“ Zur Quantenmechanik des magnetischen Elektrons,” Zeit. f. Phys. **43** (1927), 601-623.

On the quantum mechanics of magnetic electrons

By **W. PAULI, Jr.** in Hamburg

(Received on 3 May 1927)

Translated by D. H. Delphenich

It will be shown how one can arrive at a formulation of the quantum mechanics of the magnetic electron by the Schrödinger method of eigenfunctions, with no use of double-valued functions, when one, on the basis of the Dirac-Jordan general theory of transformations, introduces the components of its proper impulse moment in a fixed direction as further independent variables in order to carry out the computations of its rotational degrees freedom, along with the position coordinates of any electron. In contradiction to classical mechanics, these variables can assume only the variables $+\frac{1}{2}\frac{h}{2\pi}$ and $-\frac{1}{2}\frac{h}{2\pi}$, which is completely independent of any sort of external field. The appearance of the aforementioned new variables thus implies a simple splitting of the eigenfunctions into two position functions ψ_α, ψ_β for one electron, and more generally, for N electrons they split into 2^N functions, which are to be regarded as the “probability amplitudes” that in a well-defined stationary state of the system not only do the position coordinates of the electrons lie in a given infinitesimal interval, but also that the components of their proper moments in the chosen direction should have the given values, which are $+\frac{1}{2}\frac{h}{2\pi}$ for ψ_α and $-\frac{1}{2}\frac{h}{2\pi}$ for ψ_β. Methods will be given for constructing as many simultaneous differential equations for the ψ functions as their number suggests (thus, 2 or 2^N, resp.) from a given Hamiltonian function. These equations are completely equivalent in their consequences to the matrix equations of Heisenberg and Jordan. Furthermore, in the case of many electrons, the solutions of the differential equations that satisfy the “equivalence rule” of Heisenberg and Dirac will be characterized by their symmetry properties under the exchange of the variable values for the two electrons.

§ 1. Generalities on the nature of electronic magnetism in the Schrödinger form of quantum mechanics. The hypothesis that was first proposed by Goudsmit and Uhlenbeck in order to explain the complex structure of spectra and their anomalous Zeeman effect, according to which the electron takes on a proper impulse moment of magnitude $\frac{1}{2}\frac{h}{2\pi}$ and a magnetic moment of a magneton, was integrated into quantum mechanics by Heisenberg and Jordan [1]) with the help of matrix calculations and then made quantitatively precise. While the matrix method is mathematically equivalent to the method of eigenfunctions in many-dimensional space that was discovered by Schrödinger, one comes up against peculiar formal complications when one attempts to also treat the forces and moments that an electron experiences in an external field by the method of its proper moment. By the introduction of a further degree of freedom that

---

[1]) Zeit. f. Phys. **37** (1926), 263.

corresponds to the orientation of the proper impulse of the electron in space, one actually expresses the empirically-established fact that this momentum has two possible quantum positions in an external field, so one is next led to eigenfunctions that are many-valued, and indeed, two-valued, in the rotational angle in question – e.g., the azimuth of the impulse around a spatially fixed axis. One has often supposed that this formally possible representation by means of two-valued eigenfunctions does not do justice to the true physical nature of things and has sought the solution to the problem in another direction. Thus, Darwin [1]) has recently attempted to gather the facts that are summarized under the assumption of the electron impulse without the introduction of the top degrees of freedom for the electron that would correspond to new dimension in the configuration space, so he considered the amplitudes of the de Broglie waves as directed quantities – i.e., he considered the Schrödinger eigenfunction as vectorial. From his attempt to follow this, on first glance promising, path to its ultimate consequences, he came to complications that were again connected precisely with the number two for the positions of the electron in an external field, and which I do not believe one can surmount. On the other hand, a representation of the quantum-mechanical behavior of the magnetic electron using the method of eigenfunctions, especially in the case of atoms with many electrons, is very desirable for the fact that the variety that is realized in nature alone results for the solutions of the quantum-mechanical equations that fulfill the "equivalence rule" for all of the possible solutions of the present theory of Heisenberg [2]) and Dirac [2]) most clearly with the help of symmetry properties of the eigenfunctions under the exchange of the variable values that belong to two electrons.

We would now like to show that by a suitable use of the formulation of quantum mechanics, as described by Jordan [3]) and Dirac), which makes use of general canonical transformations of the Schrödinger functions ψ, a quantum-mechanical representation of the behavior of magnetic electrons by the method of eigenfunctions is, in fact, possible, without appealing to many-valued functions. Namely, one achieves this by adding the components of the proper impulse of each electron in a fixed direction (instead of the rotational angle that is conjugate to it) as new independent variables, along with the position coordinates q of the electron center of mass. As we will see in what follows in § 2 in the special case of a single electron, in any quantum state (in the absence of degeneracy) the eigenfunction generally splits into two functions $\psi_\alpha(q_k)$ and $\psi_\beta(q_k)$, of which the square of the absolute value, when multiplied by $dq_1, \ldots, dq_f$, yields the probability that in this state, not only should the q_k lie in the prescribed interval $(q_k, q_k + dq_k)$, but also that the components of the proper impulse in the chosen fixed direction must assume the values $+\frac{1}{2}\frac{h}{2\pi}$ $(-\frac{1}{2}\frac{h}{2\pi}$, resp.). It will be further shown how, by a suitable choice of linear operators for the components s_x, s_y, s_z of the proper moment in a prescribed coordinate axis-cross, differential equations for the eigenfunctions of the magnetic electron in an external force field can be constructed that are equivalent to the matrix equations of Heisenberg and Jordan. This will be performed in detail in § 4 for

---

[1]) Nature **119** (1927), 282.

[2]) W. Heisenberg, Zeit. f. Phys. **38** (1926), 411; **39** (1926), 499; **41** (1927), 239; P. A. M. Dirac, Proc. Roy. Soc. **112** (1926), 661.

[3]) P. Jordan, Zeit. f. Phys. **40** (1927), 809; Gött. Nachr. (1926), pp. 161; P. A. M. Dirac, Proc. Roy. Soc. (A) **113** (1927), 621; cf., also F. London, Zeit. f. Phys. **40** (1926), 193.

the case of an electron at rest in an external magnetic field and for a hydrogen atom. It will be further investigated how the eigenfunctions ψ_α, ψ_β transform under changes of the coordinate axes (§ 3).

The differential equations for the eigenfunctions of the magnetic electron that are given in the present paper can be regarded as only provisional and approximate, since they, like the Heisenberg-Jordan matrix formulation, are not written down in a relativistically-invariant way, and for the hydrogen atom they are valid only in the approximation in which the dynamical behavior of the proper moment can be considered to be a secular perturbation (in the classical theory: averaged over the orbit). In particular, it thus not possible to calculate quantum-mechanically the corrections that are proportional to higher powers of $\alpha^2 Z^2$ ($\alpha = \frac{2\pi e^2}{hc}$ = fine structure constant) in the amounts of the hydrogen fine-structure splitting, such as the empirically established amounts for the Röntgen spectra that are given so well by the Sommerfeld formula. These difficulties, which are still obstacles to the solution of this problem to this day, will be discussed briefly in § 4.

Thus, whether or not the formulation of the quantum mechanics of the magnetic electron that is communicated here is still completely unsatisfactory in that regard, on the other hand, it affords the advantage that in the case of many electrons (in contrast to the Darwin formulation), as will be shown in § 5, it gives rise to no new difficulties at all and also allows one, like Heisenberg, to easily formulate necessary symmetry properties of the eigenfunction in order for it to fulfill the "equivalence rule." In particular, on this basis, it already seems to me justified to communicate the method proposed at the present point in time, and one can perhaps hope that it will also prove useful in the unsolved problem of the calculation of the hydrogen fine structure in higher approximations.

