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# Mathematical contributions to the theory of Dirac matrices

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## § 1. Introduction

We intend that the term "Dirac matrices" should mean matrices that satisfy the relations:

$$\tag{I} \tfrac{1}{2}(\gamma^{\mu}\,\gamma^{\nu} + \gamma^{\nu}\,\gamma^{\mu}) = \delta_{\mu\nu}$$

$$(\delta_{\mu\nu} = 0 \text{ for } \mu \neq \nu, = 1;\ \mu,\ \nu = 1, 2, 3, 4).$$

In order to simplify, we always suppose that we are dealing with matrices that are comprised of four rows and four columns. One knows that all of the important properties of those matrices are independent of their numerical specialization and follow solely from the relations (1) and the fact that they have four rows and four columns. For that reason, in what follows, we shall carefully avoid any numerical specialization of the matrices that are used.

Nonetheless, there exist certain theorems for which the only known proofs up to the present are ones that appeal precisely to that numerical specialization; for example, the following theorem:

**Fundamental theorem. –** *If $\gamma^{\mu}$ and $\gamma'^{\mu}$ are two systems of matrices with four rows and four columns that satisfy both of the relations:*

$$\tag{I'} \tfrac{1}{2}(\gamma^{\mu}\,\gamma^{\nu} + \gamma^{\nu}\,\gamma^{\mu}) = \delta_{\mu\nu}, \qquad \tfrac{1}{2}(\gamma'^{\mu}\,\gamma'^{\nu} + \gamma'^{\nu}\,\gamma'^{\mu}) = \delta_{\mu\nu}$$

*then there exists a non-singular matrix S* (i.e., such that the determinant of *S* is not zero, and consequently, that the inverse $S^{-1}$ of *S* exists) *that satisfies the relation:*

$$\tag{1} \gamma'^{\mu} = S\,\gamma^{\mu}\,S^{-1}.$$

One encounters this theorem several times in the book by B. L. VAN DER WAERDEN, *Die gruppentheoretische Methode in der Quantenmechanik*, Berlin, 1932, in particular. Meanwhile, the proof that was given in that book was based upon the general theory of groups, and the theorem in question appears as a particular case of the general theorems on the representation of finite groups by matrices. It seems useful to us to give an elementary proof using the methods of the physicists, and we have found that a method of J. SCHUR ([1]) permits us to do that very simply.

We shall present that proof in detail in what follows (§ 3). We then develop some consequences of a fundamental theorem that relate the manner by which the Dirac wave functions behave under a Lorentz transformation, as well as the existence of a correspondence between the wave functions with positive energy and the ones with negative energy, which is a correspondence that is single-valued and also invariant under Lorentz transformations. The existence of that correspondence is well-known, and L. DE BROGLIE ([2]) used it in his work on the nature of light. We shall establish that correspondence (and that will be new) without having recourse to any numerical specialization of the Dirac matrices.

Finally, we prove, and under that same conditions, some quadratic identities between the Lorentz covariants that are known identities, but which have always been verified numerically, up to now, and not deduced algebraically ([3]). In that proof, we shall utilize, on the one hand, an identity that was obtained in the course of the proof of the fundamental theorem [see, eq. (II)], and on the other hand, the fact that there exists a certain matrix that is obtained in the course of considerations from the applications of the group of Lorentz transformations [§ 4, eq. (15)].

## § 2. Some preliminary statements

Consider the 16 elements:

$$(2)\qquad \left\{\begin{array}{llllll} I \\ \gamma^1 & \gamma^2 & \gamma^3 & \gamma^4 \\ i\gamma^2\gamma^3 & i\gamma^3\gamma^1 & i\gamma^1\gamma^2 & i\gamma^1\gamma^4 & i\gamma^2\gamma^4 & i\gamma^3\gamma^4 \\ i\gamma^1\gamma^2\gamma^3 & i\gamma^1\gamma^2\gamma^4 & i\gamma^3\gamma^1\gamma^4 & i\gamma^2\gamma^3\gamma^4 \\ \gamma^1\gamma^2\gamma^3\gamma^4 \end{array}\right.$$

which form a system of hypercomplex numbers. We denote the various rows in the preceding table by:

$$(2')\qquad I,\ \gamma^\mu,\ \gamma^{[\mu\nu]},\ \gamma^{[\lambda\mu\nu]},\ \gamma^5,$$

in which the brackets indicate the anti-symmetry of the quantities $\gamma$ with respect to the indices that are found between the brackets; the factor $i$ must be included in the definitions of the $\gamma^{[\mu\nu]}$ and $\gamma^{[\lambda\mu\nu]}$. Those factors are chosen in such a manner that the square of each quantity $\gamma^A$ is equal to the unit matrix:

([1]) Berliner Sitzungsberichter, Math. phys. Klasse (1905), 406.
([2]) L. DE BROGLIE, *Une nouvelle conception de la lumière*, Paris, 1934.
([3]) L. DE BROGLIE, *L'électron magnétique*, Paris, 1934. See esp. pp. 161, eq. (14); pp. 189, eq. (14); pp. 220, eq. (24); pp. 221, eq. (28).

(3) $$(\gamma^A)^2 = I .$$

(We use Latin capital letters $A$, $B$, … for indices when the index runs through the entire hypercomplex system from 1 to 16.)

**Lemma I.** – *The product of two elements $\gamma^A$ and $\gamma^B$ is always equal to a third element $\gamma^C$, up to a numerical factor $\varepsilon_{AB}$, which can have the values* $\pm 1$, $\pm i$:

(4) $$\gamma^A\gamma^B = \varepsilon_{AB}\,\gamma^C.$$

That statement is an immediate consequence of the relations (1).

**Lemma 2.** – *If $\gamma^A$ is fixed in the product $\gamma^A\gamma^B$, and $\gamma^B$ runs through the entire system of* 16 *elements then the $\gamma^C$ that is defined by equation* (4) *will also run through the entire system of* 16 *elements.*

Indeed, it results from the fact that $\gamma^A\gamma^B = \gamma^A\gamma^D$ that $\gamma^B = \gamma^D$, in such a way that if one has chosen $\gamma^A$ then the 16 elements $\gamma^A\gamma^B$ will all be distinct.

Now represent $\gamma^\mu$ (and consequently, $\gamma^A$, as well) by matrices with four rows and four columns.

**Lemma 3.** – *The sum of the diagonal elements* (which we shall call "the diagonal sum," and which we shall denote by $D$, to abbreviate) *of all the elements $\gamma^A$ is zero except when $\gamma^A$ is the identity matrix:*

$$D(\gamma^A) = 0 \qquad \text{for} \qquad \gamma^A \neq I.$$

This is a consequence of the relations (1) and the fact that the diagonal sum of a product of two matrices is independent of the order of the factors:

$$D(AB) = D(BA).$$

For example, for $\mu \neq \nu$, one has:

$$\tfrac{1}{2}[-\gamma^\nu\cdot(\gamma^\mu\gamma^\nu) + (\gamma^\mu\gamma^\nu)\cdot\gamma^\nu] = \gamma^\mu,$$

and for that reason:

$$D(\gamma^\mu) = 0.$$

That argument is also valid for $\mu = 5$ ($\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$), so:

$$D(\gamma^5) = 0.$$

On the other hand, for $\mu \neq \nu$, one has:

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu,$$

from which, it will result that:

$$D(\gamma^{[\mu\nu]}) = 0,$$

which is likewise valid for $\mu = 5$, so:

$$D(\gamma^5\,\gamma^\nu) = D(\gamma^{[\lambda\mu\nu]}) = 0. \qquad \text{Q. E. D.}$$

It then results that the 15 quantities that are not identical to $I$ can no longer be represented by the identity matrix.

**Lemma 4.** – *The matrices $\gamma^A$ are linearly independent.*

It results from:

$$\sum_{A=1}^{16} C_A\,\gamma^A = 0,$$

in which $C_A$ are ordinary numbers, that:

$$C_A = 0 \qquad \text{for any } A.$$

Indeed, in order to see that the coefficient $C_B$ of a certain arbitrarily-fixed $\gamma^B$ is zero, it suffices to multiply the equation:

$$\sum_A C_A\,\gamma^A = 0$$

by $\gamma^B$:

$$\sum_{\substack{A \\ (A\neq B)}} C_A\,\gamma^A\gamma^B + C_B = 0.$$

For $A \neq B$, the $\gamma^A\gamma^B$ will be different from the unity, and consequently, if we form the diagonal sum then it will result from Lemma 3 that $C_B = 0$.

