"Beiträge zur mathematischen Theorie der Dirac'schen Matrizen," in *Pieter Zeeman Verhandelingen*, Martinus Nijhoff, 's-Gravenhage, 1935, pp. 31-43.

Contributions to the mathematical theory of the Dirac matrices

By **W. PAULI** (Zurich)

Translated by D. H. Delphenich

§ 1. Introduction. – The goal of the present article is to fill in some lacunas that remain in the proofs of some theorems that are concerned with the hypercomplex number system that is constructed from the **Dirac** matrices γ^μ (μ = 1 to 4) with the relations:

$$\tfrac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = \delta_{\mu\nu} \cdot I \tag{1}$$

($\delta_{\mu\nu} = 0$ for $\mu \neq \nu$, while it is 1 for $\mu = \nu$). The latter is known to be constructed from the sixteen quantities:

$$I,\ \gamma^\mu,\ \gamma^{[\mu\nu]},\ \gamma^{[\lambda\mu\nu]},\ \gamma^5, \tag{2}$$

in which $\gamma^{[\mu\nu]}$ and $\gamma^{[\lambda\mu\nu]}$ are antisymmetric in all indices and are defined by:

$$\gamma^{[\mu\nu]} = (i\,\gamma^2\gamma^3,\ i\,\gamma^3\gamma^1,\ i\,\gamma^1\gamma^2;\ i\,\gamma^1\gamma^4,\ i\,\gamma^2\gamma^4,\ i\,\gamma^3\gamma^4), \tag{3a}$$

$$\gamma^{[\lambda\mu\nu]} = (i\,\gamma^2\gamma^3\gamma^4,\ i\,\gamma^3\gamma^1\gamma^4,\ i\,\gamma^1\gamma^2\gamma^4;\ i\,\gamma^1\gamma^2\gamma^3), \tag{3b}$$

in which one has set:

$$\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4. \tag{3c}$$

The factors of i are included in order to make:

$$(\gamma^{[\mu\nu]})^2 = +\,I, \quad (\gamma^{[\lambda\mu\nu]})^2 = +\,I.$$

If we also denote the sixteen quantities (2) by γ^A (here and in what follows, uppercase Latin indices will run from 1 to 16) then we will also have:

$$(\gamma^A)^2 = +\,I \tag{4}$$

for each of the sixteen quantities γ^A.

The following theorem regarding this hypercomplex number system, which follows from general algebraic theorems ([1]):

Theorem I. – *If* γ^μ *and* $\gamma^{\mu\prime}$ *are two four-rowed systems of matrices that both fulfill the same relations* (1) *then there will always be a matrix S (with non-vanishing determinant) such that:*

$$\gamma^{\mu\prime} = S\,\gamma^\mu\,S^{-1}. \tag{5}$$

An elementary proof of this shall now be given that does not involve the numerical specialization of the γ^μ, and is based upon a method of **J. Schur** ([2]). In the following section § **2**, that will lead to the identity:

$$\sum_{A=1}^{16} \gamma^A_{\rho\sigma}\,\gamma^A_{\bar\rho\bar\sigma} = 4\,\delta_{\bar\rho\sigma}\delta_{\rho\bar\sigma}\,, \tag{I}$$

which is also interesting independently of the aforementioned theorem.

Namely, if ([3]):

$$\psi^\dagger = i\,\psi^*\gamma^4, \quad \alpha^k = i\,\gamma^4\gamma^k, \quad \alpha^4 = \gamma^4 \tag{6}$$

(here and in what follows, lowercase Latin indices will run from 1 to 3, corresponding to the spatial coordinates) then if one uses the sixteen quantities γ^A to construct the scalar:

$$\Omega_1 = -\,i\,\psi^\dagger\psi = \psi^*\alpha^4\,\psi, \tag{7a}$$

the pseudo-scalar:

$$\Omega_2 = \psi^\dagger\,\gamma^5\psi = \psi^*\alpha^1\,\alpha^2\,\alpha^3\,\alpha^4\,\psi, \tag{7b}$$

the four-vector:

$$s_\mu = \psi^\dagger\,\gamma^\mu\psi \qquad (s_0 = -\,i\,s_4 = \psi^*\psi,\ s_\kappa = \psi^*\alpha^\kappa\,\psi), \tag{7c}$$

the skew-symmetric tensor:

$$M_{\mu\nu} = -\,i\psi^\dagger\gamma^{[\mu\nu]}\psi \quad (M_{[ik]} = \psi^\dagger\,i\,\alpha^i\,\alpha^k\,\alpha^4\psi,\ M_{k0} = -\,i\,M_{k4} = -\,\psi^\dagger\,i\,\alpha^k\,\alpha^4\psi), \tag{7d}$$

and the spatial vector (which is dual to a four-vector):

$$s_{[\lambda\mu\nu]} = \hat{s}_\kappa = \psi^\dagger\gamma^{[\lambda\mu\nu]}\psi \quad (s_{[ik0]} = -\,i\,s_{[ik4]} = \psi^\dagger\,i\,\alpha^i\,\alpha^k\psi,\ s_{[123]} = \psi^* i\,\alpha^1\,\alpha^2\,\alpha^3\,\psi) \tag{7e}$$

then the following quadratic identities will exist between them:

([1]) On this subject, cf., **B. L. van der Waerden**, *Die gruppentheorie Methode in der Quantenmechanik*, Berlin, 1932, esp. pp. 55.