§ 2. Introduction of the components of the proper moment of the electron in a fixed direction as independent variables for the eigenfunction. Definition of the operators that correspond to the components of the proper moment. In classical mechanics, the dynamical behavior of the electron moment can be described by the following pairs of canonical variables: The amount s of the total proper moment of the electron and the rotation angle χ around its axis. Secondly, one has the component s_z of this moment in a fixed direction z and the azimuth φ of the moment vector around the z-axis, as measured in the (xz)-plane. Since the quotient s_z / s yields the cosine of the angle between this vector and the z-axis, these x and y components are given by:

$$s_x = \sqrt{s^2 - s_z^2}\cos\varphi, \quad s_y = \sqrt{s^2 - s_z^2}\sin\varphi\,.$$

Since the rotation angle χ is always cyclic, so it does not enter into the Hamiltonian function, s remains constant, and can be regarded as a fixed number, such that only (s_z , φ) remains as the actual canonical variable pair that is determined by the dynamical behavior of the electron moment.

By an application of the original Schrödinger method, one thus has an eigenfunction for the presence of a single electron in any quantum state (which is already uniquely characterized by a well-defined energy value E by lifting the degeneracy in external fields) that depends on not just the three position coordinates of the electron center of mass (which are denoted briefly by q_k , or also q), but also on the angle φ. This then gives:

$$|\psi_E(q, \varphi)|^2\, dq_1\, dq_2\, dq_3\, d\varphi$$

as the probability that in the quantum state in question of energy E the position coordinates should lie in the intervals q_k , $q_k + dq_k$, while the angle φ should lie in (φ, $\varphi + d\varphi$). If the impulse coordinate s_z that is conjugate to φ appears in any dynamical function then it would be replaced with the operator $\frac{h}{2\pi i}\frac{\partial}{\partial\varphi}$, which is applied to the eigenfunction ψ, just as the impulse coordinate p_k of the translational motion that is conjugate to q_k will be represented by the operator $\frac{h}{2\pi i}\frac{\partial}{\partial q_k}$. As is known, the fact that the number of allowed quantum orientations for the electron moment is two implies the consequence that the function $\psi_E(q, \varphi)$ thus defined cannot return to its starting value as φ continually advances from the value 0 to 2π, but must change its sign.

Meanwhile, one can avoid the appearance of such two-valuedness, like the explicit use of any polar angle whatsoever, in such a way that one introduces the impulse component s_z as an independent variable in the eigenfunction in place of φ. Thus, an especially simplified situation appears in quantum mechanics: In classical mechanics, in general, s_z will be capable of taking on a continuum of values for a certain energy (e.g., when the moment vector precesses around a direction that is different from the z-axis), except for the special case in which s_z is precisely an integral of the equations of motion. In quantum mechanics, however, s_z can, by being conjugate to an angle coordinate, assume only the characteristic values $+\frac{1}{2}\frac{h}{2\pi}$ and $-\frac{1}{2}\frac{h}{2\pi}$; this shall mean that the function $\psi_E(q, \varphi)$ splits into two functions $\psi_{\alpha, E}(q, \varphi)$ and $\psi_{\beta, E}(q, \varphi)$ that correspond to the values $s_z = +\frac{1}{2}\frac{h}{2\pi}$ and $s_z = -\frac{1}{2}\frac{h}{2\pi}$, resp. This makes:

$$|\psi_{\alpha, E}(q, \varphi)|^2\, dq_1\, dq_2\, dq_3$$

the probability that in the stationary state considered one simultaneously has that q_k lies in (q_k, $q_k + dx_k$) and s_z has the value $+\frac{1}{2}\frac{h}{2\pi}$, and:

$$|\psi_{\beta, E}(q, \varphi)|^2\, dq_1\, dq_2\, dq_3$$

is the probability that for the same value of q_k the impulse component s_z assumes the value $-\frac{1}{2}\frac{h}{2\pi}$. Any attempt to measure the magnitude of s_z in a certain stationary state will always yield only the two values $+\frac{1}{2}\frac{h}{2\pi}$ and $-\frac{1}{2}\frac{h}{2\pi}$, and also when s_z does not represent an integral of the equations of motion. *This special case (e.g., a strong magnetic field in the z-direction) is, moreover, distinguished by the fact that here, for a well-defined energy E, only one of the two functions* $\psi_{\alpha,E}$ *or* $\psi_{\beta,E}$ *is ever different from zero.* For a well-defined choice of the coordinate system ψ_α and ψ_β are determined completely, up to a common phase factor, in any stationary state by the normalization:

$$\int (|\psi_\alpha|^2 + |\psi_\beta|^2)\, dq_1\, dq_2\, dq_3 = 1. \tag{1a}$$

The orthogonality relation:

$$\int (\psi_{\alpha,n}\psi^*_{\alpha,m} + \psi_{\beta,n}\psi^*_{\beta,m})\, dq_1 dq_2 dq_3 = 0 \qquad \text{for } n \neq m \tag{1b}$$

must also be valid. In it, the indices n, m denote two distinct quantum states and the * that is affixed (here, as in the sequel) denotes the complex conjugate value [1]).

In order to be able to later describe the differential equations that the functions ψ_α, ψ_β satisfy for a given Hamiltonian function, one can proceed in such a way that one expresses them as functions of (p_k, q_k) and (s_z, φ), and then replaces p_k with the operator $\frac{h}{2\pi i}\frac{\partial}{\partial q_k}$ and φ with the operator $-\frac{h}{2\pi i}\frac{\partial}{\partial s_z}$. The total operator would then be applied to $\psi(q_k, s_z)$, and ultimately one would pass to the limit in which y is non-zero only for $s_z = +\frac{1}{2}\frac{h}{2\pi}$ and $s_z = -\frac{1}{2}\frac{h}{2\pi}$. However, such behavior would be confusing and less convenient. The Hamilton function that actually enters in always includes the angular impulse components sx, sy, sz as variables and is therefore preferable for this purpose without the detour of introducing the operator that is appropriate to the polar angle φ.

These operators must satisfy the same commutation relations (up to a sign, cf., *infra*) as the matrices in questions, namely:

$$[\mathfrak{s}\,\mathfrak{s}] = -\frac{h}{2\pi i}\mathfrak{s}; \qquad \mathfrak{s}^2 = \left(\frac{h}{2\pi}\right)^2 s(s+1) \quad \text{with} \quad s = 1/2,$$

in which $\mathfrak{s}$ means a vector matrix with the components $\mathbf{s}_x$, $\mathbf{s}_y$, $\mathbf{s}_z$ [2]). For the sake of simplicity, in what follows if we measure $\mathfrak{s}$ in units of $\frac{1}{2}\frac{h}{2\pi}$ (i.e., one replaces $\mathfrak{s}$ with $\frac{1}{2}\frac{h}{2\pi}\mathfrak{s}$) and write out the vector equations in components then we obtain:

---

[1]) Let it be mentioned at this point that according to the Dirac-Jordan transformation theory, the aforementioned function $\psi(q, \varphi)$ is connected with the functions ψ_α, ψ_β according to the formulas:

$$\psi(q, \varphi) = \psi_\alpha(q)\, e^{i\varphi/2} + \psi_\beta(q)\, e^{-i\varphi/2}.$$

[2]) Cf., W. Heisenberg and P. Jordan, *loc. cit.*, eq. (10). – Matrices and operators (or “*q*-numbers”) will always be characterized by boldface in the sequel.

$$\left.\begin{aligned} s_x s_y - s_y s_x &= 2i s_z, \cdots, \\ s_x^2 + s_y^2 + s_z^2 &= 3, \end{aligned}\right\} \tag{2}$$

in which the … imply the equations emerge from the one written down by cyclic permutations of the coordinates [1]).

This suggests that we make the Ansatz for the operations s_x , s_y , s_z that satisfy the relations (2) that they are linear transformations of the ψ_α and ψ_β , and indeed the simplest possible Ansatz is the following one:

$$\left.\begin{aligned} s_x(\psi_\alpha) &= \psi_\beta, & s_x(\psi_\beta) &= \psi_\alpha, \\ s_y(\psi_\alpha) &= -i\psi_\beta, & s_y(\psi_\beta) &= i\psi_\alpha, \\ s_z(\psi_\alpha) &= \psi_\alpha, & s_z(\psi_\beta) &= -\psi_\beta. \end{aligned}\right\} \tag{3}$$

One can also write these relations in the symbolic matrix form:

$$s_x(\psi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \psi; \qquad s_y(\psi) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \psi; \qquad s_z(\psi) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \psi. \tag{3′}$$

Relations (2) are thus to be interpreted as saying that when the matrices (3′) are substituted in (2), with an application of the usual prescription for matrix multiplication

---

[1]) As a result of the special circumstance that the number of allowed quantum positions of $\mathfrak{s}$ has the value two (so it can be treated as a two-rowed matrix), in addition to (2), one has the further sharpened relations:

$$\left.\begin{aligned} s_x s_y - s_y s_x &= i s_z, \cdots, \\ s_x^2 + s_y^2 + s_z^2 &= 1. \end{aligned}\right\} \tag{2a}$$

One sees this most simply when one chooses s_z to be a diagonal matrix (although the relations are true in general). By contrast, for many-component matrices that fulfill (2) (in which the value 3 is replaced with $r^2 - 1$, where r is the number of rows of the matrix), $s_x\, s_y$ and s_x^2 would not have vanishing matrix elements in those positions whose row index differs from the column index by 2 (which then correspond to transitions of the quantum number that belongs to s_z by two units), so equations (2a) could be valid.

