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On the tensorial relations between the mean-value densities in Dirac's theory of the electron (II). Differential relations.

By GÉRARD PETIAU

Translated by D. H. Delphenich

Dirac's theory of the electron represents that corpuscle by wave functions ψ_i (i = 1, 2, 3, 4) that are solutions to the equation:

$$\left[\left(\frac{h}{2\pi i}\right)\frac{1}{c}\frac{\partial}{\partial t}+\sum_{p=1}^{3}\left(-\frac{h}{2\pi i}\right)\frac{\partial}{\partial x_p}\alpha_p+m_0 c\,\alpha_4\right]\psi_i = 0, \tag{1}$$

in which α_1, α_2, α_3, α_4 are matrices that are linked by the relations:

$$\alpha_\mu\,\alpha_\nu + \alpha_\nu\,\alpha_\mu = 2\delta_{\mu\nu} \qquad (\mu,\ \nu = 1, 2, 3, 4). \tag{2}$$

The products of those four matrices permit one to construct a complete system of sixteen matrices $\alpha_{\rm A}$ that we write:

$$\left\{\begin{array}{c} \alpha_0 = 1, \quad \alpha_\mu, \quad i\alpha_{\mu\nu} = i[\alpha_\mu\alpha_\nu - \alpha_\nu\alpha_\mu], \\ i\alpha_{\mu\nu\rho} = i(\alpha_\mu\alpha_\nu\alpha_\rho - \alpha_\mu\delta_{\nu\rho} + \alpha_\nu\delta_{\rho\mu} - \alpha_\rho\delta_{\mu\nu}), \\ \alpha_1\alpha_2\alpha_3\alpha_4. \end{array}\right. \qquad (\mu,\ \nu,\ \rho = 1, 2, 3, 4). \tag{3}$$

That system is irreducible, and as a result, those matrices will have rank four.

The matrices $\alpha_{\rm A}$ can be considered to be operators that act upon the ψ_i, and as such, they will represent quantities that are characteristic of the theory of the electron whose mean-value densities are written:

$$\psi^*\alpha_{\rm A}\,\psi. \tag{4}$$

Similarly, one uses the operators:

$$\frac{h}{2\pi i}\left[\frac{\overleftarrow{\partial}}{\partial x^\mu}\alpha_{\rm A} - \alpha_{\rm A}\frac{\overrightarrow{\partial}}{\partial x^\mu}\right] \qquad \left(\frac{\partial}{\partial x^4} = \frac{1}{c}\frac{\partial}{\partial t}\right)$$

to define the differential mean-value densities:

(5) $$T_{\mu,\,\mathrm{A}} = \left(\frac{h}{2\pi i}\right)\left[\left(\frac{\partial}{\partial x^\mu}\psi^*\right)\alpha_{\mathrm{A}}\psi - \psi^*\alpha_{\mathrm{A}}\left(\frac{\partial}{\partial x^\mu}\psi\right)\right].$$

One can make the space-time (t, x^p) form of equation (1) correspond to an equation that is written in world coordinates ($x^4 = ict$, x^p) by introducing the matrices:

$$\gamma_4 = \alpha_4\,, \qquad \gamma_p = i\,\alpha_4\,\alpha_p\,.$$

Equation (1) will then be replaced with the equation:

(6) $$\left[\left(-\frac{h}{2\pi i}\right)\frac{\partial}{\partial x^\mu}\gamma^\mu + im_0c\right]\psi_i = 0 \qquad (\mu = 1, 2, 3, 4),$$

and the mean-value densities that are analogous to (4) and (5) will be defined by:

(7) $$\psi^+\,\gamma_{\mathrm{A}}\,\psi,$$

(8) $$\frac{h}{2\pi i}\left[\left(\frac{\partial}{\partial x^\mu}\psi^+\right)\gamma_{\mathrm{A}}\psi - \psi^+\gamma_{\mathrm{A}}\left(\frac{\partial}{\partial x^\mu}\psi\right)\right],$$

with the following expressions for γ_{A}:

(8′) $$\left\{\begin{array}{l}\gamma_0 = 1, \quad \gamma_\mu, \quad i\gamma_{\mu\nu} = i\,(\gamma_\mu\gamma_\nu - \delta_{\mu\nu}),\\ i\gamma_{\mu\nu\rho} = i\,(\gamma_\mu\gamma_\nu\gamma_\rho - \gamma_\mu\delta_{\nu\rho} + \gamma_\nu\delta_{\mu\rho} - \gamma_\rho\delta_{\mu\nu}),\\ \gamma_{1234} = \gamma_1\gamma_2\gamma_3\gamma_4,\end{array}\right.$$

and when one sets:

$$\psi^+ = \psi^*\,i\,\alpha_4\,.$$

The densities (4) or (7) are not independent and are coupled by some relations that were the subject of numerous studies. Pauli ([1]) was the first, and he systematically deduced them from an identity that links two systems of γ_{A} matrices. Later, those relations, as well as the ones that couple the $T_{\mu,\,\mathrm{A}}$, were the object of numerous papers that utilized Pauli's method. In particular, Kofink ([2]) established a very large number of relations of various types between those quantities. However, the obscurity of his symbolism, as well as the absence of tensorial notation, made them very difficult to understand. I personally studied relations between densities of the form (4) or (7) in the course that I taught at the Collège de France as a scholar of the Peccot Foundation in 1942, while starting with Pauli's method. In a paper ([3]) that I wrote for the Seminaire de Théories Physique at l'Institut Henri Poincaré in November 1944, I once more addressed that question by starting with a method that is simpler than that of Pauli, and which involved generating the Dirac matrices by starting with the spin matrices. I could then

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([1]) Ann. Inst. H. Poincaré **6** (1936), 109.
([2]) Ann. Phys. (Leipzig) **38** (1940), 421, 436, 565, 583.
([3]) J. de Math. **25** (1946), 335.

establish all of the relations between densities of the form (4) in a systematic fashion by putting them into a general tensorial form that permitted me to show that the full set of those relations can be deduced from four of them. Costa de Beauregard likewise examined that problem in his thesis ([1]) by studying the relations that couple densities of the form (7) and (8) by the Pauli method, but he did not establish all of the relations between $T_{\mu, \mathrm{A}}$, and in the absence of a well-adapted tensorial notation, the general forms did not appear clearly.

We shall take up that question again here, and obtain all of the relations between quantities of the form (7) and (8) in a general tensorial form that we can deduce from a minimum number of them.