Lemma 4 is very important, because it permits one to conclude that it is not possible to satisfy the relations (1) with matrices whose number of rows and columns is less than 4; indeed, one cannot have 16 linearly-independent matrices when the number of rows and columns is less than 4. Conversely, there exist exactly 16 linearly-independent matrices with four rows and columns, because 16 is likewise the number of elements in those matrices. It results from this that:

**Lemma 5.** – *One can represent each arbitrary matrix $X$ with four rows and columns in the form:*

$$X = \sum_A x_A\,\gamma^A$$

*by suitably choosing the numbers $x_A$, where the $\gamma^A$ are arbitrary matrices from the system* (2).

**Lemma 6.** – *For any $\gamma^\mu$ and any $\gamma^A$, one will always have:*

$$\gamma^\mu \gamma^A \gamma^\mu = \pm \gamma^A,$$

*and one can make any given $\gamma^A$ correspond to a $\gamma^\alpha$ such that one will have, with the lower sign on the right-hand side:*

$$\gamma^\alpha \gamma^A \gamma^\alpha = - \gamma^A, \qquad \text{for } \gamma^A \neq I \text{ and a suitably-chosen } \gamma^\alpha.$$

The first statement is an immediate consequence of relations (1). As for the second, one sees that one can choose:

$$\begin{array}{ll} \gamma^\alpha \neq \gamma^\nu, & \text{for } \gamma^A = \gamma^\nu, \\ \gamma^\alpha = \gamma^\mu, & \text{for } \gamma^A = \gamma^{[\mu\nu]}, \\ \kappa \text{ different from } \lambda, \mu, \text{ and } \nu, & \text{for } \gamma^A = \gamma^{[\lambda\mu\nu]}, \\ \text{any } \gamma^\mu. & \text{for } \gamma^A = \gamma^5. \end{array}$$

One deduces the following lemma from that, which is very important:

**Lemma** 7. – *If a matrix $X$ with four rows and columns commutes with each $\gamma^\mu$ (and consequently, each $\gamma^A$, as well) then it will be a multiple of the identity matrix $I$.*

That is, if $X\gamma^\mu = \gamma^\mu X$ for all $\gamma^\mu$ then $X = c \cdot I$, where $c$ is an ordinary number.

Lemma 5 permits us to put $X$ into the form:

$$X = \sum_A x_A \gamma^A .$$

Let $\gamma^B \neq I$ be chosen arbitrarily; take $\gamma^\alpha$, as in Lemma 6, in such a manner that $\gamma^\alpha \gamma^B \gamma^\alpha = - \gamma^B$. Form:

$$\gamma^\alpha X \gamma^\alpha = - x_B \gamma^B + \sum_{\substack{A \\ (\gamma^A \neq \gamma^B)}} \pm X_A \gamma^A .$$

One the other hand, one will have:

$$X = + x_B \gamma^B + \sum_{\substack{A \\ (\gamma^A \neq \gamma^B)}} x_A \gamma^A .$$

By hypothesis, one will have:

$$\gamma^\alpha X \gamma^\alpha = X,$$

and that will be possible only if $x_B = 0$. Now, $\gamma^B$ is any of the 15 elements $\gamma^A \neq I$ ; it then results that the coefficient of $I$ in the sum $\sum x_A \gamma^A$ is the only one that is not zero. Q. E. D.

## § 3. Proof of the fundamental theorem.

Before addressing the fundamental theorem, we remark that relation (4), viz.:

$$\gamma^A \gamma^B = \varepsilon_{AB}\, \gamma^C, \tag{4}$$

will give, upon taking (3) into account:

$$\gamma^B = \varepsilon_{AB}\, \gamma^A \gamma^C,$$

and upon taking the inverse, according to equation (3):

$$\gamma^B = \frac{1}{\varepsilon_{AB}}\, \gamma^C \gamma^A,$$

so:

$$\gamma^C \gamma^A = \gamma^B \varepsilon_{AB}\,. \tag{5}$$

If we now consider a second system of matrices $\gamma'^{\mu}$ that satisfy the equations:

$$\tfrac{1}{2}(\gamma'^{\mu}\gamma'^{\nu} + \gamma'^{\nu}\gamma'^{\mu}) = \delta_{\mu\nu}$$

then we will likewise have for this system:

$$\gamma'^A \gamma'^B = \varepsilon_{AB}\gamma'^C\,, \tag{4′}$$

since these relations are simply consequences of the relations (1′).

Upon following the method of J. SCHUR, one first forms the expression:

$$\sum_{B=1}^{16} \gamma'^B F \gamma^B = S, \tag{5}$$

with an arbitrary matrix with four rows and four columns that is denoted by $F$.

Equation (4′) gives us:

$$\gamma'^A S = \sum_{B=1}^{16} \varepsilon_{AB}\, \gamma'^C F \gamma^B\,.$$

However, if $A$ is fixed, and $B$ runs through the entire system of $\gamma$ then $C$ will likewise run through the entire system and will take on each value exactly once. We then change only the notations and write:

$$\gamma'^A S = \sum_{C=1}^{16} \varepsilon_{AB}\, \gamma'^C F \gamma^B\,.$$

On the other hand, if one forms the expression $S\gamma^A$, with:

$$S = \sum_{C=1}^{16} \gamma'^{C} F \gamma^{C} ,$$

which will become, upon taking (5) into account:

$$S\gamma^{A} = \sum_{C=1}^{16} \gamma'^{C} F \gamma^{C} \varepsilon_{AB} .$$

Upon comparing these, one will find that:

(7) $$\gamma'^{A} S = S\gamma^{A}.$$

That will already give the statement of the fundamental theorem, if we exclude the possibility that the matrix $S$ or its determinant is zero, and for any $F$.

We are reduced to proving that it is always possible to avoid that singularity of $S$ with matrices $F$ that are chosen conveniently.

First, it is easy to see that it is always possible to obtain a matrix $S$ that is non-zero. Indeed, if $S$, as defined by (6), is equal to zero for any $F$ then one will have:

$$\sum_{A} \gamma'^{A}_{\rho\sigma} \gamma'^{A}_{\bar{\rho}\bar{\sigma}} = 0,$$

for all $\rho, \sigma, \bar{\rho}, \bar{\sigma}$, where the $\gamma^{A}_{\rho\sigma}$ ($\rho, \sigma$ take values from 1 to 4) are the elements of the matrix $\gamma^{A}$. However, that contradicts the fact that the $\gamma^{A}$ are linearly independent. (Lemma 4 of the preceding §) We can then assume that $F$ is such that $S \neq 0$ in the sequel.

In order to prove that the determinant is non-zero, one can utilize a lemma of SCHUR ([1]). Let $S$ be a matrix that satisfies equations (7) and whose determinant is zero, without the matrix itself being zero. The lemma permits one to construct matrices $\gamma^{A}$ from $S$ that have a number of rows and columns that is less than four, and which also satisfy the relations (1). Now, Lemma 4 shows that this is impossible, in such a way that in our case, the fact that the determinant of $S$ non-zero will already be a consequence of the fact $S \neq 0$.

Without using the cited lemma of SCHUR, one can proceed in the following manner: Upon permuting the roles of $\gamma^{A}$ and $\gamma'^{A}$, construct the matrix:

$$T = \sum_{B} \gamma^{B} G \gamma'^{B} ,$$

with a suitably-chosen matrix $G$, such that one can prove, as for $S$, that it satisfies the relation:

(7′) $$\gamma^{A} T = T \gamma'^{A} .$$

([1]) See B. L. VAN DER WAERDEN, *loc. cit.*, pp. 55.

It then results from (7) and (7′) that:

$$\gamma^A TS = TS\gamma'^A,$$

so, upon using Lemma 7:

(8) $$TS = c \cdot I,$$

in which $c$ is an ordinary number.

It is now easy to prove that if $T \neq 0$ is fixed then one can always choose $F$ in (7) in such a manner that:

$$TS \neq 0.$$

Indeed, in the contrary case, the relations:

$$\sum_B \left(T\gamma'^B\right)_{\rho\sigma} \gamma^B_{\bar\rho\bar\sigma} = 0$$

will become valid for all $\rho, \sigma, \bar\rho, \bar\sigma$. However, that contradicts Lemma 4, since the $(T\gamma'^B)_{\rho\sigma}$, which contain the $T_{\rho\sigma}$ themselves for $\gamma'^B = I$, will not all be zero.

