"Über die Formulierung der Naturgesetze mit fünf homogenen Koordinaten. Teil II: Die Diracschen Gleichung für die Materiewellen," Ann. d. Phys. **18** (1933), 337-372.

On the formulation of the laws of nature with five homogeneous coordinates.

Part II: The Dirac equation for the matter waves

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§ 1. Introduction. – § 2. Construction of the metric tensor $g_{\mu\nu}$ from 5 matrices α_μ . – § 3. Coordinate s and *S*-transformations. – § 4. Covariant differentiation of spinors. – § 5. Connection between projective and affine spinors. Statement of the wave equation. – § 6. Variational principle and field equations.

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§ 1. **Introduction**

The search for a suitable form for the Dirac equation in terms of a wave function that depends upon five homogeneous variables seems, upon setting aside many more serious criticisms, to be the corresponding search for the classical equations of gravitational and electromagnetic fields. Even when we thus ignore the fact such methods are, in general, notoriously "formal" – without definitively resolving the question of whether this judgment is correct, we can nevertheless predict, with hindsight, that the method will bring about a *logical unification* of the foundations of natural law – we must stress that, from a physical standpoint, the physical foundations of the Dirac theory are completely dubious. They lead to the extension of those laws to include opposite states of negative energy for matter waves! The following investigation shall therefore not achieve the objective of giving new support for the validity of the Dirac wave equation, but rather to show that the unification of gravitational and electromagnetic fields by means of projective differential geometry with five homogeneous coordinates is a *general* method whose consequences reach from classical field physics into quantum theory. Perhaps it is not incorrect to hope that the method will prove to be a general framework for physical laws, as well as a future, physically meaningful improvement of the foundations of the Dirac theory.

The applicability of the method of five coordinates to the Dirac wave equation rests upon the fact that it is also related to the group of orthogonal linear transformations of *five* variable quantities Ψ with *four* components that likewise transform linearly by these transformations (i.e., that a four-rowed representation of the five-dimensional rotation group exists). If one specializes the orthogonal transformations of the coordinates to a subgroup that fixes X^5 (viz., the Lorentz group) then these quantities transform just as the Dirac ones do.

We establish this fact in the following (§§ 2 and § 3) in a unified way by considering the matrix equations:

I) $$\tfrac{1}{2}(\alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu) = g_{\mu\nu} \, ?1 \qquad (m, n = 1 \text{ to } 5),$$

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† Translated by D.H. Delphenich.

which were first presented by Tetrode in the analogous four-dimensional case, and then construct the general relativistic extension of the Dirac matrix equations. IT is, however, essential now that there are *five* four-rowed matrices α_μ that satisfy the relations (I). When the determinant of $g_{\mu\nu}$ is non-null, which may be assumed, the 16 matrices:

$$1,\ \alpha_\mu,\ \ \alpha_{[\mu\nu]} \equiv \tfrac{1}{2}(\alpha_\mu\,\alpha_\nu - \alpha_\nu\,\alpha_\mu)$$

define a basis for a hypercomplex number system. It has two important properties: first, it possesses one *and only one* representation by four-rowed matrices, up to equivalence (i.e., up to a similarity transformation and a possible change of sign $\alpha'_\mu = \pm\, S^{-1}\alpha_\mu S$). Thus, if a second system of matrices α'_μ exists that satisfies the same relations (I) as the α_μ then there is a matrix S such that $\alpha'_\mu = \pm\, S^{-1}\alpha_\mu S$. The second property is that the 15 relations (I) may not be satisfied by matrices with less than four rows, so the four-rowed representation is thus irreducible. Therefore, it can be inferred that any matrix that commutes with all five matrices α_μ (it suffices that this is true for four of the matrices) is a multiple of the identity matrix.

From this, it further follows that a matrix A exists such that:

$$A\alpha_\mu$$

is Hermitian, where either A itself or iA is likewise Hermitian ([1]). This gives rise to the construction of vectors:

$$a_\mu = \Psi^*\,\alpha_\mu\,\Psi,$$

that possess real components. (We shall always employ real coordinates here.)

The transformation law for Ψ is coupled to that of α_μ, namely:

$$\alpha'_\mu = S^{-1}\alpha_\mu S, \qquad A' = S^{\dagger} A\, S, \qquad\qquad \Psi' = S^{-1}\Psi\ .$$

Under these S-transformations, there is, in particular, an associated 10-parameter group $S(D_5)$, with the property that for any coordinate transformations:

$$X'^{\mu} = a^{\mu}_{\cdot\nu} X^{\nu},$$

that leave $g_{\mu\nu}$ invariant, which can be referred to as *rotations*, an S exists in D_5 such that:

$$\alpha'_\nu = S\alpha_\nu S^{-1}, \quad A' = S^{\dagger} A\, S = A\,,$$
$$\alpha_\nu = S^{-1}\alpha'_\nu S = a^{\mu}_{\cdot\nu}\alpha'_\mu\ .$$

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[1] A matrix a with the elements a_{rs} is called *Hermitian* when a_{sr} is the complex conjugate of a_{rs} ($a_{sr} = a^*_{rs}$). It is called *symmetric* when $a_{sr} = a_{rs}$, and *skew-symmetric* (or *anti-symmetric*) when $a_{sr} = -\,a_{rs}$. The Hermitian conjugate matrix to a will be denoted by $a^\dagger$, and it is defined by $a^\dagger_{rs} = a^*_{sr}$.

It then transforms as a covariant vector, *also for fixed* α_μ:

$$a_\mu = \Psi^* A \alpha_\mu \Psi$$

with:

$$\Psi' = S^{-1}\Psi .$$

Thus, in the special case of rotations, the mutually independent *S*-transformations are connected with the coordinate transformations.

The argument against this way of thinking is often brought forth that it is unnatural top introduce a transformation law for the Ψ by rotations whose coefficients depend upon which numerical values the α_μ are allowed to have. On the contrary, we would like to put forth the statement here that this state seems to be completely natural when we introduce the idea of a four-dimensional *spin space.* This then rests on the fact that when one is given the position of an electron there are four possible states that are characterized by four linearly independent $\Psi_\rho^{(r)}$ (ρ = 1 … 4):

$$\sum_r c_r \Psi_\rho^{(r)} = 0 ,$$

only when $c_r = 0$. Just as arbitrary systems of reference are permitted in the four-dimensional spacetime continuum, so are arbitrary systems of reference permitted in spin space, and they can, moreover, vary arbitrarily from point to point in the spacetime continuum. The methodical contradiction to the van der Waerden spinor calculus that follows from this statement will be mentioned in § 3.

The method of basing the spinors of five-dimensional space (the projective spinors that are independent of the five homogeneous coordinates, resp.) that is given here is different from the one that presently exists in the literature. Recently, W. Pauli and J. Solomon ([1]) have established the existence of such spinors, and, with their help, sought to bring the Dirac equation into accord with the Einstein-Mayer statement of field theory. Here, however, the results will only be slightly unified in a formal sense, and the "Beingrößen" $h^{\mu}_{\bullet\bar{\nu}}$ (that are associated with Fock and Weyl), which further complicate the formulas, will not be explicitly introduced anywhere.

In the present article, this is avoided, and indeed the method of Schrödinger ([2]) and Bargmann ([3]) (with its use of ordinary inhomogeneous coordinates), which carries over to our own case with no further assumptions (§ 4). In particular, one first finds the general introduction in Bargmann's work of the Hermitian matrix that we denote by *A* here, and the general covariance of the equations under arbitrary *S*-transformations is first achieved there.

[1] W. Pauli and J. Solomon, Journ. de Phys. (7) **3** (1932), 452, 582.
[2] E. Schrödinger, Berl. Ber. (1932), 105.
[3] V. Bargmann, Berl. Ber. (1932), 346.

Independently of Pauli and Solomon, Schouten and van Dantzig ([1]) have examined the problem of spinors and the Dirac equation, and indeed with the use of five homogeneous coordinates. The calculus of and the foundation for projective spinors in this work can be accurately described as hard to understand and non-intuitive, since we will always use properties of the matrices that are only valid in a special reference system in spin space, although they are inessential for the results and tend to complicate one's intuitive grasp of them.

By contrast, we have adopted the form of the Dirac equations that was used by these authors in § 5 of the present article. The projective spinor Ψ will thus be set equal to:

$$\Psi = \psi F^l,$$

in which ψ is an ordinary (inhomogeneous affine) spinor and F is a real scalar of degree 1. In order to satisfy the requirement of the reality of the Lagrange function in a simple way the degree of homogeneity l of Ψ must be assumed to be pure imaginary. (In the case of a symmetric $\Gamma^{\lambda}_{\mu\nu}$, which we assign particular values, as was discussed in Part I, the dimensionless number l becomes:

$$l = i\frac{e}{h}\frac{1}{\sqrt{2\kappa}},$$

if e means the charge of the particle, h is Planck's constant divided by 2π, and κ is the Einstein gravitational constant.)

In § 6, we will seek to link the classical field theory of the vacuum (which corresponds to the absence of ponderable mass and charge) to the Dirac theory of matter wave fields by adding the associated Lagrange functions together. In order to distinguish from the previous treatments of the same problems by Pauli-Solomon and Schouten-van Dantzig (Schouten, resp.), we succeed here in presenting a formally unified expression for the symmetric projector $T_{\mu\nu}$, which combines the energy-momentum tensor T_{ik} and the current vector v^i, § 10 eq. (71). For this, the assumption of the symmetry $\Gamma^{\lambda}_{\mu\nu}$ of seems to particularly prove it worth. Additional terms appear in both tensors that are proportional to $\sqrt{\kappa}$ (which was also the case for Pauli-Solomon, but with different numerical factors), which are also non-vanishing in the absence of gravitational fields (special relativity theory). However, due to its smallness, this expansion of the theory that was developed by Dirac can hardly be conformed by experiment.

The latter theory does not apply directly to reality, but only after quantization of the wave fields, which the transition to configuration space brings with it. However, we shall not take this further step here, which inevitably leads back to the well-known unsolved problem of the self-energy of matter waves.

Also missing from the path that has ultimately been chosen here in the following is the combining of matter wave fields with classical fields (viz., the gravitational and

[1] This first came about for a special choice of signature for the metric: J.A. Schouten and D. van Dantzig, Z. f. Phys. **78** (1932), 639, which contains older literature. Moreover: Ann. Math. (2) **34** (1933), 271. Later, after hearing of the work Pauli-Solomon, the general case: J.A. Schouten, Z. f. Phys. **81** (1933), 129, 405.

electromagnetic fields), which, in all of the previous theories, was "only foreign and logically arbitrary by way of a plus sign." This seems to be connected with the fact in the previous theories (including the one that is developed here) the atomistic nature of the electric charge was not rigorously present in the foundations.

§ 2. Construction of the metric tensor $g_{\mu\nu}$ from five matrices α_μ

The Dirac theory makes use of the existence of four four-rowed matrices $\overset{0}{\gamma}_k$ ([1]) that satisfy the relations:

$$\tfrac{1}{2}(\overset{0}{\gamma}_i \overset{0}{\gamma}_k + \overset{0}{\gamma}_k \overset{0}{\gamma}_i) = \delta_{ik} , \tag{1}$$

and which can be, moreover, chosen to be Hermitian. If all of the matrices used in the sequel are assumed to be four-rowed then the following fundamental theorems are valid:

Theorem 1. If four other (four-rowed), not necessarily Hermitian, matrices $\overset{0}{\gamma}{}'_k$ *satisfy the same relations (1) then there is a matrix S (with non-vanishing determinant) such that one has:*

$$\overset{0}{\gamma}{}'_k = S^{-1} \overset{0}{\gamma}_k S . \tag{2}$$

Theorem 2. If C is a (four-rowed) matrix that commutes with all four matrices then C is a multiple of the identity matrix.

The first theorem rests upon the fact that all representations of degree 4 of the hypercomplex number system that is defined by (1) are mutually equivalent, and the second theorem rests upon the fact that all of these representations of degree 4 are irreducible ([2]).