We start with a preliminary remark. If one considers the sixteen matrices α_{A} that were pointed out in (3) then one will confirm that the matrix $\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4$ anti-commutes with the α_1, α_2, α_3, α_4, and as a result, that system can be likewise considered to be obtained from five fundamental matrices α_1, α_2, α_3, α_4, α_5, upon setting:

$$\alpha_5 = \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ .$$

If one initially introduces those five matrices, coupled by the relation (2), then one will get 32 matrices by combining them, but that system will not be irreducible, and will decompose into two groups of each of the sixteen matrices that are obtained by setting either:

$$\alpha_5 = +\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4$$

or

$$\alpha_5 = -\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ .$$

As a result, we can attach those five matrices to a wave equation in a five-dimensional space that has the form:

$$(9) \qquad \left\{ \begin{array}{l} \left[\dfrac{h}{2\pi i}\dfrac{1}{c}\dfrac{\partial}{\partial t} + \left(-\dfrac{h}{2\pi i}\right)\displaystyle\sum_{p=1}^{3}\dfrac{\partial}{\partial x^p}\alpha_p + \left(-\dfrac{h}{2\pi i}\right)\dfrac{\partial}{\partial x^5}\alpha_5 + m_0 c\,\alpha_4\right]\psi_i = 0, \\ \text{with} \quad \left(-\dfrac{h}{2\pi i}\right)\dfrac{\partial}{\partial x^5}\psi = 0. \end{array} \right.$$

The sixteen quantities $\psi^* \gamma_{\mathrm{A}}\ \psi$ must then be interpreted in either a four-dimensional space or a five-dimensional one, and the relations that couple them and are written in tensorial form must be written with indices that vary over either 1, 2, 3, 4 or 1, 2, 3, 4, 5.

In the first case, with the metric:

$$g_{44} = +\ 1, \qquad g_{pp} = -\ 1, \qquad g_{4p} = g_{p4} = 0,$$

we set:

([1]) Thesis, Paris, 1943.

(10)
$$\begin{cases} j_4 = \psi^* \alpha_0 \psi, \quad j_p = \psi^* \alpha_p \psi, \quad \omega_1 = \psi^* \alpha_4 \psi, \\ f_{4p} = \pi_p = \psi^* i\alpha_{p4} \psi, \qquad s_p = s_{4qr} = \psi^* i\alpha_{qr} \psi, \\ s_4 = s_{pqr} = \psi^* i\alpha_{pqr} \psi, \qquad f_{pq} = \mu_r = \psi^* i\alpha_{pq4} \psi, \\ \omega_2 = \psi^* \alpha_1 \alpha_2 \alpha_3 \alpha_4 \psi. \end{cases}$$

The tensor $f_{\mu\nu}$, with components f_{4p}, f_{pq}, corresponds to the dual tensor:

$$f^*_{\rho\sigma} = \tfrac{1}{2}\, \varepsilon_{\rho\sigma\mu\nu}\, f^{\mu\nu},$$

upon setting:

$$\varepsilon_{\mu\nu\rho\sigma} = \pm 1,$$

according to whether the permutation ($\mu\nu\rho\sigma$) is deduced from (1 2 3 4) by an even or odd number of inversions.

In the second case, we introduce the metric:

$$g_{44} = +1, \qquad g_{pp} = g_{55} = -1, \qquad g_{p4} = g_{p5} = g_{45} = 0,$$

and set:

(11)
$$\begin{cases} j_4 = \psi^* \alpha_0 \psi, \qquad j_p = \psi^* \alpha_p \psi, \qquad j_5 = \psi^* \alpha_5 \psi = \psi^* \alpha_1 \alpha_2 \alpha_3 \alpha_4 \psi, \\ f_{4p} = \psi^* i\alpha_{p4} \psi, \quad f_{pq} = \psi^* i\alpha_{pq4} \psi, \\ f_{p5} = \psi^* i\alpha_{qr} \psi, \quad f_{45} = \psi^* i\alpha_{pqr} \psi, \\ \omega_1 = \psi^* \alpha_4 \psi. \end{cases}$$

We further associate the tensor $f_{\alpha\beta}$, with components f_{4p}, f_{pq}, f_{p5}, f_{45}, with a dual tensor, which now has three indices, and which we write:

$$f_{\alpha\beta\gamma} = \tfrac{1}{2}\, \varepsilon_{\alpha\beta\gamma\delta\varepsilon}\, f^{\delta\varepsilon}, \qquad \text{with} \qquad \varepsilon_{\alpha\beta\gamma\delta\varepsilon} = \pm 1.$$

With the first representation, we have established [cf., the work that was cited on pp. 2, note (3)] the following relations between the densities:

(I)
$$\begin{cases} j_\mu j^\mu = s_\mu s^\mu = \omega_1^2 + \omega_2^2, \qquad j_\mu s^\mu = 0, \\ f_{\mu\nu} s^\mu = -\omega_2 j^\mu, \qquad f_{\mu\nu} j^\mu = \omega_2 s_\mu, \\ f^*_{\mu\nu} j^\nu = -\omega_1 s_\mu, \qquad f^*_{\mu\nu} s^\nu = \omega_1 j^\mu, \\ f_{\mu\nu} f^{\mu\nu} = 2(\omega_1^2 - \omega_2^2), \qquad f^*_{\mu\rho} f^{\rho\nu} = -\omega_1 \omega_2 g^\nu_\mu, \\ s_\mu j_\nu - s_\nu j_\mu = f_{\mu\nu} \omega_2 - f^*_{\mu\nu} \omega_1, \\ s_\mu s^\nu + j_\mu j^\nu = f_{\mu\rho} f^{\rho\nu} + \omega_1^2 g^\nu_\mu. \end{cases}$$

Those relations are deduced from four fundamental relations that we write:

$$\text{(II)}\qquad \begin{cases} j_\mu j^\mu = s_\mu s^\mu = \omega_1^2 + \omega_2^2, \\ j_\mu s^\mu = 0, \\ f_{\mu\nu} \;\; = \dfrac{\omega_1 s_{\mu\nu\rho} j^\rho - \frac{1}{2}\omega_{\mu\nu\rho\sigma} s^{\rho\sigma\lambda} j_\lambda}{\omega_1^2 + \omega_2^2}. \end{cases}$$