We now know that:

$$TS = c \cdot I \qquad \text{and} \qquad TS \neq 0,$$

from which, it will result that $c \neq 0$; we then conclude that:

$$\text{Det } S \neq 0, \qquad S^{-1} = \frac{1}{c} T,$$

(9) $$\gamma'^A = S\,\gamma^A\,S^{-1}.$$

The fundamental theorem is thus found to have been proved.

We finally specialize the results (6) and (7) for the case $\gamma'^A = \gamma^A$. One then has, for all $F$ such that:

$$\sum_B \gamma^B F \gamma^B = S,$$

the relation:

$$\gamma^A\,S = S\,\gamma^A,$$

so, upon using Lemma 7:

$$S = \sum_B \gamma^B F \gamma^B = c \cdot I.$$

Now, that relation must be valid for all $F$; we deduce from it that:

$$\sum_A \gamma^A_{\rho\sigma}\,\gamma^A_{\bar\rho\bar\sigma} = c_{\sigma\bar\rho}\,\delta_{\rho\bar\sigma}.$$

In order to determine the $c_{\sigma\bar\rho}$, set $\rho = \bar\sigma$ and sum over that index. We obtain:

$$\sum_A \sum_{\rho=1}^{4} \gamma^A_{\bar\rho\rho}\,\gamma^A_{\rho\sigma} = \sum_A \left(\gamma^A\right)^2_{\bar\rho\sigma} = 4c_{\sigma\bar\rho}.$$

However, from (3), one will have:

$$\left(\gamma^A\right)^2_{\bar\rho\sigma} = \delta_{\rho\bar\sigma};$$

one will then have:

$$16\,\delta_{\rho\bar\sigma} = 4c_{\sigma\bar\rho} \qquad \text{or} \qquad c_{\sigma\bar\rho} = 4\,\delta_{\bar\rho\sigma},$$

so the identities will result:

(II) $$\sum_{A=1}^{16} \gamma^A_{\sigma\rho}\,\gamma^A_{\rho\bar\sigma} = 4\,\delta_{\bar\rho\sigma}\delta_{\rho\bar\sigma},$$

which will be very useful in what follows.

We shall now pass on to the applications of the fundamental theorem.

## § 4. The definition of the matrices *A* and *B*.

In physical applications, one supposes that the matrices $\gamma^\mu$ are Hermitian; i.e., that $\gamma^\mu_{\sigma\rho}$ is equal to the complex conjugate $\gamma^{\mu *}_{\rho\sigma}$ of $\gamma^\mu_{\rho\sigma}$. In order to exhibit the fact that several of our statements about the $\gamma^\mu$ are independent of the Hermitian character of those matrices, we shall not introduce such an assumption here *a priori*.

In general we let $\gamma^{+\mu}$ denote the Hermitian conjugate matrices (i.e., transposed and conjugated) of the $\gamma^\mu$ (i.e., such that $\gamma^{+\mu}_{\rho\sigma} = \gamma^{*\mu}_{\rho\sigma}$). We remark that it results from (1) that the $\gamma^{+\mu}$ also satisfy the relations:

(9) $$\tfrac{1}{2}(\gamma^{+\mu}\gamma^{+\nu} + \gamma^{+\nu}\gamma^{+\mu}) = \delta_{\mu\nu}$$

(because $\gamma^{+\mu}\gamma^{+\nu}$ is the Hermitian conjugate of $\gamma^{\nu}\gamma^{\mu}$).

That being the case, it will result from Theorem I that there exists a matrix $A$ such that:

(10) $$\gamma^{+\mu} = A\,\gamma^\mu A^{-1} \qquad (\text{where } \gamma^{+\mu} A = A\,\gamma^\mu).$$

Lemma 7 tells us that if the $\gamma^\mu$ are given then the matrix $A$ will be determined up to a numerical factor $c$.

Upon taking the Hermitian conjugates of both sides of (10), one will then obtain:

$$A^+\gamma^\mu = \gamma^{+\mu} A^+,$$

so:

$$A^{-1}A^+\gamma^\mu = \gamma^\mu A^{-1}A^+,$$

and by virtue of Lemma 7:

$$A^{-1}A^{+} = c \cdot I, \qquad\qquad A^{+} = c\,A.$$

Upon normalizing the undetermined factor, one can always have:

(11) $$A^{+} = A;$$

i.e., the matrix $A$ is itself Hermitian (a real factor still remains undetermined in $A$). Upon substituting (11) in (10), one will find that:

$$\gamma^{+\mu}A^{+} = (A\gamma^{\mu})^{+} = A\,\gamma^{\mu},$$

or, in other words, *the matrices $A\gamma^{\mu}$ are Hermitian.* Finally, one easily verifies that for all $\gamma^{A}$:

(12) $$A\gamma^{A} \text{ is Hermitian}$$

when the $\gamma^{A}$ are defined as above (with the factors of $i$ chosen conveniently) in such a manner that $(\gamma^{A})^{2} = I$.

We further remark that for a transformation:

$$\gamma'^{\mu} = S^{-1}\gamma^{\mu}S,$$

one can set:

$$A' = S^{+}AS$$

in order to obtain:

$$\gamma^{+'\mu} = A'\gamma^{+\mu}A'^{-1}.$$

It is true that for all of the physical applications, one can assume that the $\gamma^{\mu}$ are Hermitian, and consequently, that $A = I$. I shall show that not only are certain theorems independent of that hypothesis, but also emphasize the analogy between the matrix $A$ and another matrix $B$ that is defined in the following manner ([1]): One considers the matrices $\bar{\gamma}^{\mu}$ that one obtains by starting with the $\gamma^{\mu}$ and permuting the rows and columns, and that we call the *transposed matrices* of the $\gamma^{\mu}$. One will obviously have $\bar{\gamma}^{\mu}_{\rho\sigma} = \gamma^{\mu}_{\sigma\rho}$, which is a notation that shall reserve for all matrices $a$ and $\bar{a}$ such that $\bar{a}_{\rho\sigma} = a_{\sigma\rho}$. We call a matrix *symmetric* if $\bar{a} = a$, and *skew* if $\bar{a} = -a$. Moreover, one has $\overline{(ab)} = \bar{b}\,\bar{a}$, in such a way that the operation $a \rightarrow \bar{a}$ is analogous to the operation $a \rightarrow a^{+}$; on the other hand, the complex conjugate is not necessary to define the former, which might be of interest to algebraists.

We see that they likewise verify the equations:

(14) $$\tfrac{1}{2}(\bar{\gamma}^{\mu}\bar{\gamma}^{\nu} + \bar{\gamma}^{\nu}\bar{\gamma}^{\mu}) = \delta_{\mu\nu}.$$

([1]) I have introduced that matrix before in a paper on the theory of spinors in five dimensions [Cf., Ann. de Phys. **18** (1933), 337]. Later on, I confirmed that the matrix $B$ also permits some more immediate physical applications.

One deduces from these relations, with the aid of Theorem I, as was shown previously for the $\gamma^{\dagger\mu}$, that there exists a matrix $B$ that satisfies the relations:

(15) $$\bar{\gamma}^{\mu} = B\,\gamma^{\mu}\,B^{-1} \qquad \text{(where } \bar{\gamma}^{\mu}B = B\,\gamma^{\mu}\text{)}.$$

Upon transposing that equation, one will also find that $B^{-1}\bar{B}$ commutes with all of the $\gamma^{\mu}$, and consequently, by virtue of Lemma 7:

$$\bar{B} = c\,B.$$

One then deduces that $B = c\,\bar{B}$, and consequently, $c^2 = 1$, $c = \pm 1$, with the two possibilities:

$$\bar{B} = B \qquad \text{or} \qquad \bar{B} = -B.$$

In order to distinguish between these two possibilities, (conforming to a method for which I have HAANTJES to thank) one must consider the conclusions that relate to the matrices $B\gamma^{\mu}$, $B\gamma^{[\mu\nu]}$, $B\gamma^{[\lambda\mu\nu]}$, and $B\gamma^5 = B\gamma^1\gamma^2\gamma^3\gamma^4$. Upon remarking that the permutations (12) $\to$ (21), (123) $\to$ (321) are odd, while the permutation (1234) $\to$ (4321) is even, one will see that the first possibility $\bar{B} = B$ implies that:

$$B,\ B\gamma^{\mu},\ B\gamma^5 \text{ are symmetric;} \quad B\gamma^{[\mu\nu]},\ B\gamma^{[\lambda\mu\nu]} \text{ are skew.}$$

