquad (i = 1, 2, \ldots, n)$$

and

$$K = p_1 x_1' + p_2 x_2' + \cdots + p_n x_n' - f .$$

If one replaces the x_i as functions of the p_i in F then one will find that:

$$(a) \qquad x_i' = \frac{\partial K}{\partial p_i}, \qquad -\frac{\partial K}{\partial p_i} = \frac{\partial f}{\partial x_i}.$$

In order to see that, it will suffice to repeat the argument that was made for $f = T$ (page 160), which made no assumption about the form of T.

Equations (B) can then be replaced with the following ones:

$$(C) \qquad \frac{dp_i}{dx} = -\frac{\partial K}{\partial x_i} + X_i, \quad \frac{dx_i}{dx} = \frac{\partial K}{\partial p_i} \qquad (i = 1, 2, \ldots, n).$$

If the X_i are the partial derivatives of a function $U(x, x_1, x_2, \ldots, x_n)$ then one will again have, upon setting $H = K - U$:

$$(C') \qquad \frac{dp_i}{dx} = -\frac{\partial H}{\partial x_i}, \qquad \frac{dx_i}{dx} = \frac{\partial H}{\partial p_i} \qquad (i = 1, 2, \ldots, n).$$

Any first integral of (C), say $\varphi(x, x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n) = \text{const.}$, must verify the condition:

$$(b) \qquad \frac{\partial \varphi}{\partial x} + (\varphi, K) + \sum_{i=1}^{n} \frac{\partial \varphi}{\partial p_i} X_i \equiv 0$$

(see page 172), in which the symbol (φ, K) represents $\sum_{i=1}^{n}\left(\frac{\partial \varphi}{\partial x_i}\frac{\partial K}{\partial p_i} - \frac{\partial \varphi}{\partial p_i}\frac{\partial K}{\partial x_i}\right)$, as always.

In particular, if x does not enter into f, nor as a consequence ([17]) into K, then the independent first integrals of x are characterized by the condition:

$$\frac{\partial \varphi}{\partial x} + (\varphi, K) + \sum_{i=1}^{n} \frac{\partial \varphi}{\partial p_i} X_i = 0 .$$

When the X_i are derived from a function $U(x, x_1, x_2, \ldots, x_n)$, equation ($b$) can be written:

$$\frac{\partial \varphi}{\partial x} + (\varphi, H) = 0 .$$

([17]) Conversely, if K is independent of x, moreover, then [from (a)] the same thing will be true of $f = \sum p_i x_i' - K$.

If f and U do not depend upon x then the independent first integrals of x will simply verify the condition that:

$$(\varphi, H) = 0 .$$

In the latter case, $H \equiv K - U$ = const. is obviously an integral. That integral will reduce to K = const. if $X_1 \equiv X_2 \equiv \ldots \equiv X_n \equiv 0$.

Finally, equations (C) admit unity as a last multiplier (see pp. 238). It will then suffice to know $(2n - 1)$ first integrals of (C) if one is to achieve the integration by quadratures. If x does not appear in (C) then it will suffice to know $(2n - 2)$ independent integrals of x.

As for the original system (B), it is easy to see that it admits the Hessian Δ of f as a last multiplier. Indeed, admit that one knows $(2n - 1)$ first integrals of (C), namely:

$$\varphi_i\,(x, x_1, \ldots, x_n, p_1, p_2, \ldots, p_n) = c_j \qquad (j = 1, 2, \ldots, 2n - 1).$$

If one infers $p_1, p_2, \ldots, p_n, x, x_1, \ldots, x_{n-2}$ as functions of x_{n-1} and x_n, for example, then the expression:

$$\frac{1}{\delta}\left(\frac{\partial K}{\partial p_n}\,dx_{n-1} - \frac{\partial K}{\partial p_{n-1}}\,dx_n\right) \equiv \frac{1}{\delta}\,[x_n'\,dx_{n-1} - x_{n-1}'\,dx_n]$$

will be an exact differential, in which δ denotes the functional determinant:

$$\frac{D(\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_{2n-1})}{D(x, x_1, \ldots, x_{n-2}, p_1, p_2, \ldots, p_n)} .$$

However, if one supposes that the integrals are expressed with the aid of the x_i' then if one lets δ_1 denote the determinant:

$$\frac{D(\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_{2n-1})}{D(x, x_1, \ldots, x_{n-2}, x_1', x_2', \ldots, x_n')} ,$$

one will have:

$$\delta_1 = \frac{D(\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_{2n-1})}{D(x, x_1, \ldots, x_{n-2}, p_1, p_2, \ldots, p_n)} \times \frac{D(\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_{2n-1})}{D(x, x_1, \ldots, x_{n-2}, x_1', x_2', \ldots, x_n')} = \frac{D(p_1, p_2, \ldots, p_n)}{D(x_1', x_2', \ldots, x_n')} = \delta\Delta .$$

Hence, the expression:

$$\frac{\Delta}{\delta_1}\,(x_n'\,dx_{n-1} - x_{n-1}'\,dx_n)$$

will be an exact differential when one takes into account the $(2n - 1)$ integrals:

$$\varphi_j\,(x, x_1, \ldots, x_n, x_1', x_2', \ldots, x_n') = \text{const.}$$

of the system (3). The system then admits Δ as a last multiplier.

In particular, if x does not enter explicitly into either f or the X_i then Δ will be a multiplier of the system:

$$\frac{dx_1}{x_1'} = \frac{dx_2}{x_2'} = \dots = \frac{dx_n}{x_n'} = \frac{d \cdot \dfrac{\partial f}{\partial x_1'}}{\dfrac{\partial f}{\partial x_1} + X_1} = \dots = \frac{d \cdot \dfrac{\partial f}{\partial x_n'}}{\dfrac{\partial f}{\partial x_n} + X_n}.$$

Finally, the integration of the canonical system (C') can be reduced to the study of a complete integral of the partial differential equation:

$$\frac{\partial V}{\partial x} + H\left(x, x_1, x_2, \dots, x_n, \frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n} \right) = 0 .$$

As we said in Lecture Fifteen (pp. 222-226), that makes no assumption about the form of H.

Case in which f is homogeneous with respect to the x_i'.

In particular, suppose that f is homogeneous with respect to the x_i'. The degree of homogeneity μ (which can be arbitrary, real, or imaginary, moreover) must not be equal to zero or unity. Otherwise, the Euler identities would show that the Hessian Δ would be identically zero.

If one calculates the canonical function K in that case then one will have:

$$K = \sum_{i=1}^{n} x_i' \frac{\partial f}{\partial x_i'} - f = \mu f - f = (\mu - 1) f .$$

When μ is equal to 2, K will coincide with f on the condition that one must replace the x_i' with aid of the p_i in f.

Furthermore, if x does not enter into either f or the X_i then the system (B) will imply the consequence:

$$(\mu - 1) f = \int X_1\, dx_1 + X_2\, dx_2 + \dots + X_n\, dx_n ,$$

and thus, the integral:

$$(\mu - 1) f = U + h ,$$

when the X_i are the derivatives of a function $U(x_1, x_2, \dots, x_n)$.

We shall exclusively adopt the hypothesis that x does not enter into (B), so the relations between $x_1, x_2, \dots, x_n$ will then depend upon at most $(2n - 1)$ constants. If f is homogeneous then the number of constants will be precisely $2n - 1$, as long as none of the X_i are identically zero, in which case that number will drop down to $2n - 2$.

In order to see that, it will suffice to argue as on page 234. If one observes that the coefficients of the x_i'' in equations (B) are homogeneous and of degree $\mu - 2$ with respect to the x_i' and that the other first-order terms are homogeneous of degree μ then one will see that when equations (B) are solved for the x_i'', they will have the form:

$$(B') \qquad x_i'' = x_1'^2\,\Pi_i + x_1'^{2-\mu}\,\beta_i \qquad (i = 1, 2, \ldots, n),$$

in which the Π_i , β_i are homogeneous and of degree zero with respect to the x_i' . If one calculates $d^2 x_i / dx_1^2$ using (B) then one will find that:

$$(C) \qquad \frac{d^2 x_i}{dx_1^2} = \Pi_i \cdot \frac{dx_i}{dx_1}\Pi_1 + \frac{\beta_i - \frac{dx_i}{dx_1}\beta_1}{x_1'^{\mu}} \qquad (i = 1, 2, \ldots, n).$$

It follows from this, as it does for the equations of mechanics, that the relations between x_1, x_2, …, x_n will depend upon $2n - 1$ constants, unless the ratios:

$$\frac{\beta_i - \frac{dx_i}{dx_1}\beta_1}{x_1'^{\mu}}$$

are not independent of the x_1' , which is possible only if those ratios are identically zero, which will imply that:

$$\frac{\beta_1}{x_1'} \equiv \frac{\beta_2}{x_2'} \equiv \ldots \equiv \frac{\beta_n}{x_n'} \equiv \lambda ,$$

and therefore, one will have the following values for the X_i [see pp. 235, eq. (5)]:

$$X_i = \lambda \left[\frac{\partial^2 f}{\partial x_i' \partial x_1'} x_1' + \frac{\partial^2 f}{\partial x_i' \partial x_2'} x_2' + \cdots + \frac{\partial^2 f}{\partial x_i' \partial x_n'} x_n' \right] = (\mu - 1)\,\lambda \frac{\partial f}{\partial x_i'} \qquad (i = 1, 2, \ldots, n).$$

Those equalities demand that λ and the X_i must be identically zero. In other words, the derivatives $\partial f / \partial x_i'$, when considered to be functions of the x_i' , will differ by only a constant factor, and their functional determinant Δ will be identically zero.

By definition, when the functions X_i are not all zero, the relations between x_1, x_2, …, x_n will depend upon ($2n - 1$) constants. One can form the differential equations that define the relations in the same way as in the special case where f is a quadratic form T (see pp. 239). When those equations are integrated, x can be calculated as a function of x_1 by a simple quadrature by using the relation (C) that gives dx / dx_1 .

On the contrary, if all of the functions X_i are zero then the relations between $x_1, x_2, \ldots, x_n$ will depend upon $(2n - 2)$ constants, and they will be defined by the system:

$$(D) \qquad \frac{d^2 x_i}{dx_i^2} = \Pi_i - \frac{dx_i}{dx_1}\Pi_1 \qquad (i = 2, 3, \ldots, n)$$

x will then be determined (once that system is integrated) by the first integral $f = h$, which can be written:

$$\left(\frac{dx'}{dx_1}\right)^{\mu} = \frac{f\left(x_1, x_2, \ldots, x_n, 1, \frac{dx_2}{dx_1}, \ldots, \frac{dx_n}{dx_1}\right)}{h} .$$

Finally, one can give an explicit form to the system (D), which extends to the case in which there exist functions X_i that are derivatives of $U(x_1, x_2, \ldots, x_n)$. Indeed, replace the system (B) with the last $(n - 1)$ equations of that system:

$$(E) \qquad \frac{d}{dx}\frac{\partial f}{\partial x_i'} - \frac{\partial f}{\partial x_i} = \frac{\partial U}{\partial x_i}, \quad \frac{dx_i}{dx} = x_i' \qquad (i = 2, 3, \ldots, n),$$

combined with the first integral:

$$(\mu - 1) f - U = h ,$$

which can be written:

$$(F) \qquad dx = dx_1 \sqrt[\mu]{\frac{(\mu-1) f_1}{U+h}} ,$$

when one lets f_1 denote what f will become when one replaces x_1' with 1, x_2' with $dx_2 / dx_1 = x_{(2)}'$, …, and x_n' with $dx_n / dx_1 = x_{(n)}'$.

If one expresses dx as a function of dx_1 using (F) everywhere in equations (E) then since one has:

$$\frac{\partial f}{\partial x_i'} = x_1'^{\,\mu-1}\frac{\partial f_1}{\partial x_{(i)}'}, \qquad \frac{\partial f}{\partial x_i} = x_1'^{\,\mu}\frac{\partial f}{\partial x_i'},$$

they will become (see page 270):

$$\left[\frac{U+h}{(\mu-1) f_1}\right]^{1/\mu} \times \frac{d}{dx_1}\left[\left(\frac{U+h}{(\mu-1) f_1}\right)^{1-1/\mu}\frac{\partial f_1}{\partial x_{(i)}'}\right] - \frac{U+h}{(\mu-1) f_1} \times \frac{\partial f_1}{\partial x_i} = \frac{\partial U}{\partial x_i} \qquad (i = 2, 3, \ldots, n),$$

which can be written:

$$\frac{d}{dx_1}\left[\left(\frac{U+h}{f_1}\right)^{1-1/\mu}\frac{\partial f_1}{\partial x'_{(i)}}\right]-\left(\frac{U+h}{f_1}\right)^{1-1/\mu}\frac{\partial f_1}{\partial x_i} = (\mu-1)\left(\frac{f_1}{U+h}\right)^{1/\mu}\frac{\partial U}{\partial x_i},$$

or rather, upon setting $P = (U+h)^{(1-1/\mu)}\, f_1^{1/\mu}$:

$$(G) \qquad \frac{d}{dx_1}\frac{\partial P}{\partial x'_{(i)}}-\frac{\partial P}{\partial x'_i} = 0 \qquad (i = 2, 3, \ldots, n).$$

In particular, if $X_1 \equiv X_2 \equiv \ldots \equiv X_n \equiv 0$ then equations (S), which define the relations between $x_1, x_2, \ldots, x_n$, can be put into the form (G), in which $P \equiv f_1^{1/\mu}$. Let us say *the geodesics of* $ds^\mu \equiv f(x_1, \ldots, x_n, dx_1, \ldots, dx_n)$ to mean those relations between the x_i that depend upon $(2n-2)$ constants. On the other hand, let us say *the trajectories of a system* (B) that is defined by $\{f \equiv ds^\mu / dx^\mu, U\}$ to mean the relations between the $x_1, x_2, \ldots, x_n$ that imply that system. For each value h of the constant in the integral $(\mu-1)f - U = h$, the trajectories will define a $(2n-2)$-parameter congruence that will coincide with the geodesics of $(U+h)^{(\mu-1)}ds^\mu$. That is a generalization of the principle of least action.

The trajectories will not be changed if one replaces f with $(a\,U + b)^{\mu-1} f$ and U with $\dfrac{\gamma U+\delta}{\alpha U+\beta}$ in a system (B) that is defined by $\{f, U\}$, in which f is homogeneous and of degree μ with respect to the x'_i.

One sees that the most important properties of the equations of mechanics extend to the more general equations (B), in which f is an arbitrary homogeneous function of the x'_i, rather than a quadratic form T.

LECTURE 17

PROPERTIES OF FIRST INTEGRALS WHEN THE FORCES ARE DERIVED FROM A POTENTIAL. POISSON PARENTHESES. RETURN TO THE JACOBI EQUATION.

Previously, we established some properties of the first integrals of dynamics. When there exists a force function, we can complete those properties with a remarkable proposition that is due to Poisson and whose importance was shown by Jacobi.

In the case that we shall address, the motion is defined by the canonical system:

$$(1)\qquad \begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H}{\partial p_i}, \\[2ex] \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial q_i}, \end{cases}$$

in which H is a function of the p_i, q_i, and also t when the constraints or forces depend upon time.

In order for the equality:

$$f(t, q_1, q_2, \ldots, q_k, p_1, p_2, \ldots, p_k) = \alpha$$

to be a first integral of the system (1), it is necessary and sufficient that f should be an integral of the first-order differential equation:

$$(2)\qquad \frac{\partial f}{\partial t} + (f, H) = 0 ,$$

in which the parentheses (f, H) represent the sum:

$$\sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial q_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right) ,$$

with Poisson's notation.

The integration of the system (1) is equivalent to the search for $2k$ distinct particular integrals to equation (2).

Jacobi's theory of the last multiplier teaches us that it will suffice to know $(2k - 1)$ integrals of (2) in order for the last one to be obtained by quadrature.

In the case where t does not enter into H, the first integrals that are independent of t will verify the equation:

(3) $$(f, H) = 0 .$$

It will then suffice to know $(2k - 3)$ integrals of the equation (3) that are distinct from the integral $H = h$ in order for the integration of (1) to be achieved by quadratures.

The theorem of Poisson that we shall address is then stated:

If f_1 *and* f_2 *are two first integrals of the system* (1) *then the expression* (f_1, f_2) *will again be an integral of* (1).

However, it is appropriate to observe immediately that the integral (f_1, f_2) cannot be distinct from f_1 and f_2, and in particular, it can reduce to an absolute constant.

In order to prove the theorem, we shall begin by establishing some properties of the Poisson parentheses.

First of all, the following identities will result from definition itself of the symbol (f, φ):

$$(f, \varphi) \equiv -(\varphi, f) ,$$
$$(f, f) \equiv 0 ,$$
$$(f, c) \equiv 0 , \text{ where } c \text{ denotes a constant.}$$

In the second place, let $f = F(x_1, x_2, \ldots, x_n)$, $\varphi = \Phi(x_1, x_2, \ldots, x_n)$, in which the x_i represent arbitrary functions of $q_1, \ldots, q_k, p_1, \ldots, p_k$. One has identically:

(4) $$(f, \varphi) \equiv \sum_{i,j} (x_i, x_j) \left[\frac{\partial F}{\partial x_i} \frac{\partial \varphi}{\partial x_j} - \frac{\partial F}{\partial x_j} \frac{\partial \varphi}{\partial x_i} \right] .$$

[The number of terms on the right-hand side is $n (n - 1) / 2$.]

In order to verify the identity, it will suffice to express the derivatives of f and φ with respect to the p, q in (f, φ) as functions of the derivatives of F, Φ with respect to the x_i and the derivatives of the x_i with respect to the p, q and then look for the coefficient of the product $\frac{\partial x_i}{\partial p_r} \frac{\partial x_j}{\partial q_r}$. That coefficient will coincide with its analogue in the right-hand side.