On the existence of relations (2a), I would cordially refer to P. Jordan, to whom I would also like to express my thanks at this point. He also brought to my attention the following connection with quaternion theory: If one writes a quaternion Q in the form:

$$Q = k_1 A + k_2 B + k_3 C + D$$

then the "units" k_1, k_2, k_3 satisfy the relations:

$$k_1 k_2 = -k_2 k_1 = k_3, \ \ldots,$$
$$k_1^2 = k_2^2 = k_3^2 = -1.$$

These are equivalent to the relations (2a), when one sets:

$$s_x = ik_1, \qquad s_y = ik_2, \qquad s_z = ik_3 .$$

the matrices [1]) satisfy these relations. *The corresponding operators therefore satisfy equations that emerge from* (2) *by permuting the order of all multiplications* [2]). The justification for this prescription will come to us by way of the general connection between operator algebra and matrix algebra. The last of relations (3) is obviously physically necessary when ψ_α and ψ_β mean the probability amplitudes for $\boldsymbol{s}_z$ (measured in units of $\frac{1}{2}\frac{h}{2\pi}$) to assume the value + 1 or – 1, because the operator $\boldsymbol{s}_z$ must then imply simply multiplication of the eigenfunction by the numerical value of $\boldsymbol{s}_z$. The fact that in the special choice of $\boldsymbol{s}_x$, $\boldsymbol{s}_y$ that is included in the demand that relations (2) follow from normalization implies no loss of generality will be made clear in the following paragraphs, where the behavior of the functions ψ_α, ψ_β under a shift of the axes for the coordinate system that was defined will be examined. [Cf. below, pp. 13, eq. (3″).]

Now, if any Hamiltonian function:

$$H(p_k, q_k, s_x, s_y, s_z) = E$$

is given for a special mechanical system that includes a magnetic electron then the two simultaneous differential equations for ψ_α and ψ_β that likewise determine the eigenvalue E are given by:

$$\left.\begin{aligned} H\left(\frac{h}{2\pi i}\frac{\partial}{\partial q_k}, q_k, \boldsymbol{s}_x, \boldsymbol{s}_y, \boldsymbol{s}_z\right)\psi_{E,\alpha} &= E\psi_\alpha, \\ H\left(\frac{h}{2\pi i}\frac{\partial}{\partial q_k}, q_k, \boldsymbol{s}_x, \boldsymbol{s}_y, \boldsymbol{s}_z\right)\psi_{E,\beta} &= E\psi_\beta, \end{aligned}\right\} \tag{4}$$

in which $\boldsymbol{s}_x$, $\boldsymbol{s}_y$, $\boldsymbol{s}_z$ replace the operations (3).

The matrix components of any function $f(p, q, s_x, s_y, s_z)$, of which, we would first like to assume that it either does not include the quantities s_x, s_y, s_z at all or it includes them only linearly, are defined by the simultaneous equations:

$$\boldsymbol{f}(\psi_{m\alpha}) = \sum_n f_{nm}\psi_{n\alpha}\,, \qquad \boldsymbol{f}(\psi_{m\beta}) = \sum_n f_{nm}\psi_{n\beta}\,, \tag{5}$$

if we understand $\boldsymbol{f}$ to mean the operator $f\left(\frac{h}{2\pi i}\frac{\partial}{\partial q}, q, \boldsymbol{s}_x, \boldsymbol{s}_y, \boldsymbol{s}_z\right)$. In particular, one thus has:

$$\boldsymbol{s}_x(\psi_{m\alpha}) = \psi_{m\beta} = \sum_n (s_x)_{nm}\psi_{n\alpha}\,, \qquad \boldsymbol{s}_x(\psi_{m\beta}) = \psi_{m\alpha} = \sum_n (s_x)_{nm}\psi_{n\alpha}\,, \tag{6}$$

---

[1]) Cf., rem. 1, pp. 11.

[2]) The necessity of distinguishing between operator relations and matrix relations at this point was first made evident to me after the fact on the basis of a letter from C. G. Darwin that concerned the comparison between the equations that he presented and my own. (See below, rem. 2, pp. 16) I would also like to express my deepest thanks to Darwin at this point for his encouragement.

and corresponding equations for y and z. The fact that one sums over the first index of the matrix on the right-hand sides of (5) and (6) is essential in order for one to get agreement between successive application of two operators $\boldsymbol{f}$ and $\boldsymbol{g}$ and the multiplication of matrices. It easily follows from (6), by using the orthogonality relations (1a) and (1b), that:

$$f_{nm} = \int [\boldsymbol{f}(\psi_{m\alpha})\psi_{n\alpha}^* + \boldsymbol{f}(\psi_{m\beta})\psi_{n\beta}^*]\,dq_1 dq_2 dq_3 \ldots \tag{5′}$$

In particular, one thus has:

$$\begin{aligned}(s_x)_{nm} &= \int [(\boldsymbol{s}_x\psi_{m\alpha})\psi_{n\alpha}^* + (\boldsymbol{s}_x\psi_{m\beta})\psi_{n\beta}^*]\,dq = \int (\psi_{m\beta}\psi_{n\alpha}^* + \psi_{m\alpha}\psi_{n\beta}^*)\,dq,\\ (s_y)_{nm} &= \int [(\boldsymbol{s}_y\psi_{m\alpha})\psi_{n\alpha}^* + (\boldsymbol{s}_y\psi_{m\beta})\psi_{n\beta}^*]\,dq = \int i(-\psi_{m\beta}\psi_{n\alpha}^* + \psi_{m\alpha}\psi_{n\beta}^*)\,dq,\\ (s_z)_{nm} &= \int [(\boldsymbol{s}_z\psi_{m\alpha})\psi_{n\alpha}^* + (\boldsymbol{s}_z\psi_{m\beta})\psi_{n\beta}^*]\,dq = \int (\psi_{m\alpha}\psi_{n\alpha}^* - \psi_{m\beta}\psi_{n\beta}^*)\,dq.\end{aligned} \tag{6′}$$

If one directs one’s attention to the general eigenfunctions:

$$\psi_\alpha = \sum c_n\,\psi_{n\alpha}, \qquad \psi_\beta = \sum c_n\,\psi_{n\beta},$$

with the undetermined factors c_n then the expressions:

$$\left.\begin{aligned} d_x &= \psi_\beta\psi_\alpha^* + \psi_\alpha\psi_\beta^*,\\ d_y &= -i(\psi_\beta\psi_\alpha^* - \psi_\alpha\psi_\beta^*),\\ d_z &= (\psi_\alpha\psi_\alpha^* - \psi_\beta\psi_\beta^*)\end{aligned}\right\} \tag{6″}$$

formally play the role of volume densities for the proper moment of the electron.