([2]) **J. Schur**, Berl. Ber. math.-phys. Klasse (1905), pp. 406.

([3]) Cf., the bibliography in *Handbuch der Physik*, v. 24, paper by **W. Pauli**, pp. 222, rem. [2]). Further, see **L. de Broglie**, *L'électron magnétique*, Paris, 1934, in particular: pp. 161, eq. (14), pp. 189, eq. (14); pp. 220, eq. (24); pp. 221, eq. (28).

$$-\sum_{\mu} s_{\mu}^2 \equiv s_0^2 - \sum_k s_k^2 = \Omega_1^2 + \Omega_2^2, \tag{8a}$$

$$\sum_{[\mu\nu]} M_{[\mu\nu]}^2 \equiv \sum_{[ik]} M_{[ik]}^2 - \sum_k M_{k0}^2 = \Omega_1^2 - \Omega_2^2, \tag{8b}$$

$$-\frac{i}{2}\sum_{\mu\nu} M_{\mu\nu}\hat{M}_{\mu\nu} \equiv M_{23}\, M_{10} + M_{31}\, M_{20} + M_{12}\, M_{30} = \Omega_1\, \Omega_2, \tag{8c}$$

$$\sum_{[\lambda\mu\nu]} s_{[\lambda\mu\nu]}^2 \equiv s_{[123]}^2 - \sum_{[ik]} s_{[ik0]}^2 = -(\Omega_1^2 + \Omega_2^2) \tag{8d}$$

$$\sum s_k\, s_{[\lambda\mu\nu]} \equiv \sum_{\mu} s_{\mu}\, \hat{s}_{\mu} \equiv s_0\, \hat{s}_0 - \sum_k s_k\, \hat{s}_k = 0, \tag{8e}$$

Whereas, up to now, no one had succeeded in deriving these identities without a numerical specialization of the matrices γ^{μ}, that goal will be achieved in the present note, where it will be shown (§§ **3** and **4**) that *the identities* (8a) *to* (8e) *can be obtained by starting with the identity* (I) *and applying simple transformations.*

§ 2. Proof of Theorem I. – We begin with some elementary theorems about the system of γ^A that are easy to prove and follow directly from the relations (1).

Theorem 1. When one multiplies two quantities γ^A and γ^B, one will obtain a unique third quantity γ^C:

$$\gamma^A\, \gamma^B = \varepsilon_{AB}\, \gamma^C, \tag{9}$$

up to a numerical factor that is denoted by ε_{AB} (which can assume the values ± 1, $\pm i$), and γ^C is equal to the identity I only when one has $\gamma^B = \gamma^A$, in particular. Furthermore, if γ^A is fixed and the quantities γ^B run through the entire system then γ^C will also run through the entire system.

The latter fact is implied by the fact that for a fixed γ^A, the sixteen quantities $\gamma^A\gamma^B$ will be distinct, since it will follow from $\gamma^A\gamma^B = \gamma^A\gamma^{B'}$ that $\gamma^B = \gamma^{B'}$ and $I = \gamma^B\gamma^{B'}$, due to (4).

Theorem 2. – If the quantities γ^{μ}, and correspondingly, the remaining γ^A are represented by matrices then the traces of all γ^A will vanish, except for the identity matrix I.

For example, one has $\frac{1}{2}(-\gamma^2 \cdot \gamma^1\, \gamma^2 + \gamma^1\, \gamma^2 \cdot \gamma^2) = \gamma^1$. The fact that $\mathrm{Tr}\, \gamma^1 = 0$ will then follow from the commutativity of the trace. It likewise follows that $\mathrm{Tr}\, \gamma^2 = 0$ and $\mathrm{Tr}\, \gamma^3 = 0$.

Moreover, Tr $\gamma^{[12]} = 0$ follows directly from the fact that $\gamma^1\gamma^2 = -\gamma^2\gamma^1$, and Tr $\gamma^4\gamma^5$ = Tr $\gamma^{[123]} = 0$ will likewise follow from $\gamma^4\gamma^5 = -\gamma^5\gamma^4$.

It will then follow that none of the matrices γ^A, except for just the identity matrix I itself, can be represented as a multiple of the identity matrix, and that two distinct γ^A, γ^B cannot be represented by the same matrix. The last statement is true because $\gamma^A = \gamma^B$ would imply that $I = \gamma^A\gamma^B = \varepsilon_{AB} \cdot \gamma^C$.