In particular, let $f = F(x_1, x_2, \ldots, x_{n-1})$, $\varphi = x_n$. One will have:

$$(f, \varphi) = (x_1, \varphi) \frac{\partial F}{\partial x_1} + (x_2, \varphi) \frac{\partial F}{\partial x_2} + \cdots + (x_{n-1}, \varphi) \frac{\partial F}{\partial x_{n-1}} .$$

Notably, if $f = u + v$ then one will get:

$$(u + v, \varphi) = (u, \varphi) + (v, \varphi) .$$

If $f = u\, v$ then one will get:

$$(u\, v, \varphi) = v\, (u, \varphi) + u\, (v, \varphi) .$$

However, a lemma that is much more important than that one is:

Lemma:

Let A, B, C be three functions of $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_k$ (which can contain t) ([18]). *If one sets:*

$$(B, C) = A', \quad (C, A) = B', \quad (A, B) = C'$$

then one will have:

$$(A, A') + (B, B') + (C, C') = 0$$

identically.

One must then verify that the sum:

$$S = (A, (B,C)) + (B, (C,A)) + (C, (A,B))$$

is identically zero.

(B, C) is an expression that is linear and homogeneous with respect to the first derivatives of B and C. As a result, $(A, (B, C))$ will be an expression that is linear and homogeneous with respect to the second derivatives of B and C. The sum S will then be a linear, homogeneous expression in the second derivatives of A, B, C, and everything will come down to showing that the coefficient of any of those derivatives is identically zero.

Now, one has:

$$A' = (B, C) = \sum_{i=1}^{k} \left(\frac{\partial B}{\partial q_i} \frac{\partial C}{\partial p_i} - \frac{\partial B}{\partial p_i} \frac{\partial C}{\partial q_i} \right),$$

$$(A, A') = \sum_{j=1}^{k} \left[\frac{\partial A}{\partial q_j} \sum_{i=1}^{k} \left(\frac{\partial B}{\partial q_i} \frac{\partial^2 C}{\partial p_i \, \partial p_j} - \frac{\partial B}{\partial p_i} \frac{\partial^2 C}{\partial q_i \, \partial p_j} \right) - \frac{\partial A}{\partial p_j} \sum_{i=1}^{k} \left(\frac{\partial B}{\partial q_i} \frac{\partial^2 C}{\partial p_i \, \partial q_j} - \frac{\partial B}{\partial p_i} \frac{\partial^2 C}{\partial q_i \, \partial q_j} \right) \right] + \cdots$$

The ellipses … represent the terms that do not contain the second derivatives of C.

The coefficient of $\dfrac{\partial^2 C}{\partial p_i \, \partial p_j}$ $(i, j \leq k)$ in (A, A') is:

$$\left(\frac{\partial A}{\partial q_j} \frac{\partial B}{\partial q_i} + \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial q_j} \right).$$

However, $(B, B') \equiv (B, (C, A)) \equiv -(B, (A, C))$ also contains a term in $\dfrac{\partial^2 C}{\partial p_j \, \partial p_i}$, and since $(B, (A, C))$ differs from (A, A') by only the permutation of A and B, the coefficient of that term will

([18]) It is clear that it hardly matters in all of this whether t does or does not figure in the expressions considered.

be the previous one (up to sign), or one will have to change B into A and A into B. Since that coefficient will not change under that permutation, the two terms in $\dfrac{\partial^2 C}{\partial p_i\,\partial p_j}$ will cancel in S.

Similarly, (A, A') will provide terms:

$$-\frac{\partial^2 C}{\partial p_j\,\partial q_i}\left(\frac{\partial A}{\partial q_j}\frac{\partial B}{\partial p_i}+\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_j}\right)+\frac{\partial^2 C}{\partial q_j\,\partial q_i}\left(\frac{\partial A}{\partial p_j}\frac{\partial B}{\partial p_i}+\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial p_j}\right)$$

that do not change under the permutation of A and B, and which will, as a result, cancel with the corresponding terms in (B, B') or $(B, (A, C))$.

It results from this that the sum S is identically zero.

Poisson's theorem:

Let:

$$f_1\,(t, q_1, \ldots, q_k, p_1, \ldots, p_k) = \alpha_1\,, \qquad f_2\,(t, q_1, \ldots, q_k, p_1, \ldots, p_k) = \alpha_2$$

be two first integrals of the canonical system. The expression:

$$(f_1, f_2) \equiv \sum_{i=1}^{k}\left(\frac{\partial f_1}{\partial q_i}\frac{\partial f_2}{\partial p_i}-\frac{\partial f_1}{\partial p_i}\frac{\partial f_2}{\partial q_j}\right)$$

is also an integral of that system.

If time t does not enter into either H or f_1, f_2 then the proof will be immediate. By hypothesis, one will have:

$$(f_1, H) = 0\,, \qquad (f_2, H) = 0\,.$$

Thus, one also has:

$$(f_1, (f_2, H)) = 0\,, \qquad (f_2, (f_1, H)) = 0\,.$$

However, when the preceding lemma is applied to the function f_1, f_2, H, that will give:

$$(H, (f_1, f_2)) + (f_1, (f_2, H)) + (f_2, (f_1, H)) = 0\,,$$

and as a result:

$$(H, (f_1, f_2)) = 0\,,$$

and the equality:

$$(f_1, f_2) = \alpha$$

will give an integral of (1) that is independent of t.

Now suppose that t enters into H, f_1, f_2. By hypothesis, one has:

$$\frac{\partial f_1}{\partial t} + (H, f_1) = 0\,, \qquad \frac{\partial f_1}{\partial t} + (H, f_1) = 0\,,$$

and everything comes down to showing that one also has:

$$\frac{\partial (f_1, f_2)}{\partial t} + (H, (f_1, f_2)) = 0\,. \tag{5}$$

Now, from the lemma, one can write:

$$(H, (f_1, f_2)) + (f_1, (f_2, H)) + (f_2, (f_1, H)) = 0\,,$$

or rather:

$$(H, (f_1, f_2)) + \left(f_1, \frac{\partial f_2}{\partial t}\right) - \left(f_2, \frac{\partial f_1}{\partial t}\right) = 0\,. \tag{6}$$

However, from the definition of (f_1, f_2), one concludes that:

$$\frac{\partial (f_1, f_2)}{\partial t} = \sum_{i=1}^{k}\left[\frac{\partial f_2}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial f_1}{\partial t}\right) - \frac{\partial f_2}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial f_1}{\partial t}\right)\right] + \sum_{i=1}^{k}\left[\frac{\partial f_1}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial f_2}{\partial t}\right) - \frac{\partial f_1}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial f_2}{\partial t}\right)\right],$$

i.e.:

$$\frac{\partial (f_1, f_2)}{\partial t} = \left(\frac{\partial f_1}{\partial t}, f_2\right) + \left(f_1, \frac{\partial f_2}{\partial t}\right).$$

Therefore, if one takes (6) into account then the equality (5) to be proved will have been established.

Remark. – From that theorem, it would seem that it would suffice to know two first integrals, f_1 and f_2, in general, for one to succeed in integrating the problem algebraically. Indeed, one can form the integral $f_3 = (f_1, f_2)$ from f_1 and f_2, and then the integrals $f_4 = (f_1, f_3)$, $f_5 = (f_2, f_3)$ from f_1, f_2, f_3, and so on, until one has obtained $2k$ integrals. In reality, those integrals are not distinct in most applications, and upon repeating the Poisson process, one will come to integrals that were used already or that one can recognize immediately.

In order for us to account for that, let us see what will happen if we apply Poisson's theorem to the integrals that are provided by the theorem of the motion of the center of gravity and the theorem of areas in the case of a system of n free points that are subject to only internal forces.

Let M_i or (x_i, y_i, z_i) be one of those points, and let m_i be its mass:

$$T = \sum_{i=1}^{n} \tfrac{1}{2} m_i (x_i'^2 + y_i'^2 + z_i'^2)\,.$$

Take the canonical variables to be:

$$p_i = m_i\, x_i', \qquad q_i = m_i\, y_i', \qquad r_i = m_i\, z_i' \qquad (i = 1, 2, \dots, n),$$

so:

$$H = \tfrac{1}{2}\sum_{i=1}^{n} \frac{1}{m_i}(p_i^2 + q_i^2 + r_i^2) - U(\dots, x_i, y_i, z_i, \dots) .$$

The theorem of the motion of the center of gravity gives the integrals:

$$\sum p_i = \alpha , \qquad \sum q_i = \beta , \qquad \sum r_i = \gamma .$$

The theorem of areas gives the integrals:

$$\begin{aligned} \sum (y_i\, r_i - z_i\, q_i) &= A , \\ \sum (z_i\, p_i - x_i\, r_i) &= B , \\ \sum (x_i\, q_i - y_i\, p_i) &= C . \end{aligned}$$

Those integrals will be valid whenever the force function satisfies the conditions:

$$\begin{aligned} &\sum \frac{\partial U}{\partial x_i} = 0 , \quad \sum \left(y_i \frac{\partial U}{\partial z_i} - z_i \frac{\partial U}{\partial y_i} \right) = 0 , \\ &\sum \frac{\partial U}{\partial y_i} = 0 , \quad \sum \left(z_i \frac{\partial U}{\partial x_i} - x_i \frac{\partial U}{\partial z_i} \right) = 0 , \\ &\sum \frac{\partial U}{\partial z_i} = 0 , \quad \sum \left(x_i \frac{\partial U}{\partial y_i} - y_i \frac{\partial U}{\partial x_i} \right) = 0 . \end{aligned}$$

Does Poisson's theorem allow us to deduce any new integrals from those six integrals?
More generally, one has (f_1 = const. and f_2 = const. denote two of those six integrals):

$$(f_1, f_2) = \sum_{i=1}^{n} \left(\frac{\partial f_1}{\partial x_i}\frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i}\frac{\partial f_2}{\partial x_i} \right) + \left(\frac{\partial f_1}{\partial y_i}\frac{\partial f_2}{\partial q_i} - \frac{\partial f_1}{\partial q_i}\frac{\partial f_2}{\partial y_i} \right) + \left(\frac{\partial f_1}{\partial z_i}\frac{\partial f_2}{\partial r_i} - \frac{\partial f_1}{\partial r_i}\frac{\partial f_2}{\partial z_i} \right)$$

here.

1. Make the associations:

$$f_1 = \sum p_i , \qquad f_2 = \sum q_i .$$

That will obviously give:

$$(f_1, f_2) \equiv 0 .$$

Poisson's theorem then leads to an identity.

2. Make the associations:

$$f_1 = \sum p_i\,, \qquad f_2 = \sum (y_i\, r_i - z_i\, q_i)\,.$$

That will again give:

$$(f_1, f_2) \equiv 0\,,$$

i.e., an identity.

3. Make the associations:

$$f_1 = \sum p_i\,, \qquad f_2 = \sum (z_i\, p_i - x_i\, r_i)\,.$$

That will give:

$$(f_1, f_2) = -\sum r_i\,.$$

Poisson's theorem then gives back the third integral of the motion of the center of gravity.

4. Make the associations:

$$f_1 = \sum (y_i\, r_i - z_i\, q_i)\,, \quad f_2 = \sum (z_i\, p_i - x_i\, r_i)\,.$$

That will give:

$$(f_1, f_2) = \sum (q_i\, x_i - p_i\, y_i)\,,$$

which gives back the third integral of area.

Thus, an arbitrary combination of two of the six known integrals will lead us to only one of those integrals or to an identity.

One can pose this question: Upon forming two combinations:

$$F_1\,(f_1, f_2, \ldots, f_6) = \alpha_1\,, \qquad F_2\,(f_1, f_2, \ldots, f_6) = \alpha_2\,,$$

will one obtain a new integral: $(F_1, F_2) = \text{const.}$?

The answer is no: One will arrive at a combination of known integrals. That is a general property.

Let $f_1 = \alpha_1$, $f_2 = \alpha_2$, …, $f_r = \alpha_r$ be r first integrals such that any combination (f_i, f_j) is a function of $f_1, f_2, \ldots, f_r$ $(i, j \leq r)$. The combination (F_1, F_2), where F_1 and F_2 are two arbitrary functions of $f_1, f_2, \ldots, f_r$, will also be a function of $f_1, f_2, \ldots, f_r$. (According to Lie, one then says that the r integrals $f_1, \ldots, f_r$ form a *group*.)

That results immediately from the equality (4) on page 283:

$$(F_1, F_2) \equiv \sum_{i,j} (f_i, f_j) \left[\frac{\partial F_1}{\partial f_i} \frac{\partial F_2}{\partial f_j} - \frac{\partial F_1}{\partial f_j} \frac{\partial F_2}{\partial f_i} \right]. \tag{4}$$

Path to follow in order to apply Poisson's theorem. – Suppose that one knows r distinct first integrals $f_1 = \alpha_1$, $f_2 = \alpha_2$, …, $f_r = \alpha_r$. One then forms all combinations $\varphi = (f_i, f_j)$. If φ is a function of $f_1, f_2, \ldots, f_r$ (for any $i, j \leq r$) then Poisson's theorem will give nothing when φ is identically zero or constant, in particular. Otherwise, among the $r(r-1)/2$ combinations φ, there will exist s of them, say, $f_{r+1}, \ldots, f_{r+s}$ [s can be equal to $r(r-1)/2$], such that the $r + s$ integrals $f_1, f_2, \ldots, f_r, f_{r+1}, \ldots, f_{r+s}$ are distinct and all of the other combinations φ are functions of $f_1, \ldots, f_{r+s}$. One then proceeds with the new system of integrals $f_1, \ldots, f_{r+s}$ as one did with the first one, and so on, until one arrives at a system of distinct integrals $f_1, f_2, \ldots, f_l$ ($l > r$) such that all of the combinations (f_i, f_j) are functions of $f_1, f_2, \ldots, f_l$. That will necessarily be true after a finite number of such operations, because each operation will increase the number of integrals f_i, and that number cannot exceed $2k$. One then forms, by definition, a system of distinct integrals $f_1, f_2, \ldots, f_l$ ($l \leq 2k$) such that no integral (f_i, f_j) is distinct from the first ones, and as a result, they will form a group.

One has thus exhausted all of the consequences of Poisson's theorem, because the combinations φ that were overlooked along the way are all functions of $f_1, f_2, \ldots, f_l$, and as a result, the combinations (f_i, φ), as well, from the remark that was made above. If $l = 2k - 1$ then the problem will be solved by quadratures.

For example, assume that one knows two integrals f_1, f_2. One then forms the combination $\varphi = (f_1, f_2)$. Two cases are possible according to whether φ is or is not a function of f_1, f_2. In the former case, Poisson's theorem will give nothing new ([19]), while in the latter case, φ will be a new integral f_3. One forms (f_1, f_3) and (f_2, f_3). If one supposes that those two combinations are functions of f_1, f_2 then Poisson's theorem will not permit one to add any integral to f_1, f_2, f_3 ([20]).

In the case where H is independent of t, $H = h$ will be one integral. Let $f_1 = \alpha_1$ be a second integral that is independent of t, so one will have $(f_1, H) = 0$ identically. There is then good reason to apply Poisson's theorem only if one knows at least two integrals f_1, f_2 (that are independent of time) that are distinct from that of *vis viva*. Poisson's theorem cannot provide more than $(2k - 2)$ such integrals (that are distinct from the *vis viva* integral). If it provides $2k - 3$ of them then the integration can be achieved by quadratures.

If f_1 is an integral that depends upon t then the equality:

$$\frac{\partial f_1}{\partial t} + (f_1, H) = 0$$

will show that $\partial f_1 / \partial t$ will again be an integral, and as a result, so will all of the derivatives of f_1 with respect to t.

In the applications, Poisson's theorem will rarely give results that one might not have expected from the outset. However, that theorem still plays an important role, as we shall see, in the

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([19]) That is what happens when one associates two of the integrals that are provided by the theorem of the motion of the center of gravity (see pp. 288), say, $f_1 = \sum p_i$ and $f_2 = \sum q_i$, or also the integral $f_1 = \sum p_i$ and the area integral $f_2 = \sum (y_i r_i - z_i q_i)$. One will then find that $(f_1, f_2) \equiv 0$.

([20]) This case presents itself when one associates (see pp. 288) the two integrals $f_1 = \sum p_i$ and $f_2 = \sum (z_i p_i - x_i r_i)$. One finds that $(f_1, f_2) = -\sum r_i$, so one has a new integral $f_3 = r_i$. However, (f_1, f_3) and (f_2, f_3) are zero. Similarly, if one associates the two integrals $f_1 = \sum (y_i r_i - z_i q_i)$, $f_2 = \sum (z_i p_i - x_i r_i)$ then one will find that $(f_1, f_2) = \sum (q_i x_i - r_i y_i)$, which defines a third integral f_3, but $(f_1, f_3) \equiv -f_2$, $(f_2, f_3) \equiv f_1$.

integration of first-order partial differential equations, which is a problem that one will come to in the integration of a canonical system, as one knows.

Return to Jacobi's method.

We said that the canonical system ([21]):

$$(1)\qquad \begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H}{\partial p_i}, \\ \dfrac{dq_i}{dt} = -\dfrac{\partial H}{\partial q_i} \end{cases} \qquad (i = 1, 2, \ldots, k)$$

is integrated when one knows a complete integral of the partial differential equation:

$$(A)\qquad \frac{\partial V}{\partial t} + H\left(t, q_1, q_2, \ldots, q_k, \frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \ldots, \frac{\partial V}{\partial q_k}\right) = 0 .$$

If $V(t, q_1, q_2, \ldots, q_k, p_1, p_2, \ldots, p_k, a_1, a_2, \ldots, a_k)$ denotes that complete integral then it will suffice to set:

$$(B)\qquad \frac{\partial V}{\partial a_i} = b_i , \qquad p_i = \frac{\partial V}{\partial q_i} \qquad (i = 1, 2, \ldots, k),$$

in which the a_i, b_i denote constants, in order for the motion to be well-defined.