With the second representation (α, β, γ … = 1, 2, 3, 4, 5), those relations can be assembled into the formulas:

$$\text{(III)}\qquad \begin{cases} (1) & j_\alpha j^\alpha = \omega_1^2, \\ (2) & f^{\alpha\beta} j_\beta = 0, \\ (3) & f_{\alpha\gamma} f^{\beta\gamma} = \omega_1^2 g_\alpha{}^\beta - j_\alpha j^\beta, \\ (4) & f_{\alpha\beta\gamma} f^{\gamma\delta} = \omega_1 (g_\alpha^\delta j_\beta - g_\beta^\delta j_\alpha), \\ (5) & f_{\alpha\beta\gamma} j^\gamma = \omega_1 f_{\alpha\beta}, \\ (6) & f_{\alpha\beta} j_\gamma + f_{\beta\gamma} j_\alpha + f_{\gamma\alpha} j_\beta = \omega_1 f_{\alpha\beta\gamma}, \end{cases}$$

which are all deduced from (3) and (5), as we easily confirm. We then have then reduced the independent relations to the two fundamental relations that characterize the magnitudes and relative positions of the tensors ω_1, j_α, $f_{\alpha\beta}$:

$$\text{(IV)}\qquad \begin{cases} f_{\alpha\beta\gamma} j^\gamma \;= \omega_1 f_{\alpha\beta}, \\ f_{\alpha\beta\gamma} f^{\gamma\delta} = \omega_1 (g_\alpha^\delta j_\beta - g_\beta^\delta j_\alpha), \end{cases}$$

We shall now examine the relations that couple the quantities of the form $T_{\mu,\,\mathrm{A}}$. In order to facilitate their analysis, we employ the quantities that are defined by means of the matrices γ_{A} .

The metric is then defined by:

$$g_{\alpha\alpha} = +1, \qquad g_{\alpha\beta} = 0 \qquad\qquad (\alpha, \beta = 1, 2, 3, 4, 5),$$

so we set:

$$\text{(12)}\qquad \begin{cases} \omega_1 = \psi^+ \psi, \quad j_\alpha = \psi^+ \gamma_\alpha \psi, \\ \qquad f_{\alpha\beta} = \psi^+ i \gamma_{\alpha\beta} \psi, \end{cases}$$

but ω_1, j_α, $f_{\alpha\beta}$ differ from the densities that were previously denoted by those symbols by factors of $\pm i$, ± 1. Nevertheless, one easily sees that the relations (III) and (IV) will still be valid for those densities.

The relations that we seek between the quantities:

$$T_{\mu,\,\mathrm{A}} = \frac{h}{2\pi i}\left\{\left(\frac{\partial}{\partial x^\mu}\psi^+\right)\gamma_{\mathrm{A}}\psi - \psi^+\gamma_{\mathrm{A}}\left(\frac{\partial}{\partial x^\mu}\psi\right)\right\}.$$

will be deduced from the Pauli relation that couples two systems of matrices γ_{A} , namely:

(13) $$\sum_{\mathrm{A}} (\gamma_{\mathrm{A}})_{i_1 m_1} (\gamma_{\mathrm{A}})_{i_2 m_2} = 4\delta_{i_1 m_2} \delta_{i_2 m_1} .$$

By combining them, we will infer some matrix relations that are written:

(14) $$\sum_{\mathrm{B,C}} (\gamma_{\mathrm{B}})_{i_1 m_1} (\gamma_{\mathrm{C}})_{i_2 m_2} = \sum_{\mathrm{D,E}} (\gamma_{\mathrm{D}})_{i_1 m_2} (\gamma_{\mathrm{E}})_{i_2 m_1} .$$

If we multiply this on the right by ψ_{m_1}, ψ_{m_2} and on the left by $\psi_{i_1}^{+}$, $\psi_{i_2}^{+}$ then those relations will give us the system (III).

We set:

$$\psi_\mu = \frac{\partial}{\partial x^\mu}\,\psi, \quad \psi_\mu^{\,+} = \frac{\partial}{\partial x^\mu}\,\psi^{+},$$

and multiply (14) on the left and then the right by:

$$\psi_{\mu,i_1}^{+}\psi_{i_2}^{+} \text{ and } \psi_{m_1}\psi_{m_2}, \quad \psi_{i_1}^{+}\psi_{i_2}^{+} \text{ and } \psi_{\mu,m_1}\psi_{m_2},$$
$$\psi_{i_1}^{+}\psi_{\mu,i_2}^{+} \text{ and } \psi_{m_1}\psi_{m_2}, \quad \psi_{m_1}^{+}\psi_{m_2}^{+} \text{ and } \psi_{m_1}\psi_{\mu,m_2}, \text{ resp.}$$

On the one hand, we will deduce from this that:

(15) $$\begin{cases} \displaystyle\sum_{\mathrm{B,C}} T_{\mu,\mathrm{B}}(\psi^{+}\gamma_{\mathrm{C}}\psi) + (\psi^{+}\gamma_{\mathrm{B}}\psi)T_{\mu,\mathrm{C}} \\ \displaystyle = \sum_{\mathrm{D,E}} T_{\mu,\mathrm{D}}(\psi^{+}\gamma_{\mathrm{E}}\psi) + (\psi^{+}\gamma_{\mathrm{D}}\psi)T_{\mu,\mathrm{E}}, \end{cases}$$

and on the other hand:

(16) $$\begin{cases} \displaystyle\sum_{\mathrm{B,C}} T_{\mu,\mathrm{B}}(\psi^{+}\gamma_{\mathrm{C}}\psi) - (\psi^{+}\gamma_{\mathrm{B}}\psi)T_{\mu,\mathrm{C}} \\ \displaystyle = \sum_{\mathrm{D,E}} \left(+\frac{h}{2\pi i}\right)\left\{\left(\frac{\partial}{\partial x^\mu}(\psi^{+}\gamma_{\mathrm{D}}\psi)\right)(\psi^{+}\gamma_{\mathrm{E}}\psi) - (\psi^{+}\gamma_{\mathrm{D}}\psi)\left(\frac{\partial}{\partial x^\mu}(\psi^{+}\gamma_{\mathrm{E}}\psi)\right)\right\}. \end{cases}$$

We will then be led to two types of differential relations.

The relations (15) can be easily deduced from Table (III), because one sees immediately that the simplifications that were introduced into the calculations by the compensation of the terms on the two sides when one established the relations (III) upon starting from (14) will be reproduced here.