Theorem 3. – The matrices γ^A are linearly independent of each other. That is, if:

$$\sum_A C_A \gamma^A = 0,$$

with ordinary numbers C_A, then that must imply the vanishing of all C_A :

$$\sum_A C_A \gamma^A = 0 \quad \rightarrow \quad C_A = 0. \tag{10}$$

Namely, if one multiplies (10) by a particular γ_B then it will follow that:

$$C_B + {\sum}' C_{BA}\, \varepsilon_{BA}\, \gamma^C = 0,$$

in which the identity matrix I does not appear under the Σ'. Taking the trace will yield $C_B = 0$, and thus, the vanishing of all C_A, since C_B was chosen arbitrarily.

Theorem 3 will play an essential role in what follows. It implies:

Theorem 4. – It is impossible represent the system of γ^A by matrices with less than four rows. Moreover, since there are sixteen linearly-independent four-rowed matrices, any four-rowed matrix F can be represented by a four-rowed representation of the γ_A (whose existence will be assumed to be known here) with the help of suitably-chosen ordinary numbers C_A :

$$F = \sum C_A \gamma^A . \tag{11}$$

Theorem 5. – One will always have:

$$\gamma^\mu\, \gamma^A\, \gamma^\mu = \pm\, \gamma^A, \tag{12}$$

for all γ^A and the four γ^μ, and for a given γ^A that is different from the identity I, there is always at least one γ^μ such that:

$$\gamma^\mu\, \gamma^A\, \gamma^\mu = -\, \gamma^A \qquad \text{or} \qquad \gamma^\mu\, \gamma^A = -\, \gamma^A\, \gamma^\mu. \tag{12a}$$

As far as the latter statement is concerned, one will have, e.g.:

$$\gamma^1 \cdot \gamma^{[12]}\, \gamma^1 = -\, \gamma^{[12]}, \qquad \gamma^4 \cdot \gamma^{[123]} \cdot \gamma^4 = -\, \gamma^{[123]}, \qquad \text{etc.}$$

For the sake of simplicity, we would now like to assume that all of the γ^A are represented by four-rowed matrices. One will then have:

Theorem 6. – If a four-rowed matrix F commutes with the four γ^μ in the (four-rowed) representation (and for that reason, with all γ^A) then it will be a multiple of the identity matrix.

If:

$$F\,\gamma^\mu = \gamma^\mu F \text{ for all } \mu \qquad \text{then} \qquad F = c \cdot I. \tag{13}$$

In fact, if we represent F in the form:

$$F = \sum_A C_A\,\gamma^A$$

using Theorem 4, and if we choose a γ^B that is given arbitrarily, but different from I, and a γ^μ such that $\gamma^\mu\gamma^B = -\gamma^B\gamma^\mu$, from Theorem 5, then it will follow immediately that $C_\mu = 0$, according to (10), and therefore $C_A = 0$ for all fifteen of the γ_A that are different from the identity.

We can now address the actual proof of Theorem I. According to (4), it will follow from:

$$\gamma^A\gamma^B = \varepsilon_{AB}\,\gamma^C, \tag{9}$$

that:

$$\gamma^B = \varepsilon_{AB}\,\gamma^A\,\gamma^C,$$

and when one takes the reciprocal:

$$\gamma^B = \frac{1}{\varepsilon_{AB}}\,\gamma^C\,\gamma^A,$$

that:

$$\gamma^C\gamma^A = \varepsilon_{AB}\,\gamma^B. \tag{9a}$$

Now, if γ'^A is a second four-rowed matrix representation of the system then one will likewise have:

$$\gamma'^A\,\gamma'^B = \varepsilon_{AB}\,\gamma'^C, \tag{9$'$}$$

since (9), as well as (9*a*), will follow from just (1).

We now construct the matrix:

$$\sum_{B=1}^{16} \gamma'^B F \gamma^B = S \tag{14}$$

from the matrix F, which is initially arbitrary, and then use it to prove the relation:

$$\gamma'^A S = S\,\gamma^A, \tag{15}$$

which is valid for all A.

Next, one has, from (9′):

$$\gamma'^{A} S = \sum_B \varepsilon_{AB}\, \gamma'^{C} F\, \gamma^{B},$$

in which C is associated with B (for fixed A) in a one-to-one correspondence. On the latter grounds, one can also write (14) as:

$$S = \sum_C \gamma'^{C} F \gamma^{C}$$

by merely changing the summation symbol, and from (9*a*), one will have:

$$S\, \gamma^{A} = \sum_C \gamma'^{C} F \varepsilon_{AB}\, \gamma^{B} = \sum_B \varepsilon_{AB}\, \gamma'^{C} F \gamma^{B},$$

with which, (15) is proved.

The relation (15) would already be equivalent to the statement of Theorem I if one could prove that with a suitable choice of F, it could already be arranged that $S \neq 0$ and Det $S \neq 0$, moreover. It would then be easy to see that S could not vanish identically in F. Namely, if $\gamma^{A}_{\rho\sigma}$ are the matrix elements of γ^{A} then that would be equivalent to:

$$\sum_A \gamma'^{A}_{\rho\sigma}\, \gamma^{A}_{\bar\rho\bar\sigma} = 0,$$

which is impossible, due to the linear independence of the γ^{A} (Theorem 3) and the fact that not all of the $\gamma^{A}_{\bar\rho\bar\sigma}$ are identically zero.