We have indicated (pp. 229-231) some important cases in which the substitution of the Jacobi equation for the canonical system is advantageous. However, more generally, is there any benefit to replacing the integration of the canonical system (1) with the search for a complete integral of equation (*A*)?

Theoretically, the two problems are equivalent. Indeed, once the system (1) has been integrated, it is easy to form a complete integral of (*A*). Let $p_i = a_i$ and $q_i = b_i$ for $t = t_0$ and let:

$$(C)\qquad p_i = \varphi_i(q_1, \ldots, q_k, a_1, \ldots, a_k, b_1, \ldots, b_k) , \qquad q_i = \psi_i(q_1, \ldots, q_k, a_1, \ldots, a_k, b_1, \ldots, b_k)$$

$$(i = 1, 2, \ldots, k)$$

([21]) That case presents itself when one associates (see pp. 288) the two integrals $f_1 = \sum p_i$, $f_2 = \sum(z_i p_i - x_i r_i)$. One finds that $(f_1, f_2) = -\sum r_i$, which is a new integral $f_3 = r_i$. However, (f_1, f_3) and (f_2, f_3) are zero. Similarly, if one associates the two integrals $f_1 = \sum(y_i r_i - z_i q_i)$, $f_2 = \sum(z_i p_i - x_i r_i)$ then one will find that $(f_1, f_2) = \sum(q_i x_i - r_i y_i)$, which will define a third integral f_3, but $(f_1, f_3) \equiv -f_2$, $(f_2, f_3) \equiv f_1$.

be the general integral (1). Let $\chi(t)$ be the function of t that is defined by the expression ([22]):

$$p_1 \frac{\partial H}{\partial p_1} + p_2 \frac{\partial H}{\partial p_2} + \cdots + p_k \frac{\partial H}{\partial p_k} - H$$

when one replaces the p_i, q_i as functions of t using (C).

If one sets $u(t) = \int_{t_0}^{t} \chi(t)\, dt$ and:

$$v = a_1\, b_1 + \ldots + a_k\, b_k + u\,(t)$$

then the function v will be a function of t and the constants $a_1, \ldots, a_k, b_1, \ldots, b_k$. Infer the $b_1, \ldots, b_k$ as functions of $t, q_1, \ldots, q_k, a_1, \ldots, a_k$ using equations (C) and substitute those values in v. The function $V(t, q_1, q_2, \ldots, q_k, a_1, a_2, \ldots, a_k)$ thus-obtained is a complete integral of (A). I shall confine myself to merely stating that theorem, whose proof is quite simple, but it would not be useful for our purposes to have it.

In reality, the question that one should pose is the following one: "Is there any advantage to studying the Jacobi equation directly, rather than the canonical system?" In order for us to account for that, we shall compare the various methods with the aid of which one can integrate the system (1), on the one hand, and the system (2), on the other.

When one starts from the canonical system, it will be necessary to integrate the ordinary differential equations (1) directly, or what amounts to the same thing, to find $2k$ distinct integrals f of the linear homogeneous partial differential equation:

$$\frac{\partial f}{\partial t} + (f, H) = 0 \,. \tag{2}$$

We do not need to return to the study of those integrals. We know that, in the general case, all of the difficulty involved with that comes down to forming $(2k - 1)$ integrals of (2), and in the case where t does not enter into H, one forms $(2k - 3)$ integrals other than H of the equation $(f, H) = 0$.

How can one determine a complete integral when one starts from the Jacobi equation? From a previous remark, a first method consists of integrating the corresponding canonical system (1) ([23]). In the case that we are concerned with, where V does not enter explicitly into the equation, Cauchy's method of characteristics will lead to the same system of differential equations (1). If one wishes to integrate (A) by one or the other of those two methods then there will then be no reason for replacing the equation (A) with the system (1). However, Jacobi has indicated a method for integrating (A) that is completely different, but has remained the same with several reprises,

([22]) That expression coincides with $T + U$ in the case of mechanics.

([23]) That is Jacobi's first method for integrating an arbitrary equation (A). Indeed, one knows that once one knows a complete integral of (A), it is easy to deduce a general integral from that. On the other hand, if one is given a first-order equation (A) into which V enters explicitly then one can always reduce it to an analogous equation into which V no longer enters, but in which the number of variables is increased by one unit. It suffices to replace the search for functions V with the search for functions $F(t, q_1, q_2, \ldots, q_k, V)$ that are equal to a constant that defines an integral V of the first equation.

and which Mayer and S. Lie have brought to a state of perfection. We shall briefly discuss that method, while referring to the ample presentation in the well-known book by Goursat on first-order partial differential equations.

The basic principle of the method is the following:

Let $V(t, q_1, q_2, \dots, q_k, a_1, a_2, \dots, a_k)$ be a complete integral of equation (A). The equalities:

$$p_i = \frac{\partial V}{\partial q_i} \qquad (i = 1, 2, \dots, k)$$

can always be solved for k constants (from the definition of the complete integral). If one performs that solution then one will form k first integrals of the system (1).

$$(D) \qquad f_i(t, q_1, \dots, q_k, p_1, \dots, p_k) = a_i \qquad (i = 1, 2, \dots, k),$$

and equations (D) are verified identically when one replaces the p_i with the derivatives $\partial V / \partial q_i$ of the complete integral. Since there always exists an infinitude of complete integrals of (A), one will see that there is always an infinitude of ways of forming k first integrals $f_i = a_i$ of the system (1), which can be solved for the p_i, and in such a way that the expression:

$$(E) \qquad -H\,dt + p_1\,dq_1 + \dots + p_k\,dq_k$$

is an exact total differential dV when one replaces the p_i as functions of $t, q_1, \dots, q_k, a_1, \dots, a_k$. It is clear that once those k integrals f_i have been formed, the function $V(t, q_1, \dots, q_k, a_1, \dots, a_k)$ that is given by the integration of that total differential will be a complete integral of (A), and that as a result, the system (1) will be integrated.

Jacobi's new method will then replace the search for $2k$ distinct integrals f of equation (2) with the search for only k integrals, but those k integrals must be such that the expression (E) is an exact total differential. We shall indicate the path to follow in order to effectively determine one such system of k integrals, and we verify that this determination is always possible by virtue of that fact itself, which we knew beforehand.

When t does not enter into H, one must determine $(k - 1)$ integrals $f_i = a_i$ that are independent of t, and when they are combined with the integral $H = h$ and solved for p_i, that will make the expression $p_1\,dq_1 + p_2\,dq_2 + \dots + p_k\,dq_k$ into an exact differential dW. The function $-h\,t + W(q_1, \dots, q_k, a_1, \dots, a_{k-1}, h)$ is then a complete integral of (A).

In the latter case, when one starts from the canonical system, in order to arrive at quadrature, it will be necessary to determine $(2k - 3)$ integrals f that are distinct from each other and from H, but those $(2k - 3)$ integrals are arbitrary.

There is one case in which the two methods are found to coincide, which is the one in which t does not enter into H, and the number of parameters k is equal to 2. One will then have: $2k - 3 = k - 1 = 1$. On the other hand, we said (see pp. 208-211 and pp. 226-227) that any first integral $f(q_1, q_2, p_1, p_2) = a$, when combined with $H = h$, will make $(p_1\,dq_1 + p_2\,dq_2)$ into an exact differential. The Jacobi method, like that of the last multiplier, will therefore require only quadratures when one knows an arbitrary integral f of $(f, H) = 0$. We have seen that the calculations to which those two methods lead are the same.

However, when k exceeds 2, or when t enters into H, the same thing will no longer be true. In order to present a general form for the Jacobi method, we shall not make t play a special role, and we shall consider an arbitrary equation (A):

$$F\left(q_1, q_2, \ldots, q_k, \frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \ldots, \frac{\partial V}{\partial q_k}\right) = 0\,,$$

in which the number of variables q_i is arbitrary and V does not enter explicitly.

Presentation of the method of Jacobi and Mayer.

The theorem upon which the method is based is this:

Let k relations ([24]):

$$(\alpha) \qquad f_i\,(q_1, q_2, \ldots, q_k, p_1, p_2, \ldots, p_k) = 0 \qquad (i = 1, 2, \ldots, k)$$

be soluble for $p_1, p_2, \ldots, p_k$ *and such that the functional determinant* $\frac{D(f_1, f_2, \ldots, f_k)}{D(p_1, p_2, \ldots, p_k)}$ *is not annulled identically when one replaces the* p_i *with their values* $\varphi_i\,(q_1, \ldots, q_k)$ *that one infers from* (α). *In order for the expression* $p_1\,dq_1 + p_2\,dq_2 + \ldots, p_k\,dq_k = \sum \varphi_i\,dq_i$ *to be an exact differential, it is necessary and sufficient that all of the equalities:*

$$(f_i, f_j) = 0 \qquad (i, j \le k)$$

(in which the p_i, q_i *are independent variables) must be consequences of the system* (α).

The theorem supposes only that the functional determinant $\frac{D(f_1, f_2, \ldots, f_k)}{D(p_1, p_2, \ldots, p_k)}$ is not annulled identically when one replaces the p_i as functions of the q_i using (α).

In order for the expression $\sum p_i\,dq_i$ to be an exact total differential, it is necessary and sufficient that the functions $p_i = \varphi_i\,(q_1, \ldots, q_k)$ satisfy the $k\,(k-1) / 2$ conditions:

([24]) These relations can depend upon constants.

$$\frac{\partial p_i}{\partial q_j} \equiv \frac{\partial p_j}{\partial q_i} \qquad\qquad (i, j = 1, 2, \ldots, k).$$

We shall show that those conditions are equivalent to the stated condition by appealing to the following lemma:

Lemma:

If two simultaneous partial differential equations:

$$(\beta) \qquad f_1(q_1, q_2, \ldots, q_k, p_1, p_2, \ldots, p_k) = 0 , \quad f_2(q_1, q_2, \ldots, q_k, p_1, p_2, \ldots, p_k) = 0 ,$$

in which $p_i = \partial V / \partial q_i$ *, admit a common integral* $V(q_1, q_2, \ldots, q_k)$ *then that integral will also verify the equation:*

$$(f_1, f_2) = 0 .$$

Indeed, let $p_1, \ldots, p_k$ be arbitrary functions of $q_1, \ldots, q_k$ that verify equations (β). One has:

$$\left.\begin{aligned} \frac{\partial f_1}{\partial q_i} + \sum_{j=1}^{k} \frac{\partial f_1}{\partial p_j}\frac{\partial p_j}{\partial q_i} = 0, \\ \frac{\partial f_2}{\partial q_i} + \sum_{j=1}^{k} \frac{\partial f_2}{\partial p_j}\frac{\partial p_j}{\partial q_i} = 0, \end{aligned}\right\} \qquad (i = 1, 2, \ldots, k) .$$

Multiply the first one by $\partial f_2 / \partial p_i$, the second of them by $\partial f_1 / \partial p_i$, and add corresponding sides. In the result, one then gives the values 1, 2, …, k to i in succession and takes the sum of the equations thus-obtained:

$$(f_1, f_2) + \sum_{j=1}^{k}\sum_{i=1}^{k}\left(\frac{\partial f_2}{\partial p_i}\frac{\partial f_1}{\partial p_j} - \frac{\partial f_2}{\partial p_j}\frac{\partial f_1}{\partial p_i}\right)\frac{\partial p_j}{\partial q_i} = 0 ,$$

which can be further written:

$$(f_1, f_2) + \sum_{j=1}^{k}\sum_{i=1}^{k}\frac{\partial f_2}{\partial p_i}\frac{\partial f_1}{\partial p_j}\left(\frac{\partial p_j}{\partial q_i} - \frac{\partial p_i}{\partial q_j}\right) = 0 .$$

If $p_1, \ldots, p_k$ are the derivatives $\partial V / \partial q_1, \ldots, \partial V / \partial q_k$ of a function V then the latter equality will become:

$$(f_1, f_2) = 0 ,$$

which proves the lemma in question.

An immediate consequence of that lemma is that any integral $V(q_1, \ldots, q_k)$ that is common to equations (α) will also verify the equations:

$$(f_i, f_j) = 0 \qquad\qquad (i, j = 1, 2, \ldots, k).$$

If one then infers the p_i as functions of the q_i from (α) then since one has $p_i = \partial V / \partial q_i$, the conditions $(f_i, f_j) = 0$ must be verified identically when one replaces the p_i with those values. In other words, the equalities $(f_i, f_j) = 0$ must be consequences of the system (α).

The stated conditions then appear to be necessary. It remains for us to prove that they are sufficient.

In order to do that, recall that the system of functions $p_i = \varphi_i(q_1, \ldots, q_k)$ that is defined by an arbitrary system (α) verifies the equations:

$$(\gamma) \qquad (f_r, f_s) + \sum_{j=1}^{k}\sum_{i=1}^{k} \frac{\partial f_r}{\partial p_i}\frac{\partial f_s}{\partial p_j}\left(\frac{\partial p_j}{\partial q_i} - \frac{\partial p_i}{\partial q_j}\right) = 0 .$$

Replace the variables $p_1, \ldots, p_k$ with $\varphi_1, \ldots, \varphi_k$ in (γ). By hypothesis, the expressions (f_r, f_s) are then annulled identically and equations (γ) will then become:

$$\sum_{j=1}^{k}\sum_{i=1}^{k} \frac{\partial f_r}{\partial p_i}\frac{\partial f_s}{\partial p_j}\left(\frac{\partial p_j}{\partial q_i} - \frac{\partial p_i}{\partial q_j}\right) \equiv \sum_{j=1}^{k}\frac{\partial f_r}{\partial p_j}\sum_{i=1}^{k}\frac{\partial f_s}{\partial p_i}\left(\frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i}\right) \equiv \sum_{j=1}^{k}\frac{\partial f_r}{\partial p_j}\xi_j = 0 .$$

Since the determinant $\dfrac{D(f_1, \ldots, f_k)}{D(p_1, \ldots, p_k)}$ is not zero identically when one replaces the p_i with $\varphi_i(q_1, \ldots, q_k)$, the k homogeneous linear equalities in the ξ_j that one obtains by giving the values 1, 2, …, k to r in the latter equality will imply the consequences:

$$\xi_1 = 0 , \qquad \xi_2 = 0 , \qquad \ldots, \qquad \xi_k = 0 ,$$

i.e.:

$$\sum_{i=1}^{k} \frac{\partial f_s}{\partial p_i}\left(\frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i}\right) = 0 \qquad\qquad (s = 1, 2, \ldots, k),$$

which are equalities that imply the consequences:

$$\frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i} = 0 \qquad\qquad (i = 1, 2, \ldots, k)$$

for the same reason, and that is true for any j ($j = 1, 2, \ldots, k$). The expression $\sum p_i \, dq_i$ is then indeed an exact total differential. The stated conditions are then sufficient. The theorem is thus proved completely.

Remark on the preceding theorem. – If the equalities (α) have the form:

$$(\alpha') \qquad F_i\,(p_1, p_2, \ldots, p_k, q_1, \ldots, q_k) - a_i = 0$$

(in which the a_i denote arbitrary constants, and the determinant $\dfrac{D(F_1,\ldots,F_k)}{D(p_1,\ldots,p_k)}$ is not identically zero) then in order for the expression $\sum p_i\, dq_i$ to be an exact differential, it is necessary and sufficient that the brackets (f_i, f_j) should be *identically* zero.

More generally, suppose that the f_i in the system (α) depend upon n constants a, and that equations (α) are solved for l of the variables p_i, m of the variables q_i, and n of the constants a ($l + m + n = k$): The conditions (f_i, f_j) cannot be consequences of the system (α) without being verified identically.

Before applying the theorem to the integration of the Jacobi equation, we shall establish some properties of systems of homogeneous linear partial differential equations.

Complete linear systems.

One knows that j functions $f_1, f_2, \ldots, f_j$ of the n arbitrary variables $(x_1, x_2, \ldots, x_n)$ ($j < n$) are called *independent* or *distinct* when there exists no identity relation $G\,(f_1, f_2, \ldots, f_j) \equiv 0$ between those functions. In order for $f_1, f_2, \ldots, f_j$ to not be distinct, it is necessary and sufficient that all of the functional determinants $\dfrac{D(f_1, f_2,\ldots, f_k)}{D(x_h, x_l,\ldots, x_r)}$ should be identically zero.

An arbitrary linear equation:

$$X_1(x_1,x_2,\ldots,x_n)\frac{\partial f}{\partial x_1}+X_2(x_1,x_2,\ldots,x_n)\frac{\partial f}{\partial x_2}+\cdots+X_n(x_1,x_2,\ldots,x_n)\frac{\partial f}{\partial x_n} = 0$$

admits $(n - 1)$ independent integrals $f_1, f_2, \ldots, f_{n-1}$, and its general integral has the form $f = \varphi\,(f_1, f_2, \ldots, f_{n-1})$.

In particular, let F be a given function of the $2k$ variables:

$$(q_1, \ldots, q_k, p_1, p_2, \ldots, p_k)\,.$$

The equation:

$$(F, f) = \sum\left(\frac{\partial F}{\partial q_i}\frac{\partial f}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial f}{\partial q_i}\right) = 0$$

admits $(2k - 1)$ independent integrals.