If one defines the matrix:

$$\overset{0}{\gamma}_5 = \overset{0}{\gamma}_1 \overset{0}{\gamma}_2 \overset{0}{\gamma}_3 \overset{0}{\gamma}_4 ,$$

then $\overset{0}{\gamma}_5$ is, like the $\overset{0}{\gamma}_k$, Hermitian, and it satisfies the relations:

$$\begin{aligned} \overset{0}{\gamma}_5 \overset{0}{\gamma}_k + \overset{0}{\gamma}_k \overset{0}{\gamma}_5 &= 0 \\ (\overset{0}{\gamma}_5)^2 &= I \end{aligned}$$

(in which I denotes the identity matrix). There are thus, in total, five four-rowed matrices $\overset{0}{\gamma}_\mu$ (and usually no more than five such matrices) that satisfy the relations:

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[1] Once, again, the Latin indices range from 1 to 4 and Greek ones from 1 to 5.

[2] For the proof of this, cf. B. L. van der Waerden, *Gruppentheoretische Methoden in der quantentheorie*, Berlin, 1932. Cf., in particular, pp. 55.

(3) $$\tfrac{1}{2}(\overset{0}{\gamma}_\mu \overset{0}{\gamma}_\nu + \overset{0}{\gamma}_\nu \overset{0}{\gamma}_\mu) = \delta_{\mu\nu}\,.$$

The 16 linearly independent matrices:

$$I,\ \overset{0}{\gamma}_\mu\,,\ \text{and} \qquad \overset{0}{\gamma}_{[\mu\nu]} \equiv \tfrac{1}{2}(\overset{0}{\gamma}_\mu \overset{0}{\gamma}_\nu - \overset{0}{\gamma}_\nu \overset{0}{\gamma}_\mu)$$

define the basis for a hypercomplex number system.

We generalize this result by first allowing also non-Hermitian metrices and then corresponding real coordinates that replace $\delta_{\mu\nu}$ with:

(4) $$\overset{0}{g}{}^{\mu\nu} = \overset{0}{g}_{\mu\nu} = e_\mu\, \delta_{\mu\nu}\,,$$

in which the sign $e_\mu = \pm 1$ is determined by the signature of the metric. One thus has:

(5) $$\tfrac{1}{2}(\overset{0}{\alpha}_\mu \overset{0}{\alpha}_\nu + \overset{0}{\alpha}_\nu \overset{0}{\alpha}_\mu) = \overset{0}{g}_{\mu\nu} = e_\mu\, \delta_{\mu\nu}\,.$$

If e_μ is negative then one obtains a solution of (5) in terms of the $\overset{0}{\gamma}_\mu$ of (3) by multiplying by *i*, and conversely. We next prove that it follows from (5) that the product of the five matrices $\overset{0}{\alpha}_\mu$ is given by:

(6) $$\overset{0}{\alpha}_1 \overset{0}{\alpha}_2 \overset{0}{\alpha}_3 \overset{0}{\alpha}_4 \overset{0}{\alpha}_4 = \pm\sqrt{\eta}\,,$$

in which the sign $\eta = \pm 1$ is defined by:

(7) $$\eta = e_1\, e_2\, e_3\, e_4\, e_5 = \mathrm{Det}\, \| \overset{0}{g}_{\mu\nu} \|\,.$$

Since this matrix product commutes with all of the $\overset{0}{\alpha}_\mu$, according to (5), it is therefore, from theorem 2, a multiple of the identity matrix, and furthermore, according to (5) one likewise has that the square of the matrix product is equal to η.

We can now give the generalizations of Theorem 1 and 2 for the case of the five matrices $\overset{0}{\alpha}_\mu$.

Theorem 1a. If the $\overset{0}{\alpha}_\mu$ *satisfy the relations (3) and if* $\overset{0}{\alpha}{}'_\mu$ *are five other matrices that satisfy the same relations (5) then there is a matrix S with a non-vanishing determinant such that either:*

(2a) $$\overset{0}{\alpha}{}'_\mu = S^{-1} \overset{0}{\alpha}_\mu\, S$$

or:

(2b) $$\overset{0}{\alpha}{}'_\mu = -S^{-1}\overset{0}{\alpha}_\mu S\,.$$

This follows immediately from the application of the Theorem 1 to the first four matrices, and then an application of (6). In the case of four matrices $\overset{0}{\alpha}_\mu$ one can likewise deduce that:

$$\overset{0}{\alpha}{}'_k = +S^{-1}\overset{0}{\alpha}_k S\,,$$

and also (by another choice of S, which we describe as replacing S with Σ) that:

$$\overset{0}{\alpha}{}'_k = -\Sigma^{-1}\overset{0}{\alpha}_k \Sigma\,,$$

whereas in the case of five matrices $\overset{0}{\alpha}_\mu$, as follows from (6), only one of the two equations (2a) *or* (2b) can be satisfied. The (just used) generalization of Theorem 2 for five matrices $\overset{0}{\alpha}_\mu$ is trivial and says:

Theorem 2a. If C is a four-rowed matrix that commutes with four of the five matrices $\overset{0}{\alpha}_\mu$ *then it also commutes with the fifth and is a multiple of the identity matrix.*

We can now pass from the special values $\overset{0}{g}_{\mu\nu}$ of the metric tensor to the general $g_{\mu\nu}$ by remarking that through a choice of certain fixed real coefficients $h^{\mu}_{\bullet\nu}$, by means of:

$$a_\mu = h^{\mu}_{\bullet\nu}\overset{0}{\alpha}_\mu\,,$$

any solution to (5) produces a solution of:

(8) $$\tfrac{1}{2}(\alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu) = g_{\mu\nu}\,,$$

and conversely, any solution of (8) produces a solution of (5) by means of the inverse transformation:

$$\overset{0}{\alpha}_\mu = h^{\mu}_{\bullet\nu}\alpha_\mu\,.$$

However, we must naturally assume that the quadratic form:

$$g_{\mu\nu}X^\mu X^\nu$$

can be converted into the form:

$$\overset{0}{g}_{\mu\nu}\,\overset{0}{X}{}^{\mu}\,\overset{0}{X}{}^{\nu} = \sum_{\nu} e_{\nu}\,(\overset{0}{X}{}^{\nu})^2$$

through a continuous change of coordinates, i.e., that the functional determinant:

$$\left\| \frac{\partial X^{\mu}}{\partial \overset{0}{X}{}^{\nu}} \right\|$$

is positive, so, in particular, it is non-null. From this, e.g., reflections of an odd number of coordinates are excluded. The determinant:

$$g = \mathrm{Det}\,\|\,g_{\mu\nu}\,\| \neq 0$$

is non-vanishing and has the same sign η as the $\overset{0}{g}_{\mu\nu}$. We now remark that by raising the indices of the α_μ one can define the matrices:

(9) $$\alpha^{\mu} = g^{\mu\nu}\,\alpha_{\nu}\,,$$

which satisfy the relations:

(8a) $$\tfrac{1}{2}(\alpha^{\mu}\alpha^{\nu} + \alpha^{\nu}\alpha^{\mu}) = g^{\mu\nu}\,,$$

8b) $$\tfrac{1}{2}(\alpha^{\mu}\alpha_{\nu} + \alpha_{\nu}\alpha^{\mu}) = \delta^{\mu}_{\bullet\nu}\,.$$

We can now convert all of the theorems for the case of the relations (5) into one for the case of the relations (8). In place of $\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5$ one has the anti-symmetric object:

(10) $$\alpha_{[12345]} \equiv \frac{1}{5!}\sum_{P} \varepsilon_P\,\alpha_{\mu_1}\alpha_{\mu_2}\alpha_{\mu_3}\alpha_{\mu_4}\alpha_{\mu_5}\,,$$

in which P is a permutation that always takes 1, 2, 3, 4, 5 into the mutually unequal numerals μ_1, μ_2, μ_3, μ_4, μ_5, and $\varepsilon_P = +1$ or -1, according to whether P is even or odd; the sum is over all permutations P. Then, in place of (6), one has:

(11) $$\alpha_{[12345]} = \pm\sqrt{g} = \pm\sqrt{\eta}\,\sqrt{|\,g\,|}\,.$$

We now have the following general theorem, which is completely analogous to Theorems 1a and 2a:

Theorem 1b. If the α_μ satisfy relations (8) and if α'_μ are five other matrices that satisfy the same relations (8) then there is a matrix S with a non-vanishing determinant such that either:

(12a) $$\alpha'_{\mu} = S^{-1}\alpha_{\mu}\,S$$

or:

(12b)
$$\alpha'_\mu = -S^{-1}\alpha_\mu S,$$

and indeed one has the former or the latter equation according to whether:

$$\alpha'_{[12345]} = +\alpha_{[12345]}$$

or

$$\alpha'_{[12345]} = -\alpha_{[12345]}.$$

Furthermore:

Theorem 2b. If a matrix C commutes with four of the matrices α_μ that satisfy relations (8) then it also commutes with the fifth matrix α_μ and equals a multiple of the identity matrix.

Since we have expressly not assumed the Hermiticity of the α_μ, we would now like to examine the Hermitian conjugates $\alpha_\mu^\dagger$ of the α_μ. As one immediately infers from (8), due to the reality of the $g_{\mu\nu}$, they satisfy, just as in eq. (8):

(8†)
$$\tfrac{1}{2}(\alpha_\mu^\dagger\alpha_\nu^\dagger + \alpha_\nu^\dagger\alpha_\mu^\dagger) = g_{\mu\nu}.$$

Furthermore, one has ([1]), due to (11):

(13)
$$\alpha_{[12345]}^\dagger = (\alpha_{[12345]})^\dagger = \eta\alpha_{[12345]}.$$

in which η is the sign of the determinant of g. Thus, it follows from Theorem 1b that:

Theorem 3. If the α_μ satisfy relations (8) then there is a matrix A with a non-vanishing determinant such that:

(14)
$$\alpha_\mu^\dagger = \eta A\alpha_\mu A^{-1}.$$

This matrix A will play a fundamental role in the sequel ([2]). Next, it follows from Theorem 2b that A is uniquely determined by eq. (4) up to a multiple of the identity matrix, i.e., a numerical factor. The determinant of the matrix A:

(15)
$$a = \operatorname{Det} A,$$

can thus be normalized arbitrarily, except that it may not vanish.

[1] One observes that $12345 \to 54321$ is an even permutation.

[2] It was first introduced by V. Bargmann, Berl. Ber. (1932), 345, although by making use of a special solution of (8). – In the work of W. Pauli and J. Solomon (loc. cit.) certain relations linking $A\,\alpha_\mu\,A\,\alpha_\nu$ and $A\,\alpha_\nu\,A\,\alpha_\mu$ were replaced for the matrices $A\,\alpha_\mu$, which, however, represents an unnecessary reduction in generality.

