"Ueber die Brennflächen der Strahlensysteme und die Singularitätenflächen der Compleze," J. f. reine u. angew. Math. **76** (1873), 156-169.

On the focal surfaces of ray systems and the singularity surfaces of complexes

(By M. Pasch in Giessen.)

Translated by D. H. Delphenich

Any line is associated with a two-parameter group of *polar complexes* relative to a given first-order line complex that are called *tangential complexes* when the line belongs to the complex. The congruence that is determined by the tangential complex of a complex line is of a special kind, in that both of its directrices coincide in a single line that is the complex line itself. It is of a special kind of line that *Plücker* referred to as the *singular lines* of the given complex, for which the tangential complexes are all special, and the directrices of the congruence therefore define a pencil of rays in a plane that goes through the singular lines, namely, the associated *singular plane*, that has a midpoint that lies on the singular line, namely, the associated *singular point*. *Plücker* examined the distinguished elements thus defined more closely while restricting himself to second-degree complexes. For them, he arrived at the conclusion that the associated singular point is the fixed contact point of the singular line with all of the complex curves that it contacts whose associated singular planes are the common tangential planes to the complex cone that includes all of the singular lines. He further recognized the identity of the singular lines with the lines of intersection of those pairs of planes that define a complex cone, and with the connecting lines of those point-pairs that would represent a complex curve. The vertex of the complex cone is, in any event, the associated singular point, and the plane of the complex curve is the associated singular line. In particular, *Plücker* examined the fourth-order surfaces that are defined by the singular points of second-degree complexes and the fourth-order surface that is enveloped by its singular planes, and found that these two surfaces are identical (*), in such a way that at any point of the surface, any tangent is a singular line that corresponds to the point as a singular point and the tangent plane as a singular plane.

The validity of the latter theorem of *Plücker* for complexes of arbitrary degree is proved in my Habilitationsschrift (**) by simple geometric considerations that are based upon the double definition of the focal surface of a ray system. In fact, the surface of singular points or planes defines a part of the focal surface of the congruence that is composed of the singular lines. The proof of the identity of both definitions of the singularity surfaces of a complex – or more generally, the focal surface of a ray system –

(*) *Klein* called these surfaces the *singularity surfaces* of the complex. Gött. Nachr. (1869), pp. 267.

(**) "Zur Theorie der Complexe und Congruenzen von Geraden," Giessen, 1870. [DHD: Hereinafter, referred to as *loc. cit.*]

will be communicated in what follows from the analytical representation (cf., §§ 11, 12 of *loc. cit.*), as well as the geometric interpretation of the other parts of the focal surface of singular rays and some considerations that relate to the analytical representation. Let me take this opportunity to summarize the discussion that was given in §§ 1-3, 7-8 of *loc. cit*. (§ 1).

§ 1.

Let any line $\mathfrak{x}$ be chosen from among the lines of a complex. The planes u that go through $\mathfrak{x}$ and the points ξ that lie on $\mathfrak{x}$ will be related to each other projectively by the complex, in such a way that one of the lines $\mathfrak{x}$ that are infinitely-close to the complex lines in the plane u can be drawn through the corresponding point ξ. The point ξ will be the contact point of $\mathfrak{x}$ with the complex curve that lies in the plane u, and the plane u will contact the complex cone that emanates from ξ along the generator $\mathfrak{x}$.

Now, let $\mathfrak{x}$ be a ray that is common to two given complexes, and therefore a ray of the ray system that is defined by the two complexes. The planes that go through $\mathfrak{x}$ will then be associated projectively with the points of $\mathfrak{x}$ in two ways. However, there will be two points ξ, ξ' and two planes u, u' that will be distinguished by the fact that ξ, u' and ξ', u will correspond to each other relative to the two complexes. $\mathfrak{x}$ will be intersected by infinitely many rays of the congruence at the points ξ (ξ', resp.) that lie in the planes u' (u, resp.), so ξ (ξ', resp.) will then be the focal point that is associated with the ray $\mathfrak{x}$, and u' (u, resp.) will be the corresponding focal plane of the ray system.

The surface that is composed of the focal points of the ray system – viz., the locus of all points ξ (ξ', resp.) at which two coincident rays $\mathfrak{x}$ will go through – is identical with the surface that is enveloped by the focal planes – viz., the envelope of all planes u' (u, resp.) on which two coincident rays $\mathfrak{x}$ of the system lie; it will be doubly contacted by the rays $\mathfrak{x}$, and will be called the *focal surface of the ray system* (*). However, the tangent plane to the focal surface at the point ξ is not the corresponding focal plane u', which contains the rays that go through ξ and are infinitely close to $\mathfrak{x}$. Moreover, ξ is the contact point of the focal plane u that corresponds to the other focal point of $\mathfrak{x}$, namely, the point ξ'; then, in addition to the latter, yet another tangent to the focal surface can be drawn through ξ' that is infinitely close to $\mathfrak{x}$, so it is also infinitely close to ξ.