Having recalled that, let us prove this more general proposition:

Theorem:

Let m distinct equations ($m \leq k$) be given:

$$(\delta) \qquad (f_1, f) = 0\,, \qquad (f_2, f) = 0\,, \qquad \ldots, \qquad (f_m, f) = 0\,,$$

in which $f_1, f_2, \ldots, f_m$ *are given functions of the* q_i, p_i *that verify the* $m\,(m-1)/2$ *relations:*

$$(f_i, f_j) = 0 \qquad\qquad (i, j = 1, 2, \ldots, m).$$

The system (δ) *admits* ($2k - m$) *distinct first integrals.*

[Jacobi gave the name of *complete system* to such a system ([25]).]

Equations (δ), which are linear and homogeneous in the $\frac{\partial f}{\partial q_i}, \frac{\partial f}{\partial p_i}$, will be distinct unless all of the determinants $\frac{D(f_1, f_2, \ldots, f_m)}{D(x_h, x_l, \ldots, x_m)}$ (in which $x_1, x_2, \ldots, x_m$ are any m of the variables q_i, p_j) are not identically zero. To say that equations (γ) are distinct is to say that the m functions $f_1, f_2, \ldots, f_m$ are distinct.

The number μ of distinct integrals cannot exceed $2k - m$ then. In order to see that, let $\varphi_1, \varphi_2, \ldots, \varphi_{2k-m+1}$ be ($2k - m + 1$) independent integrals such that, as a result, at least one of the determinants $\frac{D(\varphi_1, \varphi_2, \ldots, \varphi_{2k-m+1})}{D(\ldots, p_i, \ldots, q_j, \ldots)}$ is non-zero. The function $\psi = \sum a_i\, \varphi_i$ (in which the a_i are constants) is once more an integral of (δ), and one can arrange the constants a_i in such a fashion that among the derivatives $\frac{\partial \psi}{\partial q_i}, \frac{\partial \psi}{\partial p_i}$, ($2k - m + 1$) of them take arbitrary values of p_i, q_i : That is absurd is the equations (δ) are distinct. The theorem expresses the idea that μ is equal to $2k - m$ precisely.

The theorem is true for $m = 1$. Assume that it is true for m and prove that it is true for $m + 1$. Let (δ') be the new system that is obtained adding the equation:

$$(f_{m+1}, f) = 0$$

to the system (δ), in which f_{m+1} denotes a given function that verifies the conditions:

$$(f_1, f_{m+1}) = 0\,, \qquad (f_2, f_{m+1}) = 0\,, \qquad \ldots, \qquad (f_m, f_{m+1}) = 0\,,$$

by hypothesis, i.e., it is an integral of the first system (δ). Furthermore, let $f_1, f_2, \ldots, f_{m+1}, \varphi_1, \varphi_2, \ldots, \varphi_{2k-2m-1}$ be a system of ($2k - m$) distinct integrals of (δ). Any function $f = F(\ldots, f_i, \ldots, \varphi_j, \ldots)$

([25]) The term *complete system* applies to even more general systems, moreover. (See Goursat, *loc. cit.*, Chapter II.)

will also verify the system (δ), and any integral of (δ) will have that form. Let us seek to determine the function F in such a fashion that the equation:

$$0 = (f_{m+1}, f) \equiv \sum_{i=1}^{m+1} (f_{m+1}, f_i)\frac{\partial F}{\partial f_i} + \sum_{j=1}^{2k-2m-1} (f_{m+1}, \varphi_j)\frac{\partial F}{\partial \varphi_j}$$

is also verified. On the one hand, all of the combinations (f_{m+1}, f_i) are identically zero. On the other hand, from Poisson's theorem, the combinations (f_{m+1}, φ_j) are integrals of each equation (δ), and therefore of the system (δ), which then gives the identity:

$$(f_{m+1}, \varphi_j) \equiv g_j (f_2, \ldots, f_m, \varphi_1, \varphi_2, \ldots, \varphi_{2k-2m-1}) \qquad (j = 1, 2, \ldots, 2k - 2m - 1).$$

It is then necessary and sufficient that the function $F(\ldots, f_i, \ldots, \varphi_j, \ldots)$ should satisfy the relation:

$$(d) \qquad g_1 \frac{\partial F}{\partial \varphi_1} + g_2 \frac{\partial F}{\partial \varphi_2} + \cdots + g_{2k-2m-1}\frac{\partial F}{\partial \varphi_{2k-2m-1}} = 0\,,$$

in which the g are functions of $\ldots, f_i, \ldots, \varphi_j, \ldots$ Those coefficients are not identically zero, moreover, since otherwise the system (δ') would admit all of the integrals of (δ), i.e., $(2k - m)$ distinct integrals, and equations (δ') would not be distinct. Since equation (d) admits $2k - m - 1$ distinct integrals, the same thing will be true for the system (δ'). Q.E.D.

If k is equal to m then the m distinct integrals of (δ) will be $f_1, f_2, \ldots, f_m$. There can exist no more than k distinct functions f_i that satisfy the conditions that $(f_i, f_j) = 0$ $(i, j = 1, 2, \ldots, k)$.

Remark. – When at least one of the functional determinants of $f_1, f_2, \ldots, f_m$ relative to m of the variables $p_1, p_2, \ldots, p_k$ $(m < k)$ is non-zero, say $\frac{D(f_1, f_2, \ldots, f_m)}{D(p_1, p_2, \ldots, p_m)} \neq 0$, one can always find integrals f of the system (d) such that the determinant $\frac{D(f_1, f_2, \ldots, f_m, f)}{D(p_1, p_2, \ldots, p_m, p_{m+1})}$ is non-zero.

In other words, the equality:

$$\frac{D(f_1, f_2, \ldots, f_m, f)}{D(p_1, p_2, \ldots, p_m, p_{m+1})} \equiv A_1 \frac{\partial f}{\partial p_1} + A_2 \frac{\partial f}{\partial p_2} + \cdots + A_{m+1}\frac{\partial f}{\partial p_{m+1}} = 0 \qquad (A_{m+1} \neq 0)$$

will be a consequence of the system (δ), i.e., one can eliminate the $\partial f / \partial q_i$ from the m equations (δ), which is absurd, since the determinant of the coefficients of $\frac{\partial f}{\partial q_1}, \frac{\partial f}{\partial q_2}, \ldots, \frac{\partial f}{\partial q_m}$ in that system is the determinant $\frac{D(f_1, f_2, \ldots, f_m)}{D(p_1, p_2, \ldots, p_m)} \equiv A_{m+1}$, which is non-zero.

From that, let m functions $f_1, f_2, \ldots, f_m$ ($m \leq k$) be given whose functional determinant relative to m of the variables p_i is not identically zero: If those functions verify the conditions:

$$(f_i, f_j) \equiv 0 \qquad (i, j = 1, 2, \ldots, m)$$

then it will always be possible to find $(k - m)$ other functions $f_{m+1}, \ldots, f_k$ that satisfy the conditions:

$$(f_i, f_j) \equiv 0 \qquad (i, j = 1, 2, \ldots, k)$$

and are such that the system:

$$f_i = a_i \qquad (i = 1, 2, \ldots, k)$$

is soluble for the p_i . The expression $\sum p_i \, dq_i$ will then be an exact differential.

An important theorem results from that remark, which can be stated as follows:

Theorem:

Let m functions $f_1, f_2, \ldots, f_m$ ($m \leq k$) *be given that satisfy the conditions* $(f_i, f_j) = 0$ ($i, j = 1, 2, \ldots, m$) *identically. If the equalities:*

$$f_i = 0 \qquad (i = 1, 2, \ldots, m)$$

are soluble with respect to m of the variables p_i*, namely:*

$$p_1 - \varphi_1 (p_{m+1}, p_{m+2}, \ldots, p_k, q_1, q_2, \ldots, q_k) = 0 ,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$p_m - \varphi_m (p_{m+1}, p_{m+2}, \ldots, p_k, q_1, q_2, \ldots, q_k) = 0 ,$$

then the functions $F_i \equiv p_i - \varphi_i$ *will further satisfy the conditions* $(F_i, F_j) = 0$ *identically, with the only proviso being that the functional determinant:*

$$\Delta = \frac{D(f_1, f_2, \ldots, f_m)}{D(p_1, p_2, \ldots, p_m)}$$

should not be annulled identically when one replaces $p_1, p_2, \ldots, p_m$ *with* $\varphi_1, \varphi_2, \ldots, \varphi_m$ *in it.*

When the latter restriction is satisfied, the functions $\varphi_1, \ldots, \varphi_m$ will have continuous first derivatives, except for the exceptional values of $p_{m+1}, \ldots, p_k, q_1, \ldots, q_k$ that annul $\Delta(p_{m+1}, \ldots, p_k, q_1, \ldots, q_k)$. Therefore, take a system of those variables ($\Delta \neq 0$) at random and write down the equalities:

$$f_i = a_i \qquad (i = 1, 2, \ldots, m).$$

If one solves those equalities for $p_1, p_2, \ldots, p_m$, namely:

$$G_i \equiv p_i - P_i(p_{m+1}, p_{m+2}, \ldots, p_k, a_1, \ldots, a_k) = 0$$

then the function P_i and its first derivatives will tend to φ_i and its first derivatives when $a_1, a_2, \ldots, a_k$ tend to zero. The expressions (G_i, G_j) tend to (F_i, F_j) when $a_1, a_2, \ldots, a_k$ tend to zero. As a result, in order to prove the theorem, it will suffice to prove that the brackets (G_i, G_j) are identically zero.

Now, one can combine the equalities:

$$f_i = a_i \qquad (i = 1, 2, \ldots, m)$$

with $(k - m)$ other ones:

$$f_{m+1} = a_{m+1}, \quad f_{m+2} = a_{m+2}, \quad \ldots, \quad f_k = a_k,$$

such that all of the equalities $(f_i, f_j) = 0$ will be verified $(i, j = 1, 2, \ldots, k)$ and the system $f_j = a_j$ $(j = 1, 2, \ldots, k)$ will be soluble for the $p_1, \ldots, p_k$. The expression $\sum p_i\, dq_i$ will then be an exact differential. Now solve the system $f_j = a_j$ for the $p_1, p_2, \ldots, p_m, a_{m+1}, \ldots, a_k$.

That will give:

$$\begin{aligned}
G_1 &\equiv p_1 - P_1 \ (p_{m+1}, \ldots, p_k, q_1, \ldots, q_k, a_1, \ldots, a_m) = 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
G_m &\equiv p_m - P_m \ (p_{m+1}, \ldots, p_k, q_1, \ldots, q_k, a_1, \ldots, a_m) = 0, \\
G_{m+1} &\equiv a_{m+1} - P_{m+1} (p_{m+1}, \ldots, p_k, q_1, \ldots, q_k, a_1, \ldots, a_m) = 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
G_k &\equiv a_k - P_k \ (p_{m+1}, \ldots, p_k, q_1, \ldots, q_k, a_1, \ldots, a_m) = 0.
\end{aligned}$$

From a previous remark, all of the conditions $(G_i, G_j) = 0$ must be verified identically in order for $\sum p_i\, dq_i$ to be an exact differential. Thus, one will have:

$$(G_i, G_j) = 0 \qquad (i, j = 1, 2, \ldots, m),$$

in particular. Q.E.D. ([26])

We are now in a position to present the integration methods of Jacobi and Mayer.

([26]) More generally (the conditions of the theorem having been fulfilled), if $f_1, f_2, \ldots, f_m$ depend upon h constants $b_1, b_2, \ldots, b_h$, and if one solves the equations $f_i = 0$ for the j variables p_i, r variables q_i, s constants b_i $(j + r + s = m)$ then the equalities $(F_i, F_j) \equiv 0$ will be verified identically, so $F_i = 0$ will represent any of the new equations. That will result immediately from the previous argument.

Jacobi's method.

The first form that Jacobi gave to his method was the following one:

Let f_0 ($p_1, \ldots, p_k, q_1, \ldots, q_k$) = 0 be a first-order partial differential equation that one can always suppose to be solved for one of the derivatives (p_1, for example) or with respect to a constant when f_0 depends upon constants. In order to form a complete integral of that equation, it will suffice to determine (k – 1) first integrals $f_1, \ldots, f_{k-1}$ of the equation:

(1) $$(f_0, f) = 0$$

such that all of the conditions $(f_i, f_j) \equiv 0$ are verified identically, and the determinant $\frac{D(f_0, \ldots, f_{k-1})}{D(p_1, \ldots, p_k)}$ is non-zero, moreover.

One begins by determining an integral f_1 of equation (1), which one combines with f_0, and then looks for a common integral of the system:

(2) $$(f_0, f) = 0 , \qquad (f_1, f) = 0 ,$$

which is a system that admits ($2k$ – 2) distinct integrals. In order to do that, one first determines an integral f_2 of equation (1) and then forms the combination (f_1, f_2), which is again an integral of (1). If that parenthesis is identically zero then f_2 will be the desired integral. Otherwise, one sets $(f_1, f_2) = f_3$, then forms (f_1, f_3), and so on, until one arrives at an integral f_h such that (f_1, f_h) is a function of $f_0, f_1, \ldots, f_h$. One then sets $f = \varphi(f_0, f_1, \ldots, f_h)$, and seeks to determine φ in such a fashion that the parenthesis (f_1, φ) is zero. Now one has:

$$\begin{aligned}(f_1, \varphi) &\equiv \frac{\partial\varphi}{\partial f_2}(f_1, f_2) + \frac{\partial\varphi}{\partial f_3}(f_1, f_3) + \cdots + \frac{\partial\varphi}{\partial f_h}(f_1, f_h) \\ &\equiv \frac{\partial\varphi}{\partial f_2} f_3 + \frac{\partial\varphi}{\partial f_3} f_4 + \cdots + \frac{\partial\varphi}{\partial f_h} F(f_0, f_1, f_2, \ldots, f_h) = 0 .\end{aligned}$$

One sees that φ must satisfy a linear equation that is equivalent to a system of h – 2 first-order ordinary differential equations (upon regarding f_0, f_1 as parameters).

There is nonetheless one exceptional case, namely, that of h = 2, i.e., the case in which one has $(f_r, f_2) \equiv F(f_0, f_1, f_2)$, where F is not identically zero. The equation in φ will then become:

$$\frac{\partial\varphi}{\partial f_2} \cdot F(f_0, f_1, f_2) = 0 ,$$

and will give $f = \varphi(f_0, f_1)$, i.e., an integral that is distinct from f_0, f_1 . In that case, it will be necessary to calculate a third integral f_3 of equation (1). If the number h that relates to f_h is not equal to 2 then one can form an integral of (2) with f_3 by using the procedure that was just indicated. Otherwise, one sets $f = \varphi(f_0, f_1, f_2, f_3)$ and determines φ in such a fashion that (f_1, φ) is zero: That determination requires only quadratures. Indeed, φ must satisfy the equation:

$$\frac{\partial \varphi}{\partial f_2} \cdot F(f_0, f_1, f_2) + \frac{\partial \varphi}{\partial f_3} \cdot F'(f_0, f_1, f_2) = 0 ,$$

and as a result, it will suffice to take φ to be the function:

$$\varphi = \int \frac{df_2}{F(f_0, f_1, f_2)} - \int \frac{df_3}{F'(f_0, f_1, f_2)} ,$$

in which f_0 and f_1 are regarded as parameters.

Hence, assume that one has calculated an integral of the system (2), which is an integral that we shall represent by f_2, and look for an integral of the system:

$$(3) \qquad (f_0, f) = 0 , \qquad (f_1, f) = 0 , \qquad (f_2, f) = 0 .$$

In order to do that, one first calculates a new integral, namely, f_3, of the system (2), and one forms the combination (f_2, f_3), which is again an integral of (2), from Poisson's theorem. If that parenthesis is identically zero then f_3 is the desired integral. Otherwise, one sets $(f_2, f_3) = f_4$, then forms $(f_2, f_4) = f_5$, and so on, until one arrives at an integral f_h such that (f_2, f_h) is a function F of $(f_0, f_1, \ldots, f_h)$. One can then determine $\varphi(f_0, f_1, f_2, \ldots, f_h)$ in such a fashion that φ is an integral of the system (3). It will suffice that φ must verify the equation:

$$\frac{\partial \varphi}{\partial f_3} f_4 + \frac{\partial \varphi}{\partial f_4} f_5 + \cdots + \frac{\partial \varphi}{\partial f_h} F(f_1, f_2, \ldots, f_h) = 0 .$$

There is nonetheless one exceptional case, namely, the one in which f_h coincides with f_3 without F being identically zero. It will then be necessary to determine a new integral of (2), namely, f_4. If one does not have $(f_2, f_4) \equiv F'(f_0, f_1, f_2, f_4)$ then the preceding procedure will permit one to deduce an integral of (3) from f_4 . On the contrary, if that identity is satisfied then one sets $f = \varphi(f_0, f_1, f_2, f_3, f_4)$, and the function:

$$\varphi = \int \frac{df_2}{F(f_0, f_1, f_2)} - \int \frac{df_3}{F'(f_0, f_1, f_2)}$$

will be an integral of (3).

It is clear that the argument that was just made for the systems (2) and (3) can be repeated until one arrives at a system of k equations. More precisely, assume that one has learned how to find an arbitrary integral of a system:

$$(4) \qquad (f_0, f) = 0 , \qquad (f_1, f) = 0 , \qquad \ldots, \qquad (f_{i-1}, f) = 0 \qquad (i \leq k - 1),$$

in which all of the conditions $(f_i, f_j) \equiv 0$ are verified $(i, j = 0, 1, 2, \ldots, i-1)$.