By going over to the Hermitian matrix, it follows from (14) that:

(14†) $$\alpha_\mu = \eta\, A^{\dagger -1} \alpha_\mu^\dagger A^\dagger ,$$

hence:

$$\alpha_\mu = \eta\, A^{\dagger\,-1}(A\alpha_\mu\; A^{-1})A^\dagger ,$$

or:

$$\alpha_\mu\; A^{\dagger\,-1}A = A\, A^{\dagger\,-1}\alpha_\mu .$$

From Theorem 2b one thus has that $A^{\dagger\,-1}A$ is a multiple of the identity matrix, i.e.:

$$A^\dagger = cA .$$

It then follows only from the fact that $A = c^*\, A^\dagger = c^*\, c\, A$, that $c^*\, c = 1$. Furthermore, we would like to normalize A so that:

(16) $$A^\dagger = \eta\, A ,$$

which then makes $A\alpha_\mu$ Hermitian:

(17) $$(A\alpha_\mu)^\dagger = A\alpha_\mu .$$

In fact, one has:

$$(A\alpha_\mu)^\dagger = \alpha_\mu^\dagger A^\dagger = h\, \alpha_\mu^\dagger A = A\alpha_\mu\; A^{-1}A = A\alpha_\mu .$$

By means of the normalization (16), the value a for the determinant of A that was introduced in (15) is real:

(15†) $$a = a^\dagger .$$

§ 3. **Coordinate transformations and *S*-transformations.**

We shall now consider the group of coordinate transformations:

(18) $$X'^\mu = a^\mu_{\cdot\nu} X^\nu$$

that leave the values of the $g_{\mu\nu}$ in the invariant form:

$$g_{\mu\nu} X^\mu X^\nu$$

unchanged. The $a^\mu_{\cdot\nu}$ must then satisfy the conditions:

(19) $$g_{\rho\sigma} a^\rho_{\cdot\mu} a^\sigma_{\cdot\nu} = g_{\mu\nu} .$$

If we now set:

(20a) $$\alpha'^\mu = a^\mu_{\cdot\nu} \alpha^\nu ,$$

(20b) $$a^{\mu}_{\cdot\nu}\alpha'_{\mu} = \alpha_{\nu} ,$$

then it follows from (19) that:

$$\tfrac{1}{2}(\alpha'^{\mu}\alpha'^{\nu} + \alpha'^{\nu}\alpha'^{\mu}) = \tfrac{1}{2}(\alpha^{\mu}\,\alpha^{\nu} + \alpha^{\nu}\,\alpha^{\mu}) = g^{\mu\nu} ,$$
$$\tfrac{1}{2}(\alpha'_{\mu}\alpha'_{\nu} + \alpha'_{\nu}\alpha'_{\mu}) = \tfrac{1}{2}(\alpha_{\mu}\,\alpha_{\nu} + \alpha_{\nu}\,\alpha_{\mu}) = g_{\mu\nu} .$$

From Theorem 1b, it then follows that:

Theorem 4. If the coefficients $a^{\mu}_{\cdot\nu}$ *satisfy the condition (19) of the invariance of the* $g_{\mu\nu}$ *then there is a matrix S such that:*

(21a) $$S^{-1}\alpha_{\mu}\,S = a^{\mu}_{\cdot\nu}\alpha^{\nu} ,$$

(21b) $$a^{\mu}_{\cdot\nu}S^{-1}\alpha_{\mu}\,S = \alpha_{\nu} .$$

In these expressions, the + sign applies, since we assume the determinant of the $a^{\mu}_{\cdot\nu}$ (which, from (19), is necessarily + 1 or – 1) to be equal to + 1; hence, we are considering proper rotations.

From (21), S is, however, only defined up to a multiplicative numerical factor. We can fix it by the requirement that:

$$\operatorname{Det} S = 1 ,$$

and that S must continuously transform into the identity matrix when the rotation $a^{\mu}_{\cdot\nu}$ goes to the identity matrix.

With this assignment, S gives us the association:

$$(a^{\mu}_{\cdot\nu}) \rightarrow S ,$$

which is a four-rowed representation of the rotation group for five-dimensional space. The two consecutive transformations $(a^{\mu}_{\cdot\nu})$, $(a'^{\mu}_{\cdot\nu})$ correspond to a multiplication of the associated matrices S and S'. We denote the totality of all these special S-matrices by $D_5(g_{\mu\nu})$.

We shall call a four-component quantity Ψ that transforms under rotations according to the rule:

$$\Psi' = \sum_s S_{rs}\ \Psi_s ,$$

or, in matrix form:

(22) $$\Psi' = S\Psi ,$$

a Ψ-*spinor* (in five-space). We shall call a four-component quantity Φ_r that transforms according to the rule:

(22a) $$\Phi'_r = \sum_s S^{-1}_{rs}\ \Phi_s ,$$

or, in matrix form:

(23) $$\Phi' = \Phi S^{-1},$$

a Φ-*spinor*. Thus, we must regard Ψ as a column matrix and Φ as a row matrix. One then defines a scalar:

(23) $$a = \Phi\Psi$$

and a five-vector:

(23a) $$a^\mu = \Phi\alpha^\mu\,\Psi,$$

and this is the case under:

1. fixed α^μ and spinor transformations of the Φ, Ψ,
2. fixed Φ, Ψ and vector transformations of the α^μ .

This *double* covariance property of the vector a^μ is essential for physical applications.

The components of the five-vector a^μ that is thus defined, like the scalar a, are not generally real. In order to obtain a real five-vector, we remark that from Theorem 3 a matrix A exists such that:

$$A\alpha_\mu$$

is Hermitian. Now $A\alpha'_\mu$ is Hermitian, just as $A\alpha_\mu$ is, since the coefficients $a^{\mu}_{\cdot\nu}$ in (21a) are real. One then has:

$$\eta\, A\alpha_\mu\, A^{-1} = \alpha^\dagger_\mu\,,$$

$$\eta\, A\alpha'_\mu\, A^{-1} = \alpha'^\dagger_\mu\,,$$

$$\eta\, AS\;\alpha'_\mu\, S^{-1}A^{-1} = S^{\dagger\,-1}\,\alpha'^\dagger_\mu\;S^\dagger = \eta\, S^{\dagger\,-1}A\,\alpha'_\mu\, A^{-1}\, S^\dagger,$$

$$A^{-1}S^\dagger AS\;\alpha'_\mu = \alpha'_\mu\;A^{-1}S^\dagger AS\,,$$

hence, from Theorem 2b, one has:

$$S^\dagger A\, S = c\, A\,.$$

Since the determinant of S is equal to one, it follows that $c^4 = 1$, and since c must vary continuously with the coefficients $a^{\mu}_{\cdot\nu}$, and $c = 1$ for $S = I$, it generally follows that $c = 1$, so:

(24) $$S^\dagger A\, S = A\,.$$

for all S in $D_5(g_{\mu\nu})$. Due to the fact that:

$$\Psi'^{*} = \Psi^*\, S^\dagger\,,$$

it follows that:

(25) $$\Psi'^{*}\, A = \Psi^*\, S^\dagger\, A = (\Psi^* A)\, S^{-1},$$

hence:

$$\Psi^* A = \Phi$$

is a Φ-spinor. The scalar:

(26) $$a = \Psi^* A\, \Psi$$

is now real, and the five-vector:

(26a) $$a^{\mu} = \Psi^* A\alpha^{\mu} \Psi$$

has real components, since $A\alpha^{\mu}$ is Hermitian. The matrix A thus plays an essential role in insuring that one can construct real scalars and vectors from the spinor Ψ.

We would now like give the solution for the matrix S that satisfies eq. (21) when one is concerned with infinitesimal transformations (18). One thus has:

(18′) $$X^{\mu\prime} = X^{\mu} + \varepsilon^{\mu}_{\bullet\nu} X^{\nu} ,$$

in which $\varepsilon^{\mu}_{\bullet\nu}$ is regarded as small to first order. The condition (19) takes on the form:

$$g_{\rho\nu}\varepsilon^{\rho}_{\bullet\mu} + g_{\mu\sigma}\varepsilon^{\sigma}_{\bullet\nu} = 0 ,$$

or, with the usual definition of the lowering of indices:

(19′) $$\varepsilon_{\mu\nu} = -\, \varepsilon_{\nu\mu} .$$

From the Ansatz:

$$S = I + T ,$$

in which T is of first order, the equation to be solved, (21a), takes the form:

$$\alpha^{\mu} T - T \alpha^{\mu} = \varepsilon^{\mu}_{\bullet\nu} \alpha^{\nu} ,$$

or:

$$\alpha_{\mu} T - T \alpha_{\mu} = \varepsilon_{\mu\nu} \alpha^{\nu} ,$$

If we then set:

$$T = \varepsilon_{\alpha\beta} T^{\alpha\beta} ,$$

in which, in order to agree with (19′), we set:

$$T^{\alpha\beta} = -\, T^{\alpha\beta},$$

and sum over α and β independently, then, due to the fact that:

$$\varepsilon_{\mu\nu} \alpha^{\nu} = \tfrac{1}{2} \varepsilon_{\alpha\beta} (\delta^{\alpha}_{\cdot\mu} \alpha^{\beta} - \delta^{\beta}_{\cdot\mu} \alpha^{\alpha}) ,$$

one has:

(21′) $$\alpha_{\mu} T - T \alpha_{\mu} = \tfrac{1}{2} (\delta^{\alpha}_{\cdot\mu} \alpha^{\beta} - \delta^{\beta}_{\cdot\mu} \alpha^{\alpha}) .$$

A solution of this equation is:

$$T^{\alpha\beta} = \tfrac{1}{8} (\alpha^{\alpha} \alpha^{\beta} - \alpha^{\beta} \alpha^{\alpha}) = \tfrac{1}{4} \alpha^{[\alpha\beta]} .$$

One then has, with the hindsight of (8b):

$$\tfrac{1}{2}(\alpha_\mu\,\alpha^\alpha\,\alpha^\beta - \alpha^\alpha\,\alpha^\beta\,\alpha_\mu) = \tfrac{1}{2}(\alpha_\mu\,\alpha^\alpha + \alpha^\alpha\,\alpha_\mu)\alpha^\beta - \tfrac{1}{2}\alpha^\alpha(\alpha_\mu\,\alpha^\beta + \alpha^\beta\,\alpha_\mu)$$
$$= \delta^\alpha_{\cdot\mu}\alpha^\beta - \delta^\beta_{\cdot\mu}\alpha^\alpha$$

Hence, we finally have:

(27) $$S = I + \tfrac{1}{4}\,\varepsilon_{\alpha\beta}\,\alpha^{[\alpha\beta]}\,,$$

with:

(28) $$\alpha^{[\alpha\beta]} = \tfrac{1}{2}(\alpha^\alpha\,\alpha^\beta - \alpha^\alpha\,\alpha^\beta)\,.$$

It must be remarked that what the general solution to $(21')$ implies is that, from Theorem 2b, it differs from any particular solution by the additional term:

$$C^{\alpha\beta} \approx I\,.$$

The particular solution that thus obtained is uniquely determined by the demand that the trace of $S - I$ must vanish (since, as one easily shows, this is true for the trace of $\alpha^{[\alpha\beta]}$). However, this means that for the first order terms in the $\varepsilon_{\alpha\beta}$ that are included in Det S remain unchanged and equal to one (due to the group property, this follows rigorously, in general), which contradicts our previously assumed condition.

We must therefore prove that the matrix α^5 remains fixed under the subgroup of the rotations that leave X^5 fixed, which thus corresponds to the rotations of four-dimensional space. Since $(\alpha^5)^2 = g^{55}$ and $\mathrm{Tr}(\alpha^5) = 0$, α^5 has the eigenvalues $+\sqrt{g^{55}}$, $+\sqrt{g^{55}}$, $-\sqrt{g^{55}}$, $-\sqrt{g^{55}}$. Since, from (21), S commutes with α^5 in this case, S decomposes into two sub-matrices, as long as α^5 takes the diagonal form; the four-component quantities Ψ decompose then decompose into two-component ones that transform like the four-component ones. This decomposition is at the basis of the van der Waerden spinor calculus, in which, moreover, the matrices α^μ are further specialized ([1]). In the case of five-dimensional rotations, there exists the possibility of four-component quantities that decompose into two two-component quantities.

The spinor calculus seems entirely natural when one wishes to go beyond the possible representations of the rotation group to one where one does not have to consider relations of the form (1) [(5) and (8), resp.] . For this, it seems natural to us to use the relations (8) as a starting point, which corresponds to Dirac's original Ansatz. From this standpoint, it is a consequence that the numerical realization of the matrix S that is determined by the transformation law for Ψ depends upon the numerical realization of the matrices α_μ.

We refer to a choice of numerical realization of the matrices α_μ – which agrees with relations (8) – as a *reference system in (four-dimensional) spin space.* As a consequence, in any physical theory, one must require that they are not only covariant under an arbitrary coordinate transformation in spacetime (a homogeneous transformation of the

[1] The general covariant form of the Dirac wave equation in an ordinary four-dimensional continuum is treated in this manner by B. L. van der Waerden and L. Infeld, Berl. Ber. (1933), 380.

five coordinates X^μ, resp.), but also *that they are covariant under arbitrary [but consistent with (8)] transformations of the reference system of the spin space.* This demand stands in methodological contradiction to that of the van der Waerden spinor calculus, which is based upon a specialization of the reference system in spin space.