Upon setting:

$$(17) \qquad \begin{cases} T_{\mu,0} = \left(\dfrac{h}{2\pi i}\right)(\psi_\mu^+ \psi - \psi^+ \psi_\mu), \\ T_{\mu,\alpha} = \left(\dfrac{h}{2\pi i}\right)(\psi_\mu^+ \gamma_\alpha \psi - \psi^+ \gamma_\alpha \psi_\mu), \\ T_{\mu,\alpha\beta} = \left(\dfrac{h}{2\pi i}\right)(\psi_\mu^+ i\gamma_{\alpha\beta} \psi - \psi^+ i\gamma_{\alpha\beta} \psi_\mu), \end{cases}$$

we will get a system of differential relations that can be written:

$$(\mathrm{V}) \qquad \begin{cases} (1) \quad T_{\mu,\alpha} j^\alpha = T_{\mu,0}\, \omega_1, \\ (2) \quad T_{\mu,\alpha\beta} j^\beta + f_{\alpha\beta} T_{\mu,}{}^\beta = 0, \\ (3) \quad T_{\mu,\alpha\gamma} f^{\beta\gamma} + f_{\alpha\gamma} T_{\mu,}{}^{\beta\gamma} = 2T_{\mu,0}\, \omega_1\, g_\alpha^\beta - T_{\mu,\beta} j^\alpha - T_{\mu,}^{\ \beta} j_\alpha, \\ (4) \quad T_{\mu,\alpha\beta\gamma} f^{\gamma\delta} + f_{\alpha\beta\gamma} T_{\mu,}{}^{\gamma\delta} = 2T_{\mu,0}(g_\alpha^\beta j_\beta - g_\beta^\beta j_\alpha) + \omega_1 (g_\alpha^\beta T_{\mu\beta} - g_\beta^\beta T_{\mu\alpha}), \\ (5) \quad T_{\mu,\alpha\beta\gamma} j^\gamma + f_{\alpha\beta\gamma} T_{\mu,}{}^\gamma = T_{\mu,0}\ f_{\alpha\beta} + \omega_1 T_{\mu,\alpha\beta}, \\ (6) \quad T_{\mu,\alpha\beta} j_\gamma + T_{\mu,\beta\gamma} j_\alpha + T_{\mu,\gamma\alpha} j_\beta + f_{\alpha\beta} T_{\mu,\gamma} + f_{\beta\gamma} T_{\mu,\alpha} + f_{\gamma\alpha} T_{\mu,\beta} \\ \qquad\qquad = T_{\mu,0}\, f_{\alpha\beta\gamma} + \omega_1 T_{\mu,\alpha\beta\gamma}, \end{cases}$$

and we can easily see that those relations are all deduced from relations (V_4) and (V_5).

We shall now examine the relations of the type (16), and their study is more delicate.

We start from the Pauli relation (13), which we write in the form:

$$4(\gamma_0)_{i_1 m_1}(\gamma_0)_{i_2 m_2} = \sum_{\mathrm{A}} (\gamma_{\mathrm{A}})_{i_1 m_2}(\gamma_{\mathrm{A}})_{i_2 m_1}\ .$$

If we multiply this by (γ_{B}), (γ_{C}) then we will get:

$$(18) \qquad 4(\gamma_{\mathrm{B}})_{i_1 m_1}(\gamma_{\mathrm{C}})_{i_2 m_2} = \sum_{\mathrm{A}} (\gamma_{\mathrm{BA}})_{i_1 m_2}(\gamma_{\mathrm{CA}})_{i_2 m_1}\ ,$$

and we will obtain a relation of the type (16) by the combination that was pointed out above.

We remark that the symmetric terms in the right-hand side that have the form:

$$(\gamma_{\mathrm{D}})_{i_1 m_2}(\gamma_{\mathrm{E}})_{i_2 m_1} + (\gamma_{\mathrm{E}})_{i_1 m_2}(\gamma_{\mathrm{D}})_{i_2 m_1}$$

will not contribute anything to the sum:

$$\begin{aligned} &(\psi_\mu^+ \gamma_{\mathrm{D}} \psi)(\psi^+ \gamma_{\mathrm{E}} \psi) - (\psi^+ \gamma_{\mathrm{D}} \psi)(\psi_\mu^+ \gamma_{\mathrm{E}} \psi) \\ +\ &(\psi_\mu^+ \gamma_{\mathrm{E}} \psi)(\psi^+ \gamma_{\mathrm{D}} \psi) - (\psi^+ \gamma_{\mathrm{E}} \psi)(\psi_\mu^+ \gamma_{\mathrm{D}} \psi) = 0 \end{aligned}$$

when one combines them, while the terms of the form:

$$(\gamma_{\mathrm{F}})_{i_1 m_2}(\gamma_{\mathrm{E}})_{i_2 m_1} - (\gamma_{\mathrm{F}})_{i_1 m_2}(\gamma_{\mathrm{G}})_{i_2 m_1}$$

will give:

(19) $$2\left[\left(\frac{\partial}{\partial x^{\mu}}(\psi^{+}\gamma_{\mathrm{F}}\psi)\right)(\psi^{+}\gamma_{\mathrm{G}}\psi) - (\psi^{+}\gamma_{\mathrm{F}}\psi)\left(\frac{\partial}{\partial x^{\mu}}(\psi^{+}\gamma_{\mathrm{G}}\psi)\right)\right].$$

The relation (18) will then give us a relation of the form:

(20) $$T_{\mu,\mathrm{B}}\, f_{\mathrm{C}} - T_{\mu,\mathrm{C}}\, f_{\mathrm{B}} = \frac{1}{2}\left[\left(\frac{h}{2\pi i}\right)\sum_{\mathrm{F,G}}\left(\frac{\partial}{\partial x^{\mu}}(f_{\mathrm{F}})\right)(f_{\mathrm{G}}) - (f_{\mathrm{F}})\frac{\partial}{\partial x^{\mu}}(f_{\mathrm{G}})\right].$$

We will get all of the relations of this type if we consider the left-hand sides:

(21) $$\begin{cases} T_{\mu,\alpha}\,\omega_1 - T_{\mu,0}\, j_{\alpha} \\ T_{\mu,\alpha\beta}\,\omega_1 - T_{\mu,0}\, f_{\alpha\beta} \\ T_{\mu,\alpha\beta}\, j_{\gamma} - T_{\mu,\gamma}\, f_{\alpha\beta} \\ T_{\mu,\alpha\beta}\, f_{\gamma\delta} - T_{\mu,\gamma\delta}\, f_{\alpha\beta} \\ T_{\mu,\alpha}\, j_{\beta} - T_{\mu,\beta}\, j_{\alpha}. \end{cases}$$

In order to facilitate the writing, we adopt a notation of O. Costa de Beauregard and denote the derivative $\frac{\partial}{\partial x^{\mu}}\psi^{+}\gamma_{\mathrm{A}}\psi$ by $\underline{\psi^{+}\gamma_{\mathrm{A}}\psi}$ to emphasize the terms that are differentiated with respect to x^{μ}.