Here, on the basis of a lemma by **Schur** ([1]), one can further conclude that if one had $S \neq 0$, Det $S = 0$ then one could construct matrices with less than four rows from the γ^{A} that would fulfill the relations (1), which is impossible, from Theorem 3. One can also prove that independently of **Schur**'s lemma. One switches the roles of γ^{A} and γ'^{A} and defines:

$$T = \sum_A \gamma^{A} G\, \gamma'^{A}. \tag{14′}$$

One will then have:

$$\gamma^{A}\, T = T\, \gamma'^{A}, \tag{15′}$$

and when one combines this with (15):

$$\gamma^{A}\, T\, S = T\, \gamma'^{A} S = T\, S\, \gamma^{A},$$

Theorem 6 will have the consequence that:

$$T\, S = c \cdot I. \tag{16}$$

([1]) Cf., **B. L. van der Waerden**, *loc. cit.*, pp. 47.

One now thinks of G in (14′) as having been chosen such that $T \neq 0$. If one had that:

$$T\,S = 0$$

for all F then one would need to have:

$$\sum_A T_{\sigma\rho}\,\gamma'^A_{\rho\sigma}\,\gamma^A_{\bar\rho\bar\sigma} = 0,$$

so:

$$T_{\sigma\rho}\,\gamma^A_{\rho\sigma} = 0, \qquad \text{i.e.,} \qquad T\,\gamma^A = 0 \qquad\qquad \text{for all } A,$$

due to the linear independence of the $\gamma^A_{\bar\rho\bar\sigma}$ [Theorem 3].

That contradicts the assumption, since one will certainly have $T\,\gamma^A = T \neq 0$ for $\gamma^A = I$. Hence, for a fixed $T \neq 0$, one can certainly choose F such that one also has:

$$T\,S \neq 0.$$

However, it will then follow from (16) that:

$$T\,S = c \cdot I \qquad \text{with} \qquad c \neq 0$$

and

$$\text{Det } S \neq 0, \qquad S^{-1} = \frac{1}{c}\,T.$$

Theorem I is then proved.

We would now like to apply (14), (15) to $\gamma'^A = \gamma^A$, in particular.

From Theorem 6, if follows from:

$$\gamma^A\,S = S\,\gamma^A$$

that:

$$S = c \cdot I,$$

so one must have:

$$\sum_A \gamma^A F \gamma^A = c \cdot I$$

for all F.

However, that is equivalent to:

$$\sum_A \gamma^A_{\rho\sigma}\,\gamma^A_{\bar\rho\bar\sigma} = c_{\bar\rho\sigma}\,\delta_{\rho\bar\sigma}\,.$$

In order to determine the $c_{\bar\rho\sigma}$, we set $\rho = \bar\sigma$ and sum over ρ. From (4), the left-hand side will then become:

$$\sum_A \sum_\rho \gamma^A_{\bar\rho\rho}\,\gamma^A_{\rho\sigma} = \sum_A (\gamma^A)^2_{\bar\rho\sigma} = 16\,\delta_{\bar\rho\sigma}\,,$$

and the right-hand side will become $4\,c_{\bar\rho\sigma}$, so it will follow that:

$$c_{\bar{\rho}\sigma} = 4\,\delta_{\bar{\rho}\sigma}$$

and

$$\sum_A \gamma^A_{\rho\sigma}\,\gamma^A_{\bar{\rho}\bar{\sigma}} = 4\,\delta_{\bar{\rho}\sigma}\delta_{\rho\bar{\sigma}}\,, \tag{I}$$

which agrees with the identity (I) that was given above.

Here, we have restricted ourselves to four-rowed representations of the γ^A, for the sake of simplicity, and proved that they are all equivalent. We can also prove that the representations of the system with more than four rows are all reducible with the same method.

§ 3. Derivation of further identities. – We shall first write (I) in the detailed form:

$$\delta_{\rho\sigma}\delta_{\bar{\rho}\bar{\sigma}} + \gamma^5_{\rho\sigma}\gamma^5_{\bar{\rho}\bar{\sigma}} + \sum_\mu \gamma^\mu_{\rho\sigma}\gamma^\mu_{\bar{\rho}\bar{\sigma}} + \sum_{[\mu\nu]} \gamma^{[\mu\nu]}_{\rho\sigma}\gamma^{[\mu\nu]}_{\bar{\rho}\bar{\sigma}} + \sum_{[\lambda\mu\nu]} \gamma^{[\lambda\mu\nu]}_{\rho\sigma}\gamma^{[\lambda\mu\nu]}_{\bar{\rho}\bar{\sigma}} = 4\,\delta_{\bar{\rho}\sigma}\delta_{\rho\bar{\sigma}}\,. \tag{17}$$