The most general transformation of the reference system in spin space, which we shall also refer to as an *S-transformation*, is given by:

(29) $$\alpha'_\mu = S^{-1} \alpha_\mu S \,,$$

and:

(30) $$\Psi' = S^{-1}\Psi, \qquad \Phi' = \Phi S \,.$$

Then, just like the $g_{\mu\nu}$, the scalar:

$$a = \Phi\Psi$$

and the vector:

$$a^\mu = \Phi\, \alpha^\mu\, \Psi$$

remain invariant under S-transformations. [On this formal basis, the transformation law for Φ and Ψ will be changed from (22), (22a) in that S will be replaced by S^{-1}.] One sees, furthermore, that the transformation law:

$$A' = S^\dagger A\, S$$

satisfies the demand that the statements:

$$\alpha^\dagger_\mu = \eta\, A \alpha_\mu A^{-1},$$
$$A^\dagger = \eta\, A,$$
$A\alpha_\mu$ is Hermitian,
$\Psi^* A$ is a Φ-spinor,

remain unchanged (i.e., they are also valid for the primed quantities). For all of these statements, the *dependence of the matrix S on the* X^μ then remains arbitrary. When the X^μ are regarded as homogeneous coordinates, it seems to be a general consequence of assuming that S is homogeneous of null degree that the degree of homogeneity of Φ and Ψ remains invariant under S-transformation (30).

Next, the S-transformation and (homogeneous) transformations of the X^μ appear to be completely independent of each other. From what was proved earlier, there thus exists, as is also necessary for physical reasons, a connection between special coordinate transformations and special S-transformations, namely, the rotations in X^μ -space and the S-transformation of D_5 ([1]). When we replace the S in (30) with the matrix that is associated with the inverse rotation $X^{\mu\prime} = a^{\mu}_{\cdot\nu} X^\nu$, the earlier result can be formulated in the following way: *To any rotation of the coordinate space there is a unique associated*

[1] Thus, no coupling of the rotations at different spacetime points exists whatsoever, in contrast to the prior Einsteinian idea of teleparallelism.

S-transformation in D_5, in such a way that the am remain invariant under a simultaneous application of the rotation and the S-transformation. This result is, moreover, physically necessary, since for a given $g_{\mu\nu}$ (e.g., in special relativity theory) there is no coordinate system that is distinguished by a particular choice of the α_μ ([1]).

A further property of the *S*-transformations in D_5 is obtained by equating (24) and (31): *The matrix A remains invariant under S-transformation in D_5.* The matrix *A* can be regarded as a sort of *fundamental tensor for spin space*, since it has analogous properties under *S*-transformations to those of the $g_{\mu\nu}$ under coordinate transformations. (For a further analogy, cf., the following paragraphs.) From (16), the number of independent real elements of *A* is equal to $4^3 = 16$, compared to the $\frac{5\cdot 6}{2} = 15$ independent components of the $g_{\mu\nu}$.

It must be remarked, as an appendix, that, along with the matrix *A*, there is a matrix *B* in spin space that is analogous to *A*, and likewise remains invariant under *S*-transformation in D_5; however, it plays no role in physical applications. One obtains it when one considers the transposed ("flipped") matrices $\bar{\alpha}_\mu$, in which the rows and columns have been switched, instead of the Hermitian conjugates $\alpha_\mu^\dagger$. Therefore, they likewise satisfy the relations:

$$\tfrac{1}{2}(\bar{\alpha}_\mu\bar{\alpha}_\nu + \bar{\alpha}_\nu\bar{\alpha}_\mu) = g_{\mu\nu}. \tag{8*}$$

Since:

$$a_{[12345]} = (\alpha_{[12345]}) = \alpha_{[12345]},$$

there follows the existence of a matrix *B* such that:

$$\bar{\alpha}_\mu = B\,\alpha_\mu\,B^{-1}, \tag{14*}$$

(but this time, without the sign η). *B* is determined up to a numerical factor by this equation such that:

$$b = \text{Det}\,B \tag{15*}$$

still remains arbitrary. From (14*), it follows (analogously to the situation with *A* before) that $B^{-1}\bar{B}$ commutes with the α_μ, so:

$$\bar{B} = c\,B.$$

One shows, with no further assumptions, that $c^2 = 1$, hence, $c = \pm 1$, i.e.:

$$\bar{B} = +B. \qquad \text{or} \qquad \bar{B} = -B;$$

in the former case, *B* is symmetric and in the latter, it is skew-symmetric. In order to distinguish between these two possibilities, one must examine them more closely ([2]).

From (14*), it follows that:

$$\bar{\alpha}_\mu\bar{\alpha}_\nu = B\,\bar{\alpha}_\mu\bar{\alpha}_\nu\,B^{-1},$$

hence, with:

$$\alpha_{[\mu\nu]} = \tfrac{1}{2}(\alpha_\mu\,\alpha_\nu - \alpha_\nu\,\alpha_\mu),$$

[1] The work of T. Levi-Civita, Berl. Ber. (1933), 340 rests exclusively on the absence of this result, and is, for that reason, physically unacceptable. Moreover, it lacks a prescription for the construction of real vectors.

[2] For the communication of this argument, as well as the cordial permission to publish it here, I wish to thank Herrn Haantjes, Delft.

since:

$$\bar{\alpha}_{[\mu\nu]} = -\tfrac{1}{2}(\bar{\alpha}_\mu \bar{\alpha}_\nu - \bar{\alpha}_\nu \bar{\alpha}_\mu),$$
$$\bar{\alpha}_{[\mu\nu]} = -B\, a_{[\mu\nu]}\, B^{-1},$$

and with:

$$\bar{B} = \pm B, \qquad \overline{(B\alpha_{[\mu\nu]})} = \bar{\alpha}_{[\mu\nu]}\bar{B} = \mp B\alpha_{[\mu\nu]}.$$

We now prove that the $B\, a_{[\mu\nu]}$ cannot be skew-symmetric. Since the 10 matrices $\alpha_{[\mu\nu]}$ are linearly independent (i.e., from the fact that $c^{[\mu\nu]}\, \alpha_{[\mu\nu]} = 0$, with ordinary numbers $c^{[\mu\nu]}$, it must follow that $c^{[\mu\nu]} = 0$) the same is true for the 10 matrices $B\alpha_{[\mu\nu]}$. However, there are only six linearly independent skew-symmetric four-rowed matrices, compared to 10 linearly independent symmetric four-rowed matrices. *Hence, the $B\alpha_{[\mu\nu]}$ must be symmetric matrices, i.e., one has the lower sign:*

$$\overline{(B\alpha_{[\mu\nu]})} = +B\, \alpha_{[\mu\nu]}.$$

so:

(16*) $$\bar{B} = -B.$$

The matrix B is skew. The six matrices:

$$B,\ B\alpha_\mu$$

define a basis for all skew four-rowed matrices, and the 10 matrices:

$$B\alpha_{[\mu\nu]},$$

define a basis for all symmetric four-rowed matrices.

Since an S in D_5 gives us that:

$$B\alpha'_\mu = B\, S^{-1} \alpha_\mu\, S$$

is skew, just as $B\alpha_\mu$ is, it follows from an analogous argument to the proof of (24) that:

(24*) $$\bar{S}\bar{B}\, S = B$$

for all S in D_5.

The general transformation law for $\bar{B}$ under S-transformations is:

(31*) $$\bar{B}' = \bar{S}\, B\, S$$

so:

$$B'\alpha'_\mu = \bar{S}\, B\, S\, (S^{-1}\alpha_\mu\, S) = \bar{S}\, B\alpha_\mu\, S$$

is skew, just as $B\alpha_\mu$ is.

§ 4. **Covariant differentiation of spinors.**

From the relations:

$$\tfrac{1}{2}(\alpha_\mu\, \alpha_\nu + \alpha_\nu\, \alpha_\mu) = g_{\mu\nu},$$

one obtains, by differentiation:

$$\tfrac{1}{2}\left(\frac{\partial \alpha_\mu}{\partial X^\rho}\alpha_\nu + \alpha_\mu \frac{\partial \alpha_\nu}{\partial X^\rho} + \frac{\partial \alpha_\nu}{\partial X^\rho}\alpha_\mu + \alpha_\nu \frac{\partial \alpha_\mu}{\partial X^\rho}\right) = \frac{\partial g_{\mu\nu}}{\partial X^\rho}.$$

Since:

$$g_{\mu\nu;\,\rho} \equiv \frac{\partial g_{\mu\nu}}{\partial X^{\rho}} - \Gamma^{\sigma}_{\mu\rho} g_{\sigma\nu} - \Gamma^{\sigma}_{\nu\rho} g_{\mu\sigma} = 0 \,,$$

it follows, by letting the symbol $Y_{\mu\rho}$ abbreviate the matrices:

$$Y_{\mu\rho} = \frac{\partial \alpha_{\mu}}{\partial X^{\rho}} - \Gamma^{\sigma}_{\mu\rho} \alpha_{\sigma} \,,$$

that the equations:

$$Y_{\mu\rho}\, \alpha_{\nu} + \alpha_{\nu}\, Y_{\mu\rho} + Y_{\nu\rho}\, \alpha_{\mu} + \alpha_{\mu}\, Y_{\nu\rho} = 0$$

are satisfied. This is equivalent to the statement that the matrices:

$$\alpha'_{\mu} = \alpha_{\mu} + \varepsilon^{\rho}\, Y_{\mu\rho}$$

(one can understand the ε^{ρ} to be, e.g., $\varepsilon \dfrac{\bar{d}X^{\rho}}{ds}$ along a curve) satisfy, up to and including terms of first order in ε^{ρ}, the equations:

$$\tfrac{1}{2}(\alpha'_{\mu}\alpha'_{\nu} + \alpha'_{\nu}\alpha'_{\mu}) = g_{\mu\nu}$$

with unchanged $g_{\mu\nu}$. Hence, from Theorem 1, there is a:

$$S = I + \varepsilon^{\rho}\, \Lambda_{\rho} \,,$$

such that, up to this order of magnitude, one has:

$$\alpha'_{\mu} = S^{-1}\, \alpha_{\mu}\, S \,.$$

Hence ([1]):

$$Y_{\mu\rho} = -\Lambda_{\rho}\, \alpha_{\mu} + \alpha_{\mu}\, \Lambda_{\rho} \,,$$

and it follows that:

(32a) $$\alpha_{\mu\,;\,\rho} \equiv \frac{\partial \alpha_{\mu}}{\partial X^{\rho}} - \Gamma^{\sigma}_{\mu\rho} \alpha_{\sigma} + \Lambda_{\rho}\, \alpha_{\mu} - \alpha_{\mu}\, \Lambda_{\rho} = 0 \,.$$

By the use of:

$$g^{\mu\nu}{}_{;\,\rho} = \frac{\partial g^{\mu\nu}}{\partial X^{\rho}} + \Gamma^{\mu}_{\sigma\rho} g^{\sigma\nu} + \Gamma^{\nu}_{\sigma\rho} g^{\sigma\mu} = 0 \,,$$

one further deduces that:

(32b) $$\alpha^{\mu}{}_{;\,\rho} = g^{\mu\nu}\, \alpha_{\mu;\,\rho} = \frac{\partial \alpha^{\mu}}{\partial X^{\rho}} + \Gamma^{\mu}_{\sigma\rho} \alpha^{\sigma} + \chi\alpha^{\mu} - \alpha^{\mu}\, \Lambda_{\rho} = 0 \,.$$

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[1] This conclusion can be found in E. Schrödinger, Berl. Ber. (1932), 105, in particular, § 2. Our Λ_{ρ} corresponds to Schrödinger's $-\Gamma_{\rho}$.