Let f_i be an integral of the system that is distinct from $f_0, f_1, \ldots, f_{i-1}$, and consider the equations:

$$(5) \qquad (f_0, f) = 0\,, \qquad (f_1, f) = 0\,, \qquad \ldots, \qquad (f_{i-1}, f) = 0\,, \qquad (f_i, f) = 0\,.$$

If i is less than $k - 1$ then one can always determine an integral of (5) that is distinct from $f_0, f_1, \ldots, f_i$ in the following manner: One determines a new integral f_{i+1} of (4) and then forms $(f_i, f_{i+1}) = f_{i+2}$, etc., until one has obtained an integral f_h such that $(f_i, f_h) \equiv F(f_0, f_1, \ldots, f_i, f_{i+1}, \ldots, f_h)$.

If one then sets $f = \varphi(f_0, f_1, f_2, \ldots, f_i, \ldots, f_h)$, one can determine φ in such a fashion that φ is an integral of (5) that is distinct from $f_0, f_1, \ldots, f_i$. That will be impossible only when one has $(f_1, f_{i+1}) \equiv F(f_0, f_1, \ldots, f_i, f_{i+1})$ unless F is identically zero. It would then be necessary to introduce a new integral of the system (4).

When one follows that method, one will thus arrive at a system of k equations:

$$(6) \qquad (f_0, f) = 0\,, \qquad (f_1, f) = 0\,, \qquad \ldots, \qquad (f_{k-1}, f) = 0$$

such that all of the conditions $(f_i, f_j) \equiv 0$ are verified.

If one supposes, in addition, that condition is fulfilled ([27]) that the integrals $f_0, f_1, \ldots, f_{k-1}$ are distinct when they are considered to be functions of only the p_i then the system (6) will define (up to an additive constant) a complete integral of the given equation $f_0 = 0$. Indeed, it will suffice to write that:

$$(7) \qquad f_0 = 0\,, \qquad f_1 = C_1\,, \qquad f_2 = C_2\,, \qquad \ldots, \qquad f_{k-1} = C_{k-1}\,.$$

The expression $p_1\, dq_1 + p_2\, dq_2 + \ldots + p_k\, dq_k$, in which one replaces the p_i with their values that one infers from (6), will be the exact differential of a function:

$$V(q_1, \ldots, q_k, C_1, C_2, \ldots, C_{k-1})\,,$$

which is a complete integral of $f_0 = 0$.

One sees that in order to apply the preceding method, it is necessary to know *at least* $(m - 1)$ distinct integrals of the equation $(f_0, f) = 0$, in addition to the integral f_0, and to calculate the integrals of the intermediate equations in φ.

Observe that those equations in φ bear upon a number of variables that is even larger, and as a result, they are equivalent to an ordinary differential system whose order is even higher than the one for which Poisson's theorem is most advantageous. Now, Jacobi's method does not take into account the simplification that can result from knowing new integrals of (1). That gap was filled by Sophus Lie. The method of Sophus Lie is the most perfect of all of them. It requires only the minimum number of integrations that is necessary. It completes and extends the theory of the last multiplier (in the case where the forces derive from a potential). However, presenting that method

([27]) We will see that this condition is not necessary.

would require too much development. We shall confine ourselves to indicating the improvements that Jacobi himself had made to his method, along with the ones that Mayer introduced. For the results of Sophus Lie, we shall refer to the works of that author, as well as the book by Goursat that was cited before. (Chapters X, XI, and XII).

Method of Jacobi and Mayer.

Path of the calculations. – The second path that Jacobi indicated is the following one: Solve the given partial differential equation for one of its derivatives (p_1, for example), and let:

(1) $$f_0 \equiv p_1 - F_1 (p_2, p_3, \ldots, p_h, q_1, q_2, \ldots, q_h) = 0$$

be that equation. Form the equation:

(2) $$0 = (f_0, f) \equiv \frac{\partial f}{\partial q_1} + \frac{\partial F_1}{\partial q_1}\frac{\partial f}{\partial p_1} + \sum_{i=2}^{k} \frac{\partial F_1}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial F_1}{\partial p_i}\frac{\partial f}{\partial q_i} .$$

Since p_1 does not enter into (2) explicitly, calculate an integral f of (2) that is independent of p_1 . In other words, look for an integral $f(p_2, \ldots, p_h, q_1, \ldots, q_h)$ of the equation that is obtained by suppressing the term in $\partial f / \partial p_1$ from (2), which is an integral that we suppose depends upon at least one of the variables p_i, say, p_2 . Only the exceptional integrals of (2) cannot satisfy that condition. We then write down the equalities:

$$p_1 - F_1 (p_2, \ldots, p_k, q_1, \ldots, q_k) = 0 , \qquad f_1 (p_2, \ldots, p_k, q_1, \ldots, q_k) = C_1 ,$$

and solve them for p_1, p_2 : That will give:

(1′) $$\begin{cases} f_0' \equiv p_1 - F_1'(p_3, p_4, \ldots, p_k, q_1, q_2, \ldots, q_k) = 0, \\ f_1' \equiv p_2 - F_2'(p_3, p_4, \ldots, p_k, q_1, q_2, \ldots, q_k) = 0, \end{cases}$$

and from Theorem III of this lecture, the condition $(f_0', f_1') \equiv 0$ is verified, since the condition $(f_0, f_1) \equiv 0$ is itself verified ([28]).

Now form the equations:

([28]) Observe that in order to form the system (1′), it will suffice to find an integral $f_1 (p_1, p_2, \ldots, p_k, q_1, \ldots, q_k)$ of the equation $(f_0, f_1) = 0$ without having solved f_0 with respect to one of the p_i . Upon then solving the system $f_0 = 0$, $f_1 = c_1$ with respect to p_1, p_2, one will form a system (1′) for which the condition $(f_0', f_1') \equiv 0$ is once more verified: Nevertheless, it is necessary that the solution should be possible.

(2)
$$\begin{cases} 0 = -(f_0', f) \equiv \dfrac{\partial f}{\partial q_1} + \dfrac{\partial F_1}{\partial q_1}\dfrac{\partial f}{\partial p_1} + \dfrac{\partial F_1}{\partial q_2}\dfrac{\partial f}{\partial p_2} + \sum_{i=3}^{k} \dfrac{\partial F_1}{\partial q_i}\dfrac{\partial f}{\partial p_i} - \dfrac{\partial F_1}{\partial p_i}\dfrac{\partial f}{\partial q_i}, \\ 0 = -(f_1', f) \equiv \dfrac{\partial f}{\partial q_2} + \dfrac{\partial F_2}{\partial q_1}\dfrac{\partial f}{\partial p_1} + \dfrac{\partial F_2}{\partial q_2}\dfrac{\partial f}{\partial p_2} + \sum_{i=3}^{k} \dfrac{\partial F_2}{\partial q_i}\dfrac{\partial f}{\partial p_i} - \dfrac{\partial F_2}{\partial p_i}\dfrac{\partial f}{\partial q_i}. \end{cases}$$

We know that this system, into which p_1, p_2 do not enter explicitly, admits $(2k - 2)$ distinct integrals. Let us assume that we have determined an integral $f(p_3, p_4, \ldots, p_h, q_1, \ldots, q_h)$ that depends upon p_3 explicitly (we shall soon see the means for doing that). Write down the equalities:

$$f_0' = 0, \qquad f_1' = 0, \qquad f(p_3, p_4, \ldots, p_h, q_1, \ldots, q_h) = C_2 .$$

Solve the last equality for p_3 and substitute that in the first one. We thus form a system:

(1″)
$$\begin{cases} f_0'' \equiv p_1 - F_1''(p_4, \ldots, p_k, q_1, \ldots, q_k) = 0, \\ f_1'' \equiv p_2 - F_2''(p_4, \ldots, p_k, q_1, \ldots, q_k) = 0, \\ f_2'' \equiv p_3 - F_3''(p_4, \ldots, p_k, q_1, \ldots, q_k) = 0 \end{cases}$$

that satisfies the conditions:

$$(f_0'', f_1'') \equiv 0, \quad (f_0'', f_2'') \equiv 0, \quad (f_1'', f_2'') \equiv 0,$$

from a theorem that was recalled before.

One then seeks an integral $f(p_4, \ldots, p_h, q_1, \ldots, q_h)$ of the system:

(2″)
$$(f_0'', f) \equiv 0, \qquad (f_1'', f) \equiv 0, \qquad (f_2'', f) \equiv 0 .$$

One solves the equality $f = C_4$ for p_4, and so on, until one succeeds in having k equations that are solved for $p_1, \ldots, p_h$, namely:

(A)
$$\varphi \equiv p_i - P_i(p_4, \ldots, p_h, q_1, \ldots, q_h) = 0 \qquad (i = 1, 2, \ldots, k),$$

which satisfies all of the conditions $(\varphi_i, \varphi_j) \equiv 0$.

Since the P_i depend upon $(k - 1)$ constants $C_1, C_2, \ldots, C_{k-1}$ (which are, conversely, expressed as functions of the $p_2, \ldots, p_h, q_1, \ldots, q_h$, from the preceding), the equalities (A) will define the differential $\sum p_i \, dq_i$ of a complete integral $V(q_1, \ldots, q_k)$ of the given equation (1).

Integrations that must be performed. – All of the difficulty then comes down to determining an integral $f(p_{r+1}, \ldots, p_k, q_1, \ldots, q_k)$ of each intermediate system:

$$(r)\qquad \begin{cases} 0 = -(f_1, f) \equiv \dfrac{\partial f}{\partial q_1} + \displaystyle\sum_{i=1}^{r} \frac{\partial F_1}{\partial q_i}\frac{\partial f}{\partial p_i} + \sum_{j=r+1}^{k} \left(\frac{\partial F_1}{\partial q_j}\frac{\partial f}{\partial p_j} - \frac{\partial F_1}{\partial p_j}\frac{\partial f}{\partial q_j} \right), \\ 0 = -(f_2, f) \equiv \dfrac{\partial f}{\partial q_2} + \displaystyle\sum_{i=1}^{r} \frac{\partial F_2}{\partial q_i}\frac{\partial f}{\partial p_i} + \sum_{j=r+1}^{k} \left(\frac{\partial F_2}{\partial q_j}\frac{\partial f}{\partial p_j} - \frac{\partial F_2}{\partial p_j}\frac{\partial f}{\partial q_j} \right), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = -(f_{r-1}, f) \equiv \dfrac{\partial f}{\partial q_r} + \displaystyle\sum_{i=1}^{r} \frac{\partial F_r}{\partial q_i}\frac{\partial f}{\partial p_i} + \sum_{j=r+1}^{k} \left(\frac{\partial F_r}{\partial q_j}\frac{\partial f}{\partial p_j} - \frac{\partial F_r}{\partial p_j}\frac{\partial f}{\partial q_j} \right), \end{cases}$$

in which the functions ([29]) $f_l \equiv p_l - F_l\,(p_{r+1}, \ldots, p_r, q_1, \ldots, q_k)$ satisfy all of the conditions $(f_l, f_m) \equiv 0$ $(l, m = 0, 1, 2, \ldots, r-1)$.

Observe, first of all, that each system (r) (where r is equal to at most $k-1$) admits $2\,(k-r)$ distinct integrals in which $p_1, p_2, \ldots, p_r$ do not appear. In other words, the system:

$$(r')\qquad \begin{cases} \dfrac{\partial f}{\partial q_1} + \displaystyle\sum_{j=r+1}^{k} \frac{\partial F_1}{\partial q_j}\frac{\partial f}{\partial p_j} - \frac{\partial F_1}{\partial p_j}\frac{\partial f}{\partial q_j} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial f}{\partial q_r} + \displaystyle\sum_{j=r+1}^{k} \frac{\partial F_r}{\partial q_j}\frac{\partial f}{\partial p_j} - \frac{\partial F_r}{\partial p_j}\frac{\partial f}{\partial q_j} = 0 \end{cases}$$

admits $2\,(k-r)$ distinct integrals $f\,(p_{r+1}, \ldots, p_k, q_1, \ldots, q_k)$. In order to see that rigorously, it will suffice to repeat the argument on page 297. First of all, the first equation in (r) admits $2k-r-1$ distinct integrals of the form $f\,(p_{r+1}, \ldots, p_k, q_1, \ldots, q_k)$. Assume that the first l equations in (r) $(f < r)$ possess $2k-r-l$ distinct common integrals of the same form and prove that an analogous theorem is true for $l+1$. If one sets $f = \Phi\,(\varphi_1, \ldots, \varphi_{2k-r-l})$ then one will see that, as on the cited page, in order to verify the $(l+1)^{\text{th}}$ equation in (r), F must satisfy the condition:

$$G_1 \frac{\partial \phi}{\partial \varphi_1} + \cdots + G_{(2k-r-l)} \frac{\partial \phi}{\partial \varphi_{(2k-r-l)}} = 0\,,$$

in which $G_i \equiv [f_l, \varphi_i] \equiv g_i\,(\varphi_1, \ldots, \varphi_{2k-r-l})$. Furthermore, those coefficients G_i are not zero. In other words, the $(j+1)^{\text{th}}$ equation in (r) will be a consequence of the first j, which is impossible (see pp. 291).

One thus arrives at the conclusion that the system (r') admits $2\,(k-r)$ distinct integrals, namely, $\varphi_i\,(q_1, \ldots, q_k, p_{r+1}, p_{r+2}, \ldots, p_k)$ $[i = 1, 2, \ldots, 2\,(k-r)]$.

([29]) To simplify the notation, we shall suppress the upper indices (?) on the symbols f_l, F_l.

Let me add that the general integral $f = F(\varphi_1, \varphi_2, \ldots, \varphi_{2(k-r)})$ of the system (r') depends upon p_{r+1} explicitly. Indeed, at least one of the functional determinants of $\varphi_1, \varphi_2, \ldots, \varphi_{2(k-r)}$ with respect to $2(k-r)$ of the variables p_j, q_j is non-zero. Now, if the determinant:

$$\frac{D(\varphi_1, \ldots, \varphi_{2(k-r)})}{D(p_{r+1}, \ldots, p_k, q_{r+1}, \ldots, q_k)}$$

is zero then all of the other ones will be so from equations (r'). It then follows from this that not all of the derivatives $\partial\varphi_i / \partial p_{r+1}$ can be zero, and that at least one of the φ_i must depend upon p_{r+1}, and therefore f.

Finally, let $q_1^0, \ldots, q_k^0, p_{r+1}^0, \ldots, p_k^0$ be a system of numerical values for the q_i, p_i that are taken at random, and in whose neighborhood the functions f_i, F_j will be holomorphic as a result. Any integral of (r') that is holomorphic in the neighborhood of those values will be obtained by taking f to be a function $\Phi(\varphi_1, \varphi_2, \ldots, \varphi_{2(k-r)})$ that is holomorphic in the neighborhood of the corresponding values of the φ_i. I say that one can choose Φ in such a fashion that for $q_1^0, \ldots, q_r^0$, f coincides with a given function $A(q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k)$ that is holomorphic in the neighborhood of $q_{r+1}^0, \ldots, q_k^0, p_{r+1}^0, \ldots, p_k^0$. Indeed, set:

$$\varphi_i(q_{r+1}^0, \ldots, q_k^0, p_{r+1}^0, \ldots, p_k^0) = \xi_i \qquad [i = 1, 2, \ldots, 2(k-r)]$$

and solve those equalities for p_{r+j}, q_{r+j} (which is possible). That will give (in which $q_1^0, \ldots, q_r^0$ are numbers):

$$q_{r+j} = u_j(\xi_1, \ldots, \xi_{2(k-r)}), \qquad p_{r+j} = v_j(\xi_1, \ldots, \xi_{2(k-r)}) \qquad [j = 1, 2, \ldots, (k-r)].$$

If one substitutes those values in $A(\ldots, q_{r+j}, \ldots, p_{r+j}, \ldots)$ then one will obtain a function $\alpha(\xi_1, \ldots, \xi_{2(k-r)})$ that must coincide with $\Phi(\xi_1, \ldots, \xi_{2(k-r)})$:

$$\Phi(\xi_1, \ldots, \xi_{2(k-r)}) \equiv \alpha(\xi_1, \ldots, \xi_{2(k-r)}),$$

which will determine F unambiguously. There will then exist one and only one integral of the system (r) that coincides with $A(q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k)$ for $q_1^0, \ldots, q_r^0$.

In particular, there exist $2(k-r)$ distinct integrals that will reduce to $q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k$, respectively, for $q_1^0, \ldots, q_r^0$.

Having said that, we shall now indicate Jacobi's integration procedure, and then that of Mayer.

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Jacobi's integration procedure.

In order to find an integral f ($p_{r+1}, \dots, p_k, q_1, \dots, q_k$) of the system (r'), Jacobi employed a procedure that is analogous to the one that was discussed in the context of the first method (see pp. 300). One first determines an integral $f = \varphi_1$ of the first equation (r'), namely:

$$(s) \qquad \frac{\partial f}{\partial q_1} + \sum_{i=r+1}^{k} \frac{\partial F_1}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial F_1}{\partial p_i}\frac{\partial f}{\partial q_i} = 0 .$$

Since one can regard $q_1, \dots, q_k$ as parameters in that equation, equation (1) will involve 2 ($k - r$) + 1 variables, and the search for one integral f will be equivalent to the search for a first integral of a system of 2 ($k - r$) first-order equations.