When we multiply this identity by the arbitrary quantities $\psi^\dagger_\rho\,\varphi^\dagger_{\bar{\rho}}\,\psi_\sigma\varphi_{\bar{\sigma}}$ and sum over equal indices, we will already get an identity of the same type as the identities that were written down in § **1**, namely:

$$-\,\Omega_1\,\Omega'_1 + \Omega_2\,\Omega'_2 + \sum_\mu s_\mu\,s'_\mu - \sum_{[\mu\nu]} M_{[\mu\nu]}M'_{[\mu\nu]} + \sum_{[\lambda\mu\nu]} s_{[\lambda\mu\nu]}\,s'_{[\lambda\mu\nu]} = 4\,(\varphi^\dagger\psi)\cdot(\psi^\dagger\varphi)\,. \tag{18}$$

The definitions (7*a*) to (7*c*) are employed in this, and the quantities that are denoted with a prime will arise from the corresponding unprimed quantities replacing $\psi^\dagger$ and ψ with $\varphi^\dagger$ and φ, resp. We remark that in what follows, neither the connection (6) between $\psi^\dagger$ ($\varphi^\dagger$, resp.) and the complex conjugate ψ^* (φ^*, resp.), which is crucial for the reality of the quantities Ω_1, Ω_2, s_μ, etc., nor the Hermiticity of the matrices γ^μ will be employed. The identities will remain correct when $\psi^\dagger$ ($\varphi^\dagger$, resp.) are regarded as quantities that are entirely independent of the ψ (φ, resp.).

By specializing $\varphi^\dagger = \psi^\dagger$ and $\varphi = \psi$, (18) will imply that:

$$-\,\Omega_1^2 + \Omega_2^2 + \sum_\mu s_\mu^2 - \sum_{[\mu\nu]} M^2_{[\mu\nu]} + \sum_{[\lambda\mu\nu]} s^2_{[\lambda\mu\nu]} = -4\,\Omega_1^2\,. \tag{18a}$$

This identity is a consequence of the identities (8*a*), (8*b*), (8*d*), but it obviously says more than the latter.

In order to proceed, we multiply the second matrix γ^A in (17) by γ^5 on the left and right, or more precisely, one first replaces $\bar{\rho}$, $\bar{\sigma}$ with $\bar{\bar{\rho}}$, $\bar{\bar{\sigma}}$, resp., multiplies by $\gamma^5_{\bar{\rho}\bar{\bar{\rho}}}, \gamma^5_{\bar{\bar{\sigma}}\bar{\sigma}}$ and sums over $\bar{\bar{\rho}}$, $\bar{\bar{\sigma}}$. Since the γ^μ and $\gamma^{[\lambda\mu\nu]}$ anticommute with γ^5, but the remaining matrices commute with γ^5, the terms with γ^μ and $\gamma^{[\lambda\mu\nu]}$ will then change signs, and one will get:

$$\delta_{\rho\sigma}\delta_{\bar\rho\bar\sigma} + \gamma^5_{\rho\sigma}\gamma^5_{\bar\rho\bar\sigma} - \sum_\mu \gamma^\mu_{\rho\sigma}\gamma^{\mu\nu}_{\bar\rho\bar\sigma} + \sum_{[\mu\nu]} \gamma^{[\mu\nu]}_{\rho\sigma}\gamma^{[\mu\nu]}_{\bar\rho\bar\sigma} - \sum_{[\lambda\mu\nu]} \gamma^{[\lambda\mu\nu]}_{\rho\sigma}\gamma^{[\lambda\mu\nu]}_{\bar\rho\bar\sigma} = 4\gamma^5_{\bar\rho\sigma}\gamma^5_{\rho\bar\sigma}, \tag{19}$$

which will give:

$$\delta_{\rho\sigma}\delta_{\bar\rho\bar\sigma} + \gamma^5_{\rho\sigma}\gamma^5_{\bar\rho\bar\sigma} + \sum_{[\mu\nu]} \gamma^{[\mu\nu]}_{\rho\sigma}\gamma^{[\mu\nu]}_{\bar\rho\bar\sigma} = 2(\delta_{\bar\rho\sigma}\delta_{\rho\bar\sigma} + \gamma^5_{\bar\rho\sigma}\gamma^5_{\rho\bar\sigma}), \tag{20}$$

$$\sum_\mu \gamma^\mu_{\rho\sigma}\gamma^{\mu\nu}_{\bar\rho\bar\sigma} + \sum_{[\lambda\mu\nu]} \gamma^{[\lambda\mu\nu]}_{\rho\sigma}\gamma^{[\lambda\mu\nu]}_{\bar\rho\bar\sigma} = 2(\delta_{\bar\rho\sigma}\delta_{\rho\bar\sigma} - \gamma^5_{\bar\rho\sigma}\gamma^5_{\rho\bar\sigma}), \tag{21}$$

