"Sur la représentation par des systèmes différentiels non linéaires du premier ordre de modèles corpusculaires définis par l'association de champs de types microphysiques, électromagnétiques et gravitationnels," Ann. Inst. Henri Poincaré A: Phys. théor. **36** (1982), 89-125.

On the representation by first-order, nonlinear differential systems of corpuscular models that are defined by the association of fields of micro-physical, electromagnetic, and gravitational type.

by

Gérard PETIAU

Physique théorique, Université Paris VI

Translated by D. H. Delphenich

Summary. – Introduction, study, and characterization of the models for the representation of micro-physical structures by sets of associated fields. The spatio-temporal evolution of fields that constitute an individualized structure is described by a system of first-order, nonlinear, partial differential equations. These associated fields are described by systems of tensors that are irreducible with respect to the proper, orthochoronous Lorentz group, with preference given to the tensor-spinors of Van der Waerden. For the proposed models, the solutions of plane-wave type are completely determined by systems of elliptic or hyperelliptic functions.

A model that describes a microphysical structure by the set of spinors is established and analyzed by which one characterizes the linear field approximations called spin $\hbar/2$, spin $\hbar$, and $2\hbar$ (electronic, electromagnetic or mesic vectorial, gravitational fields). The set of its solutions of plane-wave type is completely determined.

In numerous previous papers that were mostly published from 1958 to 1965, I sought to construct and study the most essential elements that were susceptible to being presented by structural models of mathematical representations that I called "corpuscle-field models" and were capable of describing the spatio-temporal evolution of elementary physical objects ("corpuscles") uniquely by functional systems that represent the associated fields.

The spatio-temporal evolution of these structures is determined by the values taken in the course of time and at each point of real, three-dimensional physical space by a set of associated constitutive fields that have no possibility of being separated, except in the degenerate case.

The elementary – or "corpuscular" – physical objects are defined by the values taken by systems of fields (quantum, electromagnetic, gravitational, more general tensors) during associated forms of evolution, and are capable of being observed and measured simultaneously.

The actions (interactions) of external origin in the domain in question are introduced by constraints that are imposed on the associated fields in all of the experimental domain (or domain of existence of the representation considered).

The classical distinction between active particles and passive particles with respect to one of the types of field considered (the other being only present) no longer exists in these models. The notion of mass or electric charge that a "charged, material" point in physical space is affected with is no longer a primitive notion here, but will possibly be present as a representation of the existence of associated configurations of a gravitational field and an electromagnetic field whose representations are simplified by the introduction of quantities – mean or principal values – that are localized and permanent. Later, I will present an example of conditions that permit one to isolate a general representation that is associated with electromagnetic and gravitational fields from the particular fields that evolving with an electrostatic or gravito-static character.

I admit that the fields whose association constitutes an individualized physically fundamental element are described and determined simultaneously by starting with systems of *first-order partial differential equations* that are essentially *nonlinear* and present a *covariant tensorial structure* with respect to the transformations of the proper, orthochronous Lorentz group $(L_+^\uparrow)$, or what one calls the *causality group*.

The *associated fields*, whose *aggregate comprises the physical object*, will be individually represented by systems of *irreducible tensors* with respect to the transformations of the proper and orthochronous Lorentz group (viz., invertible, linear transformations of type $x'^\mu = a^\mu{}_\nu x^\nu$, with $| a^\mu{}_\nu | = + 1$, $a^0{}_0 = + 1$ or μ_0).

I will prefer to represent these tensors *in such a fashion as to include the notion of causality*, *a priori*, in all of its representations – the *irreducible tensor-spinors of Van der Waerden* (which will be defined by the symmetry conditions in the spinorial indices of the same type).

In order to look for representative structures for the models of the associated fields that allow one to analyze the character of the physical object that is being described, and, in turn, to specify the necessary choices for the representation of the simplest natural microphysical structure, I essentially appeal to the possible *existence* and the *complete characterization* of *particular solutions* of simple types that are capable of being *defined and constructed simultaneously* for *the set of associated fields.*

For this, I have constructed and studied some "corpuscular" models for the set of constituent fields that admit some solutions of "plane-wave" type and are determined completely.

A system of representative functions for the field $\Phi_A(x^\mu)$ will admit "plane-wave" states if one may, in that system, characterize and isolate *a complete family* of functions of the field $\Phi_A(x^\mu)$ that depend upon only one variable u that is defined by:

(1) $$u = n_\mu x^\mu,$$

n^μ being a *timelike unitary vector:*

(2) $$n_\mu n^\mu = (n^0)^2 - \mathbf{n}^2 = 1,$$

$$(n^\mu = (n^0, n^p), p = 1, 2, 3; \qquad g^{00} = +1, \qquad g^{pp} = -1, \qquad g^{pq} = g^{qp} = 0).$$

The set of associated fields $\Phi_A(x^\mu)$ that is determined by starting with a system of *first-order partial differential equations* becomes a system of functions $\Phi_A(u)$. Upon applying Hadamard's "principle of descent," these functions will be solutions of a *nonlinear system of first-order partial differential equations.*

Each of the elements $\Phi_A(u)$ will depend essentially upon the totality of integration constants that are associated with a system of initial, boundary, or asymptotic values $\Phi_A(u_0)$ of the totality of $\Phi_A(u)$.