Now, let $\mathfrak{x}$ again be an arbitrary ray of a given complex. The projective relationship between the points of $\mathfrak{x}$ and the planes that go through $\mathfrak{x}$ that is generated by the complex can be specialized in such a way that all planes will correspond to a fixed point ξ, and therefore all points, to a fixed pane u. ξ will then be the fixed contact point of $\mathfrak{x}$ with all complex curves of the pencil of planes, and u will be the common tangential planes for

(*) *Kummer*, this Journal, Bd. 57, and Math. Abh. der Berl. Akad. (1866), pp. 5.

all of the complex cones at the points that lie on $\mathfrak{x}$. Such a complex line $\mathfrak{x}$ will be called a *singular line*, ξ will be the associated *singular point*, and u will be the associated *singular plane*. One can define the same elements in other ways. One does not have a well-defined contact point with $\mathfrak{x}$ for the complex curve that lies in the plane u; i.e., $\mathfrak{x}$ will be a double tangent to that curve. Likewise, $\mathfrak{x}$ will be a double generator of the complex cone whose vertex is at ξ. The singular points will then be included as the vertices of those complex cones that possess a double generator, and the singular planes, as the planes of those complex curves that possess a double tangent. The double elements that enter into both cases will be the same, namely, the singular lines of the complex.

The singular elements of each of the three kinds define two-dimensional manifolds. The singular lines, in particular, can be considered to be the common lines of two complexes, one of which is given. The planes and points of $\mathfrak{x}$ will be projectively associated with each other on two ways. Therefore, the point ξ will correspond to the same plane u' twice, and the plane u, to the same point ξ' twice. Therefore, the singular point ξ that is associated with $\mathfrak{x}$ will prove to be one of the two focal points that is associated with $\mathfrak{x}$, and the associated singular plane u will prove to be one of the focal planes of the ray system that is composed of singular lines that are associated with $\mathfrak{x}$; however, u will not be the focal plane that corresponds to the focal point ξ. The focal surface will then decompose into two components: One of them will be, at the same time, the locus of the points ξ and the planes u, while the other one will be both the locus of the points ξ' and the envelope of the planes u'. The following theorem is then proved:

For any complex, the surface of singular points is identical with the surface of singular planes and defines a component of the focal surface of the ray system of singular lines. The singular lines are tangents to that surface at their associated singular points and along their associated singular planes.

§ 2.

I shall now given the algebraic proof of the foregoing theorem, in which I will apply the notations that were introduced in my paper "Zur Theorie der linearen Complexe" of this Journal, Bd. 75.

Let $F(\mathfrak{x})$ be an entire homogeneous function of n^{th} degree of the line coordinates $\mathfrak{x}_1, \ldots, \mathfrak{x}_6$, so $F = 0$ is the equation of a line complex of n^{th} degree. The derivatives:

$$\frac{\partial F}{\partial \mathfrak{x}_i} \text{ will be denoted by } \qquad n\, F_{i+3} \qquad (i = 1, \ldots, 6),$$

and if $\mathfrak{x}$ is a line of the complex then the quantities $\mathfrak{F}_1, \ldots, \mathfrak{F}_6$ can be considered to be coordinates of a linear complex $\mathfrak{F}$, that has the ray $\mathfrak{x}$ and the rays that are infinitely close to it in common with the complex $F = 0$ (*).

If $G = 0$ is likewise the equation of a complex of m^{th} degree, and if the derivatives of G are denoted by $m\mathfrak{G}_{i+3}$ then the common rays of the linear complexes $\mathfrak{F}$ and $\mathfrak{G}$ will define a ray system of first order and first class that has the ray $\mathfrak{x}$ and the rays that are infinitely close to it in common with the congruence of complexes $F = 0$, $G = 0$. The rays of this system will intersect two fixed lines $\mathfrak{f}$ and $\mathfrak{f}'$ that have the coordinates:

$$(1) \qquad \mathfrak{f}_i = \lambda\, \mathfrak{F}_i + \mu\, \mathfrak{G}_i, \qquad \mathfrak{f}'_i = \lambda'\mathfrak{F}_i + \mu'\mathfrak{G}_i \qquad (i = 1, \ldots, 6),$$

where $\lambda : \mu$, $\lambda' : \mu'$ will be determined by solving the equation:

$$(2) \qquad \lambda^2\, (\mathfrak{F}, \mathfrak{F}) + 2\lambda\mu\, (\mathfrak{F}, \mathfrak{G}) + \mu^2\, (\mathfrak{G}, \mathfrak{G}) = 0.$$

The lines $\mathfrak{f}$ and $\mathfrak{f}'$ will both intersect $\mathfrak{x}$; one will then have:

$$(\mathfrak{f}, \mathfrak{x}) = \lambda\, (\mathfrak{F}, \mathfrak{x}) + \mu\, (\mathfrak{G}, \mathfrak{x}) = \lambda F + \mu G = 0, \qquad (\mathfrak{f}', \mathfrak{x}) = \lambda' F + \mu' G = 0.$$

The intersection point might be called ξ (ξ', resp.), and the common planes u (u', resp.). A ray of the congruence that is infinitely close to $\mathfrak{x}$ will go through the point ξ (ξ', resp.) in the plane u (u', resp.), so u, u' will be the focal planes that are associated with the ray $\mathfrak{x}$ and ξ ξ' will be the corresponding focal points of the congruence (**). From § 5 of *loc. cit.*, the coordinates of the focal point ξ and the focal plane u will then be:

$$(3) \qquad \xi_i = \Pi_i\, (\mathfrak{x}, E(\mathfrak{f}, \alpha)), \qquad u_i = E_i\, (\mathfrak{x}, \Pi(\mathfrak{f}, \alpha)), \qquad (i = 1, \ldots, 4),$$

where the quantities a and α can be chosen arbitrarily. These equations are to be regarded as the representations of the locus of focal points (focal planes, resp.) in terms of the coordinates $\mathfrak{x}$, which are subject to the conditions $F = 0$, $G = 0$, $(\mathfrak{x}, \mathfrak{x}) = 0$.