We now have to demonstrate the proof that the matrices that are calculated from (6′) generally satisfy the relations (2) of Heisenberg and Jordan. If we denote any of the indices x, y, z by i and k then we can form:

$$(s_i\,s_k)_{nm} = \sum_l (s_i)_{nl}(s_k)_{lm}\,.$$

If we replace $(s_k)_{lm}$ with its value that follows from (6′) then this gives:

$$(s_i\,s_k)_{nm} = \int \left\{\left[\sum_l (s_i)_{nl}\psi_{l\alpha}^*\right]\boldsymbol{s}_k(\psi_{m\alpha}) + \left[\sum_l (s_i)_{nl}\psi_{l\beta}^*\right]\boldsymbol{s}_k(\psi_{m\beta})\right\} dq.$$

One now has $(s_i)_{nl} = (s_i)_{ln}^*$, since (as one easily confirms on the basis of (6′), moreover) the matrices $\boldsymbol{s}_i$ are Hermitian, so according to (6), one has:

$$\sum_l (s_i)_{nl}\psi_{l\alpha}^* = \sum_l (s_i)_{ln}^*\psi_{l\alpha}^* = [\boldsymbol{s}_i(\psi_{n\alpha})]^*,$$

and likewise:

$$\sum_l (s_i)_{nl}\psi^*_{l\beta} = [\boldsymbol{s}_i(\psi_{n\beta})]^*.$$

The final result is then:

$$(s_i\,s_k)_{nm} = \int\left\{\left[s_i(\psi_{n\alpha})\right]^*\boldsymbol{s}_k(\psi_{m\alpha}) + \left[s_i(\psi_{n\beta})\right]^*\boldsymbol{s}_k(\psi_{m\beta})\right\}dq\,.$$

On the basis of this relation, one easily confirms all of relations (2) by replacing the operators (3) and comparing with (6′), if one regards them as matrix relations. For example, for $i = x$, $k = z$, this gives:

$$(s_x\,s_y - s_y\,s_x)_{nm} = 2i\int(-\psi^*_{n\beta}\psi_{m\beta} + \psi^*_{n\alpha}\psi_{m\alpha}) = 2i\,(s_z)_{nm}\,,$$

according to (6′). One likewise verifies the remaining relations (2). With that, the choice of operators (3) is likewise justified.

Examples of equations of the form (4) will be given in § 4.

§ 3. Behavior of the functions ψ_α, ψ_β under rotations of the coordinate system.

In the Dirac-Jordan theory, one generally answers the question of how the functions ψ transform under a transition from a system of canonical variables (p, q) to a new system P, Q. If $\boldsymbol{S}$ is an operator that takes the operators $\boldsymbol{q}$ (multiplication by q) and $\boldsymbol{p} = \frac{h}{2\pi i}\frac{\partial}{\partial q}$ into the operators $\boldsymbol{P}$, $\boldsymbol{Q}$ that correspond to the new variables according to:

$$\boldsymbol{P} = \boldsymbol{S}\,\boldsymbol{p}\,\boldsymbol{S}^{-1}, \qquad \boldsymbol{Q} = \boldsymbol{S}\,\boldsymbol{q}\,\boldsymbol{S}^{-1} \tag{7}$$

then one obtains the eigenfunction $\psi_E(Q)$ that belongs to Q from the eigenfunction $\psi_E(q)$ that belongs to q simply by an application of the operator $\boldsymbol{S}$:

$$\psi_E(Q) = \boldsymbol{S}[\psi_E(Q)]. \tag{8}$$

The expression:

$$|\,\psi_E(Q)\,|^2\,dQ$$

then again represents the probability that the variable Q should lie between Q and $Q + dQ$ for a certain energy E and an arbitrary value for P ¹).

---

¹) The fact that we chose precisely the energy E to be a fixed parameter now represents a special case of the transformations that were considered by Dirac and Jordan. This author also investigated the connection between two different representations of the operators S more closely:

1. The differential representation, in which $\boldsymbol{S} = \boldsymbol{S}\left(\frac{h}{2\pi i}\frac{\partial}{\partial x}, x\right)$ is thought of as composed of the operators of differentiation with respect to a variable x and multiplication by x.
2. The integral representation of S, in which one sets:

In our case, we will not generally calculate with the canonical variables (s_z, φ) themselves, but with the components s_x, s_y, s_z of the proper moment, for which the commutation relations do not have the canonical form (2). We then have to answer the question of *how, starting with the given eigenfunctions* ψ_α, ψ_β, *and operators* $\boldsymbol{s}_x$, $\boldsymbol{s}_y$, $\boldsymbol{s}_z$ *relative to a certain axis-cross* (x, y, z), *one can calculate the eigenfunctions* ψ'_α, ψ'_β, *and operators* $\boldsymbol{s}_{x'}$, $\boldsymbol{s}_{y'}$, $\boldsymbol{s}_{z'}$ *relative to a new axis-cross* (x', y', z'). The squares of the absolute magnitudes of the new ψ'_α, ψ'_β then determine the probability that (for certain values of the position coordinates q of the electron) for an arbitrary value of the angle φ' around the z'-axis the impulse $\boldsymbol{s}_{z'}$ (measured in units of $\frac{1}{2}\frac{h}{2\pi}$) has the value + 1 (–1, resp.).

Now, for the operator equation (7), it is not essential that the commutation relations between $\boldsymbol{p}$ and $\boldsymbol{q}$, as well as $\boldsymbol{P}$ and $\boldsymbol{Q}$, have the canonical form. Moreover, this only comes down to the fact that the commutation relations preserve their form under the transformation; i.e., they remain correct when one simply writes the new variables in place of the old ones. Now, in our case, it is, in fact, known that relations (2) remain unchanged under orthogonal coordinate transformations, such that one also has for the primed quantities:

$$\left.\begin{array}{l} \boldsymbol{s}_{x'}\boldsymbol{s}_{y'} - \boldsymbol{s}_{y'}\boldsymbol{s}_{x'} = 2i\,\boldsymbol{s}_{z'}, \cdots, \\ \boldsymbol{s}_{x'}^2 + \boldsymbol{s}_{y'}^2 + \boldsymbol{s}_{z'}^2 = 3. \end{array}\right\} \tag{2'}$$

It will then also be permitted for us to set:

$$\boldsymbol{s}_{x'} = \boldsymbol{S}\,\boldsymbol{s}_x\,\boldsymbol{S}^{-1}, \quad \boldsymbol{s}_{y'} = \boldsymbol{S}\,\boldsymbol{s}_y\,\boldsymbol{S}^{-1}, \quad \boldsymbol{s}_{z'} = \boldsymbol{S}\,\boldsymbol{s}_z\,\boldsymbol{S}^{-1}. \tag{9}$$

The most comfortable formal representation of the operations that we will always have to apply to the eigenfunction pair (ψ_α, ψ_β) is the matrix representation that was used already in (3′) above. If the operator $\boldsymbol{S}$ takes the pair (ψ_α, ψ_β) to ($S_{11}\psi_\alpha + S_{12}\psi_\beta$, $S_{21}\psi_\alpha + S_{22}\psi_\beta$), in which S_{11}, S_{12}, S_{13}, S_{14} are ordinary numerical coefficients, then we can write $\boldsymbol{S}$ as the matrix:

$$\boldsymbol{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

In order for the relations (1a) and (1b) to also be true for the new pair ($\boldsymbol{S}\psi_\alpha$, $\boldsymbol{S}\psi_\beta$), $\boldsymbol{S}$ must satisfy the well-known orthogonality relation:

$$\boldsymbol{S}\bar{\boldsymbol{S}}^{*} = 1, \tag{10}$$

$$\boldsymbol{S}[f(q)] = \int S(x, q)\, f(x)\, dx,$$

in which $S(x, q)$ is an ordinary function.

in which the * means the transition to complex conjugate values and the prime means the exchange of rows and columns in the matrix. When this is written out, one has [1]:

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{pmatrix} \equiv \begin{pmatrix} S_{11}S_{11}^* + S_{12}S_{12}^* & S_{11}S_{21}^* + S_{12}S_{22}^* \\ S_{21}S_{11}^* + S_{22}S_{12}^* & S_{21}S_{21}^* + S_{22}S_{22}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (10')$$

On the other hand, if follows from the definition of the components of the proper moment that the operators that correspond to them must transform precisely like the coordinates, so, with the introduction of the Euler angles Θ, Φ, Ψ according to the formulas [2]:

$$\left.\begin{aligned} \boldsymbol{s}_x &= (\cos\Phi\cos\Psi - \sin\Phi\sin\Psi\cos\Theta)\boldsymbol{s}_{x'} + (-\sin\Phi\cos\Psi - \cos\Phi\sin\Psi\cos\Theta)\boldsymbol{s}_{y'} + \sin\Psi\sin\Theta\,\boldsymbol{s}_{z'}, \\ \boldsymbol{s}_y &= (\cos\Phi\sin\Psi + \sin\Phi\cos\Psi\cos\Theta)\boldsymbol{s}_{x'} + (-\sin\Phi\sin\Psi + \cos\Phi\cos\Psi\cos\Theta)\boldsymbol{s}_{y'} - \cos\Psi\sin\Theta\,\boldsymbol{s}_{z'}, \\ \boldsymbol{s}_z &= \sin\Phi\sin\Theta\,\boldsymbol{s}_{x'} + \cos\Phi\sin\Theta\,\boldsymbol{s}_{y'} + \cos\Theta\,\boldsymbol{s}_{z'}. \end{aligned}\right\} \qquad (11)$$