Conversely, the second possibility $\bar{B} = -B$ implies that:

(16) $$B,\ B\gamma^{\mu},\ B\gamma^5 \text{ are skew;} \qquad B\gamma^{[\mu\nu]},\ B\gamma^{[\lambda\mu\nu]} \text{ are symmetric.}$$

Now, we can exclude the first possibility by a simple counting argument. The ten matrices $B\gamma^{[\mu\nu]}$, $B\gamma^{[\lambda\mu\nu]}$ are linearly independent, like the ten matrices $\gamma^{[\mu\nu]}$, $\gamma^{[\lambda\mu\nu]}$, and the inverse $B^{-1}$ of $B$ exists (and the six matrices $B$, $B\gamma^{\mu}$, $B\gamma^5$ are likewise linearly independent). However, a skew matrix with four rows and four columns has only six linearly-independent elements, and consequently, there exist only six linearly-independent skew matrices with four rows and four columns (on the other hand, there exist ten independent symmetric matrices of the same type). It is then impossible for the ten matrices $B\gamma^{[\mu\nu]}$, $B\gamma^{[\lambda\mu\nu]}$ to all be *skew, and only the second possibility can be realized.*

The statements (16) are valid, and one will have:

(16 cont.) $$\bar{B} = -B,$$

(17) $$\overline{\gamma^{[\mu\nu]}} = -B\gamma^{[\mu\nu]}B^{-1}, \qquad \overline{\gamma^{[\lambda\mu\nu]}} = -B\gamma^{[\lambda\mu\nu]}B^{-1}, \qquad \overline{\gamma^5} = +B\gamma^5B^{-1}.$$

Under a transformation:

$$\gamma'^{\mu} = S^{-1}\gamma^{\mu}\,S,$$

one must have:

(18) $$B' = \bar{S}BS$$

in order for:

$$\gamma'^{\mu} = B'\gamma'^{\mu}B'^{-1}$$

to be valid. A numerical factor is once more undetermined in the definition of $B$.

One must remark that the equations:

$$\overline{\gamma^{\mu}}B = B\gamma^{\mu}$$

are very convenient for determining $B$ numerically, when the numerical values of the elements of the $\gamma^{\mu}$ are given. On the other hand, the matrix $B$ – which is different from $A$, by virtue of (18) – is not invariant under unitary transformations of the $\gamma^{\mu}$.

Finally, one can establish a relations between $A$ and $B$ if one takes into account the fact that the operations $\gamma^{\mu} \to \overline{\gamma^{\mu}}$ and $\gamma^{\mu} \to \gamma^{+\mu}$ commute; i.e.:

$$(\overline{\gamma^{\mu+}}) = (\overline{\gamma^{\mu}})^{+} \ (= \gamma^{*\mu}).$$

One has:

$$(\overline{\gamma^{\mu}})^{+} = (B\gamma^{\mu} B^{-1})^{+} = B^{+-1}\gamma^{\mu+}B^{+} = B^{+-1}A\gamma^{\mu}A^{-1}B^{+}$$

and

$$(\overline{\gamma^{\mu+}}) = (\overline{A\gamma^{\mu}A^{-1}}) = \bar{A}^{-1}\overline{\gamma^{\mu}}\bar{A} = \bar{A}^{-1}B\gamma^{\mu}B^{-1}\bar{A};$$

it will then result that:

$$\bar{A}^{-1}B = c\, B^{+-1}A \qquad \text{and} \qquad A^{-1}B^{+} = c\, B^{-1}\bar{A},$$

so:

$$B^{+} = c\, A\, B^{-1}\bar{A}.$$

One can normalize the factor that is contained in the definition of $B$ in such a way that $c$ is equal to unity, and that one will have:

(19) $$B^{+} = A\, B^{-1}\bar{A}.$$

In the particular case where the $\gamma^{\mu}$ are Hermitian and $A = I$, one will have simply:

(19 cont.) $$B^{+} = B^{-1};$$

i.e., $B$ will be unitary.

## § 5. Lorentz transformation of the Dirac wave functions. The quadratic covariants.

Introduce real coordinates $x_k$ for space ([1]), as usual, and the imaginary coordinate $x_4 = ict$ for time, and consider the Lorentz coordinate transformations:

(20) $$x'_\mu = \sum_\nu a_{\mu\nu} x_\nu ,$$

with the known orthogonality conditions:

(21) $$\sum_\mu a_{\mu\rho} a_{\mu\sigma} = \sum_\mu a_{\rho\mu} a_{\sigma\mu} = \delta_{\rho\sigma} ,$$

in which the coefficients $a_{ik}$ , $a_{44}$ are real, while the $a_{4i}$ are purely imaginary. In what follows, one must distinguish, on the one hand, the Lorentz transformations in the restricted sense, for which one has:

$$\text{Det} \mid a_{\mu\nu} \mid = +1,$$

and that one can obtain by a *continuous* variation when one starts from unity, from the reflections with:

$$\text{Det} \mid a_{\mu\nu} \mid = -1,$$

that one can obtain from the former by the adjunction of the reflections:

(20*a*) $$x'_k = -x_k , \qquad x'_4 = +x_4$$

and

(20*b*) $$x'_k = +x_k , \qquad x'_4 = -x_4 .$$

Moreover, we write the Dirac wave equations in the form:

(22) $$\sum_\mu \gamma^\mu \frac{\partial \psi}{\partial x_\mu} + \frac{mc}{h} \psi = 0,$$

in which the $\gamma^\mu$ again satisfy the relations (1), $\gamma^\mu \psi$ is an abbreviation for $\sum_\sigma \gamma^\mu_{\rho\sigma} \psi_\sigma$ , and $h$ is Planck's constant, divided by $2\pi$.

For the Lorentz transformation of $\psi$, set:

(23) $$\psi'_\rho = \sum_\rho \Lambda_{\rho\sigma} \psi_\sigma \qquad \text{(to abbreviate: } \psi' = \Lambda\, \psi)$$

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([1]) We utilize Latin indices $i$, $k$, … that run through the values 1 to 3 for space, while the Greek indices $\mu$, $\nu$, … will take on values from 1 to 4.

and postulate that the equation:

$$\sum_\mu \gamma^\mu \frac{\partial \psi'}{\partial x'_\mu} + \frac{mc}{h}\psi' = 0 \tag{22′}$$

will be satisfied in the system of coordinates $x'_\mu$ with the same $\gamma^\mu$. The condition for these equations to be a consequence of the old equations (22) is that:

$$\Lambda^{-1}\,\gamma^\mu\,\Lambda = \sum_\nu a_{\mu\nu}\gamma^\nu\,. \tag{24}$$

We remark that the direct proof of the existence of such a matrix $\Lambda$ is very complicated, while it can be deduced immediately from the preceding considerations as a simple special case of our Theorem I. Indeed, it is easy to see that the matrices:

$$\gamma'^\mu = \sum_\nu a_{\mu\nu}\gamma^\nu$$

likewise satisfy the relations:

$$\tfrac{1}{2}(\gamma'^\mu\gamma'^\nu + \gamma'^\nu\gamma'^\mu) = \delta_{\mu\nu}\,,$$

due to the orthogonality conditions (21), which justifies the application of the theorem that was invoked.