If one now defines the differential of ξ_i relative to the $\mathfrak{x}$ and subjects the $d\mathfrak{x}$ to the conditions:

$$dF = 0, \qquad dG = 0, \qquad d(\mathfrak{x}, \mathfrak{x}) = 0$$

then one will obtain the coordinates η_i of an arbitrary point η in the plane that contacts that locus of focal points at ξ, namely:

(*) *Plücker*, *Neue Geometrie des Raumes*, pp. 292.

(**) The two lines $\mathfrak{f}$, $\mathfrak{f}'$ that are employed here for the determination of the focal point are only a special pair from two pencils; cf., the general discussion by *Klein*, Math. Ann. Bd. V, pp. 289.

$$\eta_i = \Pi_i(\mathfrak{x}, E(d\mathfrak{f}, \alpha)) + \Pi_i(d\mathfrak{x}, E(\mathfrak{f}, \alpha)).$$

However, since:

$$(\mathfrak{f}, d\mathfrak{x}) = \lambda(\mathfrak{F}, d\mathfrak{x}) + \mu(\mathfrak{G}, d\mathfrak{x}) = \frac{\lambda}{n}\, dF + \frac{\mu}{n}\, dG = 0,$$

and thus, from formula (17), pp. 122 of Bd. 75:

$$\Pi_i(d\mathfrak{x}, E(\mathfrak{f}, \alpha)) + \Pi_i(\mathfrak{x}, E(d\mathfrak{f}, \alpha)) = -(\mathfrak{f}, d\mathfrak{x})\,\alpha_i = 0$$

this expression can be transformed into the following one:

$$\eta_i = \Pi_i(\mathfrak{x}, E(d\mathfrak{f}, \alpha)) - \Pi_i(\mathfrak{x}, E(d\mathfrak{f}, \alpha)) = \Pi_i(\mathfrak{x}, b) - \Pi_i(\mathfrak{f}, c),$$

when one writes b, c for $E(d\mathfrak{f}, \alpha)$ [$E(d\mathfrak{x}, \alpha)$, resp.], to abbreviate. In this, $\Pi(\mathfrak{x}, b)$ means a point of the line $\mathfrak{x}$, and $\Pi(\mathfrak{f}, c)$ means a point of the line $\mathfrak{f}$; therefore, $\Pi(\mathfrak{x}, b)$ and $\Pi(\mathfrak{f}, c)$ will be two points in the focal plane u. The point η will lie along their connecting line, which will then be likewise in the plane u. As a result, u will contact the surface of focal points at ξ, and the focal planes will envelop this surface.

It is easy to adapt the process that was applied to the focal surface to the congruence of singular lines of a complex $F = 0$. If $\mathfrak{x}$ is a singular line of $F = 0$ then its tangential complexes will all be special linear complexes; i.e., one will have:

(4) $$(p\mathfrak{F} + q\mathfrak{x}, p\mathfrak{F} + q\mathfrak{x}) = p^2(\mathfrak{F}, \mathfrak{F}) + 2pqF + q^2(\mathfrak{x}, \mathfrak{x}) = 0$$

for all p and q. That is, to $F = 0$, $(\mathfrak{x}, \mathfrak{x}) = 0$, one must add:

$$(\mathfrak{F}, \mathfrak{F}) = 0.$$

The intersection point of the lines $\mathfrak{x}$ and $\mathfrak{F}$ will be the singular point ξ that is associated with $\mathfrak{x}$, and the common plane will be the associated singular plane u, such that:

(5) $$\xi_i = \Pi_i(\mathfrak{x}, E(\mathfrak{F}, \alpha)), \quad u_i = E_i(\mathfrak{x}, \Pi(\mathfrak{F}, a)) \quad (i = 1, \ldots, 4)$$

for arbitrary a and α. Any line that is infinitely close to the ray $\mathfrak{x}$ and either goes through ξ or lies in u will belong to the complex $F = 0$.

The differential of ξ_i with respect to the $\mathfrak{x}$, in which $d\mathfrak{x}$ must fulfill the conditions:

(6) $$dF = 0, \qquad d(\mathfrak{F}, \mathfrak{F}) = 0, \quad d(\mathfrak{x}, \mathfrak{x}) = 0,$$

will now, as above, and with the help of the condition:

$$(\mathfrak{F}, d\mathfrak{x}) = \frac{1}{n} dF = 0,$$

be converted into:

$$d\xi_i = \Pi_i (\mathfrak{x}, E(d\mathfrak{F}, \alpha)) - \Pi_i (\mathfrak{F}, E(d\mathfrak{x}, \alpha)) \qquad (i = 1, \ldots, 4), \tag{7}$$

and it is then immediately clear that all of the singular points that are infinitely close to the point ξ will lie in the plane u. However, that is the property of the singularity surface that was to be proved.