Our objective will now be to determine the matrix $\boldsymbol{S}$ in such a way that (9) and (11) agree. If we achieve this then our question regarding the transformation of $(\psi_\alpha, \psi_\beta)$ under rotations of the coordinate system will be answered by the equations:

$$(\psi'_\alpha, \psi'_\beta) = \boldsymbol{S}(\psi_\alpha, \psi_\beta) \qquad (12)$$

or

$$\left.\begin{aligned} \psi'_\alpha &= S_{11}\psi_\alpha + S_{12}\psi_\beta, \\ \psi'_\beta &= S_{21}\psi_\alpha + S_{22}\psi_\beta. \end{aligned}\right\} \qquad (12a)$$

In order to now bring (9) and (11) into agreement with each other, it is preferable, as in the usual theory of tops, to introduce the following notations:

$$\left.\begin{aligned} \xi &= s_x + is_y, \quad \eta = -s_x + is_y, \quad \zeta = -s_z, \\ \xi' &= s_{x'} + is_{y'}, \quad \eta' = -s_{x'} + is_{y'}, \quad \zeta' = -s_{z'}, \end{aligned}\right\} \qquad (13)$$

$$\left.\begin{aligned} \alpha &= \cos\frac{\Theta}{2}\, e^{i\frac{\Phi-\Psi}{2}}, \quad \beta = i\sin\frac{\Theta}{2}\, e^{i\frac{-\Phi+\Psi}{2}}, \\ \gamma &= i\sin\frac{\Theta}{2}\, e^{i\frac{\Phi-\Psi}{2}}, \quad \delta = \cos\frac{\Theta}{2}\, e^{i\frac{-\Phi-\Psi}{2}}. \end{aligned}\right\} \qquad (14)$$

---

[1] We recall the fact that one obtains the (*n*, *m*) element of the product of two matrices by term-wise multiplication of the n^{th} row of the first matrix by the m^{th} column of the second matrix.

[2] For what follows, cf., A. Sommerfeld and F. Klein, *Theorie des Kreisels*, I, § 2 to 4, in particular, the definition of the parameters α, β, γ, δ. P. Jordan directed my attention to their meaning in the context of our problem.

The quantities α, β, γ, δ are the Cayley-Klein rotation parameters, between which exist the relations:

$$\delta = \alpha^*, \qquad \gamma = -\beta^*, \qquad \alpha\delta - \beta\gamma = 1. \tag{14′}$$

(11) is then equivalent to [1]:

$$\xi = \boldsymbol{S}^{-1}\xi'\boldsymbol{S}, \quad \eta = \boldsymbol{S}^{-1}\eta'\boldsymbol{S}, \quad \zeta = \boldsymbol{S}^{-1}\zeta'\boldsymbol{S}\,. \tag{9′}$$

We now assert that *in order to bring* (9′) *into agreement with* (11′), *we can simply identify the matrix* **S** *with the matrix* $\begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix}$ *of conjugate values to the Cayley-Klein parameters:*

$$\boldsymbol{S} = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} \qquad \text{or} \qquad S_{11} = \alpha^*, \;\; S_{12} = \beta^*, \;\; S_{21} = \gamma^*, \;\; S_{22} = \delta^*. \tag{15}$$

This is permissible, since the relation (10) is fulfilled precisely by means of (14′):

$$\begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If we further set ξ', η', ζ' in (9′) and (11′) equal to the matrices that follow from (3′), using (13):

$$\xi' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},$$

$$\eta' = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},$$

and

$$\zeta' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

then we obtain from the agreement of both equations that:

$$\xi = \begin{pmatrix} -2\alpha\beta & 2\alpha^2 \\ -2\beta^2 & 2\alpha\beta \end{pmatrix}, \qquad \eta = \begin{pmatrix} -2\gamma\delta & 2\gamma^2 \\ -2\delta^2 & 2\gamma\delta \end{pmatrix}, \qquad \zeta = \begin{pmatrix} -\alpha\delta - \beta\gamma & 2\alpha\gamma \\ -2\beta\delta & \alpha\beta + \beta\gamma \end{pmatrix}.$$

With that, the desired proof is achieved.

---

[1] *Theorie des Kreisels*, equation (9), pp. 21.

We now still have some supplementary remarks to add. The one concerns the special case of a rotation of the coordinate system around the *z*-axis, such that $\Theta = 0$, $\beta = \gamma = 0$, and with $\Phi + \Psi = 0$, one will have $\alpha = e^{i\omega/2}$, $\delta = e^{-i\omega/2}$. In this case, one obtains:

$$\boldsymbol{s}_x = \begin{pmatrix} 0 & e^{-i\omega} \\ e^{i\omega} & 0 \end{pmatrix}, \qquad \boldsymbol{s}_y = \begin{pmatrix} 0 & -ie^{-i\omega} \\ ie^{i\omega} & 0 \end{pmatrix}, \qquad \boldsymbol{s}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3''}$$

These are, at the same time, as one easily verifies, the most general matrices (linear transformations of the ψ_α, ψ_β, resp.) that are Hermitian, satisfy the commutation relations (2), and for which $\boldsymbol{s}_z$ has its normal form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, in addition. One sees from this *that the functions* (ψ_α, ψ_β) *are still not uniquely determined by just the given of the z-direction (i.e., the arbitrariness of the phase* ω*), but only when the entire* (*x*, *y*, *z*)*-axis-cross is given.* On this basis already, it scarcely seems possible to associate the magnetic electron with directed (vectorial) eigenfunctions.

The second remark relates to the question of the most general (Hermitian) linear transformations of the (ψ_α, ψ_β) that satisfy the relations (2). It is easy to see that these most general $\boldsymbol{s}_x$, $\boldsymbol{s}_y$, $\boldsymbol{s}_z$ can always be brought into the normal form (3′) by a transformation of the form (9) [in which $\boldsymbol{S}$ fulfills the relation (10)]. Here, we would like to only outline the proof. One first shows that the most general $\boldsymbol{S}$ that satisfies (10) can always be expressed in the form (14), (15) by means of angles Θ, Φ, Ψ. In any event, one can then convert $\boldsymbol{s}_z$ into a diagonal matrix by means of a transformation (9). From the relations (2), it then already follows that $\boldsymbol{s}_z$ has the desired normal form. One must then only make the phase ω in the $\boldsymbol{s}_x$, $\boldsymbol{s}_y$ equal to zero by a suitable rotation around the *z*-axis.

In summary, we can say that the independence of all the ultimate results of a special choice of axis-cross is guaranteed, despite the distinguishing of a certain coordinate system by the choice (3) of the operators $\boldsymbol{s}_x$, $\boldsymbol{s}_y$, $\boldsymbol{s}_z$, as a result of the invariance of the quantum-mechanical equations under substitutions of the form (9) and as a result of the behavior depicted for the (ψ_α, ψ_β) under rotations of the distinguished axis-cross.