Equations (32a) or (32b) can be regarded as the definitions of $\alpha_{\mu\,;\,\rho}$ ($\alpha^{\mu}{}_{;\,\rho}$, resp.) and Λ_ρ . However, Λ_ρ is not uniquely determined by this definition, but (from Theorem 2b) only up to a multiple of the identity matrix as an additive term. As a result of this, regardless of the validity of (32a) and (32b), one can define the vector field:

(33) $$\mathrm{Tr}(\Lambda_\rho) = F_\rho$$

arbitrarily. In order to normalize it, we examine the behavior of Λ_ρ under S-transformations, where S can be depend upon the X^μ arbitrarily. With the fact that:

$$\alpha'_\mu = S^{-1}\,\alpha_\mu\,S$$

one finds that:

$$\alpha'_{\mu;\rho} = S^{-1}\,\alpha_{\mu\,;\,\rho}\,S = \frac{\partial\alpha'_\mu}{\partial X^\rho} - \Gamma^\sigma_{\mu\rho}\alpha'_\sigma + \Lambda'_\rho\alpha'_\mu - \alpha'_\mu\Lambda'_\rho = 0$$

[and similarly for (32b)], when one sets:

(34) $$\Lambda'_\rho = S^{-1}\Lambda_\rho\,S + S^{-1}\frac{\partial S}{\partial X^\rho}\,.$$

In this, we have made use of the fact that:

$$-\frac{\partial S^{-1}}{\partial X^\rho}S = S^{-1}\frac{\partial S}{\partial X^\rho}\,.$$

The transformation (34) is equivalent to the statement that the operator:

(34a) $$\nabla_\rho \equiv \frac{\partial}{\partial X^\rho} + \Lambda_\rho,$$

transforms according to the rule:

(34b) $$\nabla'_\rho = S^{-1}\nabla_\rho\,S\,.$$

Furthermore, due to the fact that:

$$\mathrm{Det}\,S \approx \mathrm{Tr}\left(S^{-1}\frac{\partial S}{\partial X^\rho}\right) = \frac{\partial}{\partial X^\rho}\,\mathrm{Det}\,S$$

it follows that Tr Λ_ρ transforms according to the rule:

(35) $$F'_\rho = F_\rho + \frac{1}{\mathrm{Det}\,X^\rho}\frac{\partial(\mathrm{Det}\,S)}{\partial X^\rho} = F_\rho + \frac{\partial}{\partial X^\rho}(\log\,\mathrm{Det}\,S)\,.$$

For that reason, it seems natural to fix F_ρ:

(36) $$F_{\rho\sigma} = \frac{\partial F_\sigma}{\partial X^\rho} - \frac{\partial F_\rho}{\partial X^\sigma} = 0\ ,$$

or

(36a) $$F_\rho = \frac{\partial F}{\partial X^\rho}\ ,$$

for an appropriate choice of F. From (35), this normalization remains invariant precisely under S-transformations, since, by the validity of (36a) the transformation formula (35) assumes the simple form:

(35a) $$F' = F + \log \det S\ .$$

Condition (36) is therefore equivalent to the demand that F_ρ can be made to vanish for an appropriate S-transformation.

Another possibility is that one might set the $F_{\rho\sigma}$ field, which is invariant under S-transformations, proportional to the field $X_{\rho\sigma}$ that was defined in Part I, eq. (36). This would be more analogous to the method that was followed by Schrödinger of setting F_ρ proportional to the four-potential. By this method, however, the identification of the new $F_{\rho\sigma}$ field with the old $X_{\rho\sigma}$ field (up to a numerical proportionality factor) is not arbitrary. We shall therefore temporarily adopt the normalization condition (36).

We can now define the covariant derivatives of the spinors Ψ and Φ when we demand that the covariant derivative obeys the product rule of ordinary differentiation and that the scalar:

$$a = \Phi\,\Psi$$

and the vector:

$$a_\mu = \Phi\,\alpha_\mu\,\Psi$$

obey the ordinary rules:

$$a_{;\,\rho} = \frac{\partial a}{\partial X^\rho}$$

and:

$$a_{\mu\,;\,\rho} = \frac{\partial a_\mu}{\partial X^\rho} - \Gamma^\sigma_{\mu\rho}\alpha_\sigma\ .$$

One finds, by means of (32), that:

$$a_{\mu\,;\,\rho} = \Phi_{;\,\rho}\,\alpha_\mu\,\Psi + \Phi\alpha_\mu\,\Psi_{;\,\rho}$$

that:

(37a) $$\Phi_{;\,\rho} \equiv \frac{\partial \Phi}{\partial X^\rho} - \Phi\,\Lambda_\rho$$

(37b) $$\Psi_{;\,\rho} \equiv \frac{\partial \Psi}{\partial X^\rho} + \Lambda_\rho\,\Psi\ .$$

Under S-transformations one easily finds, from (34), that:

$$\Phi'_{;\rho} = \Phi_{;\,\rho}\, S\,, \qquad\qquad \Psi'_{;\rho} = S^{-1}\Psi_{;\,\rho}\,.$$

We would now like to determine the covariant derivative of the Hermitian matrix A by the requirement that:

$$(\Psi^* A)_{;\,\rho} = (\Psi_{;\,\rho})^*_{;\,\rho}\, A + \Psi^* A_{;\,\rho}\,,$$

and, on the other hand, as a result of the fact that Ψ^*A is a Φ-spinor, this is given by way of:

$$(\Psi^* A)_{;\,\rho} = \frac{\partial(\Psi^* A)}{\partial X^{\rho}} - \Psi^* A\, \Lambda_\rho\,.$$

One then finds that:

(38)
$$A_{;\,\rho} = \frac{\partial A}{\partial X^{\rho}} - (A\,\Lambda_\rho + \Lambda^{\dagger}_{\rho} A)\,,$$

if is the matrix that is Hermitian conjugate to Λ_ρ, and under an S-transformation, one has:

$$A^{\dagger}_{;\rho} = S^{\dagger}\, A_{;\,\rho}\, S\,.$$

We now obtain an important fact about $A_{;\,\rho}$ when we observe that, according to (14), one has:

$$\alpha^{\dagger}_{\mu} A = \eta\, A\, \alpha_\mu\,.$$

One easily conforms that this implies:

$$\alpha^{\dagger}_{\mu} A_{;\rho} + (\alpha_{\mu;\rho})^{\dagger} A = \eta\,(A_{;\,\rho}\,\alpha_\mu + A\,\alpha_{\mu\,;\,\rho})\,,$$

when one substitutes for $\alpha_{\mu\,;\,\rho}$ by way of (32). Since $\alpha_{\mu\,;\,\rho} = \alpha^{\dagger}_{\mu;\rho} = 0$, it further follows that:

$$\alpha^{\dagger}_{\mu} A_{;\rho} = \eta\, A_{;\,\rho}\, \alpha_\mu\,,$$

hence:

$$A\alpha_\mu\, A^{-1} A_{;\,\rho} = A_{;\,\rho}\, \alpha_\mu\,.$$

From Theorem 2, it then follows from this that:

(39)
$$A_{;\,\rho} = a_\rho\, A\,,$$

in which the ordinary numerical vector a_ρ is invariant under S-transformations, as usual. We can determine a_ρ when we take the trace of $A^{-1}A_{;\,\rho}$ and observe that:

$$\mathrm{Tr}\left(A^{-1}\frac{\partial A}{\partial X^{\rho}}\right) = \frac{1}{\alpha}\frac{\partial a}{\partial X^{\rho}}\,,$$

when a is the determinant of A that was introduced in (15). From (33) and (38), this yields:

(40) $$\frac{\partial \log a}{\partial X^\rho} - (F_\rho + F_\rho^*) = 4\, a_\rho$$

(since the trace of the four-rowed identity matrix is 4). Since, by the definition of A, the necessarily real determinant a still remains arbitrary, we can fix it by normalization, when we, as follows from (39), demand that:

(41) $$a_\rho = 0\,, \qquad \text{hence,}\ A_{;\rho} = 0\,.$$

This possible when and only when the $F_{\rho\sigma}$ field is pure imaginary, since it generally follows from (40) that:

(40a) $$-(F_\rho + F_\rho^*) = \left(\frac{\partial a_\sigma}{\partial X^\rho} - \frac{\partial a_\rho}{\partial X^\sigma}\right),$$

When the $F_{\rho\sigma}$ field vanishes everywhere, from (39a), one can set:

(41a) $$\log a = F + F^*\,,$$

in order to satisfy (41b). The vanishing of the covariant derivative of A is a property of A that is analogous to the vanishing of the covariant derivative of $g_{\mu\nu}$.

Completely analogous to the situation with the matrix A, one can proceed with the skew matrix B that was defined by (14*) in order to compute the covariant derivative. It is given by:

(38*) $$B_{;\mu} = \frac{\partial B}{\partial X^\rho} - (B\,\Lambda_\rho + \bar{\Lambda}_\rho B)\,,$$

which then satisfies:

$$\bar{B}_{;\rho} = \bar{S}\, B_{;\rho}\, B\,,$$

[cf., (13*)], and $\bar{\Psi}B$ is also a Φ-spinor with regard to its covariant derivative. Analogous to the situation with A, one finds that $B^{-1} B_{;\rho}$ commutes with all of the α_μ; thus:

(39*) $$B_{;\rho} = b_\rho B\,,$$

in which b_ρ is a numerical vector that is invariant under S-transformations. By taking the trace of $B^{-1} B_{;\rho}$, one finds, when b is the determinant of B that was introduced in (15*), that:

(40*) $$\frac{\partial \log b}{\partial X^\rho} - 2\,F_\rho = 4\, b_\rho\,,$$

and thus:

(40a*) $$-\,2\,F_{\rho\sigma} = 4\left(\frac{\partial b_\rho}{\partial X^\sigma} - \frac{\partial b_\sigma}{\partial X^\rho}\right).$$

Thus, if one wishes that:

(41*) $$b_\rho = 0\,, \qquad B_{;\rho} = 0\,,$$

one must necessarily have:

$$F_{\rho\sigma} = 0 .$$

One may consider this to be an argument for the previously stated demand. One can then set:

(41a*) $$\log b = 2\, F .$$

§ 5. **Connection between projective and affine spinors. Statement of the wave equation.**

Before we state the wave equation, we must first describe the relationship between the projective spinors that are associated with the five homogeneous coordinates X^μ and the affine spinors ψ that are associated with the inhomogeneous coordinates x^k, as well as the relationship between matrices Λ_μ ($\overset{R}{\Lambda}_k$, resp.) that are associated with their parallel translation. If we define:

(42) $$\alpha_k = \gamma^{\mu}_{\cdot k}\, \alpha_\mu , \qquad \alpha_0 = X^\mu\, \alpha_\mu ,$$

then it follows from (8) and eq. (7), (15), (18) in Part I:

(43a) $$\tfrac{1}{2}(\alpha_i\, \alpha_k + \alpha_k\, \alpha_i) = g_{ik} ,$$

(43b) $$\alpha_k\, \alpha_0 + \alpha_0\, \alpha_k = 0$$

(43c) $$\alpha_0^{\ 2} = \varepsilon ,$$

and just as $A\alpha_\mu$ is Hermitian, so are Aa_k and Aa_0 .

Just as one has Λ_μ, which satisfies eq. (37a), there is also a Riemannian $\overset{R}{\Lambda}_k$, which satisfies the analogous equation:

(44) $$\alpha_{k\,;\,l} \equiv \frac{\partial \alpha_k}{\partial X^l} - \begin{Bmatrix} m \\ k\,l \end{Bmatrix} \alpha_m + \overset{R}{\Lambda}_l\, \alpha_k - \alpha_k\, \overset{R}{\Lambda}_l = 0 .$$

As we already mentioned above, we can now demand that S must always be homogeneous of null degree, so the degree of Ψ does not change under S-transformations; one will then have that α_μ and Λ_μ are of degree – 1, moreover. Therefore, we must further demand that F must be homogeneous of null degree ([1]) in [cf., (33), 36a)]:

$$F_\rho = \mathrm{Tr}(\Lambda_\rho) = \frac{\partial F}{\partial X^\rho} ,$$

so F_ρ can be made to vanish by an S-transformation with an S that is homogeneous of null degree. One then has:

(45) $$X^\mu\, F_\mu = \mathrm{Tr}(\Lambda_\rho) = \mathrm{Tr}(\Lambda_0) = 0 .$$

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[1] If F were the logarithm of a homogeneous function of arbitrary degree then F_ρ would always be homogeneous of degree – 1.