One then deduces a common integral to the first two equations in (r') from the integral $f = \varphi_1$ of (s). To that effect, one forms $(f_1, \varphi_1) = \varphi_2$, $(f_1, \varphi_2) = \varphi_3$, …, which are just as many integrals of (1), until one arrives at an integral φ_h such that (f_1, φ_h) is expressed as a function of $\varphi_1, \dots, \varphi_h$, and $q_2, \dots, q_r$. One then sets $f = \Phi$ ($q_2, \dots, q_r, \varphi_1, \dots, \varphi_h$), and one determines Φ in such a fashion that the equality:

$$(\alpha) \qquad 0 = (f_1, f) \equiv -\frac{\partial \Phi}{\partial q_2} + \frac{\partial \Phi}{\partial \varphi_1}\varphi_2 + \cdots + \frac{\partial \Phi}{\partial \varphi_h} F(q_2, \dots, q_r, \varphi_1, \dots, \varphi_h)$$

is verified. Since h is equal to at most 2 ($k - r$), one will see that it suffices to determine a first integral Φ of a system of h first-order equations [$h \leq 2(k - r)$] ([30]).

It is clear that the argument can be pursued further. Assume that one has determined an integral that is common to the first l equations in (r'), say, $f = \psi_1$ ($p_{r+1}, \dots, p_h, q_2, \dots, q_r$). In order to deduce an integral of the first (l + 1) equations (r'), one forms $(f_l, \psi_1) = \psi_2$, $(f_l, \psi_2) = \psi_3$, etc., until one arrives at an integral ψ_h such that (f_l, ψ_h) is expressed as a function of $\psi_1, \dots, \psi_h$, and the variables $q_{l+1}, \dots, q_r, \dots$, which one can regard as parameters in the first l equations of (r) or (r'). One can then set:

$$f = \Psi (q_{l+1}, \dots, q_r, \psi_1, \dots, \psi_h) .$$

([30]) In order for one to be able to deduce an integral $f = \varphi$ that also depends upon p_{r+1}, it will suffice that the integral φ_1 that one starts from should depend upon p_{r+1} (which is always possible). Indeed, $\frac{\partial f}{\partial p_{r+1}} = \frac{\partial \Phi}{\partial \varphi_1}\frac{\partial \varphi_1}{\partial p_{r+1}} + \cdots + \frac{\partial \Phi}{\partial \varphi_h}\frac{\partial \varphi_h}{\partial p_{r+1}}$, and on the other hand, one can give the values of the derivatives $\frac{\partial \Phi}{\partial \varphi_1}, \dots, \frac{\partial \Phi}{\partial \varphi_h}$ of an integral Φ of the auxiliary equation (α) arbitrarily for $q_1^0, \dots, q_r^0, \varphi_1^0, \dots, \varphi_h^0$. Therefore, if $\frac{\partial \varphi_1}{\partial p_{r+1}}$ is not zero then $\frac{\partial f}{\partial p_{r+1}}$ will not be zero for any arbitrary integral Φ.

In order for f, which is a common integral of the first l equations in (r), to satisfy the $(l + 1)^{\text{th}}$ one, it is necessary and sufficient that one must have:

$$(\beta) \qquad 0 = (f_l, f) \equiv \frac{\partial \Psi}{\partial q_{l+1}} + \frac{\partial \Psi}{\partial \psi_1}\psi_2 + \cdots + \frac{\partial \Psi}{\partial \psi_h} F(q_{l+1}, \ldots, q_r, \psi_1, \ldots, \psi_h) .$$

Since h is equal to at most $2k - r - l - (r - l) \equiv 2(k - r)$, one will then once more have to find a first integral of an ordinary differential system of order $2(k - r)$. Let me add that the integral f corresponds to an arbitrary integral Ψ of (β) that will depend upon p_{r+1} if ψ_1 depends upon it, as was shown in the footnote above.

One then sees that the exceptional case that presents itself in the first method has thus been eliminated: From an arbitrary integral (that depends upon p_{r+1}) of the first equation (s), one will deduce an integral f_r of the system (r') that also depends upon p_{r+1}, is independent of $p_1, \ldots, p_r$, and is distinct from $f_0, \ldots, f_{r-1}$, as a result. One can solve the system:

$$f_0 \equiv p_1 - F_1 = 0 , \qquad f_1 \equiv p_2 - F_2 = 0 , \qquad \ldots, \qquad f_r \equiv c_r$$

for $p_1, \ldots, p_r, p_{r+1}$.

By definition, in order to find an integral $f(p_{r+1}, \ldots, p_k, q_1, \ldots, q_k)$ of the system (r), it will be necessary to find an integral of the first equation in (r'), which is equivalent to a differential system of order $2(k - r)$, and then an integral of each of the other $(r - 1)$ successive equations (β), …, (λ), …, which are each equivalent to a differential system of order equal to at most $2(k - r)$.

In order to find a complete integral of the original given equation, it is necessary to find an integral of each of the $(k - 1)$ successive systems (r) that correspond to the values of r : $r = 1$, $r = 2, \ldots, r = k - 1$.

Mayer's integration procedure.

The Mayer procedure presents a great advantage over the preceding one: It permits one to obtain an integral of the system (r) by calculating only a first integral of a single system of differential equations of order $2(k - r)$.

We know that the system (r'):

$$(r')\qquad \begin{cases} \dfrac{\partial f}{\partial q_1} + (F_1, f) = \sigma, \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ \dfrac{\partial f}{\partial q_r} + (F_r, f) = 0 \end{cases}$$

(in which the parentheses extend over the variable $p_{r+1}, q_{r+1}, \ldots, p_k, q_k$) admits an integral $f(q_1, \ldots, q_r, q_{r+1}, \ldots, q_h, p_{r+1}, \ldots, p_k)$, namely, $f = \varphi_1$, which reduces an integral A $(q_{r+1}, \ldots, q_h, p_{r+1}, \ldots, p_k)$ for $q_1^0, q_2^0, \ldots, q_r^0$. If the $q_1^0, \ldots, q_r^0$ are taken at random then the functions $F_1, \ldots, F_r$ will be holomorphic for arbitrary values of:

$$q_{r+j}, p_{r+j}, \qquad \text{namely,} \qquad q_{r+1}^0, \ldots, q_h^0, p_{r+1}^0, \ldots, p_h^0$$

(as well as A, by hypothesis).

Having done that, make the change of variables:

$$q_1 = q_1^0 + u_1, \quad q_2 = q_2^0 + u_1 u_2, \ldots, \qquad q_r = q_r^0 + u_1 u_r,$$

and the equalities:

$$\frac{\partial f}{\partial u_1} = \frac{\partial f}{\partial q_1} + u_2 \frac{\partial f}{\partial q_2} + \cdots + u_r \frac{\partial f}{\partial q_r},$$
$$\frac{\partial f}{\partial u_2} \equiv u_1 \frac{\partial f}{\partial q_2}, \ldots, \frac{\partial f}{\partial u_r} \equiv u_1 \frac{\partial f}{\partial q_r}$$

will show that the function f of the new variables will satisfy the system:

$$(R')\qquad \begin{cases} \dfrac{\partial f}{\partial u_1} + (F_1', f) + u_2 (F_2', f) + \cdots + u_r (F_r', f) = 0, \\ \dfrac{\partial f}{\partial u_2} + u_1 (F_2', f) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{\partial f}{\partial u_r} + u_1 (F_r', f) = 0, \end{cases}$$

in which F_i' denotes what F_i will become after one changes the variables, and the parentheses extend over the variables $p_{r+1}, q_{r+1}, \ldots, p_k, q_k$, as always.

The integral $f = \varphi_1$ of (r') will become a function:

$$\Psi_1 (u_1, \ldots, u_r, q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k)$$

that will reduce to A ($q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k$) for $u_1 = 0$ ($u_1, \ldots, u_r$ have arbitrary values). However, the functions F_i' are holomorphic in the domain:

$$u_1 = 0\,,\ u_2^0, \ldots, u_r^0,\ q_{r+1}^0, \ldots, q_k^0,\ p_{r+1}^0, \ldots, p_k^0,$$

so the first equation in (R') (in which $u_2, \ldots, u_r$ are parameters that can take on arbitrary values) will admit one and only one integral that reduces to A ($q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k$) for $u_1 = 0$. That integral will then coincide with ψ_1 .

It follows from this that in order to integrate the system (r'), it suffices to integrate the first equation in (R'). Indeed, it will suffice to determine the 2 $(k - r)$ integrals of the latter equation that reduce to $q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k$, respectively, for $u_1 = 0$. Upon reverting to the old variables, one will have 2 $(k - r)$ distinct integrals of (r').

That first equation in (R') can be written:

$$\text{(S)} \qquad \frac{\partial f}{\partial u_1} + \lambda_1 \frac{\partial f}{\partial p_{r+1}} + \cdots + \lambda_{k-r} \frac{\partial f}{\partial p_k} - \mu_1 \frac{\partial f}{\partial q_{r+1}} - \cdots - \mu_{(k-r)} \frac{\partial f}{\partial q_k} = 0\,,$$

in which the λ_i are functions of $\mu_1, q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k$ in which one regards $u_2, \ldots, u_r$ as parameters. In order to integrate that equation, it is necessary for one to find 2 $(k - r)$ distinct integrals of the system:

$$\frac{du_1}{1} = \frac{dp_{r+1}}{\lambda_1} = \ldots = \frac{dp_k}{\lambda_{k-r}} = -\frac{dq_{r+1}}{\mu_1} = -\frac{dq_k}{\mu_{k-r}}\,.$$

Assume that one has effectively calculated 2 $(k - r)$ distinct integrals of (S), say, $\chi_1, \ldots, \chi_{2(k-r)}$. How does one deduce the 2 $(k - r)$ integrals $\psi_1, \ldots, \psi_{2(k-r)}$ that will reduce to $\ldots, q_{r+j}, \ldots, p_{r+j}, \ldots$ for $u_1 = 0$ for any $u_2, \ldots, u_r$? In order to do that, observe that any integral of (S) such as:

$$\chi (u_1, u_2, \ldots, u_r, q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k)$$

can be expressed as a function of $\psi_1, \ldots, \psi_{2(k-r)}$, and $u_2, \ldots, u_r$:

$$\chi (u_1, u_2, \ldots, u_r, q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k) \equiv F (u_2, \ldots, u_r, \psi_1, \ldots, \psi_{2(k-r)})\,,$$

but for $u_1 = 0$, ψ_1 will coincide with q_{r+1}, etc., $\psi_{2(k-r)}$ with p_k, so one will have:

$$\chi (0, u_2, \ldots, u_r, q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k) \equiv F (u_2, \ldots, u_r, q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k)\,.$$

One can then write the equalities:

$$\chi_i = \chi_i\,(0, u_2, \ldots, u_r, \psi_1, \ldots, \psi_{2(k-r)}) \qquad (i = 1, 2, \ldots, 2\,(k-r)),$$

and upon solving them ([31]) for $\psi_1, \ldots, \psi_{2(k-r)}$, one will obtain the desired integrals of (S). In order to obtain 2 (k – r) distinct integrals of (r'), it will suffice to replace $u_2, \ldots, u_r$ with functions of $q_1, \ldots, q_r$ in it.

The integration of the system (r') is then reduced entirely to the integration of the single equation (S), which is equivalent to a system of 2 (k – r) first-order differential equations. However, we need only one particular integral of (r'). Mayer's method permits one to calculate such an integral when one knows just one integral of (S).

Indeed, let $\chi\,(u_1, u_2, \ldots, u_r, q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k)$ be an integral of equation (S) that we naturally assume is not simply a function of $u_2, \ldots, u_r$. For $u_1 = 0$, that integral will depend upon at least one of the variables $q_{r+1}, \ldots, p_k$, say q_{r+1}. Since it reduces (for $u_1 = 0$) to the form $\varpi\,(u_2, \ldots, u_r)$, it will coincide identically with the integral $\varpi\,(u_2, \ldots, u_r)$ of (S). On the other hand, we have the equality:

$$(l) \qquad \chi\,(u_1, u_2, \ldots, u_r, q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k) \equiv \chi\,(0, u_2, \ldots, u_r, \psi_1, \psi_2, \ldots, \psi_{2(k-r)})\,.$$

By hypothesis, the ψ_i satisfy not only the equation (S), but also all of equations (R'), since they will be integrals of (r') when one reverts to the original variables. The problem is then to define one of the integrals ψ_i in terms of the integral χ.

First of all, if the function χ is independent of $u_2, \ldots, u_r$ for $u_1 = 0$ then the integral χ will be the desired integral. If that is not true then one solves equation (l) for ψ_1:

$$\psi_1 = g_1\,(u_1, u_2, \ldots, u_r, q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k, \psi_1, \psi_2, \ldots, \psi_{2(k-r)})\,.$$

If g_1 is independent of $\psi_1, \ldots, \psi_{2(k-r)}$ then g_1 will define the desired integral ([32]). Otherwise, express the idea that ψ_1 satisfies all of equations (r'), while taking into account the fact that $\psi_2, \ldots, \psi_{2(k-r)}$ are also integrals of that system. Each equation in (r') will correspond to a relation of the form:

$$\frac{\partial g_1}{\partial u_1} + v_{r+1}\frac{\partial g_1}{\partial p_{r+1}} + \cdots + v_{k-r}\frac{\partial g_1}{\partial p_k} - \varpi_{r+1}\frac{\partial g_1}{\partial q_{r+1}} - \cdots - \varpi_k\frac{\partial g_1}{\partial q_k} = 0\,,$$

in which the v, ϖ are functions of $u_1, \ldots, u_r, q_{r+j}, \ldots, p_{r+j}, \ldots$ If all of the left-hand sides of those r equations are identically zero (no matter what the u, p, q, ψ) then the function g_1, in which one

([31]) In order for this solution to be possible, it is necessary and sufficient that the functional determinant $\dfrac{D(\chi_1,\ldots,\chi_{2(k-r)})}{D(q_{r+1},\ldots,q_k,p_{r+1},\ldots,p_k)}$ is not annulled identically for $u_1 = 0$. If that determinant is zero then one of the integrals, say χ_1, will be equal to $F\,(\chi_2, \ldots, \chi_{2(k-r)}, u_2, \ldots, u_r)$ for $u_1 = 0$, and as a result, it will coincide with the integral F. The integrals χ_i will not be distinct then.

([32]) g_1 never reduces to an absolute constant A. Otherwise, the identity (l), in which one sets $\psi_1 = A$ and gives arbitrary numerical values to $\psi_2, \ldots, \psi_{2(k-r)}$, would demand that χ must be simply a function of $u_2, \ldots, u_r$.

gives arbitrary numerical values to $\psi_2, \ldots, \psi_{2(k-r)}$ (which is function that cannot reduce to a constant, from the foregoing), will be the desired integral. Otherwise, at least one of those left-hand sides will be non-zero and contain the variables $\psi_2, \ldots, \psi_{2(k-r)}$, and one can solve the corresponding equation for ψ_2 (for example). Otherwise, there will exist a relation between the independent variables u, p, q. Therefore, perform that solution:

$$\psi_2 = g_2\,(u_1, \ldots, u_r, q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k, \psi_1, \psi_2, \ldots, \psi_{2(k-r)})$$

and argue with g_2 as one did with g_1 . Upon continuing in that way, either one will come to a function g_l [$l < 2\,(k-r)$] that satisfies the relations:

$$\frac{\partial g_l}{\partial u_1} + v_{r+1}\frac{\partial g_l}{\partial p_{r+1}} + \cdots - \varpi_{r+1}\frac{\partial g_l}{\partial q_{r+1}} - \cdots = 0$$

identically, and that function, in which one gives arbitrary numerical values to $\psi_{l+1}, \ldots, \psi_{2(k-r)}$, will be the desired integral ψ, or one will arrive at an equality:

$$\psi_{2(k-r)} = g_{2(k-r)}\,(u_1, \ldots, u_r, q_{r+1}, \ldots, q_k, p_{r+1}, \ldots, p_k)$$

that gives the desired integral ψ. Observe that in the latter case, the preceding relation $\psi_{2(k-r)-1} = g_{2(k-r)-1}$ will give $\psi_{2(k-r)-1}$ as a function of $\psi_{2(k-r)}$ and independent variables, so as a function of those latter variables. Upon reassembling the series of relations $\psi_l = g_l$, one will thus deduce all of the integrals $\psi_1, \ldots, \psi_{2(k-r)}$ from the integral χ.

However, the essential point is the following one:

In order to obtain an integral of the system (r'), *it will suffice to form an integral of equation* (3), *which is equivalent to a system of* $2\,(k-r)$ *first-order equations.*

Conclusion. – By definition, in order to integrate the given partial differential equation:

$$f_0 \equiv p_1 - F_1\,(p_1, \ldots, p_k, q_1, \ldots, q_k) = 0$$

when one employs the Mayer procedure, one must successively calculate a first integral of an ordinary differential system of order $2\,(k-1)$, then a system of order $2\,(k-2)$, etc., and finally, of a system of order 2. At each intermediate integration, the order of the auxiliary differential system will diminish by two units.

On the contrary, when one seeks to integrate the canonical system:

$$\frac{dq_1}{\dfrac{\partial f_0}{\partial p_1}} \equiv \ldots = \frac{dq_k}{\dfrac{\partial f_0}{\partial p_k}} = -\frac{dp_1}{\dfrac{\partial f_0}{\partial q_1}} = \ldots = -\frac{dp_k}{\dfrac{\partial f_0}{\partial q_k}}$$

directly (where $f_0 \equiv$ const. is one first integral), the search for one integral (that is distinct from f_0) will once more be a differential operation of order equal to $2k - 2$. Once that integral has been calculated, the search for a new integral will come down to the search for an integral of a system of $2k - 3$ first-order equations, and so on. The determination of an intermediate integral will permit one to lower the order of the differential system by one unit each time. One will then be led to successively look for a particular integral of a differential system of order $2k - 2$, then a system of order $2k - 3$, etc., a system of order 3, and finally a system of order 2. The calculation is then achieved by quadratures, from the theory of the last multiplier.

That comparison will suffice to establish the superiority of the method of Jacobi and Mayer (compared to the general method) for the integration of a canonical system, at least when k exceeds 2. However, one should not forget that the method supposes essentially that H exists, i.e., that the forces derive from a potential.