§ 4. Differential equations for the eigenfunctions of a magnetic electron in special force fields.

a) Electron at rest in a homogeneous magnetic field. Equations (3), (4) already give one the way that the differential equations for the eigenfunction pair (ψ_α, ψ_β) of the magnetic electron can be constructed for a given Hamiltonian function *H*. We first consider the case of the electron at rest in a magnetic field whose field strength might possess the components H_x, H_y, H_z. Since the electron is at rest, the eigenfunctions do not depend upon the position coordinates of the electron here. If *e* and m_0 denote the charge and mass of the electron, respectively, and:

$$\mu_0 = \frac{eh}{4\pi m_0 c}$$

is the magnitude of the Bohr magneton then the Hamiltonian function here reads:

$$H = \mu_0\,(H_x\,s_x + H_y\,s_y + H_z\,s_z),$$

if we omit the constant translational energy and once more measure s_x, … in units of $\frac{1}{2}\frac{h}{2\pi}$. If one replaces s_x, s_y, s_z with the operators (3) (while μ_0, H_x, H_y, H_z naturally remain ordinary numbers) then one obtains the system of equations for (ψ_α, ψ_β):

$$\left.\begin{aligned} \mu_0[(H_x - iH_y)\psi_\beta + H_z\psi_\alpha] = E\psi_\alpha, \\ \mu_0[(H_x + iH_y)\psi_\alpha - H_z\psi_\beta] = E\psi_\beta. \end{aligned}\right\} \tag{16}$$

We have deliberately not made the direction of the magnetic field coincide with the z-axis [that is distinguished by the choice of operators (3)] from the outset, in order to be able to explain the physical meaning of our quantities ψ_α, ψ_β and their transformation properties that were derived in the previous paragraphs by an example.

The eigenvalue E follows from (16) by means of the determinant condition:

$$\begin{vmatrix} \mu_0 H_z - E & \mu_0(H_x - iH_y) \\ \mu_0(H_x + iH_y) & -\mu_0(H_z + E) \end{vmatrix} = 0$$

or

$$-(\mu_0^2 H_z^2 - E^2) - \mu_0^2(H_x^2 + H_y^2) = 0,$$

namely:

$$E = \pm\mu_0\sqrt{H_x^2 + H_y^2 + H_z^2} = \pm\,\mu_0\,|\,H\,|,$$

which will be demanded in this case from now on. It further follows from (16), if one denotes the angle between the field direction and the z-axis by Θ and normalizes (ψ_α, ψ_β) by way of $|\psi_\alpha|^2 + |\psi_\beta|^2 = 1$, and for $E = +\,\mu_0\,|\,H\,|$ that:

$$|\psi_\alpha|^2 = \frac{\sin^2\Theta}{\sin^2\Theta + (1-\cos\Theta)^2} = \frac{\sin^2\Theta}{2(1-\cos\Theta)} = \cos^2\frac{\Theta}{2},$$

$$|\psi_\beta|^2 = \frac{(1-\cos\Theta)^2}{2(1-\cos\Theta)} = \sin^2\frac{\Theta}{2},$$

and analogously for $E = -\,\mu_0\,|\,H\,|$, one has:

$$|\psi_\alpha|^2 = \sin^2\frac{\Theta}{2}, \qquad |\psi_\beta|^2 = \cos^2\frac{\Theta}{2}.$$

This result is also in harmony with the transformation properties (12), (14), (15) of (ψ_α, ψ_β). It can be interpreted physically in, e.g., the following way: The external magnetic field originally has a direction that is given by H_x, H_y, H_z, and we let only those electrons be present that are directed parallel to the field, but none that are anti-parallel; one then suddenly rotates the field in the z-direction. One will then find that $\cos^2\frac{\Theta}{2}$ is the fraction of all electrons with moments that are directed parallel to the z-axis and $\sin^2\frac{\Theta}{2}$ is the fraction of all electrons with moments that are directed anti-parallel to the z-axis, and conversely, when only electrons that are oriented anti-parallel to the field direction are originally present.

b) A magnetic electron in a Coulomb field (hydrogen atom). If we would now like to go on to the presentation of the equations for the eigenfunction pair ψ_α, ψ_β of the magnetic electron in an atomic nucleus then we would consequently like to place ourselves at the point of view where the higher relativistic and magnetic corrections are neglected and the terms that arise from the theory of relativity and the proper moment of the electron can be regarded as perturbing functions. Analogous to the previous example, we likewise assume that a homogeneous, external, magnetic field with the components H_x, H_y, H_z is present, in order to address the theory of the anomalous Zeeman effect. We still expressly emphasize that the equations presented here are completely equivalent, mathematically and physically, to the matrix equations that were given by Heisenberg and Jordan 1). We also adopt the form of the Hamiltonian function that was given by these authors.

One first has the Hamiltonian function of the unperturbed atomic nucleus with one electron:

$$H_0 = \frac{1}{2m_0}(p_x^2 + p_y^2 + p_z^2) - \frac{Ze^2}{r}$$

(p_x, p_y, p_z = translational impulse, Z = atomic number), or, written as an operator:

$$\boldsymbol{H}_0(\psi) = -\frac{1}{2m_0}\frac{h^2}{4\pi^2}\Delta\psi - \frac{Ze^2}{r}\psi\,, \tag{17}$$

in which one sets $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, as usual. One then comes to the terms that already appear for an electron with no proper moment as a result of the action of the external magnetic field and the relativistic corrections:

$$H_1 = -\frac{1}{2m_0c^2}\left(E_0^2 + 2E_0 Ze^2\frac{1}{r} + Z^2e^4\frac{1}{r^2}\right) + \frac{e}{2m_0c}(\mathfrak{H}[\mathfrak{r}\,\mathfrak{p}])\,,$$

---

1) Zeit. f. Phys., *loc. cit.*, cf., in particular, equations (2), (3), (4) of that paper.

in which E_0 means the unperturbed eigenvalue, $\mathfrak{H}$ means the vector of external magnetic field, $\mathfrak{p}$ means the translational impulse, and $\mathfrak{r}$ is the radius vector that points from the nucleus to the electron.

When written as an operator, this gives:

$$\boldsymbol{H}_1(\psi) = -\frac{1}{2m_0c^2}\left(E_0^2 + 2E_0Ze^2\frac{1}{r} + Z^2e^4\frac{1}{r^2}\right) - i\,\mu_0\,(\mathfrak{H}\ [\mathfrak{r}\ \mathrm{grad}\ \psi]). \tag{18}$$

The operators $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$ work the same way for ψ_α and ψ_β; they do not alter the index α or β. Characteristic terms now appear for the proper moment of the electron that correspond, firstly, to the interaction terms between the proper moment and the external field that were already written out in the previous example, and secondly, to the interaction terms that follow from the theory of relativity for a moving electron with a proper moment with the Coulomb electrical field. We adopt the latter, without the new basis of Thomas 1) and Frenkel 1); in particular, as far as the factor of ½ is concerned. Both terms together give, when likewise written as an operator:

$$\boldsymbol{H}_2(\psi) = \frac{1}{4}\frac{h^2}{4\pi^2}\frac{Ze^2}{m_0^2c^2}\frac{1}{r^3}\frac{1}{i}(\boldsymbol{k}_x\,\boldsymbol{s}_x + \boldsymbol{k}_y\,\boldsymbol{s}_y + \boldsymbol{k}_z\,\boldsymbol{s}_z)(\psi) + \mu_0(H_x\,\boldsymbol{s}_x + H_y\,\boldsymbol{s}_y + H_z\,\boldsymbol{s}_z)(\psi), \tag{19}$$

in which $\boldsymbol{k}_x$, $\boldsymbol{k}_y$, $\boldsymbol{k}_z$ are written as an abbreviation for the operators that belong to the orbital impulse moment (multiplied by $2\pi i / h$) that are written:

$$\boldsymbol{k}_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad \boldsymbol{k}_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \qquad \boldsymbol{k}_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}. \tag{20}$$

If we finally replace $\boldsymbol{s}_x$, $\boldsymbol{s}_y$, $\boldsymbol{s}_z$ with the operators that are given by (3) then, according to the general prescription (4) for $\psi_\alpha(x, y, z)$ and $\psi_\beta(x, y, z)$, we obtain, in our case, the simultaneous differential equations:

$$\left.\begin{aligned}
(\boldsymbol{H}_0 + \boldsymbol{H}_1)(\psi_\alpha) + \frac{1}{4}\frac{h^2}{4\pi^2}\frac{Ze^2}{m_0^2c^2}\frac{1}{r^3}[-(i\boldsymbol{k}_x + \boldsymbol{k}_y)\psi_\beta - i\boldsymbol{k}_z\psi_\alpha] - \mu_0[(H_x - iH_y)\psi_\beta + H_z\psi_\alpha] = E\psi_\alpha,\\
(\boldsymbol{H}_0 + \boldsymbol{H}_1)(\psi_\beta) + \frac{1}{4}\frac{h^2}{4\pi^2}\frac{Ze^2}{m_0^2c^2}\frac{1}{r^3}[-(i\boldsymbol{k}_x + \boldsymbol{k}_y)\psi_\alpha + i\boldsymbol{k}_z\psi_\beta] + \mu_0[(H_x + iH_y)\psi_\alpha - H_z\psi_\beta] = E\psi_\beta,
\end{aligned}\right\} \tag{21}$$

in which $\boldsymbol{H}_0$, $\boldsymbol{H}_1$, and $\boldsymbol{k}_x$, $\boldsymbol{k}_y$, $\boldsymbol{k}_z$ are given by (17), (18), and (20). In particular, if one sets $H_x = H_y = 0$ in this then these equations go over to the ones that Darwin 2) already

---

1) L. H. Thomas, Nature **117** (1926), 514; Phil. Mag. **3** (1927), 1; J. Frenkel, Zeit. f. Phys. **37** (1926), 243.