We shall now examine the matrices $\Lambda$ in more detail that satisfy equations (24) and depend upon the $a_{\mu\nu}$, and in particular, we shall establish the conditions that distinguish the general $S$ matrices. In the first place, a numerical factor remains arbitrary in the definition of $\Lambda$; we can normalize it by demanding that the determinant of $\Lambda$ should be equal to unity:

$$\operatorname{Det}\Lambda = 1. \tag{25}$$

In the second place, consider the consequences of the reality character of the $a_{\mu\nu}$ that was mentioned above. One sees that the coefficients of the transformations that relate the:

$$\gamma'^k,\ i\gamma'^4 \qquad \text{to the} \qquad \gamma^k,\ i\,\gamma^4$$

will be real. Now, if one defines:

$$\beta = A\gamma^4$$

(in the particular case where the $\gamma^\mu$ are Hermitian and $A = I$, one will then have $\beta = \gamma^4$) then the matrices:

$$i\beta\,\gamma^k,\quad \beta\,\gamma^4 = \Lambda$$

will be Hermitian then one will have, by virtue of (10) and (1):

$$(\gamma^k)^+ = -\,\beta\,\gamma^k\beta^{-1}, \qquad (i\gamma^4)^+ = -\,\beta\,(i\,\gamma^k)\beta^{-1},$$

because $\beta^{-1} = \gamma^4 A^{-1}$. One concludes from the reality of the coefficients of the transformation:

$$(\gamma^k, i\,\gamma^4) \to (\gamma'^k, i\gamma'^4)$$

that one will likewise have:

$$(\gamma'^k)^+ = -\,\beta\,\gamma'^k\beta^{-1}, \qquad (i\gamma'^4)^+ = -\,\beta\,(i\gamma'^4)\beta^{-1}.$$

On the other hand, one concludes from (24) – as one did for equation (13) above – that:

$$\beta' = \Lambda^+\,\beta\,\Lambda$$

also satisfies the conditions:

$$(\gamma'^k)^+ = -\beta'\,\gamma'^k\beta'^{-1}, \qquad (i\gamma'^4)^+ = -\beta'\,(i\gamma'^4)\beta'^{-1},$$

from which, one deduces that:

$$\beta' = \Lambda^+\beta\,\Lambda = c\,\beta.$$

As for the factor $c$, one concludes from (24) that $c^4 = 1$. For reasons of continuity, it results that one will have $c = 1$ for the Lorentz transformations in the restricted sense. Consequently:

(26) $$\Lambda^+\beta\,\Lambda = \beta \qquad (\text{where } \Lambda^+A\gamma^4\Lambda = A\gamma^4).$$

The train of argument in the case of the matrix $B$ is even simpler, since the reality character of the Lorentz transformation does not enter into it. It results immediately from:

(15) $$\bar{\gamma}^\mu = B\,\gamma^\mu\,B^{-1}$$

that for the $\gamma'^\mu = \sum_\nu a_{\mu\nu}\gamma^\nu$ with the same $B$:

$$\bar{\gamma}'^\mu = B\gamma'^\mu B^{-1};$$

on the other hand, from (23) and (18), for:

$$B' = \bar{\Lambda} B \Lambda,$$
$$\overline{\gamma'^\mu} = B'\gamma'^\mu B'^{-1},$$

and one will conclude that:

$$B' = \bar{\Lambda} B \Lambda = c\,B,$$

and from (24), one will further have that $c^4 = 1$. For continuity reasons, one will always have $c = 1$ for Lorentz transformations in the restricted sense:

(27) $$\bar{\Lambda} B \Lambda = B.$$

As for the reflections, we can set:

(28$a$) $$\Lambda = \gamma^4 \qquad \text{for} \qquad x'_k = -\,x_k, \qquad x'_4 = +\,x_4,$$

(28*b*) $$\Lambda = \gamma^5\gamma^4 = \gamma^1\gamma^2\gamma^3 \quad \text{for} \quad x'_k = + x_k, \quad x'_4 = - x_4 .$$

One sees that (27) is likewise valid for reflections, but that (26) is no longer valid for $x'_k = - x_k$, $x'_4 = + x_4$, and that finally one has the exception:

(26′) $$\Lambda^+\beta\Lambda = - \beta \quad \text{for} \quad x'_4 = - x_4 .$$

In what follows, we shall exclude the latter transformations, to simplify.

We obtain a final condition for $\Lambda$ by setting:

$$\gamma'^5 = \gamma'^1\gamma'^2\gamma'^3\gamma'^4 = \text{Det} \mid a_{\mu\nu} \mid \gamma^5,$$

so it will result that:

(28) $$\Lambda\, \gamma^5 = \pm\, \gamma^5\, \Lambda,$$

in which the + sign is valid for transformations in the restricted sense, and the – sign is valid for the reflections.

Conditions (24), (26), (27), (28) for $\Lambda$ are complete in the sense that every matrix $\Lambda$ that satisfies these conditions will provide a:

$$\gamma'^\mu = \Lambda^{-1}\gamma^\mu\, \Lambda,$$

which satisfies the equation:

$$\gamma'^\mu = \sum_\nu a_{\mu\nu}\gamma^\nu ,$$

with the coefficients $a_{\mu\nu}$ of a Lorentz transformation.

We shall simply sketch out the proof. One can write:

$$\Lambda^{-1}\gamma^\mu\, \Lambda = \sum_A c_{\mu,A}\, \gamma^A .$$

Since the diagonal sum of the $\gamma'^\mu$ is the same as that of $\gamma^\mu$ and the latter is zero, the coefficient of the identity must disappear. From (27), one deduces that $B\gamma'^\mu$ is skew, like $B\gamma^\mu$, and it will result that the coefficients of $\gamma^{[\kappa\lambda]}$ and $\gamma^{[\kappa\lambda\nu]}$ must also disappear. One deduces from (28) that $\gamma'^\mu\gamma^5 = - \gamma^5\gamma'^\mu$, and consequently, the coefficient $c_{\mu,5}$ of $\gamma^5$ will be equal to zero. One will then have:

$$\gamma'^\mu = \sum_\nu a_{\mu\nu}\gamma^\nu ,$$

and one will deduce that $a_{ik}$, $a_{44}$, and $ia_{k4}$ are real from (26) [or (26′)]. Finally, it results from:

$$\tfrac{1}{2}(\gamma'^\mu\gamma'^\nu + \gamma'^\nu\gamma'^\mu) = \delta_{\mu\nu}$$

that the $a_{\mu\nu}$ will satisfy the orthogonality conditions (21).

One knows that with the aid of relations (23) and for an arbitrary function $\psi^+$ that satisfies the wave equation:

$$\frac{\partial \psi^+}{\partial x^\mu}\gamma^\mu - \frac{mc}{h}\psi^+ = 0 \tag{22$^+$}$$

and transforms by a Lorentz transformation according to the law:

$$\psi_\sigma^{+\prime} = \sum_\rho \psi_\rho^+ \Lambda_{\rho\sigma}^{-1} \qquad \text{(abbr.: } \psi^{+\prime} = \psi^+\Lambda^{-1}\text{)}, \tag{23$^+$}$$

one can construct the following covariants:

An invariant:

$$\psi^+\psi = i\,\Omega_1 , \tag{29$_1$}$$

a vector:

$$\psi^+\gamma^\mu\psi = S_\mu , \tag{29$_2$}$$

an anti-symmetric areal tensor:

$$\psi^+\gamma^{[\mu\nu]}\psi = -\,iM_{[\mu\nu]} , \tag{29$_3$}$$

an anti-symmetric volume tensor (which is dual to a vector):

$$\psi^+\gamma^{[\lambda\mu\nu]}\psi = S_{[\lambda\mu\nu]} , \tag{29$_4$}$$

and finally, a "pseudo-scalar":

$$\Omega_2 = \psi^+\gamma^5\psi. \tag{29$_5$}$$

One obtains a particular $\psi^+$ that satisfies the conditions (22$^+$), (23$^+$) by starting with the complex conjugate $\psi^*$ of $\psi$, and forming:

$$\psi^+ = i\,\psi^*\beta = i\,\psi^*\Lambda\,\gamma^4, \tag{30}$$

due to the relations (26). Indeed, one has the transformation law for $\psi^+$:

$$\psi^{*\prime} = \psi^*\Lambda^+, \tag{23}$$

and the wave equation (not losing sight of the imaginary character of $x_4$ !):

$$-\frac{\partial \psi^*}{\partial x^4}\gamma^4 + \sum_{k=1}^{3}\frac{\partial \psi^*}{\partial x^k}\gamma^{+k} + \frac{mc}{h}\psi^* = 0, \tag{22$^*$}$$

or

$$-\frac{\partial \psi^*}{\partial x^4}\Lambda\gamma^4 + \sum_{k=1}^{3}\frac{\partial \psi^*}{\partial x^k}\Lambda\gamma^{+k} + \frac{mc}{h}\psi^*\Lambda = 0.$$

The last equation will become identical with (22$^+$) after the substitution (30).

The factors of $i$ in $(29_1)$ and $(29_3)$ are chosen in such a way that after the substitution (30), the components of all of the quantities that were formed will be real or purely imaginary according to whether an even number of indices is equal to 4 or an odd number, respectively. The reader is referred to the corresponding publications ([1]) for the physical significance of the quantities are defined by $(29_1)$ up to $(29_5)$.