When one eliminates the $d\mathfrak{x}$ from equations (6) and (7), one will obtain a linear relationship between the differentials $d\xi_i$. Thus, when one replaces them with point coordinates η_i, one will obtain the equation of the tangent planes to the locus of singular points at ξ, and indeed in the form of a seventh-degree determinant that is set to zero (*). From the above, and when one considers equation (4), this determinant can differ from the function $u_\eta = \sum u_i \eta_i$, where u is taken from (5), only by a factor that is independent of η. In fact, there would be no difficulty in eliminating this factor by means of the relations that were presented in § 4 of *loc. cit.*

By means of formula (14) of the cited treatise, one easily brings the coordinates of an arbitrary tangent at the point ξ, and thus, the determinant $(\xi, d\xi)_k$ in the quantities:

$$\xi_i = \Pi_i (\mathfrak{x}, E(\mathfrak{F}, \alpha)) = -\Pi_i (\mathfrak{F}, E(\mathfrak{x}, \alpha)), \qquad d\xi_i = \Pi_i (\mathfrak{x}, E(d\mathfrak{F}, \alpha)) - \Pi_i (\mathfrak{F}, E(d\mathfrak{x}, \alpha)),$$

into the form $p\,\mathfrak{F}_k + q\,\mathfrak{x}_k$, where:

$$p = \sum_\lambda \mathfrak{F}_\lambda (E(\mathfrak{x}, \alpha), E(d\mathfrak{x}, \alpha))_\lambda, \qquad q = \sum_\lambda \mathfrak{x}_\lambda (E(\mathfrak{F}, \alpha), E(d\mathfrak{F}, \alpha))_\lambda .$$

The determinants in the u and du can be brought into the same form. In fact, one will have $(u, du)_{k+3} = p'\,\mathfrak{F}_k + q'\,\mathfrak{x}_k$, where:

$$p' = \sum_\lambda \mathfrak{F}_\lambda (\Pi(\mathfrak{x}, a), \Pi(d\mathfrak{x}, a))_{\lambda+3}, \qquad q' = \sum_\lambda \mathfrak{x}_\lambda (\Pi(\mathfrak{F}, a), \Pi(d\mathfrak{F}, a))_\lambda .$$

§ 3.

In order to find the focal points and focal planes of the ray system of singular lines that are associated with the ray $\mathfrak{x}$, one must substitute the roots of equation (2), in which one then assumes that:

$$2(n-1)\,\mathfrak{G}_{i+3} = \frac{\partial(\mathfrak{F}, \mathfrak{F})}{\partial \mathfrak{x}_i} \qquad (i = 1, \ldots, 6),$$

(*) Cf., Habilitationsschrift, pp. 14.

into the expression (1), such that $\mathfrak{F}_i$ and:

(8) $$\mathfrak{f}_i = \lambda\, \mathfrak{F}_i + \mu\, \mathfrak{G}_i \qquad (i = 1, \ldots, 6)$$

are the coordinates of two lines $\mathfrak{F}$ and $\mathfrak{f}$, which determine the focal points and focal planes, along with $\mathfrak{x}$; one will then have:

$$2\lambda\,(\mathfrak{F}, \mathfrak{G}) + \mu\,(\mathfrak{G}, \mathrm{G}) = 0.$$

The locus of points ξ' at which $\mathfrak{x}$ and $\mathfrak{f}$ intersect will be identical to the envelope of planes u' that include both the rays $\mathfrak{x}$ and $\mathfrak{f}$, and together with the singularity surface, it will define the complete focal surface of the singular lines. The relationship between this locus and the complex $F = 0$ shall now be sought.

To that end, I establish a generalization of the concept of polar complex. If one differentiates the equation $F = 0$ with respect to $\mathfrak{x}$ m times in succession and then replaces the $d\mathfrak{x}$ with running line coordinates $\mathfrak{y}$ then one will obtain the equation of a line complex of m^{th} degree, namely, that of the $(n - m)^{\text{th}}$ polar complex of $\mathfrak{x}$ relative to $F = 0$. The $(n - 1)^{\text{th}}$ polar will be linear. The complex curve of the linear polars in any plane that $\mathfrak{x}$ lies in reduces to a point, namely, the pole of the line $\mathfrak{x}$ relative to the complex curve of $F = 0$ (*) that one finds in the same plane, so, in particular, it will be the contact point of this complex curve with $\mathfrak{x}$, as long as $\mathfrak{x}$ itself will be a complex line, and therefore the linear polar complex will be a tangential complex. This property will be easily generalized to the $(n - m)^{\text{th}}$ polar complex. The complex curve of the $(n - m)^{\text{th}}$ polar complex of $\mathfrak{x}$ in any plane that $\mathfrak{x}$ lies in is a curve of class m that coincides with the $(n - m)^{\text{th}}$ polar of $\mathfrak{x}$ relative to the complex curve of $F = 0$ that lies in the same plane. The complex cone of the $(n - m)^{\text{th}}$ polar complex of any point $\mathfrak{x}$ that goes through $\mathfrak{x}$ will have the same relationship with the complex cone of $F = 0$ that emanates from the same point.

The coordinates of the linear complex $\mathfrak{G}$ will be:

$$(n - 1)\,\mathfrak{G}_{i+3} = \tfrac{1}{2}\frac{\partial(\mathfrak{F},\mathfrak{F})}{\partial \mathfrak{x}_i} = \sum_{k=1}^{6} \mathfrak{F}_k \frac{\partial \mathfrak{F}_{k+3}}{\partial \mathfrak{x}_i} = \frac{1}{2}\sum_{k=1}^{6} \mathfrak{F}_k \frac{\partial^2 F}{\partial \mathfrak{x}_i\, \partial \mathfrak{x}_k} \qquad (i = 1, \ldots, 6).$$