In the representations that are introduced and utilized in linear quantum mechanics for the description of the states of particles and fields, all of the elements of "plane-wave" type are expressed by circular functions ($\sin \omega_0 u$, $\cos \omega_0 u$, $e^{i\omega_0 u}$) or by combinations or superpositions of circular functions.

In the first models of corpuscle-fields that I introduced and studied in 1958-1965, I essentially sought the simple representations of the association of tensors (in the restricted sense of $L_+^\uparrow$) by covariant systems of partial differential equations for which the differential equations that corresponded to the case of plane wave *were determined completely* and *expressed by starting with Jacobi's elliptic functions* (or hyperelliptic functions) when one imposed the condition on them that they remain of bounded amplitude.

The development of numerous models that I have carried out and the very numerous international efforts of these last years that was directed to the representation of the elements of microphysics exhibited the necessity of utilizing in that domain, functional elements that were attached to elliptic functions, whose extensions and degeneracies constitute elements that are qualified by the term "solitons."

The first system of the associated fields that I introduced and studied is comprised of *two scalars* and *one vector*, whose evolution is determined simultaneously by a *system of three first-order partial differential equations* that are nonlinear.

Two simple models realize that association.

In the first model, two scalars $I_1(x^\lambda)$, $I_2(x^\lambda)$, and a vector $S^\mu(x^\lambda)$ are associated with each other by way of three equations:

$$(S_1) \qquad \begin{aligned} \partial_\mu I_1 &= \kappa_1 I_2 S_\mu , \\ \partial_\mu I_2 &= -\kappa_2 I_1 S_\mu , \\ \partial_\mu S^\mu &= -\kappa_0 I_1 I_2 . \end{aligned}$$

κ_1, κ_2, κ_0 are three *real* positive constants $\left(\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}\right)$.

In the second model, two scalars $J_1(x^\lambda)$, $J_2(x^\lambda)$, and a vector $S^\mu(x^\lambda)$ are associated by three equations:

$$(S_2) \qquad \partial_\mu J_1 = \kappa_1 J_2 S_\mu , \qquad \partial_\mu J_2 = \kappa_2 J_1 S_\mu , \qquad \partial_\mu S^\mu = -\kappa_0 J_1 J_2 .$$

(κ_1, κ_2, κ_0 are *real* positive constants.)

The choice of the system of signs (+, –, –), (+, +, –) in these models, to the exclusion of the systems of signs (+, +, +) or (–,–,–), ultimately seems necessary for the existence of solutions of the type of *plane-waves of bounded amplitudes.*

In the case of plane waves, the solutions of (S_1) and (S_2) can be put into correspondence with each other.

The system (S_1) admits the first integral:

$$\frac{1}{\kappa_1} I_1^2 + \frac{1}{\kappa_2} I_2^2 = \lambda_0^2 ,$$

(λ_0 is a positive real constant for real I_1, I_2), which *limits the amplitudes of the fields* $I_1(x^\lambda)$, $I_2(x^\lambda)$.

This relation leads one to associate, *a priori*, I_1, I_2 with a function $\Phi(x^\lambda)$ by the relations:

$$I_1(x^\lambda) = \lambda_0 \sqrt{\kappa_1} \sin \Phi(x^\lambda),$$
$$I_2(x^\lambda) = \lambda_0 \sqrt{\kappa_2} \cos \Phi(x^\lambda),$$

and allows one to reduce the system (S_1) to *two first-order partial differential equations:*

(3) $$\partial_\mu \Phi = \sqrt{\kappa_1 \kappa_2} S_\mu , \qquad \partial_\mu S^\mu = - \kappa_0 \lambda_0^2 \sqrt{\kappa_1 \kappa_2} \sin \Phi \cos \Phi.$$

As a consequence, $\Phi(x^\lambda)$ will be *associated with solutions* of the *second-order* equation:

(4) $$\partial_\mu \partial^\mu \Phi + \lambda_0^2 \kappa_0 \kappa_2 \kappa_2 \sin \Phi \cos \Phi = 0.$$

In the case of two or three spatial dimensions, this equation was studied in the development of the *theory of mappable surfaces.*

G. Darboux devoted several chapters to it in his *Théorie des surfaces* (v. III, chap. XII, XIII, XIV), while significantly developing the work of J. Weingarten (1887, 1891) and Enneper.

