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On the tensorial relations between the mean-value densities in Dirac's theory of the electron (I)

By GÉRARD PETIAU

(Institut Henri Poincaré)

Translated by D. H. Delphenich

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1. The study of relations between the tensorial quantities in Dirac's theory of the electron was done systematically for the first time by W. Pauli ([1]) and then reprised by Kofink ([2]) in a series of papers. I presented those results, while simplifying and completing them, in the course that I taught at the Collège de France as a scholar of the Peccot Foundation in 1942. Since then, Costa de Beauregard ([3]) has summarized them in his thesis. Here, I propose to recover those results by a much simpler method and to examine the independence of the relations thus-obtained.

2. Construction and representation of the mean-value densities in Dirac's theory of the electron. – Consider the Dirac equation:

$$\left[\left(\frac{h}{2\pi i}\right)\frac{1}{c}\partial_t\,\alpha_0+\sum_{p=1}^{3}\left(-\frac{h}{2\pi i}\right)\partial_p\,\alpha_p+m_0 c\alpha_4\right]\psi=0,$$

in which α_0, α_p (p = 1, 2, 3), α_4 represent the fourth-rank matrices:

$$\alpha_0 = 1, \qquad (\alpha_p)_{ik,mp} = (\sigma_p)_{im}\,(\rho_1)_{kp}\,, \qquad (\alpha_4)_{ik,mp} = (\sigma_0)_{im}\,(\rho_3)_{kp}\,,$$

$$(i, k, m, p = 1, 2),$$

and the matrices σ_p, ρ_p are written:

$$\sigma_1 \text{ or } \rho_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 \text{ or } \rho_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \sigma_3 \text{ or } \rho_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},$$

with

$$\sigma_p\;\sigma_q = -\,i\;\sigma_r\,, \qquad \rho_p\;\rho_q = -\,i\;\rho_r\,.$$

([1]) Ann. Inst. H. Poincaré **6** (1936), pp. 109.
([2]) Ann. Phys. (Leipzig) **30** (1937), pp. 91; *ibid.* **38** (1940), pp. 421, 436, 565, 583.
([3]) *Thèse*, Paris, 1943, pp. 56.

The matrices α_p, α_4 permit one to define sixteen covariant tensorial quantities by way of their products:

$$
\begin{aligned}
j_0 &= \psi^* \alpha_0\, \psi, & j_p &= \psi^* \alpha_p\, \psi,\\
s_0 &= \psi^*\, i\alpha_1\, \alpha_2\, \alpha_3\, \psi, & s_p &= \psi^*\, i\alpha_q\, \alpha_r\, \psi,\\
\pi_p &= \psi^*\, i\alpha_p\, \alpha_4\, \psi, & \mu_p &= \psi^*\, i\alpha_q\, \alpha_r\, \alpha_4 \psi,\\
\omega_1 &= \psi^* \alpha_4\, \psi, & \omega_2 &= \psi^* \alpha_1\, \alpha_2\, \alpha_3\, \alpha_4 \psi,
\end{aligned}
$$

$$(p, q, r = 1, 2, 3).$$

The matrices in these expressions that take the form of products of the α can likewise be expressed as products of the matrices σ and ρ. We will then have:

$$\alpha_0 = \sigma_0\, \rho_0 = 1, \qquad \alpha_p = \sigma_p\, \rho_1, \qquad i\alpha_p\alpha_4 = -\,\sigma_p\, \rho_2,$$
$$i\alpha_p\alpha_q = \sigma_r, \quad i\alpha_p\alpha_q\alpha_4 = \sigma_r\, \rho_3, \quad i\alpha_p\alpha_q\alpha_r = \rho_1, \quad \alpha_p\alpha_q\alpha_r\alpha_4 = \rho_2.$$

Those results are assembled into Table I.

The corresponding densities are written:

$$\psi_{ik}^*(\sigma_{\mathrm{A}})_{i,m}(\rho_{\mathrm{B}})_{k,p}\psi_{m,p},$$

in which the matrices σ and ρ permit one to construct the densities that are given by Table II.

Table I.

	σ_0	σ_p
ρ_0 …….	1	$i\,\alpha_q\,\alpha_r$
ρ_1 …….	$i\,\alpha_p\,\alpha_q\,\alpha_r$	α_p
ρ_2 …….	$\alpha_p\,\alpha_q\,\alpha_r\,\alpha_4$	$-\,\alpha_p\,\alpha_4$
ρ_3 …….	α_4	$i\,\alpha_q\,\alpha_r\,\alpha_4$

Table II.