Thus, $\mathrm{Tr}(\overset{R}{\Lambda}_k)$ and $\mathrm{Tr}(\Lambda_\mu)$ can be simultaneously made to vanish by *the same S*-transformation, and, from (35), one has:

(46) $$\mathrm{Tr}(\overset{R}{\Lambda}_k) = \gamma^{\mu}_{\cdot\nu}\ \mathrm{Tr}(\Lambda_\mu)\,.$$

According to (40), (41) one can also assume that A has null degree, so $A = \overset{R}{A}$. We can now compute:

(47) $$\Lambda_\mu - \gamma_\mu^{\cdot\nu}\ \overset{R}{\Lambda}_k \equiv \Lambda_\mu - \overset{R}{\Lambda}_\mu = \Delta_\mu\,.$$

Then, from (32a) and (44), it follows, by means of eq. (34), Part I, and the use of the general $\Gamma^\lambda_{\mu\nu}$, which were characterized by eq. (III′), Part I, § 7:

(48) $$\begin{cases} (\Lambda_\rho\alpha^\mu - \alpha^\mu\Lambda_\rho) - (\overset{R}{\Lambda}_\rho\,\alpha^\mu - \alpha^\mu\,\overset{R}{\Lambda}_\rho)\gamma^{\mu}_{\cdot k} \\ \qquad +\varepsilon\left(\dfrac{\partial\alpha_0}{\partial X^\rho} - \dfrac{q}{2}X_{\rho\nu}\alpha^\nu\right)X^\mu \\ \quad +\varepsilon\alpha_0\dfrac{q}{2}X_\rho^{\cdot\mu} + \varepsilon\dfrac{p}{2}X_\rho X_\sigma^{\cdot\mu}\alpha^\sigma = 0. \end{cases}$$

By multiplying with $\gamma_\mu^{\cdot k}$, it then follows that:

(48a) $$\Delta_\rho\,\alpha^k - \alpha^k\,\Delta_\rho = -\varepsilon\alpha_0\frac{q}{2}X_\rho^{\cdot k} - \varepsilon\frac{p}{2}X_\rho X_l^{\cdot k}\alpha^l\,.$$

Δ_ρ is thus uniquely determined since $\mathrm{Tr}(\Delta_\rho) = 0$. If one recalls (43) then one finds, through a similar argument to the one used for the solution of (21′), that:

(49) $$\Delta_\rho = -\varepsilon\frac{q}{2}X_{\rho l}\alpha_0\alpha^l - \varepsilon\frac{p}{8}X_\rho X_{kl}\alpha^{[kl]},$$

in which:

$$\alpha^{[kl]} \equiv \tfrac{1}{2}(\alpha^k\,\alpha^l + \alpha^l\,\alpha^k)\,.$$

For what follows, we point out an expression that follows from (49):

(50) $$\alpha^\rho\,\Delta_\rho = \varepsilon\frac{2q-p}{2}X_{kl}\alpha_0\alpha^{[kl]}\,.$$

On the other hand, by multiplying (48) by X^μ, one obtains:

(48b) $$\frac{\partial \alpha_0}{\partial X^\rho} + \Lambda_\rho \alpha_0 - \alpha_0 \Lambda_\rho - \frac{q}{2} X_{\rho\sigma} \alpha^\rho = 0 \,,$$

from which, it follows, by means of (49), that:

(50a) $$\overset{R}{\alpha}_{0;k} \equiv \frac{\partial \alpha_0}{\partial x^k} + \overset{R}{\Lambda}_k \alpha_0 - \alpha_0 \overset{R}{\Lambda}_k = 0 \,.$$

In order to further connect with the presentation of the Dirac theory, we describe, in the case of special relativity, the connection between the matrices α_k, α_0 and the five Hermitian matrices of the Dirac theory, which satisfy the relations:

$$\tfrac{1}{2}(\overset{0}{\gamma}_\mu \overset{0}{\gamma}_\nu + \overset{0}{\gamma}_\nu \overset{0}{\gamma}_\mu) = \delta_{\mu\nu} \,,$$
$$\overset{0}{\gamma}_1 \overset{0}{\gamma}_2 \overset{0}{\gamma}_3 \overset{0}{\gamma}_4 \overset{0}{\gamma}_5 = 1 \,.$$

This connection is essentially different in the cases $\varepsilon = +1$ and $\varepsilon = -1$ ([1]).

In the case of $\varepsilon = +1$, since:

$$\overset{0}{g}_{11} = \overset{0}{g}_{22} = \overset{0}{g}_{33} = +1, \qquad \overset{0}{g}_{44} = -1,$$

the α_μ satisfy the equations:

$$\overset{0}{\alpha}_\mu \overset{0}{\alpha}_\nu + \overset{0}{\alpha}_\nu \overset{0}{\alpha}_\mu = 0 \qquad \text{for} \qquad \mu \neq \nu,$$
$$\left(\overset{0}{\alpha}_1\right)^2 = \left(\overset{0}{\alpha}_2\right)^2 = \left(\overset{0}{\alpha}_3\right)^2 = \left(\overset{0}{\alpha}_0\right)^2 = +I \,,$$
$$\left(\overset{0}{\alpha}_4\right)^2 = -I \,.$$

These equations are satisfied by the matrices:

$$i\overset{0}{\gamma}_5 \overset{0}{\gamma}_1 \,, \quad i\overset{0}{\gamma}_5 \overset{0}{\gamma}_2 \,, \quad i\overset{0}{\gamma}_5 \overset{0}{\gamma}_3 \,, \quad -i\overset{0}{\gamma}_5 \,, \quad i\overset{0}{\gamma}_5 \overset{0}{\gamma}_4 \,,$$

and one can set:

$$A = -i\overset{0}{\gamma}_5 \,,$$

so one has:

$$[A\overset{0}{\alpha}_k, A\overset{0}{\alpha}_0] = [\overset{0}{\gamma}_1, \overset{0}{\gamma}_2, \overset{0}{\gamma}_3, -I, \overset{0}{\gamma}_4],$$

i.e.:

$$s_k = (\psi^* A \,\alpha_k\, \psi)$$

[1] On the other hand, the previously-introduced sign η is inessential, since inverting the sign of all $g_{\mu\nu}$ would simply multiply the matrices α_μ and A by i .

turns into the Dirac current vector. The special solution of A and $\overset{0}{\alpha}_k, \overset{0}{\alpha}_0$ that is assumed can then always be arrived at by an appropriate S-transformation.

Things are different in the case of $\varepsilon = -1$, since, one then has:

$$\left(\overset{0}{\alpha}_0\right)^2 = -1 ,$$

whereas the remaining relations for the $\overset{0}{\alpha}$ remain the same. In this case:

$$(\overset{0}{\alpha}_k, \overset{0}{\alpha}_0) = (-i\,\overset{0}{\gamma}_5\,\overset{0}{\gamma}_1, -i\,\overset{0}{\gamma}_5\,\overset{0}{\gamma}_2, -i\,\overset{0}{\gamma}_5\,\overset{0}{\gamma}_3, i\,\overset{0}{\gamma}_5, -\overset{0}{\gamma}_5\,\overset{0}{\gamma}_4) ,$$
$$A = \overset{0}{\gamma}_4$$

is a solution for the, such that one has:

$$(A\overset{0}{\alpha}_k, A\overset{0}{\alpha}_0) = (-i\,\overset{0}{\gamma}_4\,\overset{0}{\gamma}_5\,\overset{0}{\gamma}_1, -i\,\overset{0}{\gamma}_4\,\overset{0}{\gamma}_5\,\overset{0}{\gamma}_2, -i\,\overset{0}{\gamma}_4\,\overset{0}{\gamma}_5\,\overset{0}{\gamma}_3, i\,\overset{0}{\gamma}_4\,\overset{0}{\gamma}_5, +\overset{0}{\gamma}_5) ,$$

and, on the other hand:

$$iA\overset{0}{\alpha}_0\,\overset{0}{\alpha}_k = (\overset{0}{\gamma}_1, \overset{0}{\gamma}_2, \overset{0}{\gamma}_3, -I) , \qquad iA\overset{0}{\alpha}_0\,\overset{0}{\alpha}_0 = -i\,\overset{0}{\gamma}_4 .$$

Thus, in this case:

$$s_k = i\ \psi^*\ A\ \alpha_0\ \alpha_k\ \psi$$

turns into the Dirac current vector.

With these preparations, we can proceed with the statement and discussion of the wave equation. It reads like ([1]):

(I) $$\alpha^\mu\,(\Psi_{;\,\mu} + k\,X^\mu\,\mathrm{Y}) = 0 .$$

We shall likewise present reality conditions for the degree of homogeneity l of Ψ and the coefficient k .

Explicitly writing out (I) gives:

(I′) $$\alpha^\mu\left(\frac{\partial\Psi}{\partial X^\mu} + \Lambda_\mu\Psi + kX_\mu\Psi\right) = 0$$

or:

[1] The plus sign that appears in this equation, perhaps somewhat illogically, can be avoided if one sets the $F_{\rho\sigma}$ field that is defined by (36) not equal to zero, but proportional to $X_{\rho\sigma}$. However, the definition for $\Psi_{;\,\mu}$ that was chosen in the text seems more convenient for the computations.

$$\text{(I}''\text{)} \qquad \alpha^{\mu}\left(\frac{\partial \Psi}{\partial X^{\mu}} + \overset{R}{\Lambda}_{\mu}\Psi + kX_{\mu}\Psi\right) + \varepsilon\frac{2q-p}{8}X_{kl}\alpha_0\alpha^{[kl]}\Psi = 0\,.$$

We can further transform this equation by extending the null degree homogeneous spinor *y* by setting:

(51)
$$\Psi = \psi \cdot F^{l}\,,$$

in which F is a real scalar that is homogeneous of degree 1.

From (I″), it then follows that:

$$(52) \qquad \alpha^{k}\left(\frac{\partial \psi}{\partial X^{k}} + \overset{R}{\Lambda}_{k}\psi\right) + k\alpha_0\psi + l\alpha^{\mu}\frac{1}{F}\frac{\partial F}{\partial X^{\mu}}\psi + \varepsilon\frac{2q-p}{8}X_{kl}\alpha_0\alpha^{[kl]}\psi = 0\,.$$

Now, one has:

(53)
$$\frac{1}{F}\frac{\partial F}{\partial X^{\mu}} = e\,(X_{\mu} - \gamma_{\mu}^{\cdot k} f_k)\,,$$

since:

$$X^{\mu}\frac{\partial F}{\partial X^{\mu}} = F\,;$$

it then follows that:

$$X_{\mu\nu} = \gamma_{\mu}^{\cdot k}\gamma_{\nu}^{\cdot l}\left(\frac{\partial f_l}{\partial x^k} - \frac{\partial f_k}{\partial x^l}\right),$$

so:

$$X_{kl} = \frac{\partial f_l}{\partial x^k} - \frac{\partial f_k}{\partial x^l}\,.$$

Thus, the f_k in (53) is identical with the vector f_i that was defined in eq. (41), Part I, which is related to the potentials Φ_i by way of eq. (46), Part I :

(I, 46)
$$f_i = r\frac{\sqrt{\kappa}}{c}\Phi_i\,,$$

From (53), it finally follows that:

$$\text{(I}'''\text{)} \qquad \alpha^{k}\left(\frac{\partial \psi}{\partial X^{k}} + \overset{R}{\Lambda}_{k}\psi - \varepsilon l f_k\psi\right) + (k+\varepsilon l)\alpha_0\psi + \varepsilon\frac{2q-p}{8}X_{kl}\alpha_0\alpha^{[kl]}\psi = 0\,.$$

Further discussion of the equation will depend upon whether we set $\varepsilon = +1$ or $\varepsilon = -1$. Above all, we are interested in the case of $\varepsilon = +1$, since choosing $\varepsilon = +1$ and $p = q = 1$ leads to a symmetric $\Gamma^{\lambda}_{\mu\nu}$. In this case, we arrive at the fact that, from (I) [(I‴), resp.], it must follow that ([1]):

[1] We denote the determinant of g_{ik} by $\bar{g}$ and the determinant of $g_{\mu\nu}$ by g .