The matrix relation (13) then gives us:

$$4(\gamma_{\alpha})_{i_1 m_1}(\gamma_0)_{i_2 m_2} = \sum_{\beta}(\gamma_{\alpha\beta})_{i_1 m_2}(\gamma_{\beta})_{i_2 m_1} - (\gamma_{\beta})_{i_1 m_2}(\gamma_{\beta})_{i_2 m_1} + \text{symmetric terms,}$$

so we infer:

$$T_{\mu,\,\alpha}\,\omega_1 - T_{\mu,\,0}\, j_a = \frac{h}{2\pi}\frac{1}{2}\left\{\underline{f_{\alpha\beta}}\, j^{\beta} - \underline{j^{\beta}}\, f_{\alpha\beta}\right\}.$$

However, the relation (III_2) gives us:

$$\underline{f_{\alpha\beta}}\, j^{\beta} + \underline{j^{\beta}}\, f_{\alpha\beta} = 0.$$

We will then have:

(VI_1) $$T_{\mu,\,\alpha}\,\omega_1 - T_{\mu,\,0}\, j_a = \left(-\frac{h}{2\pi}\right)\underline{f_{\alpha\beta}}\, j^{\beta} = +\frac{h}{2\pi}\,\underline{j^{\beta}}\, f_{\alpha\beta}.$$

Similarly, when (13) is multiplied by $\gamma_{\alpha\beta}$, γ_0, that will give us:

$$4(\gamma_{\alpha\beta})_{i_1 m_1}(\gamma_0)_{i_2 m_2} =$$

$$(\gamma_\alpha)_{i_1 m_2}(\gamma_\beta)_{i_2 m_1} - (\gamma_\beta)_{i_1 m_2}(\gamma_\alpha)_{i_2 m_1} + \sum_\gamma (\gamma_{\beta\gamma})_{i_1 m_2}(\gamma_{\alpha\gamma})_{i_2 m_1} - (\gamma_{\alpha\gamma})_{i_1 m_2}(\gamma_{\beta\gamma})_{i_2 m_1}$$

+ symmetric terms,

so:

$$T_{\mu,\,\alpha\beta}\,\omega_1 - T_{\mu,\,0}\,f_{\alpha\beta} = \frac{h}{2\pi}\frac{1}{2}\Big[\underline{j_\alpha}\,j_\beta - \underline{j_\beta}\,j_\alpha - (\underline{f_{\beta\rho}}\,f_{\alpha\sigma} - \underline{f_{\alpha\sigma}}\,f_{\beta\rho})g^{\rho\sigma}\Big].$$

Taking (III$_5$) into account, we obtain:

(VI$_2$)
$$T_{\mu,\,\alpha\beta}\,\omega_1 - T_{\mu,\,0}\,f_{\alpha\beta} = \frac{h}{2\pi}\Big[\underline{j_\alpha}\,j_\beta + \underline{f_{\alpha\rho}}\,f_{\beta\sigma}g^{\rho\sigma}\Big],$$
$$= -\frac{h}{2\pi}\Big[\underline{j_\beta}\,j_\alpha + \underline{f_{\beta\rho}}\,f_{\alpha\sigma}g^{\rho\sigma}\Big].$$

If we multiply (13) by (γ_α), (γ_β) then we will get:

$$4(\gamma_\alpha)_{i_1 m_2}(\gamma_\beta)_{i_2 m_1} =$$
$$(\gamma_{\alpha\beta})_{i_1 m_2}(\gamma_0)_{i_2 m_1} - (\gamma_0)_{i_1 m_2}(\gamma_{\alpha\beta})_{i_2 m_1} + \sum_\gamma (\gamma_\gamma)_{i_1 m_2}(\gamma_{\alpha\beta\gamma})_{i_2 m_1} - (\gamma_{\alpha\beta\gamma})_{i_1 m_2}(\gamma_\gamma)_{i_2 m_1}$$

+ symmetric terms,

so

$$T_{\mu,\,\alpha}\,j_\beta - T_{\mu,\,\beta}\,j_\alpha = \left(-\frac{h}{2\pi}\right)\frac{1}{2}\Big\{\underline{f_{\alpha\beta}}\,\omega_1 - \underline{\omega_1}\,f_{\alpha\beta} + \underline{j^\gamma}\,f_{\alpha\beta\gamma} - \underline{f_{\alpha\beta\gamma}}\,j^\gamma\Big\},$$

which will give:

(VI$_3$)
$$T_{\mu,\,\alpha}\,j_\beta - T_{\mu,\,\beta}\,j_\alpha = \left(-\frac{h}{2\pi}\right)\Big[\underline{f_{\alpha\beta}}\,\omega_1 - \underline{f_{\alpha\beta\gamma}}\,j^\gamma\Big] = +\frac{h}{2\pi}\Big[\underline{\omega_1}\,f_{\alpha\beta} - \underline{j^\gamma}\,f_{\alpha\beta\gamma}\Big],$$

upon taking (III$_5$) into account.

If we apply j^β to that relation then we will get:

$$T_{\mu,\,\alpha}\,\omega_1^2 - T_{\mu,\,\beta}\,j^\beta j_\alpha = -\frac{h}{2\pi}\,\underline{j^\gamma}\,f_{\alpha\beta\gamma}\,j^\beta,$$

and upon taking into account that:

$$T_{\mu,\,\beta}\,j^\beta = T_{\mu,\,0}\,\omega_1,$$

and from (III), what will remain is:

$$T_{\mu,\,\alpha}\,\omega_1 - T_{\mu,\,0}\,j_\beta = \frac{h}{2\pi}\,f_{\alpha\gamma}\,\underline{j^\gamma},$$

i.e., (VI$_1$), which then appears as a consequence of (VI$_2$).