	σ_0	σ_p
ρ_0 …….	j_0	s_p
ρ_1 …….	s_0	j_p
ρ_2 …….	ω_2	$-\,\pi_p$
ρ_3 …….	ω_1	μ_p

3. Fundamental relations. – Consider two systems of matrices σ_{A} (A = 0, 1, 2, 3) – namely, $(\sigma_{\mathrm{A}})_{i_1 m_1}$ and $(\sigma_{\mathrm{A}})_{i_2 m_2}$ – and propose to calculate the sum:

$$S_{\mathrm{A}} = \sum_{\mathrm{A}} (\sigma_{\mathrm{A}})_{i_1 m_1}(\sigma_{\mathrm{A}})_{i_2 m_2}$$

We can write:

$$
\begin{aligned}
(\sigma_0)_{i,m} &= \delta_{i,m}, & (\sigma_3)_{i,m} &= -\,(-1)^i\, \delta_{i,m},\\
(\sigma_1)_{i,m} &= \delta_{i,m-1}, & (\sigma_2)_{i,m} &= -\,(-1)^i\, i\delta_{i,m+1} & &\pmod 2.
\end{aligned}
$$

We will then obtain:

$$S_{\mathrm{A}} = [\delta_{i_1 m_1}\delta_{i_2 m_2} + (-1)^{i_1+i_2}\,\delta_{i_1 m_1}\delta_{i_2 m_2}] + [\delta_{i_1\, m_1+1}\delta_{i_2\, m_2+1} - (-1)^{i_1+i_2}\,\delta_{i_1\, m_1+1}\delta_{i_2\, m_2+1}].$$

If $i_1 + i_2 = 1$ (mod 2) then what will remain is:

$$S_{\mathrm{A}} = 2\,\delta_{i_1\,m_1+1}\delta_{i_2\,m_2+1}\,,$$

and that will be zero unless:

$$m_1 + 1 = i_1\,,\quad m_2 + 1 = i_2\,,\quad \text{or}\quad m_1 + i_1 + i_2 = i_1\,,\quad m_2 + i_1 + i_2 = i_2\,;$$
$$\text{i.e.,}\quad m_1 = i_2\,,\quad m_2 = i_1\,.$$

One then deduces that:

$$m_1 + 1 = m_2\,,\quad m_2 + 1 = m_1\,,$$

and the sum is written:

$$S_{\mathrm{A}} = 2\,\delta_{i_1\,m_2}\delta_{i_2\,m_1}\,.$$

If $i_1 + i_2 = 0$ (mod 2) then sum can be written:

$$S_{\mathrm{A}} = 2\,\delta_{i_1\,m_1}\delta_{i_2\,m_2}\,,$$

and that will be zero unless:

$$m_1 = i_1 \quad \text{and} \quad m_2 = i_2\,,$$

or

$$m_1 = i_2 = m_2\,,\qquad m_2 = i_1 = m_1\,.$$

One will then obtain:

$$S_{\mathrm{A}} = 2\,\delta_{i_1\,m_2}\delta_{i_2\,m_1}\,,$$

and as a result, one will have:

(1) $$\boxed{\sum_{\mathrm{A}} (\sigma_{\mathrm{A}})_{i_1\,m_1} (\sigma_{\mathrm{A}})_{i_2\,m_2} = 2\delta_{i_1\,m_2}\delta_{i_2\,m_1}\,,}$$

in any case.

That relation can likewise be written between two systems of matrices ρ, thus:

(2) $$\sum_{\mathrm{A}} (\rho_{\mathrm{A}})_{k_1\,p_1} (\rho_{\mathrm{A}})_{k_2\,p_2} = 2\delta_{k_1\,p_2}\delta_{k_2\,p_1}\,.$$

Upon direct multiplying (1) and (2), we will get:

(3) $$\sum_{\mathrm{A,B}} [(\sigma_{\mathrm{A}})_{i_1\,m_1} (\rho_{\mathrm{B}})_{k_1\,p_1}][(\sigma_{\mathrm{A}})_{i_2\,m_2} (\rho_{\mathrm{B}})_{k_2\,p_2}] = 4\delta_{i_1\,m_2}\delta_{i_2\,m_1}\delta_{k_1\,p_2}\delta_{k_2\,p_1}\,.$$

Since A, B take the values 0, 1, 2, 3, the products $\sigma_{\mathrm{A}}\,\rho_{\mathrm{B}}$ will successively take on the values of the sixteen matrices α and products of the α – namely, α_{C} – in such a way that (3) can be further written:

(4) $$\sum_{\mathrm{C}} (\alpha_{\mathrm{C}})_{r_1\,s_1} (\alpha_{\mathrm{C}})_{r_2\,s_2} = 4\delta_{r_1\,s_2}\delta_{r_2\,s_1} \qquad (r, s = 1, 2, 3, 4).$$

That relation is Pauli's fundamental relation, and it is generally taken to be the basis for obtaining the relations between the tensorial quantities. However, that relation is not the simplest one that one can obtain between the sixteen matrices α_{C} .