Remarks on the methods of Jacobi and Mayer.

When one employs Jacobi's procedure to integrate an intermediate system (r), the integration is even more complicated than is advantageous when one applies Poisson's theorem, so it would hardly seem that one might profit from some simplifications that might result from it in the integration. When one employs Mayer's procedure, the integration is not complicated when the Poisson parentheses give results that are not illusory, but one does not employ those results. As for the first Jacobi method (see pp. 304), there is a gap in it that was filled by Sophus Lie. On that point, we shall refer to the works that were cited before.

Examples:

I. – The motion of a free material point x, y, z of mass 1 that is subject to a force that derives from a potential $U = yz + zx + xy$.

1. – Applying Poisson's theorem.

Since H is equal to $\frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - (yz + zx + xy)$ here, the canonical equations of motion will be:

$$(A) \quad \begin{cases} \dfrac{dx}{dt} = p_1, \\ \dfrac{dy}{dt} = p_2, \\ \dfrac{dz}{dt} = p_3, \end{cases} \qquad (B) \quad \begin{cases} \dfrac{dp_1}{dt} = y + z, \\ \dfrac{dp_2}{dt} = z + x, \\ \dfrac{dp_3}{dt} = x + y. \end{cases}$$

It will suffice to know five first integrals of motion that are independent of time. One already has:

$$f_0 \equiv (p_1^2 + p_2^2 + p_3^2) - 2(yz + zx + xy) = h .$$

If one adds corresponding sides of equations (B) then one will find that:

$$\frac{d}{dt}(p_1 + p_2 + p_3) = 2\,(x + y + z)\,,$$

so upon multiplying the two sides of that by $p_1 + p_2 + p_3 \equiv d\,(x + y + z) / dt$, one will get the integral:

$$f \equiv (p_1 + p_2 + p_3)^2 - 2\,(x + y + z)^2 = C .$$

Upon subtracting the first two of equations (B), one will then find:

$$f_1 \equiv (p_1 - p_2)^2 + (x - y)^2 = c_1 ,$$

which is an integral that will imply two others by permutation, namely, f_2 and f_3 . Those three integrals f_1, f_2, f_3 are distinct, but the integral f is a consequence of them, since one has:

$$f \equiv 3\,f_1 - f_1 - f_2 - f_3 . \quad (?)$$

Let us see whether Poisson's theorem will give us any new integrals. Let us first associate f and f_1. If we note that p_3 and z do not enter into f_1 and that the derivatives $\frac{\partial f_1}{\partial p_1}$, $\frac{\partial f_1}{\partial p_2}$, on the one hand, and $\frac{\partial f_1}{\partial x}$, $\frac{\partial f_1}{\partial y}$, on the other, are equal and of opposite sign, moreover, while p_1, p_2, and x, y enter into f symmetrically, then we will see that (f, f_1) is identically zero. Let us then associate (f_1, f_2):

$$\varphi \equiv (f_1, f_2) \equiv p_1\,(z - y) + p_2\,(x - z) + p_3\,(y - x) .$$

Is that integral (which can be obtained directly by summing the three area equalities) distinct from the preceding ones? It is easy to see that it is not. Indeed, write the integrals f_1, f_2, f_3 thus:

$$p_2 - p_3 = \sqrt{c_1 - (y-z)^2}\,, \qquad p_3 - p_1 = \sqrt{c_2 - (z-x)^2}\,, \qquad p_1 - p_2 = \sqrt{c_3 - (x-y)^2}\,.$$

If one adds their corresponding sides then one will find that:

$$0 = \sqrt{c_1 - (y-z)^2} + \sqrt{c_2 - (z-x)^2} + \sqrt{c_3 - (x-y)^2}\,,$$

which is a relation that is independent of the p_i . On the other hand, if one replaces p_1 and p_2 with $p_3 - \sqrt{c_2 - (z-x)^2}$ and $p_3 + \sqrt{c_1 - (y-z)^2}$, respectively, in φ then that will give:

$$C = -\ (z-y)\sqrt{c_2 - (z-x)^2} + (x-z)\sqrt{c_1 - (y-z)^2}\,,$$

which is a relation that must coincide with the preceding one, since otherwise y and z would be defined as functions of x and only four arbitrary constants c_1, c_2, c_3, C. Furthermore, one will soon see that those two relations depend upon only the differences of the variables, namely, $x - y$, $y - z$. Upon eliminating one of those differences, the other one will disappear, and one will effortlessly arrive at the condition:

$$4C^2 = 2c_2\,c_3 + 2c_3\,c_1 + 2c_1\,c_2 - c_1^2 - c_2^2 - c_3^2\,,$$

or rather:

$$4\varphi^2 = 2\,f_2\,f_3 + 2\,f_3\,f_1 + 2\,f_1\,f_2 - f_1^2 - f_2^2 - f_3^2\,.$$

The integral φ is symmetric with respect to the three variables p and the three variables x, y, z, so one will obtain the same integral by forming the combinations (f_2, f_3) and (f_3, f_1). Poisson's theorem will not provide any new integral then: The three integrals f_1, f_2, f_3 then form a group.

Nonetheless, observe that if one regards only the two integrals f_1, f_2 (combined with f_0) as known then Poisson's theorem will provide a new integral φ, but only one.

We have thus obtained, by definition, four distinct integrals f_0, f_1, f_2, f_3 (or φ). The theory of the last multiplier then teaches us that the problem can be solved by quadratures.

2. – Applying Jacobi's method.

The problem is to find a complete integral of the equation:

$$(C) \qquad f_0 \equiv p_1^2 + p_2^2 + p_3^2 - 2(y\,z + z\,x + x\,y) = h\,.$$

In order to do that, employ the first form of Jacobi's method and first determine an integral of the canonical system (A), (B), namely, the integral:

$$(D) \qquad f \equiv (p_1 + p_2 + p_3)^2 - 2\,(x + y + z)^2 = C\,.$$

Now calculate a new integral f_1 of the system (A), (B) and try to deduce an integral Ψ that satisfies the condition that $(f, \Psi) \equiv 0$. Take:

$$(E) \qquad f_1 \equiv (p_1 - p_2)^2 + (x - y)^2 = c_1 \ .$$

One will find that (f, f_1) is identically zero. Therefore, equations (C), (D), (E) define a complete integral of (C). One infers p_2 as a function of p_1 from equation (E), substitutes that in (D), and then infers p_3 as a function of p_1. Finally, one substitutes that in (C), and one then obtains p_1, p_2, p_3 as a function of x, y, z, and h, c, c_1 . The complete integral $V = \int p_1\, dx + p_2\, dy + p_3\, dz$ is given by some quadratures that one can carry out until the conclusion with the aid of the change of variables $x + y + z = u$, $y - z = v$, $z - x = w$.

Instead of the integral f_1, one can combine f_0 and f with the integral φ :

$$(E') \qquad \varphi \equiv p_1\,(z - y) + p_2\,(x - z) + p_3\,(y - x) = C \ .$$

Indeed, an immediate calculation shows that (f, φ) is identically zero. The three equations (C), (D), (E'), are then symmetric with respect to the variables.

Finally, suppose that one starts from the integral f_1, and not f, and one then attempts to deduce an integral Ψ from the second integral f_2 such that (f_1, Ψ) is identically zero. In order to do that, one forms $(f_1, f_2) \equiv \varphi$, and then (f_1, φ), which must be expressed as a function of f_1, f_2, φ, from the foregoing, and indeed one will get:

$$(f_1, \varphi) \equiv f_2 - f_3 \equiv - f_1 - 2\sqrt{f_1\, f_2 - \varphi^2} \ .$$

One then sets $\Psi = F\,(f_1, f_2, \varphi)$, and F must satisfy the equation:

$$\frac{\partial F}{\partial f_2}\varphi - \frac{\partial F}{\partial \varphi}(f_1 + 2\sqrt{f_1\, f_2 - \varphi^2}), \qquad (?)$$

which is an equation for which one will easily find an integral F_1 . Upon replacing f_1 with c_1 in F_1 and f_2 and φ with their expressions in terms of p_1, x, y, z, one will obtain a relation $\Psi = c'$ that will determine a complete integral when it is combined with $f_0 = h$, $f_1 = c_1$.

Remark. – If one had studied the system of Lagrange equations directly:

$$x'' = y + z\,, \qquad y'' = z + x\,, \qquad z'' = x + y\,,$$

which define the motion of the point, then one would have had to integrate some homogeneous linear equations with constant coefficients that would have shown that in the present case, all of the integrations can be pushed to the limit, which is a result that the preceding method does not exhibit in terms of the variables x, y, z. More generally, before one employs Jacobi's method, it is important to change the variables if one is to perceive the means by which one can separate at least some of the variables in the new equation in the same way that one did in the examples that were treated up to now. For example, an orthogonal change of variables x, y, z will preserve T here and reduce U to the form $\lambda x^2 + \mu y^2 + \nu z^2$. Whenever U is a polynomial of degree two (whether homogeneous or not) with respect to x, y, z, an orthogonal change of variables will reduce U to the form:

$$\lambda x^2 + \mu y^2 + \nu z^2 + l\,x + m\,y + n\,z + d\,,$$

and the integration will then be immediate.

II. – Motion of a material point that moves in a plane and is subject to a force that derives from the potential $U = \lambda\,(x + y + t\,(x - y))$.

In order to minimize the numerical coefficients, we suppose that the point has a mass of 1/2 and that λ is equal to 1/4.

We will then have:

$$H = \tfrac{1}{4}\,(x'^2 + y'^2) - \tfrac{1}{4}\,[x + y + t\,(x - y)] = p_1^2 + p_2^2 - \tfrac{1}{4}\,[x + y + t\,(x - y)]\,,$$

and we can find a complete integral of the equation:

$$p + p_1^2 + p_2^2 = \tfrac{1}{4}\,[x + y + t\,(x - y)]\,, \tag{1}$$

in which p, p_1, p_2 are the derivatives $\dfrac{\partial V}{\partial t}$, $\dfrac{\partial V}{\partial x}$, $\dfrac{\partial V}{\partial y}$ of a function $V\,(t, x, y)$.

We shall use the method of Jacobi and Mayer and first attempt to find a first integral $f\,(p_1, p_2, t, x, y)$ of the system:

$$\frac{4\,dp_1}{1+t} \equiv \frac{dx}{2\,p_1} = \frac{4\,dp_2}{1+t} = \frac{dy}{2\,p_2}\,.$$

The combination:

$$\frac{2\,(dp_1 + dp_2)}{1} = \frac{dx + dy}{2\,(p_1 + p_2)}$$

will immediately give the integral:

$$(p_1 + p_2)^2 - \tfrac{1}{2}\,(x + y) = c_1\,. \tag{2}$$

Solve equations (1) and (2) with respect to p and p_1. That will give:

$$(3)\qquad \begin{cases} p = -2p_2^2 + 2p_2\sqrt{\tfrac{1}{2}(x+y)+c_1} + \tfrac{1}{4}[t(x-y)-(x+y)] - c_1, \\ p_1 = -p_2 + \sqrt{\tfrac{1}{2}(x+y)+c_1}\,. \end{cases}$$

Now look for an integral $f(p_2, t, x, y)$ that is common to the two equations:

$$(4)\qquad \begin{cases} \dfrac{\partial f}{\partial t} + \dfrac{1}{2}\dfrac{\partial f}{\partial p_2}\left[\dfrac{p_2}{\sqrt{\tfrac{1}{2}(x+y)+c_1}} - \tfrac{1}{2}(1+t)\right] + \dfrac{\partial f}{\partial y}\left[4p_2 - 2\sqrt{\tfrac{1}{2}(x+y)+c_1}\right] = 0, \\ \dfrac{\partial f}{\partial x} + \dfrac{1}{4}\dfrac{\partial f}{\partial p_2}\dfrac{1}{\sqrt{\tfrac{1}{2}(x+y)+c_1}} + \dfrac{\partial f}{\partial y} = 0. \end{cases}$$

Let us apply Mayer's procedure. Since the coefficients in equations (4) are holomorphic for $t = 0$ (in which p_2, x, y, z are arbitrary), we set $t = t$, $x = t\,\xi$. If F denotes what f will become under a change of variables then one will have:

$$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial t} + \xi\,\frac{\partial f}{\partial x},$$

and one must determine an integral F to the single equation:

$$(5)\qquad \frac{\partial F}{\partial t} + \frac{1}{4}\frac{\partial F}{\partial p_2}\left[\frac{2p_2 - \xi}{\sqrt{\tfrac{1}{2}(t\,\xi+y)+c_1}} - (1+t)\right] + \frac{\partial F}{\partial y}\left[4p_2 + \xi - 2\sqrt{\tfrac{1}{2}(t\,\xi+y)+c_1}\right] = 0,$$

or rather a first integral of the system:

$$dt = \frac{4\,dp_2}{\dfrac{2p_2 - \xi}{\sqrt{\tfrac{1}{2}(t\,\xi+y)+c_1}} - (1+t)} = \frac{dy}{4p_2 + \xi - 2\sqrt{\tfrac{1}{2}(t\,\xi+y)+c_2}}\,.$$

The change of variables $F = -2p_2 + \sqrt{\tfrac{1}{2}(t\,\xi+y)+c_1}$ will immediately exhibit the integral of (5):

$$F = -\tfrac{1}{4}t^2 - 2p_2 + \sqrt{\tfrac{1}{2}(t\,\xi+y)+c_1}\,.$$

That integral will reduce to $2p_2 + \sqrt{\tfrac{1}{2}y + c_1}$ for $t = 0$, which is an expression that does not depend upon the variable ξ. As a result, it will be an integral of the system (4) when one replaces ξ with x / t in it. Thus, write down the equation:

$$(6)\qquad 2p_2 + \tfrac{1}{4}t^2 - \sqrt{\tfrac{1}{2}(x+y)+c_1} = c_2$$

and solve equations (3), (6) for p, p_1, p_2. That will give:

(7)
$$\begin{cases} p = -\frac{1}{2}c_1 + \frac{1}{4}t(x+y) - \frac{1}{2}(\frac{1}{4}t^2 - c_2)^2, \\ p_1 = -\frac{1}{2}c_2 + \frac{1}{8}t^2 + \frac{1}{2}\sqrt{\frac{1}{2}(x+y)+c_1}, \\ p_2 = \quad \frac{1}{2}c_2 - \frac{1}{8}t^2 + \frac{1}{2}\sqrt{\frac{1}{2}(x+y)+c_1}, \end{cases}$$

and those equations must define the partial derivatives of a function V (t, x, y). Indeed, one immediately sees that the function:

$$V = \frac{2}{3}[\frac{1}{2}(x+y)+c_1]^{3/2} + \frac{1}{2}(x-y)(\frac{1}{4}t^2 - c_2) - \frac{t^5}{2^5 \cdot 5} + \frac{1}{12}c_2\,t^3 - \frac{1}{2}(c_1 + c_2^2)\,t + \text{const.}$$

admits p, p_1, p_2 as derivatives.

The problem is then solved, moreover. The motion is defined by the equations:

$$\frac{\partial V}{\partial c_1} = b_1\,, \qquad \frac{\partial V}{\partial c_2} = b_2\,,$$

i.e.:

(8)
$$[\frac{1}{2}(x+y)+c_1]^{1/2} - \frac{1}{2}t = b_1\,, \quad -\frac{1}{2}(x+y) + \frac{1}{12}t^3 - \frac{1}{2}c_1\,t = b_2\,,$$

here, which are equalities that can be written:

(9)
$$x + y = \frac{1}{2}t^2 + \alpha\,t + \beta\,, \qquad x - y = \frac{1}{6}t^3 + \alpha'\,t + \beta'$$

when one sets:

$$b_1 = -\frac{1}{2}\alpha\,, \quad c_1 = \frac{1}{4}(\alpha^2 - 2\beta), \qquad b_2 = -\beta'\,, \qquad c_2 = -\alpha'.$$

Remark. – If one appeals to the ordinary equations of motion, instead of using the method of Jacobi and Mayer, then one will have to integrate the very simple system:

$$x'' = \frac{1}{2}(1+t)\,, \qquad y'' = \frac{1}{2}(1-t)\,,$$

which will immediately give back the equalities (9) upon adding and subtracting and will further simplify when one makes the change of variables $x + y = x_1$, $x - y = y_1$, moreover. If one employs the Jacobi method after the latter change of variables then the partial differential equations can be integrated immediately by separating the variables.

In regard to that, it is fitting to observe that, in theory, the method of Jacobi and Mayer is the most advantageous general method, but it can present some grave inconveniences in the applications. Solving for p_i can rapidly imply some complications when the variables are not distinguished from each other by having some special character. Furthermore, the single equation to which Mayer's procedure will lead can depend upon some parameters, and above all, it results from combining several distinct equations, which is a situation that generally masks the simplification, and in particular, any possible separation of variables.

It is then appropriate to employ the method of Jacobi and Mayer only after one has chosen the variables most judiciously and taken into account all of the simplifications that will reveal the most direct procedure. The very elementary example that we just treated shows how complications will quickly appear when one does not take any prior precautions.

Remark on the canonical systems that are deduced from a first integral.