2) C. G. Darwin, *loc. cit.*, equation (3).

presented. In contradiction to Darwin, however, we regard the commutation relations (2) [the sharpened relations (2a), resp.] as the ultimate source of these equations, but not the idea that amplitudes of the de Broglie waves are directed quantities. We further remark that the equations (21) are invariant under rotations of the coordinate system when the function pair (ψ_α, ψ_β) is transformed using the prescription of the previous paragraph. We will not need to go into the integration of the differential equations (21), because this can be accomplished using the methods of Heisenberg and Jordan without any difficulties, and it leads to nothing new beyond the results of those authors. Let it also be briefly mentioned that equations (21) can also be derived from a variational principle, in which the quantities d_x, d_y, d_z play a role. Since this does not yield any new physical insight, this will not be pursued further.

As was already mentioned in the introduction, the theory that is formulated here is to be regarded as only provisional, since one must demand of an ultimate theory that it be formulated in a relativistically-invariant way from the outset and that it also allows the higher corrections to be calculated. Now, it presents no complications to extend the angular impulse vector $\mathfrak{s}$ to a skew-symmetric tensor (six-vector) in the four-dimensional space-time world with the components s_{ik} , and to present commutation relations for it that are invariant under Lorentz transformations and which can be regarded as the natural generalization of (2) [or also of (2a)]. One then confronts another complication that already appears in the aforementioned theories of Thomas and Frenkel, which are based in classical electrodynamics. In these theories, one needs special constraint forces in the higher approximations in order to arrive at the fact that the electric dipole moment of the electron vanishes in a coordinate system in which it is instantaneously at rest. Indeed, in the successive approximations these constraint forces are proportional to likewise higher spatial differential quotients of the field strengths that act on the electron. It seems that these complications remain in quantum mechanics, and to date I have still not arrived at a relativistically-invariant formulation of the quantum mechanics of the magnetic electron on this basis that can be regarded as sufficiently natural and inevitable. One will actually be led, on the basis of the behavior of the constraint forces that was described, as well as on other grounds, to doubt whether such a formulation of the theory is even possible at all as long as one retains the idealization of the electron by an infinitely small magnetic dipole (while neglecting quadrupole and higher moments), or whether a more precise model of the electron is required for such a theory. Thus, we shall not go further into this still-unsolved problem.

§ 5. The case of many electrons. From our physical starting point of the method of eigenfunctions, the case in which many – say, N – electrons with proper moments are present in the mechanical system under consideration raises no new complications, when compared to the case of a single electron.

Here, we must inquire about the probability that in a certain stationary state of the system that is characterized by the value E of the total energy, the position coordinates of the electrons lie in a certain infinitesimal interval and the components of their proper moments in a z-direction that is chosen to be fixed have either the value + 1 or – 1, when measured in units of $\frac{1}{2}\frac{h}{2\pi}$. We denote the electrons by an index k that runs from 1 to N, the position coordinate of the k^{th} electron will be denoted briefly by q_k (for x_k, y_k, z_k), and their infinitesimal volume element by dq_k (for dx_k, dy_k, dz_k), and furthermore, we shall use the index α_k or β_k depending upon whether the component of the proper moment in the z-direction for the k^{th} electron is positive or negative. We then have the state of the system being characterized by the 2^N functions:

$$\psi_{\alpha_1\cdots\alpha_N}(q_1,\ldots,q_N)\,,\ \psi_{\beta_1,\alpha_2\cdots\alpha_N}(q_1,\ldots,q_N)\,,\ \psi_{\alpha_1,\beta_2,\alpha_3\cdots\alpha_N}(q_1,\ldots,q_N)\,,\ \psi_{\alpha_1,\alpha_2\cdots\alpha_N}(q_1,\ldots,q_N)\,,$$
$$\psi_{\beta_1,\beta_2,\alpha_3\cdots\alpha_N}(q_1,\ldots,q_N)\,,\ \ldots,\ \psi_{\alpha_1\ldots\alpha_{N-2},\beta_{N-1},\beta_N}(q_1,\ldots,q_N)\,,\ \ldots,\ \psi_{\beta_1\ldots\beta_N}(q_1,\ldots,q_N)\,.$$

One then has, e.g.:

$$\left|\psi_{\beta_1\beta_2\alpha_3\ldots\alpha_N}(q_1\ldots q_N)\right|^2 dq_1\cdots dq_N$$

for the probability that the first electron s_z equals – 1 and q is in (q_1, $q_1 + dq_1$), for the second electron s_z equals – 1 and q is in (q_2, $q_2 + dq_2$), and for the third to N^{th} electron s_z equals + 1 and q is in (q_3, $q_3 + dq_3$) [(q_N, $q_N + dq_N$), resp.]. The sequence in which the suffix α_k or β_k is written shall be irrelevant, while the variables q, like the index $k = 1, \ldots, N$, shall refer to a certain sequence of electrons. We can carry over the operators (3) directly for the components s_{kx}, s_{ky}, s_{kz} of the proper moments of the k^{th} electron when we make the convention that only the indices α_k or β_k of this k^{th} electron shall change, but those of the remaining electrons $\alpha_{k'}$ or $\beta_{k'}$ (for $k' \neq k$) shall remain unchanged. We then have, e.g.:

$$\left.\begin{aligned}
&\boldsymbol{s}_{kx}(\psi_{\alpha_1\cdots\alpha_k\cdots\beta_N}(q_1,\ldots,q_N)) = \psi_{\alpha_1\cdots\alpha_k\cdots\beta_N}\,, && \boldsymbol{s}_{kx}(\psi_{\cdots\beta_k\cdots}) = \psi_{\cdots\alpha_k\cdots}\,,\\
&\boldsymbol{s}_{ky}(\psi_{\cdots\alpha_k\cdots}) = -i\psi_{\cdots\beta_k\cdots}\,, && \boldsymbol{s}_{ky}(\psi_{\cdots\beta_k\cdots}) = i\psi_{\cdots\alpha_k\cdots}\,,\\
&\boldsymbol{s}_{kz}(\psi_{\cdots\alpha_k\cdots}) = \psi_{\cdots\alpha_k\cdots}\,, && \boldsymbol{s}_{kz}(\psi_{\cdots\beta_k\cdots}) = -\psi_{\cdots\beta_k\cdots}
\end{aligned}\right\} \tag{22}$$

If, as usual, we associate the impulse coordinates p_k with the operator $\dfrac{h}{2\pi i}\dfrac{\partial}{\partial q_k}$ then any function:

$$f(p_1,\ldots,p_N,q_1,\ldots,q_N,s_{1x},s_{1y},s_{1z},\ldots,s_{Nx},s_{Ny},s_{Nz})$$

now corresponds to an operator:

$$f\left(\frac{h}{2\pi i}\frac{\partial}{\partial q_1},\ldots,\frac{h}{2\pi i}\frac{\partial}{\partial q_1},q_1,\ldots,q_N,\boldsymbol{s}_{1x},\boldsymbol{s}_{1y},\boldsymbol{s}_{1z},\ldots,\boldsymbol{s}_{Nx},\boldsymbol{s}_{Ny},\boldsymbol{s}_{Nz}\right).$$