when it is combined with (15). When this is multiplied by $\psi^\dagger_\rho\,\varphi^\dagger_{\bar\rho}\,\psi_\sigma\,\varphi_{\bar\sigma}$ and summed over ρ, $\bar\rho$, σ, $\bar\sigma$, that will give:

$$-\Omega_1\,\Omega_1' + \Omega_2\,\Omega_2' - \sum_{[\mu\nu]} M_{[\mu\nu]}M'_{[\mu\nu]} = 2\,[(\varphi^\dagger\psi)\cdot(\psi^\dagger\varphi) + [(\varphi^\dagger\gamma^5\,\psi)\cdot(\psi^\dagger\gamma^5\varphi)], \tag{22}$$

$$\sum_\mu s_\mu\,s'_\mu + \sum_{[\lambda\mu\nu]} s_{[\lambda\mu\nu]}\,s'_{[\lambda\mu\nu]} = 2\,[(\varphi^\dagger\psi)\cdot(\psi^\dagger\varphi) - [(\varphi^\dagger\gamma^5\,\psi)\cdot(\psi^\dagger\gamma^5\varphi)]. \tag{23}$$

When one specializes to $\varphi^\dagger = \varphi$, $\psi^\dagger = \psi$ one will get:

$$-\Omega_1^2 + \Omega_2^2 - \sum_{[\mu\nu]} M^2_{[\mu\nu]} = -2(-\Omega_1^2 + \Omega_2^2), \tag{22′}$$

which already agrees with (8*b*), and:

$$\sum_\mu s_\mu^2 + \sum_{[\lambda\mu\nu]} s^2_{[\lambda\mu\nu]} = -2(\Omega_1^2 + \Omega_2^2), \tag{23′}$$

which coincides with the difference of (8*b*) and (8*d*).

We shall now further form the expression:

$$\tfrac{1}{2}\sum_A \gamma^A_{\rho\sigma}(\gamma^5\gamma^A + \gamma^A\gamma^5)_{\bar\rho\bar\sigma} = 2(\gamma^5_{\bar\rho\sigma}\delta_{\rho\bar\sigma} + \delta_{\bar\rho\sigma}\,\gamma^5_{\rho\bar\sigma})$$

from (17) [(I), resp.].

In that way, the terms on the left-hand side with γ^μ and $\gamma^{[\lambda\mu\nu]}$ will be annulled, and furthermore, with the introduction of quantities $\hat\gamma^{[\mu\nu]}$ that are dual to $\gamma^{[\mu\nu]}$ and are defined by:

$$\begin{aligned} &\hat\gamma^{[23]} = \gamma^{[14]}, && \hat\gamma^{[31]} = \gamma^{[24]}, && \hat\gamma^{[12]} = \gamma^{[34]}, \\ &\hat\gamma^{[14]} = \gamma^{[23]}, && \hat\gamma^{[24]} = \gamma^{[31]}, && \hat\gamma^{[34]} = \gamma^{[12]}, \end{aligned}$$

in analogy to the relation between $\hat M_{[\mu\nu]}$ and $M_{\mu\nu}$, one will have:

$$\tfrac{1}{2}(\gamma^5\gamma^{[\mu\nu]} + \gamma^{[\mu\nu]}\gamma^5) = -\hat\gamma^{[\mu\nu]}. \tag{27}$$

That will then yield that:

$$\delta_{\rho\sigma}\gamma^5_{\bar\rho\bar\sigma} + \gamma^5_{\rho\sigma}\delta_{\bar\rho\bar\sigma} - \sum_{[\mu\nu]}\gamma^{\mu\nu}_{\rho\sigma}\,\hat\gamma^{[\mu\nu]}_{\bar\rho\bar\sigma} = 2(\gamma^5_{\bar\rho\sigma}\delta_{\rho\bar\sigma} + \delta_{\bar\rho\sigma}\,\gamma^5_{\rho\bar\sigma}), \tag{25}$$

which will give rise to:

$$i(\Omega_1\,\Omega'_2 + \Omega_2\,\Omega'_1) + \sum_{[\mu\nu]} M_{[\mu\nu]}\hat M'_{[\mu\nu]} = 2\,[(\varphi^\dagger\gamma^5\psi)(\varphi^\dagger\psi) + (\varphi^\dagger\psi)(\varphi^\dagger\gamma^5\psi)], \tag{26}$$

in a manner that is analogous to what we have done up to now.

By specializing to $\varphi^\dagger = \psi^\dagger$, $\varphi = \psi$, this will imply that:

$$2i\,\Omega_1\,\Omega_2 + \sum_{[\mu\nu]} M_{[\mu\nu]}\hat M_{[\mu\nu]} = 4i\,\Omega_1\,\Omega_2, \tag{26′}$$

$$-\frac{i}{2}\sum_{[\mu\nu]} M_{[\mu\nu]}\hat M_{[\mu\nu]} = \Omega_1\,\Omega_2, \tag{8c}$$

which agrees with (8*c*).