We write the relations (1) and (2), by extension, in the form:

$$(5)\qquad \begin{cases} \sum_{\mathrm{A}} (\sigma_{\mathrm{A}})_{i_1 m_1} (\rho_0)_{k_1 p_1} (\sigma_{\mathrm{A}})_{i_2 m_2} (\rho_0)_{k_2 p_2} = 2\delta_{i_1 m_2} \delta_{i_2 m_1} \delta_{k_1 p_1} \delta_{k_2 m_2} , \\ \sum_{\mathrm{A}} (\rho_{\mathrm{A}})_{k_1 p_2} (\sigma_0)_{i_1 m_2} (\rho_{\mathrm{A}})_{k_2 p_1} (\sigma_0)_{i_2 m_1} = 2\delta_{k_1 p_1} \delta_{k_2 m_2} \delta_{i_1 m_2} \delta_{i_2 m_1} . \end{cases}$$

Upon equating those two relations, we will obtain:

$$(6)\qquad \boxed{\sum_{\mathrm{A,B}} (\sigma_{\mathrm{A}})_{i_1 m_1} (\rho_0)_{k_1 p_1} (\sigma_{\mathrm{A}})_{i_2 m_2} (\rho_0)_{k_2 p_2} = \sum_{\mathrm{A}} (\sigma_0)_{i_1 m_2} (\rho_{\mathrm{A}})_{k_1 p_2} (\sigma_0)_{i_2 m_1} (\rho_{\mathrm{A}})_{k_2 p_1} .}$$

This is the relation that must serve as the basis for obtaining the relations between the mean-value densities.

Whereas the relation (4) consists of seventeen terms, the relation (6) consists of only eight.

We will obtain some relations between the quantities $\psi^* \sigma_{\mathrm{A}}\, \rho_{\mathrm{B}} \psi$ upon multiplying the relation (6) by a matrix:

$$(\sigma_{\mathrm{C}}\, \rho_{\mathrm{D}})\ (\sigma_{\mathrm{C}'}\, \rho_{\mathrm{D}'}),$$

and the relation between the matrices will be obtained when one left-multiplies this by $\psi^*_{i_1 k_1} \psi^*_{i_2 k_2}$ and right-multiplies by $\psi_{m_1 p_1} \psi_{m_2 p_2}$.

If one remarks that:

$$\psi^*_{i_1 k_1} (\sigma_{\mathrm{A}})_{i_1 m_1} (\rho_{\mathrm{B}})_{k_1 p_1} \psi_{m_1 , p_1} = \psi^*_{i_1 k_1} (\sigma_{\mathrm{A}})_{i_1 m_2} (\rho_{\mathrm{B}})_{k_1 p_2} \psi_{m_2 , p_2}$$

then one will see that the terms that are provided by:

$$\delta_{i_1 m_1} \delta_{k_1 p_1} \delta_{i_2 m_2} \delta_{k_2 p_2}$$

and

$$\delta_{i_1 m_2} \delta_{k_1 p_2} \delta_{i_2 m_1} \delta_{k_2 p_1}$$

will cancel, and that relation (6), *when considered from the standpoint of quadratic forms, will give the same relations as:*

$$(7)\qquad \boxed{\sum_{p=1,2,3} (\sigma_p)_{i_1 m_1} (\rho_0)_{k_1 p_1} (\sigma_p)_{i_2 m_2} (\rho_0)_{k_2 p_2} = \sum_{\mathrm{A}} (\sigma_0)_{i_1 m_2} (\rho_p)_{k_1 p_2} (\sigma_0)_{i_2 m_1} (\rho_p)_{k_2 p_1}}$$

by linear combinations, and therefore:

All of the relations between quadratic forms will have at most six terms, and if two terms cancel by compensation then those relations will reduce to four terms.

4. Relations between mean-value densities. – We shall systematically study all of the relations that one can deduce from (7).

We must form all of the possible linear combinations of the relation (7) after multiplying it by the 256 matrices:

$$(\sigma_{\mathrm{C}}\,\rho_{\mathrm{D}})_{(1)}\,(\sigma_{\mathrm{C}'}\,\rho_{\mathrm{D}'})_{(2)}\,,$$

but the relation (7) is symmetric, so the matrices:

$$(\sigma_{\mathrm{C}}\,\rho_{\mathrm{D}})_{(1)}\,(\sigma_{\mathrm{C}'}\,\rho_{\mathrm{D}'})_{(2)} \quad \text{and} \quad (\sigma_{\mathrm{C}'}\,\rho_{\mathrm{D}'})_{(1)}\,(\sigma_{\mathrm{C}}\,\rho_{\mathrm{D}})_{(2)}$$

will give the same relations, and in turn, the number of combinations that one must form comes down to 136.