In particular, when the operator of the Hamiltonian function H is applied to the 2^N functions ψ, …, this yields the 2^N simultaneous differential equations:

$$H\left(\frac{h}{2\pi i}\frac{\partial}{\partial q_1},\ldots,\frac{h}{2\pi i}\frac{\partial}{\partial q_1},q_1,\ldots,q_N,\boldsymbol{s}_{1x},\boldsymbol{s}_{1y},\boldsymbol{s}_{1z},\ldots,\boldsymbol{s}_{Nx},\boldsymbol{s}_{Ny},\boldsymbol{s}_{Nz}\right)\psi_{i_1\cdots i_N} = E\psi_{i_1\cdots i_N} \tag{23}$$

with $i_k = \alpha_k$ or β_k. If the indices n or m refer to the various stationary states then one has the orthogonality relation:

$$\int \sum_{i_k=\alpha_k \text{ or } \beta_k} (\psi_{n,i_1\dots i_N}\,\psi^*_{m,i_1\dots i_N})\,dq_1\dots dq_N = \delta_{nm}\,, \qquad \delta_{nm} = \begin{cases} 1 & \text{for } n=m, \\ 0 & \text{for } n\neq m, \end{cases} \tag{24}$$

and each function f of the type written down above corresponds to the matrices:

$$f_{nm} = \int \sum_{i_k=\alpha_k \text{ or } \beta_k} \{\boldsymbol{f}(\psi_{n,i_1\dots i_N})\cdot\psi^*_{m,i_1\dots i_N}\}\,dq_1\dots dq_N\,. \tag{25}$$

Here, $\boldsymbol{f}$ means the operator that belongs to f as defined above and a sum over 2^N terms is found in the integrands of (24), as well as (25).

The Hamiltonian functions that occur in reality, like all of the functions f that occur in a matrix representation, now have, due to the equality of the electrons, the property that their value does not change when the coordinates of two electrons are exchanged with each other, and indeed, this is true for q_k as well as $\mathfrak{s}_k$; H and f can be assumed to be symmetric in the N systems of variables (q_k, s_{kx}, s_{ky}, s_{kz}). For Heisenberg and Dirac, this had the consequence that the terms subdivided into different groups that were not combined with each other, and which were characterized by the symmetry properties of the eigenfunctions under permutation of the electrons. Thus, one must essentially observe that the exchange of two electrons – say, the first and second one – implies a simultaneous exchange of the coordinate values q_1 and q_2 and the suffixes a and b that belong to the indices 1 and 2 (i.e., the values of s_{z_1} and s_{z_2}).

In particular, there is a symmetric solution. For any two indices k and j for an unchanged q and suffixes for the remaining indices, one has:

$$\left.\begin{aligned}
\psi^{\text{sym.}}\dots\alpha_k\alpha_j\dots(\dots q_k\dots q_j\dots) &= \psi^{\text{sym.}}\dots\alpha_k\alpha_j\dots(\dots q_j\dots q_k\dots),\\
\psi^{\text{sym.}}\dots\alpha_k\beta_j\dots(\dots q_k\dots q_j\dots) &= \psi^{\text{sym.}}\dots\beta_k\alpha_j\dots(\dots q_j\dots q_k\dots),\\
\psi^{\text{sym.}}\dots\beta_k\beta_j\dots(\dots q_k\dots q_j\dots) &= \psi^{\text{sym.}}\dots\beta_k\beta_j\dots(\dots q_j\dots q_k\dots).
\end{aligned}\right\} \tag{26}$$

Moreover, there is an anti-symmetric solution, for which any index pair (i.e., electron pair) k and j implies a sign change under permutation:

$$\left.\begin{aligned}
\psi^{\text{antis.}}\dots\alpha_k\alpha_j\dots(\dots q_k\dots q_j\dots) &= -\psi^{\text{antis.}}\dots\alpha_k\alpha_j\dots(\dots q_j\dots q_k\dots),\\
\psi^{\text{antis.}}\dots\alpha_k\beta_j\dots(\dots q_k\dots q_j\dots) &= -\psi^{\text{antis.}}\dots\beta_k\alpha_j\dots(\dots q_j\dots q_k\dots),\\
\psi^{\text{antis.}}\dots\beta_k\beta_j\dots(\dots q_k\dots q_j\dots) &= -\psi^{\text{antis.}}\dots\beta_k\beta_j\dots(\dots q_j\dots q_k\dots).
\end{aligned}\right\} \tag{27}$$

If follows easily from this that symmetric operators $\boldsymbol{f}$ leave invariant the symmetry character of the functions to which they are applied. Moreover, the non-combination of symmetric and anti-symmetric classes follows simply from (25).

It would be interesting to adapt the group-theoretic investigation of Wigner [1] in the case of N electrons with no proper moment to ones with a proper moment, and likewise establish how the terms that correspond to the different symmetry classes that one obtains by neglecting the proper moment are distributed over the symmetry classes of electrons with proper moments. In the case of 2 electrons, there are only symmetric and anti-symmetric classes, which are thus characterized in this case ($N = 2$), from (26), (27), by the equations:

$$\left.\begin{aligned}&\psi^{\text{sym.}}\alpha_1\alpha_2(q_1,q_2)=\psi^{\text{sym.}}\alpha_1\alpha_2(q_2,q_1),\ \psi^{\text{sym.}}\alpha_1\beta_2(q_1,q_2)=\psi^{\text{sym.}}\beta_1\alpha_2(q_2,q_1),\\&\psi^{\text{sym.}}\beta_1\alpha_2(q_1,q_2)=\psi^{\text{sym.}}\beta_1\alpha_2(q_2,q_1),\ \psi^{\text{sym.}}\beta_1\beta_2(q_1,q_2)=\psi^{\text{sym.}}\beta_1\beta_2(q_2,q_1),\end{aligned}\right\}\tag{26′}$$

$$\left.\begin{aligned}&\psi^{\text{antis.}}\alpha_1\alpha_2(q_1,q_2)=-\psi^{\text{antis.}}\alpha_1\alpha_2(q_2,q_1),\ \psi^{\text{antis.}}\alpha_1\beta_2(q_1,q_2)=-\psi^{\text{antis.}}\beta_1\alpha_2(q_2,q_1),\\&\psi^{\text{antis.}}\beta_1\alpha_2(q_1,q_2)=-\psi^{\text{antis.}}\beta_1\alpha_2(q_2,q_1),\ \psi^{\text{antis.}}\beta_1\beta_2(q_1,q_2)=-\psi^{\text{antis.}}\beta_1\beta_2(q_2,q_1).\end{aligned}\right\}\tag{27′}$$

On the contrary, in general there exists no simple relation between the function values $\psi_{\alpha_1,\beta_2}(q_1,q_2)$ and $\psi_{\alpha_1,\beta_2}(q_2,q_1)$. They then correspond to two configurations with different potential energies. Namely, in the one case, the electron with a positive s_z has the position coordinates q_1 and the one with a negative s_z has the position coordinate q_2. In the other case, conversely, the electron with a positive s_z is at the spatial point that corresponds to q_1 and the electron with a negative s_z is at the spatial point that corresponds to q_1.

The skew-symmetric solution is also the one that fulfills the "equivalence rule" in the general case of N electrons, and is the only one that occurs in nature [2]. It seems to me to be an advantage of the method of eigenfunctions that this solution can be characterized in such a simple way, and for that reason precisely, it seems to me that the formal extension of this method to electrons with proper moments is not without meaning, even if it cannot lead to any new results when compared to the Heisenberg matrix methods. Moreover, the intensities of the inter-combination lines between singlet and triplet terms, for which new results of Ornstein and Burger [3] are at hand, can be calculated quantum-mechanically by these methods in a lucid way.

---

[1] E. Wigner, Zeit. f. Phys. **40** (1927), 883.

[2] On this occasion, I would like to emphasize that the exclusive appearance of the skew-symmetric solution is required by experiments only for electrons, and indeed by considering their proper moments. In a previous paper (Zeit. f. Phys. **41** (1927), 81), the Fermi statistics were likewise implied only for the electron gas by comparing with experiment. The possibility of other types of statistics with other material gases still remains open, which was not, unfortunately, sufficiently stressed in that paper. Cf., on this, also F. Hund, Zeit. f. Phys. **42** (1927), 93.

[3] L. S. Ornstein and H. C. Burger, Zeit. f. Phys. **40** (1926), 403.