(54) $$\frac{\partial}{\partial X^{\mu}}(\sqrt{g}\,\Psi^* A\,\alpha^{\mu}\,\Psi) = 0$$

and:

(54a) $$\frac{\partial}{\partial x^{k}}(\sqrt{g}\,\psi^* A\,\alpha^{\mu}\,\psi) = 0\,.$$

Upon multiplying (I‴) by $\psi^* A$ on the left (the conjugate equation by $\psi^* A$ on the right, resp.), since:

$$\alpha^{k\dagger} A^{\dagger} = A\,\alpha^{k}$$

and (cf., 38):

$$A_{;\,k} \equiv \frac{\partial \Lambda}{\partial x^{k}} - (A\overset{R}{\Lambda}_{k} + \overset{R}{\Lambda}{}^{\dagger}_{k}\,A) = 0\,,$$

hence, recalling (44), we have:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{l}}(\sqrt{g}\,\psi^* A\,\alpha^{l}\,\psi) - \varepsilon\,(l + l^*)\,f_k\,(\psi^* A\,\alpha^{k}\,\psi) + (k + l + k^* + l^*)(\psi^* A\,\alpha_0\,\psi)$$
$$= 0\,.$$

In this, we have made essential use of the fact that:

$$(A\,\alpha_0\,\alpha^{[kl]})^{\dagger} = -\,A\,\alpha_0\,\alpha^{[kl]}\,,$$

to make the additional term disappear. Thus, (54a) is valid when:

(55) l is imaginary and k is imaginary .

We then obtain agreement with the Dirac equation, up to an additional term that we shall discuss later when we set:

(56) $$l = +\frac{ie}{\hbar c}\frac{c}{\sqrt{\kappa}}\frac{1}{r}$$

and:

(57) $$k = -\frac{imc}{\hbar} - \frac{ie}{\hbar c}\frac{c}{\sqrt{\kappa}}\frac{1}{r}\,.$$

Then, with $p = q = 1$, (I‴) assumes the form:

(58) $$\alpha^{k}\left(\frac{\partial \psi}{\partial X^{k}} + \overset{R}{\Lambda}_{k}\,\psi - \frac{ie}{\hbar c}\Phi_{k}\psi\right) - i\,\frac{mc}{\hbar}\alpha_0\psi + \frac{r}{8}\frac{\sqrt{\kappa}}{c}F_{kl}\alpha_0\alpha^{[kl]}\psi = 0\,.$$

One easily sees that (54) is therefore also valid. As a Lagrange function, we can use:

(59) $$L_m = \mathrm{Re}\,\frac{\hbar c}{i}\,2\,(\Psi^* A\,\alpha_{\mu}\,\Psi_{;\,\mu} + k\,\Psi^* A\,\alpha_0\,\Psi)$$

or also:

(59a) $L_m =$

$$\mathrm{Re}\,2\frac{\hbar c}{i}\left(\psi^* A\alpha_\mu\psi_{;k} - \frac{ie}{\hbar c}\psi^* A\alpha^k\psi\,\Phi_k - \frac{imc}{\hbar}\psi^* A\alpha_0\psi + \frac{r}{8}\frac{\sqrt{\kappa}}{c}F_{kl}\psi^* A\alpha_0\alpha^{[kl]}\psi\right),$$

in which the notation Re is understood to mean that the real part is taken.

From (35), (53), the gauge transformations:

$$\Phi'_k = \Phi_k + \frac{\partial F}{\partial x^k}$$

are identical with the *S*-transformations, for which:

$$\psi' = e^{+\frac{ie}{\hbar c}F}\psi, \qquad S = e^{-\frac{ie}{\hbar c}F}\cdot I\,.$$

Here, we shall briefly mention the case of $\varepsilon = -1$. We first multiply the wave equation (I''') by α_0 and obtain:

$$\alpha_0\alpha^k\left(\frac{\partial\psi}{\partial x^k} + \overset{R}{\Lambda}_k\,\psi + lf_k\right) + (k-l)\psi + \frac{2q-p}{8}X_{kl}\alpha^{[kl]}\psi = 0\,,$$

and deduce that it must follow that:

$$\frac{\partial}{\partial x^k}(\sqrt{g}\;\psi^* A\;\alpha_0\;\alpha^k\;\psi) = 0\,.$$

If we introduce $\bar{A} = \sqrt{\eta}\,A$ into this expression then $\bar{A}$ is Hermitian and $\bar{A}\,\alpha_0\,\alpha^k$, as well as $\bar{A}\,\alpha^{[kl]}$, is skew-symmetric. We thus obtain:

$$\frac{\partial}{\partial x^k}(\sqrt{g}\psi^*\bar{A}\alpha_0\alpha^k\psi) + (l+l^*)\,f\,(\psi^*\bar{A}\alpha_0\alpha^k\psi) + [(l-l^*)-(k-k^*)]\psi^*\bar{A}\psi + \frac{2q-p}{4}\psi^*\bar{A}\alpha^{[kl]}\psi X_{kl}$$
$$= 0.$$

From this, one next infers that the additional term becomes significant and must be made to vanish by the assumption that:

(55′) $$2q = p\,.$$

Thus, l is, moreover, purely imaginary, and one must set:

(56′) $$l = \frac{ie}{\hbar c}\frac{c}{\sqrt{\kappa}}\frac{1}{r}$$

which means that $k - l$ is real and one must set k equal to:

(57′) $$k = \frac{mc}{\hbar} - \frac{ie}{\hbar c}\frac{c}{\sqrt{\kappa}}\frac{1}{r}\,.$$

The Lagrange function becomes:

$$(51') \quad \begin{cases} L_m = \mathrm{Re}\, 2\dfrac{\hbar c}{i}(\Psi * i\bar{A}\alpha_\mu \Psi_{;k} - k\Psi * i\bar{A}\Psi) \\ = \mathrm{Re}\, 2\dfrac{\hbar c}{i}\left(\psi * i\bar{A}\alpha_0\alpha^k\psi_{;k} - \dfrac{ie}{\hbar c}\psi * i\bar{A}\alpha_0\alpha^k\psi \cdot \Phi_k - \dfrac{imc}{\hbar}\psi * \bar{A}\psi\right). \end{cases}$$

This case was originally considered by Schouten and van Dantzig, and is noteworthy for the fact that the additional term vanishes there. We therefore regard this case as singular and not very natural.

§ 6. **Variational principle and field equations**

The principle of deriving the field equations from a variational principle is the same as the one that was used in Part I. We must now combine the Lagrange function for the vacuum (i.e., the absence of matter):

$$L^{(v)} = P$$

[cf., Part I, eq. (38)] with the Lagrange function of matter:

$$L^{(v)}.$$

We must require that:

$$(60) \qquad \delta\int\left(L^{(v)} + \frac{\kappa}{c^2}L^{(m)}\right)\sqrt{g}\; dX^{(1)} \ldots dX^{(5)} = 0\,,$$

when the $\delta g_{\mu\nu}$ satisfy the additional condition:

$$(61) \qquad \delta g_{\mu\nu}X^\mu X^\nu = 0\,.$$

If one has, in general:

$$(62) \qquad \delta\int L^{(m)}\sqrt{g}\;\; dX^{(1)} \ldots dX^{(5)} = \int T_{\mu\nu}\sqrt{g}\; dg^{\mu\nu}\, dX^{(1)} \ldots dX^{(5)},$$

in which $T_{\mu\nu} = T_{\nu\mu}$ is symmetric, then, from Part I (79) and (72), the field equations become:

$$(\mathrm{II}) \qquad K_{\mu\nu} - \bar{K}X_\mu X_\nu = -\frac{\kappa}{c^2}(K_{\mu\nu} - \bar{T}X_\mu X_\nu)$$

in which we have set:

$$(63\mathrm{a}) \qquad \bar{K} = K_{\mu\nu}X^\mu X^\nu\,, \qquad \bar{T} = T_{\mu\nu}X^\mu X^\nu\,.$$

With:

$$(63\mathrm{b}) \qquad T_{ik} = \gamma^{\mu}_{\cdot i}\gamma^{\nu}_{\cdot k}\,T_{\mu\nu}\,, \qquad T_{i(0)} = \gamma^{\mu}_{\cdot i}X^\nu\,T_{\mu\nu}\,,$$

and corresponding expressions for $K_{\mu\nu}$, eq. (II) splits into:

$$(\mathrm{IIa}) \qquad K_{ik} = -\frac{\kappa}{c^2}T_{ik}\,,$$

(IIb) $$K_{i(0)} = -\frac{\kappa}{c^2} T_{i(0)} \,.$$

Analogous to eq. (73) and (76a,b) in Part I, one has the following identities:

(64) $$T_{\mu\,;\nu}^{\cdot\,\nu} \equiv 0 \,,$$

which splits into:

(64a) $$T_{i\,;k}^{\cdot\,k} - X_{ik} T_{(0)}^{\cdot\,k} \equiv 0 \,,$$

(64b) $$T^{k}_{\cdot\,(0)\,;k} \equiv 0 \,.$$

Here, the ≡ is intended to mean that the equations in question are valid when the field equations (58) are assumed to be valid, but not, however, equation (II). Then, they only vanish under variation of the coordinate system and an additional variation of the matrices α_μ and A_μ that corresponds to an infinitesimal S-transformation.

The projector $T_{\mu\nu}$ combines the energy-momentum tensor T_{ik} and the current vector:

(65) $$v^i = r\frac{\sqrt{\kappa}}{c^2} T^{i}_{\cdot\,(0)} \,.$$

The factor $r\frac{\sqrt{\kappa}}{c^2}$ is justified by the fact that from I, eq. (84c) [(99), resp.], one has:

$$K^{k}_{\cdot\,(0)} = -\frac{1}{r}\frac{\sqrt{\kappa}}{c} F^{ik}{}_{;\,k} \,,$$

and the field equation (IIb), with the introduction of (65), takes on the form:

$$-\frac{1}{r}\frac{\sqrt{\kappa}}{c} F^{ik}{}_{;\,k} = -\frac{\kappa}{c^2}\frac{1}{r}\frac{c}{\sqrt{\kappa}} v^i \,,$$

or:

(65a) $$F^{ik}{}_{;\,k} = v^i \,.$$

In the sequel, we would like pursue only the case of a symmetric $\Gamma^{\lambda}_{\mu\nu}$, which corresponds to $p = q = 1$, $\varepsilon = +1$, and $r = \sqrt{2}$ [cf., Part I, eq. (86)]. One then has, from (59):

(59) $$L^{(m)} = \mathrm{Re}\, 2\frac{\hbar c}{i} (\Psi^* A\, \alpha_\mu\, \Psi_{;\,\mu} + k\, \Psi^* A\, \alpha_0\, \Psi).$$

In order to construct the $T_{\mu\nu}$ from the prescription (62), it is permissible to assume that the wave equation:

$$\alpha_\mu\, \Psi_{;\,\mu} + k\, \alpha_0\, \Psi = 0$$

or:

$$\alpha^k\,(\overset{R}{\psi}_{;k} - f_k) + (k+l)\,\alpha_0\,\psi + X_{kl}\,\alpha_0\,\alpha^{[kl]}\,\psi = 0$$

is valid for the variation. Since an infinitesimal *S*-transformation of the α_μ then produces no change in the action integral, one may replace the variations $\delta g_{\mu\nu}$ with any expressions that satisfy the relations:

$$\tfrac{1}{2}\,\delta(\alpha^\mu\,\alpha^\nu + \alpha^\nu\,\alpha^\mu) = \delta g^{\mu\nu}\,.$$

The simplest solution of this equation is:

(66) $$\delta\alpha^\mu = \tfrac{1}{2}\,\alpha_\nu\,\delta g^{\mu\nu}\,,$$

which we shall use in what follows. It then follows that:

(66a) $$\delta\alpha_0 = -\tfrac{1}{4}\,(\alpha_\nu\,X_\mu + a_\mu\,X_\nu)\;\delta g^{\mu\nu}\,.$$