Consider an arbitrary canonical system:

$$\text{(1)} \qquad \frac{dq_1}{\frac{\partial H}{\partial p_1}} = \frac{dp_1}{-\frac{\partial H}{\partial q_1}} = \frac{dq_2}{\frac{\partial H}{\partial p_2}} = \ldots = \frac{dp_k}{-\frac{\partial H}{\partial q_k}},$$

and let $f(q_1, \ldots, q_k, p_1, \ldots, p_k)$ be a first integral of that system. We know that we can determine a complete integral V of the equation:

$$H\left(q_1, \ldots, q_k, \frac{\partial V}{\partial q_1}, \ldots, \frac{\partial V}{\partial q_k}\right) = h$$

that satisfies the equation:

$$f\left(q_1, \ldots, q_k, \frac{\partial V}{\partial q_1}, \ldots, \frac{\partial V}{\partial q_k}\right) = \alpha .$$

From that, the integral of the canonical system:

$$\text{(2)} \qquad \frac{dq_1}{\frac{\partial f}{\partial p_1}} = \frac{dp_1}{-\frac{\partial f}{\partial q_1}} = \frac{dq_2}{\frac{\partial f}{\partial p_2}} = \ldots = \frac{dp_k}{-\frac{\partial f}{\partial q_k}}$$

can be deduced from that same function $V(q_1, \ldots, q_k, h, a, a_1, \ldots, a_{k-2})$. The integral of the system (1) will be defined by the equalities:

$$\frac{\partial V}{\partial a} = b , \qquad \frac{\partial V}{\partial a_1} = b_1 , \qquad \ldots, \qquad \frac{\partial V}{\partial a_{k-2}} = b_{k-2} , \text{ and } \qquad p_i = \frac{\partial V}{\partial q_i},$$

while the integral of the system (2) will be defined by the analogous equalities:

$$\frac{\partial V}{\partial h} = c\,, \qquad \frac{\partial V}{\partial a_1} = b_1\,, \qquad \ldots, \qquad \frac{\partial V}{\partial a_{k-2}} = b_{k-2}\,, \text{ and } \qquad p_i = \frac{\partial V}{\partial q_i}.$$

One sees that the relations between the q_i that are determined by the systems (1) and (2) admit the common system of integrals:

$$\frac{\partial V}{\partial a_1} = b_1\,, \qquad \frac{\partial V}{\partial a_2} = b_2\,, \qquad \ldots, \qquad \frac{\partial V}{\partial a_{k-2}} = b_{k-2}\,,$$

from which one can generally eliminate $(k-3)$ of the variables q_i, which will give $(k-2)$ relations:

$$\varphi_1 (q_1, \ldots, q_k, h, a, a_1, \ldots, a_{k-2}) = 0\,, \quad \varphi_2 (q_1, \ldots, q_k, h, a, a_1, \ldots, a_{k-2}) = 0\,, \ \ldots,$$

$$\varphi_{k-2} (q_1, \ldots, q_k, h, a, a_1, \ldots, a_{k-2}) = 0$$

that are common to (1) and (2).

In particular, assume that the system (1) corresponds to a problem in dynamics ($H = T - U$), and that f is likewise of the form $\mathcal{T} - V$, in which $\mathcal{T}$ is homogeneous of degree two with respect to the p_i, which do not enter into V. At the same stroke, one will solve the given problem of dynamics and the problem for which the canonical function is $\mathcal{T} - V + C\,(T - U)$, in which C denotes a constant, by finding a complete integral V that is common to the system $H = h$, $f = \alpha$.

Having done that, if one considers an arbitrary system (1) and an arbitrary integral f then one can say that in general the integration of the system (1) and the integration of the system (2) are two equivalent problems. However, one cannot conclude that when one has determined a certain integral f of a particular system, it will suffice for one to integrate the system (2) in order to know how to integrate the first one. When one has obtained the general integral of (2) in any manner, one can indeed deduce a complete integral of the equation:

$$f\left(q_1, \ldots, q_k, \frac{\partial V}{\partial q_1}, \ldots, \frac{\partial V}{\partial q_k}\right) = \alpha\,,$$

but that integral will not generally satisfy the equation $H = h$, and knowing the general integral of the system (2) cannot be of any use in the determination of an integral V that is common to the equations $H = h$, $f = \alpha$.

For example, suppose that H does not depend upon q_1. The system (1) will admit the integral $f = p_1$, and the system (2) that is deduced from f will reduce to the following one:

$$\frac{dp_1}{dq_1} = 0\,, \qquad \frac{dq_2}{dq_1} = 0\,, \qquad \ldots, \qquad \frac{dp_h}{dq_1} = 0\,,$$

whose general integral is:

$$p_1 = \text{const.},\qquad p_2 = \text{const.},\qquad \ldots,\qquad p_k = \text{const.}$$

However, one cannot deduce a function V from that integral that will verify both of the equations:

$$H\left(q_1,\ldots,q_k,\frac{\partial V}{\partial q_1},\ldots,\frac{\partial V}{\partial q_k}\right) = h, \qquad \frac{\partial V}{\partial q_1} = a\,,$$

so one will be reduced to merely finding a complete integral $W(q_2, \ldots, q_k)$ of the equation:

$$H\left(q_2,\ldots,q_k,a,\frac{\partial W}{\partial q_2},\ldots,\frac{\partial W}{\partial q_k}\right) = h\,.$$

Specialized first integrals.

When a function $f(t, q_1, \ldots, q_k, p_1, \ldots, p_k)$ is a first integral of the canonical system:

$$dt = \frac{dq_1}{\dfrac{\partial H}{\partial p_1}} = \frac{-dp_1}{\dfrac{\partial H}{\partial q_1}} = \ldots = \frac{dq_k}{\dfrac{\partial H}{\partial p_k}} = \frac{-dp_k}{\dfrac{\partial H}{\partial q_k}}\,, \tag{1}$$

it will satisfy the condition:

$$\frac{\partial f}{\partial t} + (f, H) = 0$$

identically.

Now assume that the preceding relation is not true identically, but is a consequence of the equation $f = 0$:

$$\frac{\partial f}{\partial t} + (f, H) = M \cdot f\,.$$

Any motion $q_i(t)$, $p_i(t)$ that corresponds to the initial conditions t_0, q_i^0, p_i^0 and satisfies the condition that $f(t^0,\ldots,q_i^0,\ldots,p_i^0) = 0$ will satisfy that condition for any t. Indeed, replace the p_i, q_i in f with functions of t. That will give:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (f, H) = M_1(t)\, f\,,$$

since the differential equation $df / dt = M_1 f$ admits no other integral that is annulled for $t = t_0$ than $f \equiv 0$, so the function $f(t, \ldots, q_i(t), \ldots, p_i(t), \ldots)$ will be identically zero if it is zero for $t = t_0$.

Conversely, if the relation $f = 0$ cannot be verified at the instant t_0 without being verified during all of its motion then one will necessarily have:

$$\frac{\partial f}{\partial t} + (f, H) = M f .$$

Indeed, any motion that satisfies the condition:

$$f(t^0, \ldots, q_i^0, \ldots, p_i^0, \ldots) = 0$$

must satisfy the condition that:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (f, H) = 0$$

for the same values $t^0, \ldots, q_i^0, \ldots, p_i^0, \ldots$, i.e., that relation must be a consequence of the relation $f = 0$.

Such a relation $f = 0$ is called a *specialized first integral*. Indeed, one can regard it as something that provides a first integral in which one has given a particular value to the constant. In order to see that, suppose that one has formed the $2k$ first integrals:

$$p_i^0 = \varphi_i(t_0, t, \ldots, p_i, \ldots, q_i, \ldots) ,$$
$$q_i^0 = \psi_i(t_0, t, \ldots, p_i, \ldots, q_i, \ldots) ,$$

and consider the first integral:

$$f(t^0, \ldots, q_i^0, \ldots, p_i^0, \ldots) = f(t_0, \ldots, \varphi_i, \ldots, \psi_i, \ldots) = F(t, \ldots, p_i, \ldots, q_i, \ldots) ,$$

in which t_0 is a number that generally enters into F. An arbitrary motion will verify the condition $F = \text{const.}$, and since the constant is annulled with $f(t^0, \ldots, q_i^0, \ldots, p_i^0, \ldots)$, any motion whose initial conditions annul f will satisfy the condition that:

$$F(t, \ldots, p_i, \ldots, q_i, \ldots) = 0 ,$$

but, by hypothesis, it will also verify the equality:

$$f(t, \ldots, p_i, \ldots, q_i, \ldots) = 0 .$$

It follows from this that those two equalities cannot be distinct, since the motions are subject to the condition that $f(t^0, \ldots, q_i^0, \ldots, p_i^0, \ldots) = 0$, which once more depends upon $2k - 1$ constants. One can then replace the equality $f = 0$ with the equality $F = 0$. If one so desires, F can again contain f as a factor.

Having made those remarks, suppose that one has determined a specialized first integral $f = 0$. From the theorem on page 293, the two equations:

(2) $$\begin{cases} \dfrac{\partial V}{\partial t} + H\left(t, q_1, \ldots, q_k, \dfrac{\partial V}{\partial q_1}, \ldots, \dfrac{\partial V}{\partial q_k}\right) = 0, \\ f\left(t, q_1, \ldots, q_k, \dfrac{\partial V}{\partial q_1}, \ldots, \dfrac{\partial V}{\partial q_k}\right) = 0 \end{cases}$$

will admit an infinitude of common integrals, and in particular, an infinitude of complete integrals (viz., integrals that depend upon $k - 1$ arbitrary constants that permit one to give arbitrary values to $k - 1$ of the derivatives $\partial V / \partial q_i$). Knowing one such integral can serve to determine the motions that satisfy the condition that $f = 0$.

Indeed, let $F = c$ be a first integral that gives the specialized integral $f = 0$ when $c = 0$. Write down the equations:

(3) $$\begin{cases} \dfrac{\partial V}{\partial t} + H\left(t, q_1, \ldots, q_k, \dfrac{\partial V}{\partial q_1}, \ldots, \dfrac{\partial V}{\partial q_k}\right) = 0, \\ F\left(t, q_1, \ldots, q_k, \dfrac{\partial V}{\partial q_1}, \ldots, \dfrac{\partial V}{\partial q_k}\right) = c, \end{cases}$$

and let V_1 $(t, q_1, \ldots, q_k, c, c_1, \ldots, c_{k-1})$ be a complete integral of that system. All of the complete integrals V of (2) are obtained by setting $c = 0$ in the complete integrals V_1 of (3). On the other hand, the general motion is defined by the equalities:

$$\frac{\partial V_1}{\partial a} = b, \qquad \frac{\partial V_1}{\partial a_1} = b_1, \qquad \ldots, \qquad \frac{\partial V_1}{\partial a_{k-1}} = b_{k-1}, \quad \text{and} \qquad p_i = \frac{\partial V_1}{\partial q_i}.$$

Therefore, when one has determined a complete integral of the system (2), say, V_1 $(t, q_1, \ldots, q_k, a_1, \ldots, a_{k-1})$, one can write down the equalities (4), which will determine $q_2, \ldots, q_k$, for example, and the p_i as functions of t and q_1 [and $2(k - 1)$ arbitrary constants]. It then remains for one to determine q_1 as a function of t, which is a problem that depends upon a second-order equation. When one knows a complete integral of (2), the determination of the motions that satisfy the condition $f = 0$ will then come down to the integration of a second-order equation.

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Applying the Legendre transformation to the Jacobi equation.

I will conclude this study of Jacobi's method with a remark concerning the manner by which the parameters q_i, p_i enter into an arbitrary canonical system and into the corresponding partial differential equation.

It is clear that the variables p_i, q_i play a symmetric role in the canonical system (1):

(1) $$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad (i = 1, 2, \ldots, k),$$

since it will suffice to change H into $-H$ if one wishes to replace the p_i with the q_i, and *vice versa*. From that, it is legitimate to replace the Jacobi equation:

(2) $$\frac{\partial V}{\partial t} + H\left(t, q_1, \ldots, q_k, \frac{\partial V}{\partial q_1}, \ldots, \frac{\partial V}{\partial q_k}\right) = 0$$

with the following one:

(3) $$\frac{\partial W}{\partial t} - H\left(t, \frac{\partial W}{\partial p_1}, \ldots, \frac{\partial W}{\partial p_k}, p_1, \ldots, p_k\right) = 0 .$$

If one knows a complete integral $W(t, p_1, \ldots, p_k, \alpha_1, \ldots, \alpha_k)$ of the latter equation then one can set:

$$\frac{\partial W}{\partial \alpha_1} = \beta_1, \ldots, \frac{\partial W}{\partial \alpha_k} = \beta_k, \quad q_i = \frac{\partial W}{\partial p_i} \qquad (i = 1, 2, \ldots, k),$$

and those equalities will define the general integral of (1).

Moreover, it is quite easy to pass from equation (2) to equation (3) by replacing the variables q_i and the function V with the new variables p_i and the new function W that are coupled with the first ones by the equalities:

(4) $$\begin{cases} p_i = \dfrac{\partial V}{\partial q_i}, \\ W = \displaystyle\sum_{i=1}^{k} q_i \frac{\partial V}{\partial q_i} - V. \end{cases}$$

Upon repeating the entirely elementary calculation with the aid of which we converted the Lagrange equations to the canonical form (see pp. 120), we will see that those formulas imply the following ones:

(5) $$\begin{cases} q_i = \dfrac{\partial W}{\partial p_i}, \\ V = \displaystyle\sum_{i=1}^{k} p_i \frac{\partial W}{\partial p_i} - W, \\ \dfrac{\partial V}{\partial t} = -\dfrac{\partial W}{\partial t}. \end{cases}$$

The change of variables (4) then transforms equation (2) into equation (3). A complete integral $V(t, q_1, \ldots, q_k, \alpha_1, \ldots, \alpha_k)$ of (2) will then correspond to a complete integral $W(t, p_1, \ldots, p_k, a_1, \ldots, a_k)$ of (3), and one will have the relations:

$$\frac{\partial V}{\partial a_i} = -\frac{\partial W}{\partial \alpha_i}.$$

The equations $\frac{\partial V}{\partial a_i} = b_i$, $p_i = \frac{\partial V}{\partial q_i}$ (i = 1, 2, …, k) imply the equations:

$$\frac{\partial W}{\partial \alpha_i} = -b_i, \quad q_i = \frac{\partial W}{\partial p_i}.$$

The transformation (4) is called the *Legendre transformation.* It is the simplest of the *contact transformations*. One can summarize the foregoing by saying that the change of variables $q_i = \varpi_i$, $p_i = \chi_i$ will preserve the canonical form of the system (1).

More generally, one can study all of the changes of variables:

$$\varpi_i = \varphi_i\,(t, p_1, \dots, p_k, q_1, \dots, q_k), \qquad \chi_i = \psi_i\,(t, p_1, \dots, p_k, q_1, \dots, q_k) \qquad (i = 1, 2, \dots, k)$$

that preserve the canonical form of an arbitrary canonical system, or (in another form) all of the transformations:

$$\varpi_i = \varphi_i\left(t, q_1, \dots, q_k, V, \frac{\partial V}{\partial q_1}, \dots, \frac{\partial V}{\partial q_k}\right) \qquad (i = 1, 2, \dots, k),$$
$$W = F\left(t, q_1, \dots, q_k, V, \frac{\partial V}{\partial q_1}, \dots, \frac{\partial V}{\partial q_k}\right)$$

that transform an arbitrary equation (2) (in which V does not occur) into another first-order equation in which W does not occur. Those transformations, which constitute the contact transformations precisely, have been the subject of considerable work by Jacobi and Sophus Lie. One can find the presentation of the latter in the second volume of Lie's *Theorie der Transformationsgruppen* and a presentation of the former in the previously-cited book by Goursat (Chapter XI).

Let us now address the particular case in which H has the form $T - U$, in which T denotes a homogeneous quadratic form in the p_i whose coefficients, as well as U, are second-order polynomials in the q_i that are independent of t and have no first-degree terms. The function H can be regarded as the canonical function of two distinct problems in mechanics, according to whether one takes the q_i or the p_i to be the parameters that define the position of the system.

Those two problems can be solved simultaneously because the canonical equations of the first one will coincide with those of the second one when one changes t into $-t$. For example, let:

$$H = L\,p_1^2 + 2M\,p_1\,p_2 + N\,p_2^2,$$

with

$$L = Aq_1^2 + 2Bq_1\,q_2 + Cq_2^2, \qquad M = A'q_1^2 + 2B'q_1\,q_2 + C'q_2^2, \quad N = A''q_1^2 + 2B''q_1\,q_2 + C''q_2^2 .$$

That function H has the canonical form that corresponds to the ds^2:

$$(\alpha) \qquad ds^2 = \frac{N\,dq_1^2 - 2M\,dq_1\,dq_2 + L\,dq_2^2}{L\,N - M^2} .$$

Now permute the variables p and q. H will become:

$$H' = L_1\,p_1^2 + 2M_1\,p_1\,p_2 + N_1\,p_2^2 ,$$

with

$$L_1 = Aq_1^2 + 2A'q_1\,q_2 + A''q_2^2, \quad M_1 = Bq_1^2 + 2B'q_1\,q_2 + B''q_2^2, \quad N_1 = Cq_1^2 + 2C'q_1\,q_2 + C''q_2^2 ,$$

and will be the canonical function that corresponds to the ds^2:

$$(\beta) \qquad ds_1^2 = \frac{N_1\,dq_1^2 - 2M_1\,dq_1\,dq_2 + L_1\,dq_2^2}{L_1\,N_1 - M_1^2} .$$

Therefore, whenever one has determined the geodesics of the first ds^2, one will have determined those of the second (and conversely). Indeed, once the geodesics of ds^2 have been calculated, the canonical system that corresponds to H will be found to have been integrated. One will then know p_1 and p_2 as functions of t. When one eliminates t from p_1 and p_2, the relations will depend upon only two arbitrary constants $p_2 = F(p_1, a, b)$, and if one replaces p_1 with q_1 and p_2 with q_2 then the new relation $q_2 = F_1(q_1, a, b)$ will define the geodesics of ds_1^2.

END OF LECTURE 17