Finally, we construct:

$$\tfrac{1}{2}\sum_A \gamma^A_{\rho\sigma}(\gamma^5\gamma^A - \gamma^A\gamma^5)_{\bar\rho\bar\sigma} = 2(\gamma^5_{\bar\rho\sigma}\delta_{\rho\bar\sigma} - \delta_{\bar\rho\sigma}\,\gamma^5_{\rho\bar\sigma}).$$

Only the terms in γ^μ and $\gamma^{[\lambda\mu\nu]}$ will remain in this case, and in fact, when one introduces the notation:

$$-\hat\gamma^1 = \gamma^{[234]}, \quad -\hat\gamma^2 = \gamma^{[314]}, \quad -\hat\gamma^3 = \gamma^{[124]}, \quad \hat\gamma^4 = \gamma^{[123]},$$

one will have:

$$\tfrac{1}{2}(\gamma^5\gamma^\mu - \gamma^\mu\gamma^5) = -i\,\hat\gamma^\mu, \qquad \tfrac{1}{2}(\gamma^5\hat\gamma^\mu - \hat\gamma^\mu\gamma^5) = +i\,\gamma^\mu.$$

That will then yield:

$$-i\sum_A \gamma^A_{\rho\sigma}\hat\gamma^A_{\bar\rho\bar\sigma} + i\sum_A \hat\gamma^A_{\rho\sigma}\gamma^A_{\bar\rho\bar\sigma} = 2(\gamma^5_{\bar\rho\sigma}\delta_{\rho\bar\sigma} - \delta_{\bar\rho\sigma}\,\gamma^5_{\rho\bar\sigma}), \tag{27}$$

$$-i\sum_\mu s_\mu\hat s'_\mu + i\sum_\mu \hat s_\mu s'_\mu = 2\,[(\varphi^\dagger\gamma^5\psi)(\varphi^\dagger\psi) + (\varphi^\dagger\psi)(\varphi^\dagger\gamma^5\psi)]. \tag{28}$$

By specializing to $\varphi^\dagger = \psi^\dagger$, $\varphi = \psi$, the latter identity will give 0 = 0.

We are now still lacking *one* equation between the identities (8*a*) and (8*d*), as well as the identity (8*e*). The use of the matrix $\hat\gamma^5$ will not suffice to derive that equation from the relation (I), either.

§ 4. Introducing the matrix *B*. The remaining identities. – Elsewhere ([1]), the author proved that Theorem I will imply the existence of a matrix B such that:

$$\bar{\gamma}^{\mu} = B\,\gamma^{\mu}\,B^{-1} \qquad \text{or} \qquad \bar{\gamma}^{\mu}B = B\,\gamma^{\mu}, \tag{29}$$

when $\bar{\gamma}^{\mu}$ means the so-called transposed matrix that arises from γ^{μ} by switching the rows and columns:

$$\bar{\gamma}^{\mu}_{\rho\sigma} = \gamma^{\mu}_{\sigma\rho}. \tag{30}$$

The matrices $\bar{\gamma}^{\mu}$ satisfy the same relations (1) as the γ^{μ}.

It is important for us that (29) implies that:

$$\overline{\gamma^{[\mu\nu]}} = -B\,\gamma^{[\mu\nu]}, \qquad \overline{\gamma^{[\lambda\mu\nu]}} = -B\,\gamma^{[\lambda\mu\nu]}, \qquad \overline{\gamma^{5}} = +B\,\gamma^{5}. \tag{29a}$$

The signs originate in the facts that (12) $\to$ (21) and (123) $\to$ (321) are odd permutations, while (1234) $\to$ (4321) is an even permutation.

As was shown in *loc. cit.*, by going over to the transposed matrix:

$$\bar{B}\gamma^{\mu} = \bar{\gamma}^{\mu}B, \quad B^{-1}\bar{B}\gamma^{\mu} = \bar{\gamma}^{\mu}B^{-1}\bar{B},$$

it will follow from (29) that:

$$\bar{B} = c\,B,$$

as it does from Theorem 6 [eq. (13)]. That will be possible only when either:

$$\bar{B} = B \qquad \text{or} \qquad \bar{B} = -B.$$

In the latter case, from (29) and (29*a*):

$$\text{the six matrices } B,\ B\,\gamma^{\mu},\ B\,\gamma^{5} \text{ would be skew,} \tag{31a}$$

$$\text{the ten matrices } B\,\gamma^{[\mu\nu]},\ B\,\gamma^{[\lambda\mu\nu]} \text{ would be symmetric,} \tag{31b}$$

while in the former case, the opposite would be true. However, the last situation is impossible, since the ten matrices $B\,\gamma^{[\mu\nu]}$, $B\,\gamma^{[\lambda\mu\nu]}$ are linearly-independent, but there are only six linearly-independent skew four-rowed matrices (as opposed to ten linearly-independent symmetric four-rowed matrices). Hence, the former case applies, and one concludes [from **Haantjes**, cf., *loc. cit.*] that:

$$\bar{B} = -B, \tag{32}$$

in particular.

([1]) **W. Pauli**, Ann. Phys. (Leipzig) **18** (1933), 337; esp. pp. 354.