The relation (13) will give us the two equivalent relations:

$$(\gamma_{\alpha\beta})_{i_1 m_1}(\gamma_\alpha)_{i_2 m_2} =$$
$$(\gamma_0)_{i_1 m_2}(\gamma_\beta)_{i_2 m_1} - (\gamma_\beta)_{i_1 m_2}(\gamma_0)_{i_2 m_1} + \sum_\varepsilon (\gamma_{\alpha\beta\varepsilon})_{i_1 m_2}(\gamma_{\alpha\varepsilon})_{i_2 m_1} - (\gamma_{\alpha\varepsilon})_{i_1 m_2}(\gamma_{\alpha\beta\varepsilon})_{i_2 m_1}$$
+ symmetric terms,

or further:

$$\begin{aligned} -i\,(\gamma_{\gamma\delta\varepsilon})_{i_1 m_1}(\gamma_\alpha)_{i_2 m_2} = {} & (\gamma_0)_{i_1 m_2}(\gamma_{\gamma\delta\varepsilon\alpha})_{i_2 m_1} - (\gamma_{\gamma\delta\varepsilon\alpha})_{i_1 m_2}(\gamma_0)_{i_2 m_1} \\ & + (\gamma_{\varepsilon\gamma})_{i_1 m_2}(\gamma_{\delta\alpha})_{i_2 m_1} - (\gamma_{\delta\alpha})_{i_1 m_2}(\gamma_{\varepsilon\gamma})_{i_2 m_1} \\ & + (\gamma_{\delta\varepsilon})_{i_1 m_2}(\gamma_{\gamma\alpha})_{i_2 m_1} - (\gamma_{\gamma\alpha})_{i_1 m_2}(\gamma_{\delta\varepsilon})_{i_2 m_1} \\ & + (\gamma_{\gamma\delta})_{i_1 m_2}(\gamma_{\varepsilon\alpha})_{i_2 m_1} - (\gamma_{\varepsilon\alpha})_{i_1 m_2}(\gamma_{\gamma\delta})_{i_2 m_1} \\ & + \text{symmetric terms.} \end{aligned}$$

We will then get, on the one hand:

$$T_{\mu,\,\alpha\beta}\, j_\alpha - T_{\mu,\,\alpha}\, f_{\alpha\beta} = \frac{h}{2\pi}\left[\underline{\omega_1}\, j_\beta - \underline{j_\beta}\,\omega_1 - (\underline{f_{\alpha\beta\rho}}\, f_{\alpha\sigma} - f_{\alpha\beta\rho}\,\underline{f_{\alpha\sigma}})\, g^{\rho\sigma}\right],$$

which, with (III_4), will reduce to:

(VI$_4$)
$$\begin{aligned} T_{\mu,\,\alpha\beta}\, j_\alpha - T_{\mu,\,\alpha}\, f_{\alpha\beta} &= \frac{h}{2\pi}(\underline{\omega_1}\, j_\beta g_{\alpha\alpha} + f_{\alpha\beta\rho}\,\underline{f_{\alpha\sigma}}\, g^{\rho\sigma}) \\ &= -\frac{h}{2\pi}(\omega_1\,\underline{j_\beta}\, g_{\alpha\alpha} + \underline{f_{\alpha\beta\rho}}\, f_{\alpha\sigma}\, g^{\rho\sigma}), \end{aligned}$$

and, on the other hand:

$$\begin{aligned} -(T_{\mu,\,\gamma\delta\varepsilon}\, j_\alpha - T_{\mu,\,\alpha}\, f_{\gamma\delta\varepsilon}) = \frac{1}{2}\frac{h}{2\pi}\Big(&\underline{\omega_1}\, f_{\gamma\delta\varepsilon\alpha} - \omega_1\,\underline{f_{\gamma\delta\varepsilon\alpha}} + \underline{f_{\gamma\alpha}}\, f_{\delta\varepsilon} + \underline{f_{\delta\alpha}}\, f_{\varepsilon\gamma} + \underline{f_{\varepsilon\alpha}}\, f_{\gamma\delta} \\ & - f_{\gamma\alpha}\,\underline{f_{\delta\varepsilon}} - f_{\delta\alpha}\,\underline{f_{\varepsilon\gamma}} - f_{\varepsilon\alpha}\,\underline{f_{\delta\gamma}}\Big), \end{aligned}$$

upon introducing the dual $f_{\gamma\delta\varepsilon\alpha}$ of j^β by the formula:

$$f_{\alpha\beta\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta\varepsilon}\, j^\varepsilon.$$

Formula (III_4) will then give:

$$\omega_1\, f_{\alpha\beta\gamma\delta} + f_{\delta\varepsilon}\, f_{\alpha\gamma} + f_{\delta\alpha}\, f_{\varepsilon\gamma} + f_{\varepsilon\alpha}\, f_{\gamma\delta} = 0,$$

which will reduce the preceding relation to:

(VI$_5$)
$$\begin{aligned} -(T_{\mu,\,\gamma\delta\varepsilon}\, j_\alpha - T_{\mu,\,\alpha}\, f_{\gamma\delta\varepsilon}) &= -\frac{h}{2\pi}\left[\underline{\omega_1}\, f_{\gamma\delta\varepsilon\alpha} - (\underline{f_{\gamma\alpha}}\, f_{\delta\varepsilon} + \underline{f_{\delta\alpha}}\, f_{\varepsilon\gamma} + \underline{f_{\varepsilon\alpha}}\, f_{\gamma\delta})\right] \\ &= \frac{h}{2\pi}\left[\omega_1\,\underline{f_{\gamma\delta\varepsilon\alpha}} - (f_{\gamma\alpha}\,\underline{f_{\delta\varepsilon}} + f_{\delta\alpha}\,\underline{f_{\varepsilon\gamma}} + f_{\varepsilon\alpha}\,\underline{f_{\gamma\delta}})\right]. \end{aligned}$$

That formula is valid only when $\gamma \neq \delta \neq \varepsilon \neq \alpha$, and as a result, it will have no general tensorial validity.