If we classify those matrices according to the presence of the matrices σ_0 and ρ_0 then we will get the following groups:

(1) $(\sigma_0\,\rho_0)\,(\sigma_0\,\rho_0)$

(2) $(\sigma_0\,\rho_0)\,(\sigma_0\,\rho_p)$, $(\sigma_p\,\rho_0)\,(\sigma_0\,\rho_0)$,

(3) $\begin{cases} (\sigma_0\rho_p)\,(\sigma_0\rho_p), \quad (\sigma_0\rho_p)\,(\sigma_0\rho_q), \quad (\sigma_p\rho_0)\,(\sigma_0\rho_{p'}), \quad (\sigma_p\rho_0)\,(\sigma_p\rho_0), \\ (\sigma_p\rho_0)\,(\sigma_q\rho_0), \quad (\sigma_0\rho_0)\,(\sigma_p\rho_{p'}), \end{cases}$

(4) $(\sigma_p\,\rho_{p'})\,(\sigma_0\,\rho_{p'})$, $(\sigma_p\,\rho_{p'})\,(\sigma_0\,\rho_{q'})$, $(\sigma_p\,\rho_0)\,(\sigma_p\,\rho_{p'})$, $(\sigma_p\,\rho_0)\,(\sigma_q\,\rho_{q'})$,

(5) $(\sigma_p\,\rho_{p'})\,(\sigma_p\,\rho_{p'})$, $(\sigma_p\,\rho_{p'})\,(\sigma_p\,\rho_{q'})$, $(\sigma_p\,\rho_{p'})\,(\sigma_q\,\rho_{p'})$, $(\sigma_p\,\rho_{p'})\,(\sigma_q\,\rho_{q'})$.

Not all of these matrices give distinct relations, and one confirms that all of the relations that are obtained can be obtained by means of just the matrices:

$(\sigma_0\,\rho_0)\,(\sigma_0\,\rho_0)$, $(\sigma_0\,\rho_0)\,(\sigma_0\,\rho_p)$, $(\sigma_p\,\rho_0)\,(\sigma_0\,\rho_0)$, $(\sigma_0\,\rho_p)\,(\sigma_0\,\rho_p)$, $(\sigma_0\,\rho_p)\,(\sigma_0\,\rho_q)$,
$(\sigma_p\,\rho_0)\,(\sigma_0\,\rho_{p'})$, $(\sigma_p\,\rho_0)\,(\sigma_p\,\rho_0)$, $(\sigma_p\,\rho_0)\,(\sigma_q\,\rho_0)$, $(\sigma_p\,\rho_{p'})\,(\sigma_0\,\rho_{q'})$.

1. The relation (7), when multiplied by $\psi^*_{i_1 m_1}\psi^*_{i_2 m_2}$, $\psi_{k_1 p_1}\psi_{k_2 p_2}$, will give:

$$\sum_p [\psi^*(\sigma_p\,\rho_0)\psi]^2 = [\psi^*(\sigma_0\,\rho_1)\,\psi]^2 + [\psi^*(\sigma_0\,\rho_2)\,\psi]^2 + [\psi^*(\sigma_0\,\rho_3)\,\psi]^2,$$

which can be further written:

$$\sum_p s_p^2 = s_0^2 + \omega_1^2 + \omega_2^2\,. \tag{8}$$

2. The relation (7), when multiplied by $(\sigma_0\,\rho_{p'})$, $(\sigma_0\,\rho_0)$, will give:

$$\sum_p (\sigma_p \rho_{p'})(\sigma_p \rho_0) = (\sigma_0 \rho_0)(\sigma_0 \rho_p) - i\,(\sigma_0 \rho_r)(\sigma_0 \rho_q) + i\,(\sigma_0 \rho_q)(\sigma_0 \rho_r),$$

and after multiplying by $\psi^*_{(1)}\psi^*_{(2)}$ and $\psi_{(1)}\,\psi_{(2)}$:

$$\sum_p [\psi^*(\sigma_p\,\rho_{p'})\psi][\psi^*(\sigma_p\,\rho_0)\psi] = [\psi^*(\sigma_0\,\rho_p)\;\psi][\psi^*\psi].$$

If p' takes the values 1, 2, 3 then we will get:

(9*a*) $$\sum_p j_p\, s_p = j_0\, s_0\,,$$

(9*b*) $$\sum_p \pi_p\, s_p = -\,\omega_2\, j_0\,,$$

(9*c*) $$\sum_p \mu_p\, s_p = \omega_1\, j_0\,.$$

3. If one multiplies (7) by $(\sigma_0\,\rho_{p'})\,(\sigma_0\,\rho_{p'})$ then one will get:

$$\sum_p (\sigma_p \rho_{p'})(\sigma_p \rho_{p'}) = (\sigma_0 \rho_0)(\sigma_0 \rho_0) - i\,(\sigma_0 \rho_{q'})(\sigma_0 \rho_{q'}) - i\,(\sigma_0 \rho_{r'})(\sigma_0 \rho_{r'}),$$

which will give:

$$\sum_p [\psi^*(\sigma_p\,\rho_{p'})\psi]^2 = [\psi^*(\sigma_0\,\rho_0)\;\psi]^2 - [\psi^*(\sigma_0\,\rho_{q'})\;\psi]^2 - [\psi^*(\sigma_0\,\rho_{r'})\;\psi]^2,$$

so if p' takes the values 1, 2, 3 then:

(10*a*) $$\sum_p j_p^2 = j_0^2 - \omega_1^2 - \omega_2^2\,,$$

(10*b*) $$\sum_p \pi_p^2 = j_0^2 - s_0^2 - \omega_1^2\,,$$

(10*c*) $$\sum_p \mu_p^2 = j_0^2 - s_0^2 - \omega_2^2\,.$$

4. If one multiplies (7) by $(\sigma_0\,\rho_{p'})\,(\sigma_0\,\rho_{q'})$ then one will get:

$$\sum_p [\psi^*(\sigma_p\,\rho_{p'})\psi][\psi^*(\sigma_p\,\rho_{q'})\psi] = [\psi^*(\sigma_0\,\rho_p)\;\psi]\;[\psi^*(\sigma_0\,\rho_q)\,\psi],$$

so:

(11*a*) $$\sum_p j_p\, \pi_p = -s_0\ \omega_2\ ,$$

(11*b*) $$\sum_p \pi_p\, \mu_p = -\omega_1\ \omega_2\ ,$$

(11*c*) $$\sum_p \mu_p\ j_p = \quad \omega_1\ s_0\ .$$

5. If one multiplies (7) by $(\sigma_p\,\rho_{p'})$ $(\sigma_0\,\rho_{q'})$ then one will get:

$$\begin{aligned} & [\psi^*(\sigma_p\,\rho_{p'})\ \psi]\ [\psi^*(\sigma_r\,\rho_{q'})\psi] - [\psi^*(\sigma_r\,\rho_{p'})\ \psi]\ [\psi^*(\sigma_q\,\rho_{q'})\psi] \\ = \ & [\psi^*(\sigma_p\,\rho_0)\ \psi]\ [\psi^*(\sigma_0\,\rho_{r'})\psi] - [\psi^*(\sigma_p\,\rho_{r'})\ \psi]\ [\psi^*(\sigma_0\,\rho_0)\psi], \end{aligned}$$

so one will infer that:

(12*a*) $$j_r\ \pi_q\ - j_q\ \pi_r\ = s_p\ \omega_1 - \mu_p\, j_0\ ,$$
(12*b*) $$\pi_r\ \mu_q - \pi_q\ \mu_r = s_p\, j_1\ - j_p\, j_0\ ,$$
(12*c*) $$\mu_q\, j_r\ - \mu_r\, j_q\ = s_p\ \omega_1 + \pi_p\, j_0\ .$$

6. Multiplication by $(\sigma_p\,\rho_0)$ $(\sigma_0\,\rho_q)$ will give:

$$\begin{aligned} & [\psi^*(\sigma_q\,\rho_0)\ \psi]\ [\psi^*(\sigma_r\,\rho_{q'})\psi]\ - [\psi^*(\sigma_r\,\rho_0)\ \psi]\ [\psi^*(\sigma_q\,\rho_{q'})\psi] \\ = \ & [\psi^*(\sigma_p\,\rho_{r'})\ \psi]\ [\psi^*(\sigma_0\,\rho_{q'})\psi] - [\psi^*(\sigma_p\,\rho_{q'})\ \psi]\ [\psi^*(\sigma_0\,\rho_{r'})\psi], \end{aligned}$$

from which, we will infer the relations:

(13*a*) $$s_q\, j_r\ - s_r\, j_q\ \ = \ \mu_p\ \omega_2 + \pi_p\ \omega_1\ ,$$
(13*b*) $$s_r\ \pi_q - s_q\ \pi_r\ \ = \ \ j_p\ \omega_1 - \mu_p\ s_0\ ,$$
(13*c*) $$s_q\ \mu_r\ - s_r\ \mu_q\ = -\ \pi_p\ s_0 - j_p\ \omega_1\ .$$

7. Multiplication by $(\sigma_p\,\rho_0)$ $(\sigma_p\,\rho_0)$ will give:

$$[\psi^*(\sigma_p\,\rho_0)\ \psi][\psi^*(\sigma_0\,\rho_0)\psi] = \sum_r [\psi^*(\sigma_0\,\rho_r)\psi][\psi^*(\sigma_p\,\rho_r)\psi]\ ,$$

so:

(14) $$s_p\, j_0\ - s_0\, j_p\ = \omega_1\ \mu_p - \omega_2\ \pi_p\ .$$

8. Multiplication by $(\sigma_p\,\rho_0)$ $(\sigma_p\,\rho_0)$ will give:

$$[\psi^*\psi]^2 - [\psi^*(\sigma_r\,\rho_0)\ \psi]^2 - [\psi^*(\sigma_q\,\rho_0)\ \psi]^2 = \sum_p [\psi^*(\sigma_p\,\rho_{r'})\psi]^2\ ,$$

so:

(15) $$j_0^2 - s_r^2 - s_q^2 = \ j_p^2 + \pi_p^2 + \mu_p^2\ .$$

9. Multiplication by $(\sigma_0\,\rho_0)$ $(\sigma_q\,\rho_0)$ will give:

$$[\psi^*(\sigma_q\rho_0)\ \psi][\psi^*(\sigma_p\rho_0)\ \psi] = \sum_{r'}[\psi^*(\sigma_p\,\rho_{r'})\psi][\psi^*(\sigma_q\,\rho_{r'})\psi],$$

so

(16) $$s_p\, s_q\ - j_p\, j_q\ = \pi_p\, \pi_q + \mu_p\, \mu_q\ .$$

3. Tensorial forms of the relations between the mean-value densities. – When the preceding relations are established in a particular reference system, they will exhibit the tensorial and spatial components of the tensorial quantities.

In order to reestablish the relativistic symmetry and to simply the representation of those relations, we set:

$$g_{00} = 1, \qquad g_{pp} = -1, \qquad g_{pq} = g_{p0} = 0.$$

We can then introduce the contravariant components, and we will get, in turn:

$$j^{\,0} = j_0\,, \qquad j^{\,p} = -j_p\,,$$
$$s_0 = s_{pqr} = -s^{pqr} = \varepsilon_{pqr0}\, s^0 = -\varepsilon_{0pqr}\, s^0 = -s^0,$$
$$s_p = s_{0qr} = \ \ s^{0qr} = \varepsilon_{0qrp}\, s^p = \ \ \varepsilon_{0pqr}\, s^p = \ \ s^p.$$

Since $\varepsilon_{\mu\nu\rho\sigma}$ is the indicator of duality and is antisymmetric in μ, ν, ρ, σ, and equal to ±1 according to whether the permutation $\mu\nu\rho\sigma$ is deduced from $0pqr$ by an even or odd number of transpositions, respectively:

$$\pi_p = f_{0p} = -f^{\,0p}, \qquad \mu_p = f_{qr} = f^{\,qr},$$
$$\omega_1 = \omega^1, \qquad \omega_2 = \omega_{0pqr} = -\omega^{0pqr} = |\,\omega_2\,|\,\varepsilon_{0pqr}\,.$$

We define the dual of $f^{\,\mu\nu}$ by the relations:

$$f^*_{\mu\nu} = \tfrac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\, f^{\,\rho\sigma}, \qquad f^{\,\mu\nu *} = \tfrac{1}{2}\,\varepsilon^{\,\mu\nu\rho\sigma}\, f_{\rho\sigma}\,,$$

so

$$f^*_{pq} = \tfrac{1}{2}\,\varepsilon_{pq0r}\, f^{\,0r} = f^{\,0r} = -f_{\,0r}\,,$$
$$f^*_{0p} = \tfrac{1}{2}\,\varepsilon_{0pqr}\, f^{\,qr} = f^{\,qr} = \ \ f_{\,qr}\,,$$
$$f^{\,pq*} = \tfrac{1}{2}\,\varepsilon^{\,pq0r}\, f_{0r} = \tfrac{1}{2}\,\varepsilon^{\,0pqr}\, f_{0r} = -f_{\,0r}\,,$$
$$f^{\,0p*} = \tfrac{1}{2}\,\varepsilon^{\,0pqr}\, f_{qr} = -f_{\,qr}\,.$$

Relations (8) and (16) can then be written:

(17) $$s_\mu\, s^\mu \ = \ \omega_1^2 + |\,\omega_2\,|^2\,,$$

(18) $$j_\mu\, s^\mu \ = 0,$$

(19) $$f_{\mu\nu}\, s^\nu = -\,\omega_2\, j_\mu \quad \text{or} \quad f^{\,\mu\nu}\, s_\nu \ = \ \ \omega_2\, j^{\,\mu}\,,$$

(20) $$f_{\mu\nu}\, j^\nu = \ \ \omega_2\, s_\mu \quad \text{or} \quad f^{\,\mu\nu}\, j_\nu \ = -\,\omega_2\, s^{\,\mu}\,,$$