The matrix $B$ permits one to answer two questions. The first one, which plays a role in *Fermi's* theory of radioactive $\beta$-disintegration, is the following one: Let two wave functions $\psi_\rho$ and $\varphi_\rho$ have the *same* transformation law:

$$\psi' = \Lambda\psi, \qquad \varphi' = \Lambda\,\varphi$$

under the Lorentz group. How is it possible to form covariant quantities (e.g., scalar, vector, tensor, etc.) that are bilinear in $\psi$ and $\varphi$ [in a manner that is analogous to $(29_2)$-$(29_5)$]? We remark that the law $\varphi' = \Lambda\varphi$ is *identical* to the other one:

$$\overline{(23)} \qquad \varphi' = \varphi\,\bar{\Lambda},$$

because $\varphi' = \varphi\,\bar{\Lambda}$ means that $\varphi'_\sigma = \sum_\rho \varphi_\rho \bar{\Lambda}_{\rho\sigma}$ and $\bar{\Lambda}_{\rho\sigma} = \Lambda_{\sigma\rho}$. Now, from (27), for each matrix $F$, one will have:

$$\varphi\bar{\Lambda}BF\psi = \varphi\,B\,\Lambda^{-1}F\psi,$$

so

$$\varphi' BF\psi' = \varphi\,B\,\Lambda^{-1}F\psi.$$

Consequently, one has the scalar:

$$(31_1) \qquad i\,\Omega_1 = \varphi\,B\,\psi,$$

the vector:

$$(31_2) \qquad S_\mu = \varphi\,B\,\gamma^\mu\,\psi,$$

the areal tensor:

$$(31_3) \qquad -\,i\,M_{[\mu\nu]} = \varphi\,B\,\gamma^{[\mu\nu]}\,\psi,$$

the volume tensor:

$$(31_4) \qquad S_{[\lambda\mu\nu]} = \varphi\,B\,\gamma^{[\lambda\mu\nu]}\psi,$$

and the pseudo-scalar:

$$(31_5) \qquad \Omega_2 = \varphi\,B\gamma^5\psi.$$

It is permissible to specialize this and set: $\varphi = \psi$. $\Omega_1$, $S_\mu$, and $\Omega_2$ will then be annulled because the corresponding matrices are skew.

The second question is the following one: One knows that the Dirac wave equation has solutions that correspond to positive-energy states and other ones that correspond to negative-energy states. One seeks to establish a single-valued correspondence that is invariant from the relativistic viewpoint between a certain negative-energy solution and a

([1]) See, for example, L. DE BROGLIE, *L'électron magnétique*, Paris, 1934.

positive-energy solution, and conversely. The problem will be solved if one can find a matrix $C$ in such a manner that, with the relation:

$$\varphi^* = C\psi \qquad \text{or} \qquad \psi = C^{-1}\,\varphi^*, \tag{32}$$

$\varphi$ obeys the same equation (22) as $\psi$, and the same transformation law (23) as $\psi$, moreover, at least under transformations in the restricted sense. In fact, if $\psi$ contains only positive-energy states then it will result from (32) that $\psi$ will contain only negative-energy states, and conversely.

In order to arrive at a solution to the problem, we first remark that from (22*), when it is applied to $\varphi$, and (32), it will result that:

$$-\gamma^{*4}\frac{\partial\varphi^*}{c\,\partial t} + \sum_k \gamma^{*k}\frac{\partial\varphi^*}{\partial x^k} + \frac{mc}{h}\varphi^* = 0,$$

$$-\gamma^{*4}C\frac{\partial\psi^*}{c\,\partial t} + \sum_k \gamma^{*k}C\frac{\partial\psi^*}{\partial x^k} + \frac{mc}{h}C\psi^* = 0,$$

$$-C^{-1}\gamma^{*4}C\frac{\partial\psi^*}{c\,\partial t} + \sum_k C^{-1}\gamma^{*k}C\frac{\partial\psi^*}{\partial x^k} + \frac{mc}{h}\psi^* = 0.$$

A comparison with (22) will yield:

$$C^{-1}\gamma^{*4}C = -\gamma^4, \qquad C^{-1}\gamma^{*k}C = \gamma^k,$$

where:

$$\gamma^{*4} = -C^{-1}\gamma^4 C, \qquad \gamma^{*k} = C^{-1}\gamma^k C.$$

We saw above that:

$$\gamma^{*\mu} = \bar{A}^{-1}B\gamma^\mu B^{-1}\bar{A};$$

on the other hand, the matrix $\gamma^4\gamma^5$ commutes with $\gamma_k$, but anti-commutes with $\gamma^4$. Consequently, the solution will be (up to an arbitrary numerical factor):

$$C = \bar{A}^{-1}B\gamma^4\gamma^5 = -\bar{A}^{-1}B\gamma^1\gamma^2\gamma^3 \tag{33}$$

(where $C = B\,\gamma^4\gamma^5$ for the case in which the $\gamma^\mu$ are Hermitian and $A = I$).

In addition, it results from (19) that:

$$C^*C = I, \qquad C^* = C^{-1}; \tag{33a}$$

consequently, the relations (32) remain valid if one permutes $\psi$ and $\varphi$:

$$\psi^* = C\,\varphi; \qquad \varphi = C^{-1}\psi^*. \tag{32a}$$

Finally, it results from (26), (27), (28) (in the case of Lorentz transformations in the restricted sense) that the invariance postulate is fulfilled likewise.

By virtue of:

$$\psi' = \Lambda \psi, \qquad \varphi' = \Lambda \varphi,$$

one deduces from (32) that:

$$\varphi'^{*} = C\psi', \qquad \psi' = C^{-1}\,\varphi'^{*}. \tag{32$'$}$$

## § 6. Quadratic identities between the covariants.

One knows ([1]) that there exist identities between the quantities $\Omega_1$, $\Omega_2$, $S_\mu$, $M_{[\mu\nu]}$, $S_{[\lambda\mu\nu]}$ that are defined by equations ($29_1$) to ($29_5$) and are quadratic in the wave functions $\psi$ and $\psi^+$. They have the following form:

$$-\sum_\mu S_\mu^2 \equiv S_0^2 - \sum_{k=1}^{3} S_k^2 = \Omega_1^2 + \Omega_2^2, \tag{$34_1$}$$

$$\sum_{[\mu\nu]} M_{[\mu\nu]}^2 \equiv \sum_{ik} M_{[ik]}^2 - \sum_k M_{[k0]}^2 = \Omega_1^2 - \Omega_2^2, \tag{$34_2$}$$

$$-\sum_{[\lambda\mu\nu]} S_{[\lambda\mu\nu]}^2 \equiv -S_{[123]}^2 + \sum_{[ik]} S_{[ik0]}^2 = \Omega_1^2 + \Omega_2^2, \tag{$34_3$}$$

$$-\frac{i}{2}\sum_{[\kappa\lambda],[\mu\nu]} M_{[\kappa\lambda]}, M_{[\mu\nu]} \equiv M_{23}\,M_{10} + M_{31}\,M_{20} + M_{12}\,M_{30} = \Omega_1\,\Omega_2, \tag{$34_4$}$$

$$\sum_{\kappa,[\lambda\mu\nu]} S_\kappa S_{[\lambda\mu\nu]} = 0. \tag{$34_5$}$$

We have always set $a_{\dots 4} = i\,a_{\dots 0}$ ; moreover, the indices $\kappa$, $\lambda$, $\mu$, $\nu$ in ($34_1$) and ($34_5$) must be distinct and form an even permutation of the numerals 1, 2, 3, 4.

Upon introducing the tensors that are dual to $M_{[\mu\nu]}$ and $S_{[\lambda\mu\nu]}$ according to:

$$\hat{M}_{[\kappa\lambda]} \equiv M_{[\mu\nu]}, \qquad \hat{S}_\kappa \equiv S_{[\lambda\mu\nu]}, \tag{35}$$

one can also put ($34_4$) and ($34_5$) into the form:

$$-\frac{i}{2}\sum_{[\mu\nu]} \hat{M}_{[\mu\nu]} M_{[\mu\nu]} = \Omega_1\,\Omega_2, \tag{$34_4$}$$

and

$$\sum_\kappa S_\kappa \hat{S}_\kappa = 0. \tag{$34_5$}$$

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([1]) See, for example, L. DE BROGLIE, *L'électron magnétique*, Paris, 1934.