On the other hand, we will get:

$$\begin{aligned} T_{\mu,\,\alpha\beta}\, j_\alpha - T_{\mu,\,\alpha}\, f_{\alpha\beta} &= -\,(T_{\mu,\,\gamma\delta\varepsilon}\, j_\gamma - T_{\mu,\,\gamma}\, f_{\gamma\delta\varepsilon}) \\ &= \frac{h}{2\pi}\frac{1}{2}\left\{\left[(\underline{f_{\varepsilon\rho}}\, f_{\delta\sigma} - f_{\varepsilon\rho}\, \underline{f_{\delta\sigma}})g^{\rho\sigma} + \underline{j_\delta}\, j_\varepsilon - j_\varepsilon\, \underline{j_\delta}\right] g_{\gamma\gamma}\right\} + \frac{h}{2\pi}(\underline{f_{\delta\gamma}}\, f_{\varepsilon\gamma} - \underline{f_{\varepsilon\gamma}}\, f_{\delta\gamma}) \end{aligned}$$

in the same fashion, and if we take (III_3) into account then:

$$(\mathrm{VI}_6) \qquad -\,(T_{\mu,\,\gamma\delta\varepsilon}\, j_\gamma - T_{\mu,\,\gamma}\, f_{\gamma\delta\varepsilon}) = \frac{h}{2\pi}\left[(\underline{f_{\varepsilon\rho}}\, f_{\delta\sigma} g^{\rho\sigma} + \underline{j_\delta}\, j_\varepsilon) + (\underline{f_{\delta\gamma}}\, f_{\varepsilon\gamma} - \underline{f_{\varepsilon\gamma}}\, f_{\delta\gamma})\right].$$

We will then have two relations with no general tensorial validity, but those two formulas can be summarized in a single formula that does have general tensorial validity, and which we write:

$$\begin{aligned} (\mathrm{VI}_7) \qquad T_{\mu,\,\gamma\delta\varepsilon}\, j_\alpha - T_{\mu,\,\alpha}\, f_{\gamma\delta\varepsilon} = -\frac{h}{2\pi}\Big\{ & \underline{f_{\gamma\alpha}}\, f_{\delta\varepsilon} + \underline{f_{\delta\alpha}}\, f_{\varepsilon\gamma} + \underline{f_{\varepsilon\alpha}}\, f_{\gamma\delta} - \underline{\omega_1}\, f_{\gamma\delta\varepsilon\alpha} \\ & + g_{\alpha\gamma}\,(\underline{f_{\varepsilon\rho}}\, f_{\delta\sigma} g^{\rho\sigma} + \underline{j_\gamma}\, j_\delta) \\ & + g_{\alpha\delta}\,(\underline{f_{\gamma\rho}}\, f_{\varepsilon\sigma} g^{\rho\sigma} + \underline{j_\varepsilon}\, j_\gamma) + g_{\alpha\varepsilon}\,(\underline{f_{\delta\rho}}\, f_{\gamma\sigma} g^{\rho\sigma} + \underline{j_\gamma}\, j_\delta)\Big\} \\ = \frac{h}{2\pi}\Big\{ & f_{\gamma\alpha}\, \underline{f_{\delta\varepsilon}} + f_{\delta\alpha}\, \underline{f_{\varepsilon\gamma}} + f_{\varepsilon\alpha}\, \underline{f_{\gamma\delta}} - \omega_1\, \underline{f_{\gamma\delta\varepsilon\alpha}} \\ & + g_{\alpha\gamma}\,(f_{\varepsilon\rho}\, \underline{f_{\delta\sigma}}\, g^{\rho\sigma} + j_\gamma\, \underline{j_\delta}) \\ & + g_{\alpha\delta}\,(f_{\gamma\rho}\, \underline{f_{\varepsilon\sigma}}\, g^{\rho\sigma} + j_\varepsilon\, \underline{j_\gamma}) + g_{\alpha\varepsilon}\,(f_{\delta\rho}\, \underline{f_{\gamma\sigma}}\, g^{\rho\sigma} + j_\gamma\, \underline{j_\delta})\Big\}. \end{aligned}$$

We shall show that this relation implies the relation (VI_3) as a consequence.

Indeed, if we multiply the relation (VI_7) by $f^{\,\varepsilon\alpha}$ then we will get:

$$\begin{aligned} T_{\mu,\,\gamma\delta\varepsilon}\, j_\alpha f^{\,\varepsilon\alpha} - T_{\mu,\,\alpha}\, f_{\gamma\delta\varepsilon} f^{\,\varepsilon\alpha} &= -\,T_{\mu,\,\alpha}\,(\omega_1\, g_\gamma{}^\alpha\, j_\delta - \omega_1\, g_\delta{}^\alpha\, j_\gamma) \\ &= \omega_1\,(T_{\mu,\,\delta}\, j_\gamma - T_{\mu,\,\gamma}\, j_\delta) \end{aligned}$$

on the left-hand side, and after a calculation that poses no difficulty and utilizes the relation (III), we will get:

$$-\frac{h}{2\pi}\,\omega_1\left(\underline{\omega_1}\, f_{\gamma\delta} - f_{\gamma\delta\rho}\, \underline{j^\rho}\right)$$

on the right-hand side; i.e., (VI_3).

The product of (13) with $(\gamma_{\alpha\beta})\,(\gamma_{\alpha\beta\gamma}) = -\,(\gamma_{\alpha\beta})\,(\gamma_{\delta\varepsilon})$ will give us:

$$
\begin{aligned}
-4(\gamma_{\alpha\beta})_{i_1 m_1}(\gamma_{\delta\varepsilon})_{i_2 m_2} = & [(\gamma_{\beta\delta\varepsilon})_{i_1 m_2}(\gamma_\alpha)_{i_2 m_1} - (\gamma_\alpha)_{i_1 m_2}(\gamma_{\beta\delta\varepsilon})_{i_2 m_1}] \\
& - [(\gamma_{\alpha\delta\varepsilon})_{i_1 m_2}(\gamma_\beta)_{i_2 m_1} - (\gamma_\beta)_{i_1 m_2}(\gamma_{\alpha\delta\varepsilon})_{i_2 m_1}] \\
& + [(\gamma_{\alpha\beta\delta})_{i_1 m_2}(\gamma_\varepsilon)_{i_2 m_1} - (\gamma_\varepsilon)_{i_1 m_2}(\gamma_{\alpha\beta\delta})_{i_2 m_1}] \\
& - [(\gamma_{\alpha\beta\varepsilon})_{i_1 m_2}(\gamma_\delta)_{i_2 m_1} - (\gamma_\delta)_{i_1 m_2}(\gamma_{\alpha\beta\varepsilon})_{i_2 m_1}] \\
& + \text{symmetric terms,}
\end{aligned}
$$

and thus, the relation:

$$
\begin{aligned}
T_{\mu,\,\alpha\beta} f_{\delta\varepsilon} - T_{\mu,\,\delta\varepsilon} f_{\alpha\beta} = -\frac{h}{2\pi}\frac{1}{2}\Big[& \left(\underline{f_{\beta\delta\varepsilon}}\, j_\alpha - \underline{j_\alpha}\, f_{\beta\delta\varepsilon}\right) - \left(\underline{f_{\alpha\delta\varepsilon}}\, j_\beta - \underline{j_\beta}\, f_{\alpha\delta\varepsilon}\right) \\
& + \left(\underline{f_{\alpha\beta\delta}}\, j_\varepsilon - \underline{j_\varepsilon}\, f_{\alpha\beta\delta}\right) - \left(\underline{f_{\alpha\beta\varepsilon}}\, j_\delta - \underline{j_\delta}\, f_{\alpha\delta\varepsilon}\right)\Big].
\end{aligned}
$$