The matrix B is physically-meaningful, since it makes a relativistically-invariant association of state of positive energy with states of negative energy possible, and also plays a role in **Fermi**'s theory of β-decay. However, we shall not go further into that here.

In our case, the use of the matrix B leads to the goal of deriving the remaining identities, which is based essentially upon the fact that when $\varphi^\dagger = \psi^\dagger$, $\varphi = \psi$, a symmetrization will result in the matrix elements that are written down in regard to $\bar{\rho}$ and ρ, as well as to $\bar{\sigma}$ and σ, in addition.

We next construct the expression:

$$\sum_A \gamma^A_{\rho\sigma}(B\gamma^A B^{-1})_{\bar{\sigma}\bar{\rho}} = 4B_{\bar{\sigma}\sigma}B^{-1}_{\rho\bar{\rho}}$$

from (I) [(17), resp.], and when we recall (29) and (29*a*), we will get:

$$\delta_{\rho\sigma}\delta_{\bar{\rho}\bar{\sigma}} + \gamma^5_{\rho\sigma}\gamma^5_{\bar{\rho}\bar{\sigma}} + \sum_\mu \gamma^\mu_{\rho\sigma}\gamma^\mu_{\bar{\rho}\bar{\sigma}} - \sum_{[\mu\nu]} \gamma^{[\mu\nu]}_{\rho\sigma}\gamma^{[\mu\nu]}_{\bar{\rho}\bar{\sigma}} - \sum_{[\lambda\mu\nu]} \gamma^{[\lambda\mu\nu]}_{\rho\sigma}\gamma^{[\lambda\mu\nu]}_{\bar{\rho}\bar{\sigma}} = 4B_{\bar{\sigma}\sigma}B^{-1}_{\rho\bar{\rho}}, \tag{33}$$

which will yield:

$$\delta_{\rho\sigma}\delta_{\bar{\rho}\bar{\sigma}} + \gamma^5_{\rho\sigma}\gamma^5_{\bar{\rho}\bar{\sigma}} + \sum_\mu \gamma^\mu_{\rho\sigma}\gamma^\mu_{\bar{\rho}\bar{\sigma}} = 4(\delta_{\bar{\rho}\sigma}\delta_{\rho\bar{\sigma}} + B_{\bar{\sigma}\sigma}B^{-1}_{\rho\bar{\rho}}) \tag{34}$$

when combined with (18). If we now multiply by $\psi^\dagger_\rho\psi^\dagger_{\bar{\rho}}\psi_\sigma\psi_{\bar{\sigma}}$ and sum over $\bar{\rho}$, ρ, σ, $\bar{\sigma}$ then *the terms in B on the right will be annulled*, since $B_{\bar{\sigma}\sigma} = -B_{\sigma\bar{\sigma}}$ and likewise $B^{-1}_{\bar{\rho}\rho} = -B^{-1}_{\rho\bar{\rho}}$. We will then get:

$$-\Omega_1^2 + \Omega_2^2 + \sum_\mu s_\mu^2 = -2\Omega_1^2$$

or

$$-\sum_\mu s_\mu^2 = \Omega_1^2 + \Omega_2^2, \tag{8a}$$

which agrees with (8*a*). *The identity* (8*d*) *will also follow further from the previously-proved equation* (23) *then.*

In order to also prove (8*e*), we multiply the second matrix in (27) on the left by B and on the right by B^{-1}, or more precisely: We first replace $\bar{\rho}$, $\bar{\sigma}$ with $\bar{\bar{\rho}}$, $\bar{\bar{\sigma}}$, resp., and then multiply by $B_{\bar{\sigma}\bar{\bar{\rho}}}B^{-1}_{\bar{\bar{\sigma}}\bar{\rho}}$ and sum over $\bar{\bar{\rho}}$, $\bar{\bar{\sigma}}$. From (29*a*), the first term will then change sign, while the second one will keep its sign, and that will yield:

$$i\sum_\mu \gamma^\mu_{\rho\sigma}\hat{\gamma}^\mu_{\bar{\rho}\bar{\sigma}} + i\sum_\mu \hat{\gamma}^\mu_{\rho\sigma}\gamma^\mu_{\bar{\rho}\bar{\sigma}} = 2[(B\gamma^5)_{\bar{\sigma}\sigma}B^{-1}_{\rho\bar{\rho}} - B_{\bar{\sigma}\sigma}(B\gamma^5)^{-1}_{\rho\bar{\rho}}]. \tag{35}$$

The matrices that appear on the right-hand side are all skew, such that it will follow that:

$$\sum_{\mu} s_\mu \, \hat{s}_\mu = 0 \tag{8e}$$

in analogy with the above, and that will coincide with (8*e*).

All of the identities (8*a*) to (8*e*) are then proved with that. The application of the matrix *B* to the remaining relations that were derived in the previous section will then give rise to no further identities. The ones that were given are the only ones of the kind that relate *relativistically-invariant* sums to each other.

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