In a manner that is analogous to (35), we can define the matrices:

$$\hat{\gamma}^{[\kappa\lambda]} \quad \text{and} \quad \hat{\gamma}^{\kappa} \quad \text{dual to} \quad \gamma^{[\mu\nu]} \quad \text{and} \quad \hat{\gamma}^{[\kappa\lambda]},$$

with:

(36*a*)
$$\begin{cases} \hat{\gamma}^{[\kappa\lambda]} = \gamma^{[\mu\nu]} \quad (\hat{\gamma}^{[12]} = \gamma^{[34]}, \quad \hat{\gamma}^{[31]} = \gamma^{[24]}, \quad \hat{\gamma}^{[23]} = \gamma^{[14]}; \\ \qquad\qquad\qquad \hat{\gamma}^{[34]} = \gamma^{[12]}, \quad \hat{\gamma}^{[24]} = \gamma^{[31]}, \quad \hat{\gamma}^{[14]} = \gamma^{[23]}). \end{cases}$$

(36*b*)
$$\begin{cases} \hat{\gamma}^{\kappa} = \gamma^{[\lambda\mu\nu]} \quad (\hat{\gamma}^{1} = \gamma^{[234]}, \quad \hat{\gamma}^{2} = \gamma^{[314]}, \quad \hat{\gamma}^{3} = \gamma^{[124]}; \\ \qquad\qquad\qquad \hat{\gamma}^{4} = \gamma^{[321]} = -\gamma^{[123]}). \end{cases}$$

Finally, for the sake of what follows, we note the relations (in which one always has $\gamma^5 = \gamma^1\,\gamma^2\,\gamma^3\,\gamma^4$):

(37*a*)
$$\hat{\gamma}^{[\kappa\lambda]} = -\,\gamma^5\gamma^{[\kappa\lambda]} = -\,\gamma^{[\kappa\lambda]}\,\gamma^5,$$

(37*b*)
$$\hat{\gamma}^{\kappa} = -\,\gamma^5\gamma^{\kappa} = +\,i\gamma^{\kappa}\gamma^5.$$

It is important to emphasize the fact that the validity of the identities is independent of the assumption that the $\gamma_\mu$ are Hermitian, as well as the relation (30) between $\psi^+$ and $\psi^*$; on the contrary, we suppose that the $\psi$ and $\psi^+$ are quantities that are completely arbitrary and mutually independent.

In order to prove the identities ($34_1$) to ($34_5$), we start with the identity (II) at the end of § 3:

(II)
$$\sum_{A=1}^{16} \gamma^A_{\rho\sigma}\,\gamma^A_{\bar{\rho}\bar{\sigma}} = 4\,\delta_{\bar{\rho}\sigma}\delta_{\rho\bar{\sigma}},$$

so:

(38)
$$\delta_{\rho\sigma}\delta_{\bar{\rho}\bar{\sigma}} + \gamma^5_{\rho\sigma}\,\gamma^5_{\bar{\rho}\bar{\sigma}} + \sum_{\mu}\gamma^{\mu}_{\rho\sigma}\,\gamma^{\mu}_{\bar{\rho}\bar{\sigma}} + \sum_{[\mu\nu]}\gamma^{[\mu\nu]}_{\rho\sigma}\,\gamma^{[\mu\nu]}_{\bar{\rho}\bar{\sigma}} + \sum_{\lambda\mu\nu}\gamma^{[\lambda\mu\nu]}_{\rho\sigma}\,\gamma^{[\lambda\mu\nu]}_{\bar{\rho}\bar{\sigma}} = 4\,\delta_{\bar{\rho}\sigma}\delta_{\rho\bar{\sigma}}.$$

After multiplying by $\varphi^+_\rho$, $\psi^+_{\bar{\rho}}$, $\varphi_\sigma$, $\psi_{\bar{\sigma}}$, and summing over the indices $\rho$, $\bar{\rho}$, $\sigma$, $\bar{\sigma}$ (upon putting a prime on the quantities that are formed with $\varphi^+$, $\varphi$ in a manner that is analogous to the ones that are formed by starting with $\psi^+$, $\psi$), we will obtain:

(39)
$$-\,\Omega_1'\,\Omega_1 + \Omega_2'\,\Omega_2 + \sum_{\mu} S_\mu'\,S_\mu - \sum_{[\mu\nu]} M'_{[\mu\nu]}\,M_{[\mu\nu]} + \sum_{[\lambda\mu\nu]} S'_{[\lambda\mu\nu]}\,S_{[\lambda\mu\nu]} = 4(\psi^+\,\varphi)\cdot(\varphi^+\,\psi),$$

and for the particular case in which $\varphi^+ = \psi^+$ and $\varphi = \psi$:

(39*a*)
$$-\,\Omega_1^2 + \Omega_2^2 + \sum_{\mu} S_\mu^2 - \sum_{[\mu\nu]} M^2_{[\mu\nu]} + \sum_{[\lambda\mu\nu]} S^2_{[\lambda\mu\nu]} = -4\Omega_1^2.$$

One sees that (39*a*) is a consequence of the relations (34), but that conversely, (39*a*) is not sufficient to prove the relations (34). Meanwhile, one can utilize the operation of multiplication (once or twice) by $\gamma^5$ to deduce some new identities from (II) or (38). In order to do the former, construct:

$$\sum_A \gamma^A_{\rho\sigma} (\gamma^5 \gamma^A \gamma^5)_{\bar\rho\bar\sigma} = 4\gamma^5_{\bar\rho\sigma} \gamma^5_{\rho\bar\sigma},$$

so:

$$\delta_{\rho\sigma}\delta_{\bar\rho\bar\sigma} + \gamma^5_{\rho\sigma}\gamma^5_{\bar\rho\bar\sigma} - \sum_\mu \gamma^\mu_{\rho\sigma}\gamma^\mu_{\bar\rho\bar\sigma} + \sum_{[\mu\nu]} \gamma^{[\mu\nu]}_{\rho\sigma}\gamma^{[\mu\nu]}_{\bar\rho\bar\sigma} - \sum_{[\lambda\mu\nu]} \gamma^{[\lambda\mu\nu]}_{\rho\sigma}\gamma^{[\lambda\mu\nu]}_{\bar\rho\bar\sigma} = 4\gamma^5_{\bar\rho\sigma}\gamma^5_{\rho\bar\sigma}, \tag{40}$$

which will give, upon combining with (38):

$$\delta_{\rho\sigma}\delta_{\bar\rho\bar\sigma} + \gamma^5_{\rho\sigma}\gamma^5_{\bar\rho\bar\sigma} + \sum_{[\mu\nu]} \gamma^{[\mu\nu]}_{\rho\sigma}\gamma^{[\mu\nu]}_{\bar\rho\bar\sigma} = 2\left(\delta_{\bar\rho\sigma}\delta_{\rho\bar\sigma} + \gamma^5_{\bar\rho\sigma}\gamma^5_{\rho\bar\sigma}\right), \tag{41}$$

$$\sum_\mu \gamma^\mu_{\rho\sigma}\gamma^\mu_{\bar\rho\bar\sigma} + \sum_{[\lambda\mu\nu]} \gamma^{[\lambda\mu\nu]}_{\rho\sigma}\gamma^{[\lambda\mu\nu]}_{\bar\rho\bar\sigma} = 2\left(\delta_{\bar\rho\sigma}\delta_{\rho\bar\sigma} - \gamma^5_{\bar\rho\sigma}\gamma^5_{\rho\bar\sigma}\right). \tag{42}$$

One will deduce the following identities from those relations and from treating them like the relations (39):

$$-\Omega'_1\Omega_1 + \Omega'_2\Omega_2 - \sum_{[\mu\nu]} M'_{[\mu\nu]} M_{[\mu\nu]} = 2\,[(\psi^+\varphi)(\varphi^+\psi) + (\psi^+\gamma^5\varphi)(\varphi^+\gamma^5\psi)], \tag{43}$$

$$\sum_\mu S'_\mu S_\mu + \sum_{[\lambda\mu\nu]} S'_{[\lambda\mu\nu]} S_{[\lambda\mu\nu]} = 2\,[(\psi^+\varphi)(\varphi^+\psi) - (\psi^+\gamma^5\varphi)(\varphi^+\gamma^5\psi)], \tag{44}$$

and for the particular case in which $\varphi^+ = \psi^+$ and $\varphi = \psi$:

$$-\Omega_1^2 + \Omega_2^2 - \sum_{[\mu\nu]} M^2_{[\mu\nu]} = 2(-\Omega_1^2 + \Omega_2^2), \tag{43$a$}$$

which is an expression that is identical to $(34_2)$, and:

$$-\sum_\mu S_\mu^2 - \sum_{[\lambda\mu\nu]} S^2_{[\lambda\mu\nu]} = 2(\Omega_1^2 + \Omega_2^2). \tag{44$a$}$$

One sees that the last relation is exactly the sum of $(34_1)$ and $(34_3)$.