However:

$$f_{\beta\gamma} j^{\beta} = 0$$

gives:

$$f_{\beta\delta\varepsilon} j_\alpha - f_{\alpha\delta\varepsilon} j_\beta - f_{\alpha\beta\delta} j_\varepsilon + f_{\alpha\beta\varepsilon} j_\delta = 0$$

by duality, and what will remain is:

(VI$_8$)
$$
\begin{aligned}
T_{\mu,\,\alpha\beta} f_{\delta\varepsilon} - T_{\mu,\,\delta\varepsilon} f_{\alpha\beta} & = \frac{h}{2\pi}\left(\underline{j_\alpha}\, f_{\beta\delta\varepsilon} - \underline{j_\beta}\, f_{\beta\delta\varepsilon} + \underline{f_{\alpha\beta\varepsilon}}\, j_\delta - \underline{f_{\alpha\beta\delta}}\, j_\varepsilon\right) \\
& = -\frac{h}{2\pi}\left(j_\alpha\, \underline{f_{\beta\delta\varepsilon}} - j_\beta\, \underline{f_{\beta\delta\varepsilon}} + f_{\alpha\beta\varepsilon}\, \underline{j_\delta} - f_{\alpha\beta\delta}\, \underline{j_\varepsilon}\right).
\end{aligned}
$$

That formula is valid only when $\alpha \neq \beta \neq \gamma \neq \delta \neq \varepsilon$, and as a result, it will not have any general tensorial validity.

If we likewise calculate:

$$T_{\mu,\,\alpha\beta} f_{\alpha\gamma} - T_{\mu,\,\alpha\gamma} f_{\alpha\beta}$$

then we will get the relation:

$$T_{\mu,\,\alpha\beta} f_{\alpha\gamma} - T_{\mu,\,\alpha\gamma} f_{\alpha\beta} = \frac{h}{2\pi}\frac{1}{2}\left\{\underline{\omega_1}\, f_{\beta\gamma} - \underline{f_{\beta\gamma}}\, \omega_1 - \underline{f_{\beta\gamma\rho}}\, j^{\rho} - f_{\beta\gamma\rho}\, \underline{j^{\rho}}\right\} g_{\alpha\alpha} + \frac{h}{2\pi}\left(\underline{f_{\beta\gamma\alpha}}\, j_\alpha - \underline{j_\alpha}\, f_{\beta\gamma\alpha}\right),$$

and upon utilizing (III$_5$):

(VI$_9$)
$$
\begin{aligned}
T_{\mu,\,\alpha\beta} f_{\alpha\gamma} - T_{\mu,\,\alpha\gamma} f_{\alpha\beta} & = \frac{h}{2\pi}\left[\left(\underline{\omega_1}\, f_{\beta\gamma} - \underline{f_{\beta\gamma\rho}}\, j^{\rho}\right) g_{\alpha\alpha} + \left(\underline{f_{\beta\gamma\alpha}}\, j_\alpha - f_{\beta\gamma\alpha}\, \underline{j_\alpha}\right)\right] \\
& = -\frac{h}{2\pi}\left[\left(\omega_1\, \underline{f_{\beta\gamma}} - f_{\beta\gamma\rho}\, \underline{j^{\rho}}\right) g_{\alpha\alpha} + \left(f_{\beta\gamma\alpha}\, \underline{j_\alpha} - \underline{f_{\beta\gamma\alpha}}\, j_\alpha\right)\right].
\end{aligned}
$$

That relation is valid only when $\alpha \neq \beta \neq \gamma$. However, we summarize (VI$_8$) and (VI$_9$) in the relation:

(VI$_{10}$)
$$T_{\mu,\,\alpha\beta} f_{\alpha\gamma} - T_{\mu,\,\alpha\gamma} f_{\alpha\beta} = \frac{h}{2\pi}\left\{\left(\underline{f_{\beta\gamma\varepsilon}}\, j_\delta - \underline{f_{\alpha\beta\delta}}\, j_\varepsilon + f_{\beta\delta\varepsilon}\, \underline{j_\alpha} - f_{\alpha\delta\varepsilon}\, \underline{j_\beta}\right)\right.$$

$$+ g_{\alpha\beta}\left(\underline{\omega_1}\, f_{\beta\varepsilon} - \underline{f_{\beta\varepsilon\rho}}\, j^{\rho}\right) - g_{\alpha\varepsilon}\left(\underline{\omega_1}\, f_{\beta\delta} - \underline{f_{\beta\delta\rho}}\, j^{\rho}\right)$$
$$- g_{\beta\delta}\left(\underline{\omega_1}\, f_{\alpha\varepsilon} - \underline{f_{\alpha\varepsilon\rho}}\, j^{\rho}\right) + g_{\beta\varepsilon}\left(\underline{\omega_1}\, f_{\alpha\delta} - \underline{f_{\alpha\delta\rho}}\, j^{\rho}\right)\Big\},$$

which does have general tensorial validity.

If we multiply the last relation by $f^{\varepsilon\delta}$ and take into account the facts that:

$$f_{\delta\varepsilon} f^{\varepsilon\delta} = -4\,\omega_1^2, \qquad T_{\mu,\,\delta\varepsilon} f^{\varepsilon} = -4\,T_{\mu,\,0}\,\omega_1$$

then we will easily get the expression (VI_2) for:

$$T_{\mu,\,\delta\varepsilon}\,\omega_1 - T_{\mu,\,0}\,f_{\alpha\beta}\,.$$

We have thus obtained all of the differential relations of the type under study in formulas (VI_1), (VI_2), (VI_3), (VI_7), and (VI_{10}), and we have shown that they can be deduced from two of them, namely, (VI_7) and (VI_{10}).

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