By a somewhat tedious calculation, the variation of (59) gives rise to a term:

$$\mathrm{Re}\;2\frac{\hbar c}{i}(\Psi^*\,A\,\alpha^\mu\,\delta A_\mu\,\Psi)$$

that originates in the variation of A_μ. The expression δA_μ must be determined in such a way that the relations (32a) or (32b) remain valid under the variation when (65) is substituted for the $\delta\alpha^\mu$. One has, in turn:

$$\frac{\partial(\alpha_\nu\delta g^{\mu\nu})}{\partial X^\rho} + \Gamma^\mu_{\nu\rho}\alpha_\sigma g^{\mu\nu} + 2\delta\Gamma^\mu_{\sigma\rho}\alpha^\sigma + (\Lambda_\rho\alpha_\nu - \alpha_\nu\Lambda_\rho)\delta g^{\mu\nu}$$
$$= -\,2(\delta\Lambda_\rho\,\alpha^\mu - \alpha^\mu\;\delta\Lambda_\rho)\,,$$

and by considering the fact that:

$$0 = \delta g^{\mu\nu}{}_{;\,\rho} = \frac{\partial\delta g^{\mu\nu}}{\partial X^\rho} + \delta(\Gamma^\mu_{\sigma\rho}g^{\sigma\nu} + \Gamma^\nu_{\sigma\rho}g^{\sigma\mu})\,,$$

we finally have that:

$$\delta\Lambda_\rho\,\alpha^\mu - \alpha^\mu\;\delta\Lambda_\rho = \tfrac{1}{2}(g^{\mu\sigma}\delta\Gamma^\nu_{\sigma\rho} - g^{\nu\sigma}\delta\Gamma^\mu_{\sigma\rho})\;\alpha_\nu\,.$$

From this, one finds that:

(67) $$\delta\Lambda_\rho = \tfrac{1}{8}(g_{\mu\sigma}\delta\Gamma^\sigma_{\nu\rho} - g_{\nu\sigma}\delta\Gamma^\sigma_{\mu\rho})\;\alpha^{[\mu\nu]}\,.$$

In this, we have set [cf., (33)]:

$$\mathrm{Tr}(\delta\Lambda_\rho) = \delta F_\rho = 0\,.$$

One sees that, from (36a) and (54), the variation:

$$\delta F_\rho = \frac{\partial \delta F}{\partial X^\rho}$$

of F_ρ makes no contribution to the variation of the action integral.

We now compute:

$$\mathrm{Re}\,\tfrac{1}{i}\,\Psi^* A\ \delta\Lambda_\mu\,\Psi = \mathrm{Re}\,\tfrac{1}{i}\,\Psi^* A\ \alpha^\mu\ \alpha^{[\rho\sigma]}\ \Psi\ ?\,(g_{\rho\nu}\delta\Gamma^\nu_{\sigma\mu} - g_{\sigma\nu}\delta\Gamma^\nu_{\rho\mu})\,.$$

If:

$$\alpha^{[\mu\rho\sigma]} = \tfrac{1}{3}\,(\alpha^\mu\ \alpha^{[\rho\sigma]} + \alpha^\rho\ \alpha^{[\sigma\mu]} + \alpha^\sigma\ \alpha^{[\mu\rho]}\,)$$

is an anti-symmetric linear combination of the products of three matrices α^μ then one finds, by a transformation and the use of the fact that:

$$g^{\mu\nu} = \tfrac{1}{2}\,(\alpha^\mu\ \alpha^\nu + \alpha^\nu\ \alpha^\mu\,)$$

that:

(68) $$\alpha^\mu\ \alpha^{[\rho\sigma]} = \alpha^{[\mu\rho\sigma]} + (g^{\mu\rho}\ \alpha^\sigma - g^{\mu\sigma}\alpha^\rho)\,.$$

Upon taking the real part, the contribution from *one* of the α matrices vanishes, since $A\alpha^\sigma$ is Hermitian:

$$\mathrm{Re}\,\tfrac{1}{i}\,\Psi^* A\ \alpha^\sigma\Psi = 0,$$

and what remains is:

(69) $$\mathrm{Re}\,\tfrac{1}{i}\,\Psi^* A\,\delta\Lambda_\mu\Psi = \mathrm{Re}\,\tfrac{1}{i}\,\Psi^* A\alpha^{[\mu\rho\sigma]}\Psi\cdot\tfrac{1}{8}(g_{\rho\nu}\delta\Gamma^\nu_{\sigma\mu} - g_{\sigma\nu}\delta\Gamma^\nu_{\rho\mu})\,.$$

When one now assumes that $\Gamma^\nu_{\rho\mu}$ is symmetric in ρ, μ ($p = q = 1$) (which was not used up till now), and only in this case, the expression vanishes, since, as one sees, $\alpha^{[\mu\rho\sigma]}$ is anti-symmetric in μ and ρ, but $\delta\Gamma^\nu_{\sigma\mu}$ is symmetric. In this case, one thus has:

(70) $$\mathrm{Re}\,\tfrac{1}{i}\,\Psi^* A\ \delta\Lambda_\mu\Psi = 0\,.$$

The computation of $T_{\mu\nu}$ on the basis of (62), and using (65) and (66), now becomes simples, and one obtains:

(71) $$T_{\mu\nu} = \mathrm{Re}\frac{1}{2}\frac{\hbar c}{i}\,[\Psi^* A(\alpha_\nu\,\Psi_{;\,\mu} + \alpha_\mu\,\Psi_{;\,\nu}) - k\,\Psi^* A(\alpha_\nu\,X_\mu + \alpha_\mu\,X_\nu\,].$$

This expression combines the energy-momentum tensor T_{ik} and the current vector $T_{i(0)} \equiv v_i$ into a single entity ([1]). In order to split the projector $T_{\mu\nu}$ into T_{ik} and $T_{i(0)}$ (the scalar $\overline{T}$ does not enter into the physical statement) one must substitute:

[1] This unification was not achieved in the earlier work of W. Pauli and J. Solomon, loc. cit., since homogeneous coordinates were not used there.

$$\Psi_{;\,\mu} = F^l \left[\gamma_\mu^{\cdot\, i} (\overset{R}{\psi}_{;i} - l\, f_i) + l\, X_\mu\, \psi + \Delta_\mu\, \psi \right],$$
$$\alpha_\nu = \gamma_\nu^{\cdot\, k}\, \alpha_k + X_\nu\, \alpha_0\, .$$

Δ_μ is defined by (47), (49).

(72a) $$\mathrm{Re}\,\tfrac{1}{i}\,\Psi^*\, A\ \alpha_k\, \Delta_\mu\ \gamma^{\mu}_{\cdot\, i}\, \Psi = \frac{1}{4}\frac{1}{i}\ \psi^*\, A\ \alpha^{[kl]}\ \alpha_0\, \psi\ X_i^{\cdot\, l}\, ,$$

(72b) $$\mathrm{Re}\,\tfrac{1}{i}\,\Psi^*\, A\ \alpha_i\, (\Delta_\mu\, X^\nu)\Psi = -\frac{1}{8}\frac{1}{i}\ \psi^*\, A\ \alpha_{[ikl]}\ \psi\ X^{kl}\, ,$$

and:

(72c) $$\mathrm{Re}\,\tfrac{1}{i}\,\Psi^*\, A\ \alpha_0\, \Delta_\mu\ \gamma^{\mu}_{\cdot\, i}\, \Psi = 0\, .$$

In this, we have made use of a formula for $\alpha_i\ \alpha_{[kl]}$ that is analogous to (62). By substituting:

$$X_{ik} = r\frac{\sqrt{\kappa}}{c} F_{ik}\, , \qquad r = \sqrt{2}\, ,$$

(56) $$l = \frac{ie}{\hbar c}\frac{c}{\sqrt{\kappa}}\frac{1}{r}\, ,$$

one finds, in this way, that:

(73) $$\begin{cases} T_{ik} = \tfrac{1}{2}\,\mathrm{Re}\,\dfrac{\hbar c}{i}\psi^* A\left[\alpha_k\left(\overset{R}{\psi}_{;i} - \dfrac{ie}{\hbar c}\Phi_i\psi\right) + \alpha_i\left(\overset{R}{\psi}_{;k} - \dfrac{ie}{\hbar c}\Phi_k\psi\right)\right] \\ \qquad + \dfrac{\sqrt{2\kappa}}{c}\dfrac{\hbar c}{i}\dfrac{1}{8}\psi^* A(\alpha_{[kl]}\alpha_0 F_i^{\cdot\, l} + \alpha_{[il]}\alpha_0 F_k^{\cdot\, l})\psi, \end{cases}$$

(74) $$\begin{cases} \dfrac{c}{\sqrt{\kappa}}\dfrac{1}{r} v^i \equiv T^i_{\cdot\,(0)} \\ = \dfrac{1}{2}\hbar c\left\{(l-k)\tfrac{1}{i}\psi^* A\alpha^i\psi - \tfrac{1}{8}\tfrac{1}{i}\psi^* A\alpha^{[ikl]}\psi X_{kl} + \mathrm{Re}\,\tfrac{1}{i}\, g^{ik}\psi^* A\alpha_0(\overset{R}{\psi}_{;k} - l f_k\psi)\right\}, \end{cases}$$

The last expression can be essentially simplified by a transformation. If one multiplies the wave equation (58) on the left by $\tfrac{1}{i}\ \psi^*\ A\ \alpha_0\ \alpha^j$ and takes the real part then this yields:

(75) $$\begin{cases} \dfrac{1}{2}\dfrac{1}{i}\dfrac{1}{\sqrt{g}}\dfrac{\partial}{\partial x^k}\sqrt{g}(\psi^* A\alpha_0\alpha^{[ik]}\psi) + \mathrm{Re}\,\dfrac{1}{i}\psi^* A\alpha_0(\overset{R}{\psi}_{;k} - l f_k\psi) g^{ik} \\ \qquad - \dfrac{1}{i}(l+k)\psi^* A\alpha^i\psi - \dfrac{1}{i}\psi^* A\alpha^{[ikl]}\psi X_{kl} = 0, \end{cases}$$

which then makes:

$$\frac{c}{\sqrt{\kappa}}\frac{1}{r}v^i = \frac{\hbar c}{i} l\ \psi^* A\ \alpha^i\ \psi - \frac{1}{4}\frac{\hbar c}{i}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\sqrt{g}\ (\psi^* A\ \alpha_0\ \alpha^{[ik]}\ \psi)$$

or:

$$v^i = e\ \psi^* A\ \alpha^i\ \psi - \frac{1}{4}\frac{\sqrt{2\kappa}}{c}\frac{\hbar c}{i}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\sqrt{g}\ (\psi^* A\ \alpha_0\ \alpha^{[ik]}\ \psi)\ . \tag{76}$$

Since the extra term has an identically vanishing divergence, the relation that is required by (64b) follows from (54a):

$$v^k{}_{;\,k} \equiv \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}v^k}{\partial x^k} = 0\ . \tag{64b}$$

The field equation (IIb) now assumes the form:

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}F^{ik}}{\partial x^k} = v^i = e\psi^* A\alpha^i\psi - \frac{1}{4}\frac{\sqrt{2\kappa}}{c}\frac{\hbar c}{i}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\sqrt{g}\ (\psi^* A\ \alpha_0\ \alpha^{[ik]}\ \psi)\ . \tag{77}$$

The extra term in (73) [(77), resp.] represents a deviation of the present theory from the Dirac theory. Since this term is, however, proportional to $\sqrt{2\kappa}$, it can hardly contradict the physical experiments (be empirically demonstrable, resp.). From the extra term in (77), it can be inferred that electrically neutral masses with a resulting spin moment (a resulting linear momentum will not suffice) must possess a small magnetic moment (which, is possibly not without interest – possibly in regard to the problem of geomagnetism).

As was emphasized in the introduction, the field equations that were presented in this paragraph must be subjected to second quantization, which the transition to configuration space brings with it, in order to describe the interactions of charged particles. It is well known, however, that the problem of the infinite self-energy of the particle still remains unsolved.

The more provisional character of five-dimensional form of the Dirac theory that was developed here, compared to the contents of first part, which related only to the purely classical theory, finds its expression in the fact that the Lagrange function of matter was simply added to that of the vacuum, without any logical connection existing between them. In contrast to the coupling of the electromagnetic and gravitational fields, a direct logical connection of the matter wave fields with the fields that are described by the formulation of the theory that was developed here has not be attained.

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