Now, form:

$$\sum_A \gamma^A_{\rho\sigma}\, \tfrac{1}{2}\left(\gamma^5\gamma^A + \gamma^A\gamma^5\right)_{\bar\rho\bar\sigma} = 2\left(\gamma^5_{\bar\rho\sigma}\delta_{\rho\bar\sigma} + \delta_{\bar\rho\sigma}\gamma^5_{\rho\bar\sigma}\right)$$

and

$$\sum_A \gamma^A_{\rho\sigma}\, \tfrac{1}{2}\left(\gamma^5\gamma^A - \gamma^A\gamma^5\right)_{\bar\rho\bar\sigma} = 2\left(\gamma^5_{\bar\rho\sigma}\delta_{\rho\bar\sigma} - \delta_{\bar\rho\sigma}\gamma^5_{\rho\bar\sigma}\right),$$

and obtain, while taking (37*a*) and (37*b*) into account:

$$\delta_{\rho\sigma}\gamma^5_{\bar\rho\bar\sigma} + \gamma^5_{\rho\sigma}\delta_{\bar\rho\bar\sigma} - \sum_{[\mu\nu]} \gamma^{[\mu\nu]}_{\rho\sigma}\,\hat\gamma^{[\mu\nu]}_{\bar\rho\bar\sigma} = 2\left(\gamma^5_{\bar\rho\sigma}\delta_{\rho\bar\sigma} + \delta_{\bar\rho\sigma}\gamma^5_{\rho\bar\sigma}\right) \tag{45}$$

and

$$i\sum_\mu \gamma^\mu_{\rho\sigma}\,\hat\gamma^\mu_{\bar\rho\bar\sigma} - i\sum_\mu \hat\gamma^\mu_{\rho\sigma}\,\gamma^\mu_{\bar\rho\bar\sigma} = 2\left(\gamma^5_{\bar\rho\sigma}\delta_{\rho\bar\sigma} - \delta_{\bar\rho\sigma}\gamma^5_{\rho\bar\sigma}\right), \tag{46}$$

so

$$i\,\Omega_1'\,\Omega_1 + i\,\Omega_2'\,\Omega_2 + \sum_{[\mu\nu]} M'_{[\mu\nu]}\,\hat M_{[\mu\nu]} = 2\,[(\psi^+\gamma^5\varphi)(\varphi^+\psi) + (\psi^+\varphi)(\varphi^+\gamma^5\psi)], \tag{47}$$

and

$$i\sum_\mu S'_\mu\,S_\mu - i\sum_\mu \hat S'_\mu\,S_\mu = 2\,[(\psi^+\gamma^5\varphi)(\varphi^+\psi) - (\psi^+\varphi)(\varphi^+\gamma^5\psi)]. \tag{48}$$

For the particular case $\psi^+ = \varphi^+$ and $\psi = \varphi$, it will result from (47) that:

$$2i\,\Omega_1\,\Omega_2 + \sum_{[\mu\nu]} M_{[\mu\nu]}\,\hat M_{[\mu\nu]} = 4i\,\Omega_1\,\Omega_2, \tag{47$a$}$$

which is identical to ($34_4$). The relation (48) would give simply 0 = 0 for $\psi^+ = \varphi^+$ and $\psi = \varphi$.

Let me point out that I have not encountered the more general identities (43), (44), (47), (48) in any publication.

It now remains for us to deduce one of the two relations ($34_1$) or ($34_3$), and the relation ($34_5$). I have confirmed that one can do that only by introducing the matrix $B$ that was defined in § 3. Indeed, upon taking into account the relations (15) and (17) from § 3, whose signs have a decisive importance for the following conclusions, we will find that:

$$\sum_A \gamma^A_{\rho\sigma}\left(B\gamma^A B^{-1}\right)_{\bar\sigma\bar\rho} = 4B_{\bar\sigma\sigma}B^{-1}_{\rho\bar\rho},$$

$$\delta_{\rho\sigma}\delta_{\bar\rho\bar\sigma} + \gamma^5_{\rho\sigma}\,\gamma^5_{\bar\rho\bar\sigma} + \sum_\mu \gamma^\mu_{\rho\sigma}\,\gamma^\mu_{\bar\rho\bar\sigma} - \sum_{[\mu\nu]} \gamma^{[\mu\nu]}_{\rho\sigma}\,\gamma^{[\mu\nu]}_{\bar\rho\bar\sigma} - \sum_{[\lambda\mu\nu]} \gamma^{[\lambda\mu\nu]}_{\rho\sigma}\,\gamma^{[\lambda\mu\nu]}_{\bar\rho\bar\sigma} = 4B_{\bar\sigma\sigma}B^{-1}_{\rho\bar\rho}, \tag{49}$$

which will give:

$$\delta_{\rho\sigma}\delta_{\bar\rho\bar\sigma} + \gamma^5_{\rho\sigma}\,\gamma^5_{\bar\rho\bar\sigma} + \sum_\mu \gamma^\mu_{\rho\sigma}\,\gamma^\mu_{\bar\rho\bar\sigma} = 2\left(\delta_{\bar\rho\sigma}\delta_{\rho\bar\sigma} + B_{\bar\sigma\sigma}B^{-1}_{\rho\bar\rho}\right), \tag{50}$$

when it is combined with (38). Upon multiplying this by $\psi^+_\rho$, $\psi^+_{\bar\rho}$, $\psi_\sigma$, $\psi_{\bar\sigma}$ and summing over the indices $\rho$, $\bar\rho$, $\sigma$, $\bar\sigma$ *the terms in the right-hand side that contain B will disappear*, because $B$ is skew, and we have set $\psi^+ = \varphi^+$ and $\psi = \varphi$ *a priori*. One will then have:

$$-\,\Omega_1^2 + \Omega_2^2 + \sum_\mu S_\mu^2 = -2\Omega_1^2, \tag{51}$$

*which is identical to* ($34_1$). ($34_3$) results from (51) and (44$a$).

Finally, in order to obtain the last relation ($34_5$), start with (46):

$$i\sum_\mu \gamma^\mu_{\rho\sigma}\left(B\,\hat{\gamma}^\mu B^{-1}\right)_{\bar{\sigma}\bar{\rho}} - i\sum_\mu \hat{\gamma}^\mu_{\rho\sigma}\left(B\,\gamma^\mu B^{-1}\right)_{\bar{\sigma}\bar{\rho}} = 2\left[\left(B\gamma^5\right)_{\bar{\sigma}\sigma} B^{-1}_{\rho\bar{\rho}} - B_{\bar{\sigma}\sigma}\left(B\gamma^5\right)^{-1}_{\rho\bar{\rho}}\right],$$

or, upon taking into account the sign in (17):

(52) $$-i\sum_\mu \gamma^\mu_{\rho\sigma}\hat{\gamma}^\mu_{\bar{\sigma}\bar{\rho}} - i\sum_\mu \hat{\gamma}^\mu_{\rho\sigma}\gamma^\mu_{\bar{\sigma}\bar{\rho}} = 2\left[\left(B\gamma^5\right)_{\bar{\sigma}\sigma} B^{-1}_{\rho\bar{\rho}} - B_{\bar{\sigma}\sigma}\left(B\gamma^5\right)^{-1}_{\rho\bar{\rho}}\right].$$

Since the matrices $B\gamma^5B$ and their inverses are skew, one will obtain (345) immediately upon multiplying the last relation by $\psi^+_\rho$, $\psi^+_{\bar{\rho}}$, $\psi_\sigma$, $\psi_{\bar{\sigma}}$ and summing over the indices $\rho$, $\bar{\rho}$, $\sigma$, $\bar{\sigma}$.

The identities that are deduced above are the only ones of the type considered that contain expressions that are invariant from the relativistic viewpoint.