However, one also arrives at the same complex when one defines the first polar ($P = 0$) of the lines $\mathfrak{F}$ relative to $F = 0$, thus, the complex of $(n - 1)^{\text{th}}$ degree:

$$P = \sum_{k=1}^{6} \mathfrak{F}_k \frac{\partial F(\mathfrak{y})}{\partial \mathfrak{y}_i} = 0,$$

(*) *Plücker, Neue Geometrie des Raumes*, pp. 299.

and the linear polar of $\mathfrak{x}$ relative to this. The coordinates of the latter are, in fact, obtained by replacing the $\mathfrak{y}$ with $\mathfrak{x}$ in:

$$\frac{\partial P}{\partial \mathfrak{y}_i} = \frac{1}{2}\sum_{k=1}^{6} \mathfrak{F}_k \frac{\partial^2 F(\mathfrak{y})}{\partial \mathfrak{y}_i \, \partial \mathfrak{y}_k} \qquad (i = 1, \ldots, 6),$$

and are therefore proportional to the values $\mathfrak{G}_{i+3}$.

The singular plane u that is associated with the singular line $\mathfrak{x}$ includes both of the rays $\mathfrak{x}$ and $\mathfrak{F}$. From the assumptions, the complex curve of $P = 0$ will therefore be the first polar of $\mathfrak{F}$ relative to the complex curve of $F = 0$ in the same plane, and the point to which the complex curve of $\mathfrak{G}$ will reduce in u will be the polar of $\mathfrak{x}$ relative to that first polar of $\mathfrak{F}$. Now, due to (8), that point will be the one to which the complex curves of the special complex $\mathfrak{f}$ will reduces in u; i.e., the intersection point of the rays $\mathfrak{f}$ and $\mathfrak{x}$, and thus the focal point ξ. With that, when one considers that the equation $P = 0$ will be satisfied by $\mathfrak{x}$ itself, ξ will prove to be the contact point of the line $\mathfrak{x}$ with the first polar of the line $\mathfrak{F}$ relative to the curve of class n that is enveloped by the lines of the complex $F = 0$ in the plane u.

In our case, the latter curve will contact the line $\mathfrak{x}$ at two points η, η'. The contact point ξ' of the double tangent with the first polar of the line $\mathfrak{F}$ relative to that curve will thus be preserved (*) when one constructs the (conjugate to ξ) fourth harmonic point to η, η', and the intersection point of $\mathfrak{x}$ with $\mathfrak{F}$, and thus, to η, η', ξ'. One can carry out an analogous argument for the complex cone of the point ξ. If one denotes the two planes that contact that cone along the double generator $\mathfrak{x}$ by v, v' then one will find that the planes v, v' are separated harmonically by the planes u, u'. This then yields the theorem:

Among the points of any singular line of a complex, two of them will be distinguished by being the contact points with the complex curve that lies in that associated singular plane, one of which will be the associated singular point, and one of which will be the fourth harmonic point. Likewise, among the planes that go through the singular line, two of them will be distinguished by being the tangential planes to the complex cone that emanates from the associated singular point, one of which will be the associated singular plane and the other of which will be the fourth harmonic plane (**). *The surface that is defined by the fourth harmonic points will be identical with the one that is enveloped by the fourth harmonic planes, and together with the singularity surface, it will represent the complete focal surface of the singular lines.*

(*) *Salmon*, *Higher Plane Curves*, art. 67.

(**) *Plücker* considered this point and these planes on the singular lines of a second-degree complex. Cf., in particular, "N. Geom. d. R.," pp. 213.

§ 4.

For the surfaces that we have considered, the equations that were obtained from the given analytical representation by means of elimination must be developed. However, one shortly arrives at the following path:

Let a point ξ be given by three planes a, b, c that intersect at it, such that for arbitrary u (*):

$$u_\xi = (u, a, b, c).$$

Any two planes that go through ξ will then be represented by coordinates of the form:

$$p\, a_i + q\, b_i + r\, c_i\,, \qquad p'\, a_i + q'\, b_i + r'\, c_i \qquad (i = 1, \ldots, 4),$$

so as a result, an arbitrary line $\mathfrak{x}$ that goes through ξ, as the line of intersection of such planes, will be represented by coordinates (**):

$$\text{(9)} \qquad \mathfrak{x}_k = \lambda\,(bc)_{k+3} + \mu\,(ca)_{k+3} + \nu\,(ab)_{k+3} \qquad (k = 1, \ldots, 6),$$

when one sets:

$$qr' - rq' = \lambda, \quad rp' - pr' = \mu, \quad pq' - qp' = \nu.$$

A simple manifold of lines of a given complex of n^{th} degree $F(\mathfrak{x}) = 0$ that one calls a "complex cone" will go through the point ξ. The lines of such a complex cone will be represented by an equations of n^{th} degree in λ, μ, ν when one substitutes the values (9) for the $\mathfrak{x}$ in $F(\mathfrak{x}) = 0$. In an analogous way, one will express the totality of complex lines that lie in a plane – viz., a "complex curve" – by one equation in three parameters.

In order to abbreviate, set:

$$\text{(10)} \qquad \mathfrak{u}_k = (bc)_{k+3}\,, \quad \mathfrak{v}_k = (ca)_{k+3}\,, \quad \mathfrak{w}_k = (ab)_{k+3} \quad (k = 1, \ldots, 6).$$

One then must construct the discriminant Δ of the n^{th}-degree form in λ, μ, ν:

$$F(\lambda\,\mathfrak{u} + \mu\,\mathfrak{v} + \nu\,\mathfrak{w})$$

in order to present the condition by which the complex cone with the vertex ξ will possess a double generator. The discriminant Δ will be a combinant of the linear forms $(\mathfrak{u}, \mathfrak{x})$, $(\mathfrak{v}, \mathfrak{x})$, $(\mathfrak{w}, \mathfrak{x})$, and will thus include the $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$ only in certain combinations, namely, the third-degree determinants of the system (***):

(*) The notations are the ones that were applied in "Zur Th. d. lin. Compl."