(21) $$f^*_{\mu\nu}\, s^\nu = \ \ \omega_1\, j_\mu \quad \text{or} \quad f^{\,\mu\nu*}\, s_\nu = -\,\omega_1\, j^{\,\mu}\,,$$

(22) $f^*_{\mu\nu} j^\nu = -\omega_1 s_\mu$ or $f^{\mu\nu *} j_\nu = \omega_1 s^\mu$,

(23) $j_\mu j^\mu = \omega_1^2 + |\omega_2|^2$,

(24) $f_{\mu\nu} f^{\mu\nu} = \frac{1}{2}[\omega_1^2 - |\omega_2|^2]$,

(25) $f^*_{\mu\nu} f^{\nu\mu} = -4\,\omega_1\,\omega_2$ or $f^*_{\mu\rho} f^{\rho\nu} = -\omega_1\,\omega_2\, g_\mu{}^\nu$,

(26) $s_\mu j_\nu - s_\nu j_\mu = f_{\mu\nu}\,\omega_2 - f^*_{\mu\nu}\,\omega_1$,

(27) $s_\mu s^\nu + j_\mu j^\nu = f_{\mu\rho} f^{\rho\nu *} + \omega_1^2\, g_\mu{}^\nu$.

If we write the relation (26) in the form:

$$s_{\mu\nu\rho}\, j^\rho = \omega_1\, f_{\mu\nu} + \tfrac{1}{2}\,\omega_{\mu\nu\rho\sigma} f^{\rho\sigma}$$

then we can infer that:

$$\tfrac{1}{2}\,\omega_{\mu\nu\rho\sigma}\, s^{\rho\sigma\alpha} j_\alpha = \tfrac{1}{2}\,\omega_1\,\omega_{\mu\nu\rho\sigma} f^{\rho\sigma} + \tfrac{1}{4}\,\omega_{\mu\nu\rho\sigma}\,\omega^{\rho\sigma\alpha\beta} f_{\alpha\beta},$$
$$\omega_1\, s_{\mu\nu\rho}\, j^\rho = \tfrac{1}{2}\,\omega_1\,\omega_{\mu\nu\rho\sigma} f^{\rho\sigma} + (\omega_1)^2 f_{\mu\nu};$$

now:

$$\tfrac{1}{4}\,\omega_{\mu\nu\rho\sigma}\,\omega^{\rho\sigma\alpha\beta} f_{\alpha\beta} = -f_{\mu\nu}\,(\omega_1)^2,$$

so:

(28) $$f_{\mu\nu} = \frac{\omega_1\, s_{\mu\nu\rho}\, j^\rho - \frac{1}{2}\,\omega_{\mu\nu\rho\sigma} s^{\rho\sigma\alpha} j_\alpha}{\omega_1^2 + \omega_2^2},$$

and conversely, that relation will permit one to recover (26).

6. Independence of the relations. – We shall now show that the eleven relations from (17) to (28) are not independent and can be deduced from the four fundamental relations:

(I) $s_\mu s^\mu = \omega_1^2 + \omega_2^2$,

(II) $j_\mu j^\mu = \omega_1^2 + \omega_2^2$,

(III) $s_\mu j^\mu = 0$,

(IV) $$f_{\mu\nu} = \frac{\omega_1\, s_{\mu\nu\rho}\, j^\rho - \frac{1}{2}\,\omega_{\mu\nu\rho\sigma} s^{\rho\sigma\lambda} j_\lambda}{\omega_1^2 + \omega_2^2}.$$

Indeed, the relation (IV) gives us:

$$(\omega_1^2 + \omega_2^2)\; f_{\mu\nu} j^\nu = \omega_1\, s_{\mu\nu\rho}\, j^\rho j^\nu - \tfrac{1}{2}\,\omega_{\mu\nu\rho\sigma}\, s^{\rho\sigma\lambda} j_\lambda\, j^\nu;$$

now $s_{\mu\nu\rho} j^\rho j^\nu = 0$, by antisymmetry, and:

$$-\tfrac{1}{2}\,\omega_{\mu\nu\rho\sigma}\, s^{\rho\sigma\lambda} j_\lambda\, j^\nu = -\tfrac{1}{2}\,\omega_2\,\varepsilon_{\mu\nu\rho\sigma}\,\varepsilon^{\rho\sigma\lambda\tau} s_\tau\, j_\lambda\, j^\nu$$

$$= \omega_2 \, s_\mu \, j_\nu \, j^\nu = \omega_2 \, s_\mu \, (\omega_1^2 + \omega_2^2) \, ,$$

so:

(20) $$f_{\mu\nu} \, j^\nu = \omega_2 \, s_\mu \, .$$

Similarly:

$$(\omega_1^2 + \omega_2^2) \; f_{\mu\nu} \, j^\nu = \omega_1 \, s_{\mu\nu\rho} \, j^\rho \, s^\nu - \tfrac{1}{2} \, \omega_{\mu\nu\rho\sigma} \, s^{\rho\sigma\lambda} \, s^\nu \, j_\lambda \, .$$

Now:

$$\omega_1 \, s_{\mu\nu\rho} \, j^\rho \, s^\nu = \omega_1 \, \varepsilon_{\mu\nu\rho\sigma} \, s^{\rho\sigma\lambda} \, s^\nu \, j_\lambda = 0,$$

by antisymmetry:

$$\tfrac{1}{2} \, \omega_{\mu\nu\rho\sigma} \, s^{\rho\sigma\lambda} \, j_\lambda = \tfrac{1}{2} \, \omega_2 \, \varepsilon_{\mu\nu\rho\sigma} \, \varepsilon^{\rho\sigma\lambda\tau} \, s_\tau \, s^\nu \, j_\lambda \, ,$$

so:

(19) $$f_{\mu\nu} \, s^\nu = - \, \omega_2 \, j_\mu \, .$$

Upon starting with (IV) in the form:

$$s_\mu \, j_\nu - s_\nu \, j_\mu = \omega_2 \, f_{\mu\nu} - \omega_1 \, f^*_{\mu\nu} \, ,$$

we will get:

$$s_\mu \, j_\nu \, s^\nu - s_\nu \, s^\nu \, j_\mu = \omega_2 \, f_{\mu\nu} \, s^\nu - \omega_1 \, f^*_{\mu\nu} \, s^\nu,$$

so:

(21) $$f^*_{\mu\nu} \, s^\nu = \omega_1 \, j_\mu \, ,$$

and similarly:

(22) $$f^*_{\mu\nu} \, j^\nu = - \, \omega_1 \, s^\mu \, .$$

The relation (IV) will also give us:

$$\begin{aligned}(\omega_1^2 - \omega_2^2) \; f^*_{\mu\nu} \, f^{\mu\nu} &= \omega_1 \, f^*_{\mu\nu} \, s^{\mu\nu\rho} \, j_\rho - \tfrac{1}{2} \, f^*_{\mu\nu} \, \omega^{\mu\nu\rho\sigma} \, s_{\rho\sigma\lambda} \, j^\alpha \\ &= \omega_1 \, f^*_{\mu\nu} \, \varepsilon^{\mu\nu\rho\lambda} \, s_\lambda \, j_\rho - \tfrac{1}{2} \, f^*_{\mu\nu} \, \varepsilon^{\mu\nu\rho\sigma} \, \varepsilon_{\rho\sigma\alpha\beta} \, s^\beta \, j^\alpha \\ &= 2 \, \omega_1 \, f^{\rho\lambda} \, s_\lambda \, j_\rho - f^{\rho\lambda} \varepsilon_{\rho\sigma\alpha\beta} \, s^\beta \, j^\alpha \\ &= 2 \, \omega_1 \, \omega_2 \, j^\rho \, j_\rho + 2 \omega_2 \, f^*_{\mu\nu} \, s^\beta \, j^\alpha = 4 \omega_1 \, \omega_2 \, (\omega_1^2 + \omega_2^2) \, ,\end{aligned}$$

so:

(24) $$f^*_{\mu\nu} \, f^{\mu\nu} = - \, \omega_1 \, \omega_2 \, g_\mu{}^\nu .$$

The relation (4), in the form (26), will likewise give us:

$$f^{\rho\mu} \, s_\mu \, j_\nu - f^{\rho\mu} \, s_\nu \, j_\mu = f^{\rho\mu} \, f_{\mu\nu} \, \omega_2 - f^{\rho\mu} \; f^*_{\mu\nu} \, \omega_1 \, ,$$

or

$$\omega_2 \, (j^\rho \, j_\nu + s^\rho \, s_\nu) = \omega_2 \, f^{\rho\mu} \, f_{\mu\nu} + \omega_1^2 \, \omega_2 \, g_\mu{}^\nu,$$

and we will thus recover the relation:

(27) $$j^{\mu} j_{\nu} + s^{\mu} s_{\nu} = f^{\mu\rho} f_{\rho\nu} + \omega_1^2 g_{\mu}{}^{\nu},$$

and, by contraction:

$$2(\omega_1^2 + \omega_2^2) = f^{\mu\rho} f_{\rho\mu} + 4\omega_1^2 ,$$

or further:

(24) $$f^{\mu\rho} f_{\rho\mu} = 2(\omega_1^2 - \omega_2^2) .$$

All of the relations between mean-value densities can then be deduced from the four fundamental relations (I), (II), (III), *and* (IV).

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