(**) Cf., art. 51 of the previously-cited article.

(***) *Gordan*, Math. Ann., Bd. V, pp. 95.

$$\begin{vmatrix} \mathfrak{u}_1 & \mathfrak{u}_2 & \mathfrak{u}_3 & \mathfrak{u}_4 & \mathfrak{u}_5 & \mathfrak{u}_6 \\ \mathfrak{v}_1 & \mathfrak{v}_2 & \mathfrak{v}_3 & \mathfrak{v}_4 & \mathfrak{v}_5 & \mathfrak{v}_6 \\ \mathfrak{w}_1 & \mathfrak{w}_2 & \mathfrak{w}_3 & \mathfrak{w}_4 & \mathfrak{w}_5 & \mathfrak{w}_6 \end{vmatrix}.$$

The calculation of these determinants for the special values (10) is given in art. 33 of the treatise "Zur Th. d. lin. Compl."; four of them will vanish, four of the others will be equal to the squares of the coordinates ξ_1, ξ_2, ξ_3, ξ_4, and the remaining ones will be the products of any two of these quantities. Thus, the equation $\Delta = 0$ will represent the locus of the singular points of the complex $F = 0$.

If the quantities $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$ are given the following meaning:

$$\text{(11)} \qquad \mathfrak{U}_k = (\beta\gamma)_k\,, \qquad \mathfrak{V}_k = (\gamma\alpha)_k\,, \qquad \mathfrak{W}_k = (\alpha\beta)_k\,, \quad (k = 1, \ldots, 6),$$

and the plane of the point α, β, γ is denoted by u, such that for arbitrary ξ:

$$u_\xi = (\xi, \alpha, \beta, \gamma),$$

then the discriminant D will be the form of n^{th} degree in λ, μ, ν:

$$F(\lambda\,\mathfrak{U} + \mu\,\mathfrak{B} + \nu\,\mathfrak{W}),$$

which will be set equal to zero if $\mathfrak{U}$ is to be a singular plane. However, one will obtain D from Δ when one switches all of the indices in the coefficients of F with the adjoint ones and replaces the coordinates a, b, c with α, β, γ, resp. The equation for the singularity surface in plane coordinates will thus arise from its equation in point coordinates when one converts $\xi_1, \ldots, \xi_4$ into $u_1, \ldots, u_4$, and all of the indices in the coefficients of F to the adjoint ones.

If one introduces a symbolic representation for the form $F(\mathfrak{x})$:

$$F(\mathfrak{x}) = (\mathfrak{a}, \mathfrak{x})^n = (\mathfrak{b}, \mathfrak{x})^n = (\mathfrak{c}, \mathfrak{x})^n = \ldots$$

then one can, on the basis of the proposed considerations, immediately present the equations of the singularity surface in symbolic form from the symbolic expression for the discriminant of a ternary form of n^{th} degree. The discriminant of the form $(\lambda(\mathfrak{a}, \mathfrak{u}) + \mu(\mathfrak{a}, \mathfrak{v}) + \nu(\mathfrak{a}, \mathfrak{w}))^n$, as a function of λ, μ, ν, is known to be a linear combination of products of third-degree determinants of the form:

$$\begin{vmatrix} (\mathfrak{a},\mathfrak{u}) & (\mathfrak{a},\mathfrak{v}) & (\mathfrak{a},\mathfrak{w}) \\ (\mathfrak{b},\mathfrak{u}) & (\mathfrak{b},\mathfrak{v}) & (\mathfrak{b},\mathfrak{w}) \\ (\mathfrak{c},\mathfrak{u}) & (\mathfrak{c},\mathfrak{v}) & (\mathfrak{c},\mathfrak{w}) \end{vmatrix} = \begin{vmatrix} \mathfrak{a}_1 & \mathfrak{a}_2 & \mathfrak{a}_3 & \mathfrak{a}_4 & \mathfrak{a}_5 & \mathfrak{a}_6 \\ \mathfrak{b}_1 & \mathfrak{b}_2 & \mathfrak{b}_3 & \mathfrak{b}_4 & \mathfrak{b}_5 & \mathfrak{b}_6 \\ \mathfrak{c}_1 & \mathfrak{c}_2 & \mathfrak{c}_3 & \mathfrak{c}_4 & \mathfrak{c}_5 & \mathfrak{c}_6 \end{vmatrix} \begin{vmatrix} (bc)_1 & (bc)_2 & (bc)_3 & (bc)_4 & (bc)_5 & (bc)_6 \\ (ca)_1 & (ca)_2 & (ca)_3 & (ca)_4 & (ca)_5 & (ca)_6 \\ (ab)_1 & (ab)_2 & (ab)_3 & (ab)_4 & (ab)_5 & (ab)_6 \end{vmatrix}.$$

Now, such a determinant can be easily expressed as a quadratic form in the ξ, as is shown in § 8 of *loc. cit.*; it if it were denoted by $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \xi)$, to abbreviate, then one would have:

$$(12) \qquad (\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \xi) = \sum_{k=1}^{6} \mathfrak{a}_k (E(\mathfrak{b}, \xi), E(\mathfrak{c}, \xi))_k \, .$$

In this way, one will thus arrive at the equation for the singularity surface in point coordinates, and in order to convert the equation into plane coordinates, it will only be necessary to replace the determinant $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \xi)$ with $[\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, u]$, where:

$$(13) \qquad [\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, u] = \sum_{k=1}^{6} \mathfrak{a}_k (\Pi(\mathfrak{b}, u), \Pi(\mathfrak{c}, u))_{k+3} \, .$$

In the simplest case, for $n = 2$, one will have:

$$6\Delta = (\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \xi)^2, \qquad 6D = [\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, u]^2.$$

From § 8 of *loc. cit.*, one can, with no further assumptions, recognize the agreement between these results with the ones that *Clebsch* presented for the normal form of the complex equations that he introduced by means of the elimination of superfluous factors, first, for $n = 2$ (*), and then for arbitrary degree, and extended it to certain generally covariant surfaces of complexes (**).

In the same way that the equation for the singularity surface was derived from the discriminant of a plane curve, the equation for the focal surface of a congruence $F = 0$, $G = 0$ will be developed from the condition for the contact of two plane curves. The point ξ will be, in fact, a focal point, and the plane u will be a focal plane when two of the mn systems of values λ, μ, ν that satisfy the two equations:

$$F(\lambda\,\mathfrak{u} + \mu\,\mathfrak{v} + \nu\,\mathfrak{w}) = 0 \qquad \text{and} \qquad G(\lambda\,\mathfrak{u} + \mu\,\mathfrak{v} + \nu\,\mathfrak{w}) = 0,$$

or

$$F(\lambda\,\mathfrak{U} + \mu\,\mathfrak{V} + \nu\,\mathfrak{W}) = 0 \qquad \text{and} \qquad F(\lambda\,\mathfrak{U} + \mu\,\mathfrak{V} + \nu\,\mathfrak{W}) = 0,$$

respectively, coincide with each other. Therefore, if F and G are of second degree, and one symbolically sets:

$$F = (\mathfrak{a}, \mathfrak{x})^2 = (\mathfrak{b}, \mathfrak{x})^2 = (\mathfrak{b}, \mathfrak{x})^2, \qquad G = (\mathfrak{A}, \mathfrak{x})^2 = (\mathfrak{B}, \mathfrak{x})^2 = (\mathfrak{C}, \mathfrak{x})^2$$

then the equation for the focal surface in point coordinates will be:

$$(14) \qquad 4(3\Delta\Theta' - \Theta^2)(3\Theta\Delta' - \Theta'^2) - (9\Delta\Delta' - \Theta\Theta')^2 = 0,$$

with the use of the usual notation:

(*) Math. Ann., Bd. II, pp. 1.
(**) Gött. Nachr. (1872), no. 3, and Math. Ann. Bd. V, pp. 435.

(15) $6\Delta = (\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \xi)^2, \quad 2\Theta = (\mathfrak{a}, \mathfrak{b}, \mathfrak{A}, \mathfrak{x})^2, \quad 2\Theta' = (\mathfrak{a}, \mathfrak{A}, \mathfrak{B}, \xi)^2, \quad 6\Delta' = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \xi)^2.$

For the same case, the equation of the focal surface in plane coordinates reads:

(16) $$4(3DT' - T'^2)(3TD' - T'^2) - (3DD' - TT')^2 = 0,$$

where D, T, T', D' have the following meanings:

(17) $6D = [\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, u]^2, \quad 2T = [\mathfrak{a}, \mathfrak{b}, \mathfrak{A}, u]^2, \quad 2T' = [\mathfrak{a}, \mathfrak{A}, \mathfrak{B}, u]^2, \quad 6D' = [\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, u]^2.$

By assumption, the congruence of complexes $F = 0$, $G = 0$ goes to the congruence of singular lines of the complex $F' = 0$ under:

$$G = (\mathfrak{F}, \mathfrak{F}) = (\mathfrak{a}, \mathfrak{b})\,(\mathfrak{a}, \mathfrak{x})^{n-1}\,(\mathfrak{b}, \mathfrak{x})^{n-1},$$

and we have seen that in this case the singularity surface will define a part of the focal surface. The extraction of the singularity surface from equation (14) or (16) for $n = 2$, thus:

$$G = (\mathfrak{A}, \mathfrak{x})^2 = (\mathfrak{a}, \mathfrak{b})(\mathfrak{a}, \mathfrak{x})(\mathfrak{b}, \mathfrak{x}),$$

will be, in fact, performed in the following way: From formula (16) of *loc. cit.*, one will have:

$$\begin{aligned} \tfrac{1}{2} u_\xi\,(\mathfrak{A}, \mathfrak{x})^2 &= \tfrac{1}{2}(\mathfrak{F}, \mathfrak{F})(a, b, c, u) \\ &= (\mathfrak{F}, \mathfrak{u}) \sum \mathfrak{F}_k\,(au)_k + (\mathfrak{F}, \mathfrak{v}) \sum \mathfrak{F}_k\,(bu)_k + (\mathfrak{F}, \mathfrak{w}) \sum \mathfrak{F}_k\,(cu)_k, \\ &= (\mathfrak{c}, \mathfrak{x})\,(\mathfrak{d}, \mathfrak{x})\{(\mathfrak{c}, \mathfrak{u}) \sum \mathfrak{d}_k\,(au)_k + (\mathfrak{c}, \mathfrak{v}) \sum \mathfrak{d}_k\,(bu)_k + (\mathfrak{c}, \mathfrak{w}) \sum \mathfrak{d}_k\,(cu)_k\}. \end{aligned}$$

One will now get:

$$2\Theta = \begin{vmatrix} (\mathfrak{a}, \mathfrak{u}) & (\mathfrak{a}, \mathfrak{v}) & (\mathfrak{a}, \mathfrak{w}) \\ (\mathfrak{b}, \mathfrak{u}) & (\mathfrak{b}, \mathfrak{v}) & (\mathfrak{b}, \mathfrak{w}) \\ (\mathfrak{A}, \mathfrak{u}) & (\mathfrak{A}, \mathfrak{v}) & (\mathfrak{A}, \mathfrak{w}) \end{vmatrix}^2$$

from $(\mathfrak{A}, \mathfrak{x})^2$ when one replaces $\mathfrak{x}_k$ with:

$$\begin{vmatrix} (\mathfrak{a}, \mathfrak{u}) & (\mathfrak{a}, \mathfrak{v}) & (\mathfrak{a}, \mathfrak{w}) \\ (\mathfrak{b}, \mathfrak{u}) & (\mathfrak{b}, \mathfrak{v}) & (\mathfrak{b}, \mathfrak{w}) \\ \mathfrak{u}_k & \mathfrak{v}_k & \mathfrak{w}_k \end{vmatrix},$$

which will yield:

$$\Theta \cdot u_\xi = (\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \xi)(\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \xi)\{(\mathfrak{c}, \mathfrak{u}) \sum \mathfrak{d}_k\,(au)_k + (\mathfrak{c}, \mathfrak{v}) \sum \mathfrak{d}_k\,(bu)_k + (\mathfrak{c}, \mathfrak{w}) \sum \mathfrak{d}_k\,(cu)_k\}.$$

If one adds the two expressions that arise from switching $\mathfrak{a}$ and $\mathfrak{b}$ with $\mathfrak{c}$ to this then, if one considers the identity:

$$(\mathfrak{a}, \mathfrak{u})(\mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \xi) + (\mathfrak{b}, \mathfrak{u})(\mathfrak{c}, \mathfrak{a}, \mathfrak{d}, \xi) + (\mathfrak{c}, \mathfrak{u})(\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \xi) = (\mathfrak{d}, \mathfrak{u})(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \xi)$$

then one will succeed in proving the divisibility of Θ by Δ, so one will get:

$$3\,\Theta \cdot u_\xi = (\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \xi)^2 \{(\mathfrak{d}, \mathfrak{u}) \sum \mathfrak{d}_k\,(au)_k + (\mathfrak{d}, \mathfrak{v}) \sum \mathfrak{d}_k\,(bu)_k + (\mathfrak{d}, \mathfrak{w}) \sum \mathfrak{d}_k\,(cu)_k\}.$$

However, from the formula that was cited above, one will have:

$$(\mathfrak{d}, \mathfrak{u}) \sum \mathfrak{d}_k\,(au)_k + (\mathfrak{d}, \mathfrak{v}) \sum \mathfrak{d}_k\,(bu)_k + (\mathfrak{d}, \mathfrak{w}) \sum \mathfrak{d}_k\,(cu)_k = \tfrac{1}{2}(\mathfrak{d}, \mathfrak{d})(a, b, c, u) = \tfrac{1}{2}(\mathfrak{d}, \mathfrak{d})\,u_\xi.$$

One will then ultimately find the following value for Θ, and analogously for *T*:

$$\Theta = (\mathfrak{d}, \mathfrak{d})\,\Delta, \qquad T = (\mathfrak{d}, \mathfrak{d})\,D. \tag{18}$$

Therefore, for the singular lines of the second-degree complex, the part of the focal surface that remains after removing the singularity surface will be expressed in point (plane, resp.) coordinates by the equations:

$$\left\{\begin{aligned} 4(3\Theta' - (\mathfrak{d}, \mathfrak{d})\Theta)(3\Theta\Delta' - \Theta'^2) &= (9\Delta' - (\mathfrak{d}, \mathfrak{d})\Theta')^2\,\Delta, \\ 4(3T' - (\mathfrak{d}, \mathfrak{d})T)(3TD' - T'^2) &= (9D' - (\mathfrak{d}, \mathfrak{d})T')^2\,D. \end{aligned}\right. \tag{19}$$

The linear invariant:

$$(\mathfrak{d}, \mathfrak{d}) = \frac{\partial^2 F}{\partial \mathfrak{x}_1 \partial \mathfrak{x}_4} + \frac{\partial^2 F}{\partial \mathfrak{x}_2 \partial \mathfrak{x}_5} + \frac{\partial^2 F}{\partial \mathfrak{x}_3 \partial \mathfrak{x}_6}$$

appears as a factor in the coefficients of *F* in Θ and *T*. Therefore, the equation of the complex *F* = 0 will take the normal form (*), such that $(\mathfrak{d}, \mathfrak{d}) = 0$, so the forms Θ and *T* will vanish identically, and equations (19) will reduce to:

$$4\Theta'^3 + 27\,\Delta\Delta'^2 = 0, \qquad 4T'^3 + 27\,DD'^2 = 0.$$

Giessen, in December 1872.

(*) *Clebsch*, Math. Ann., Bd. II, pp. 1.