In the case of the system (S_2), a first integral of the same type is written:

$$\frac{J_1^2}{\kappa_1} - \frac{J_2^2}{\kappa_2} = \lambda_0^2$$

(with an indexing of convenient order for the scalars J_1, J_2). This relation leads one to write, *a priori, for* $J_1 \neq J_2$:

$$J_1(x^\lambda) = \lambda_0 \sqrt{\kappa_1} \cosh \Phi(x^\lambda), \qquad J_2(x^\lambda) = \lambda_0 \sqrt{\kappa_1} \sinh \Phi(x^\lambda),$$

and permits one to reduce the system (S_2) to *two equations:*

$$(S_2') \qquad \partial_\mu \Phi = \sqrt{\kappa_1 \kappa_2} S_\mu \,, \qquad \partial_\mu S^\mu = -\, \kappa_0 \lambda_0^2 \sqrt{\kappa_1 \kappa_2} \,\sinh \Phi \cosh \Phi.$$

The function $\Phi(x^\lambda)$ will be *associated* with the *solutions of the second-order equation:*

$$(5) \qquad \partial_\mu \partial^\mu \Phi + \lambda_0^2 \;\kappa_0\; \kappa_2\; \kappa_2 \sinh \Phi \cosh \Phi = 0.$$

The systems (S_2) or (S_2') admit a *simple degeneracy* that corresponds to the case $\lambda_0 = 0$. Let:

$$(6) \qquad \frac{J_1}{\sqrt{\kappa_1}} = \frac{J_2}{\sqrt{\kappa_2}} = J_0 \,.$$

(S_2) will then reduce to the two equations:

$$(7) \qquad \partial_\mu J_0 = \sqrt{\kappa_1 \kappa_2} J_0 S_\mu \,, \quad \partial_\mu S^\mu = -\, \kappa_0 \sqrt{\kappa_1 \kappa_2} J_0^2 \,,$$

and upon setting:

$$(8) \qquad J_0(x^\lambda) = K_0 e^{K_1 \Phi(x^\lambda)} \,,$$

$\Phi(x^\lambda)$ will be determined by the system:

$$(9) \qquad \partial_\mu \Phi = \frac{\sqrt{\kappa_1 \kappa_2}}{K_1} S_\mu \,, \qquad \partial_\mu S^\mu = -\, \kappa_0 \sqrt{\kappa_1 \kappa_2} K_0^2 e^{2K_1 \Phi} \,.$$

$\Phi(x^\lambda)$ will be determined by *starting with solutions* of the second-order equation:

$$(10) \qquad \partial_\mu \partial^\mu \Phi + \kappa_0\; \kappa_2\; \kappa_2\; \frac{K_0^2}{K_1^2} e^{2K_1 \Phi} = 0.$$

One then finds the Liouville equation here, whose *general solution* he gave in the case of two dimensions (*t* and *z*) (J. Liouville, *C. R. Acad. Sci.*, v. **36**, 28 February 1853, pp. 371-373; *J. de Math. pures et appliquées*, 1st series, v. **18**, 1853, pp. 71).

Later, we shall specify that solution in the case of the first-order system that is considered here.

The complete solutions of the systems (S_1), (S_2) from their associated or degenerate forms are easily obtained in the case of "plane-waves" with $u = n_\mu x^\mu$, $n_\mu n^\mu = +1$. In the case of the system (S_1), one sets:

$$(11) \qquad \begin{aligned} I_1 \to I_1(u) &= \frac{1}{\sqrt{\kappa_0 \kappa_2}} Y_1(u) \,, \\ I_2 \to I_2(u) &= \frac{1}{\sqrt{\kappa_0 \kappa_1}} Y_2(u) \,, \end{aligned}$$

$$S_\mu \to n^\mu S_0(u) = \frac{n^\mu}{\sqrt{\kappa_1 \kappa_2}} Y_3(u) .$$

The three functions $Y_1(u)$, $Y_2(u)$, $Y_3(u)$ will be determined by the system that I will take to be the *principal system:*

$$(Y) \qquad \frac{dY_1}{du} = Y_2 Y_3 , \qquad \frac{dY_2}{du} = - Y_1 Y_3 , \qquad \frac{dY_3}{du} = - Y_1 Y_2 .$$

This system admits the *two principal first integrals:*

$$(12) \qquad \begin{aligned} Y_1^2 + Y_2^2 &= \lambda_1^2 = (Y_1^0)^2 + (Y_2^0)^2 , \\ Y_1^2 + Y_3^2 &= \lambda_2^2 = (Y_1^0)^2 + (Y_3^0)^2 . \end{aligned}$$

We assume that $\lambda_2^2 > \lambda_1^2$ (which corresponds to the choice of indexing I_1, I_2).

One deduces a third first integral from this:

$$(13) \qquad Y_3^2 - Y_2^2 = \lambda_2^2 - \lambda_1^2 = \lambda_3^2 = (Y_3^0)^2 - (Y_2^0)^2 \geq 0,$$

and, in turn:

$$0 \leq Y_1^2 \leq \lambda_1^2 , \quad 0 \leq Y_2^2 \leq \lambda_1^2 , \quad \lambda_3^2 \leq Y_3^2 \leq \lambda_2^2 .$$

Introducing these constants λ_1, λ_2, λ_3, one obtains the following differential relations *for a convenient neighborhood* of a system of three *positive* values $Y_1(u)$, $Y_2(u)$, $Y_3(u)$:

$$(14) \qquad \begin{aligned} \frac{dY_1(u)}{du} &= +\sqrt{(\lambda_1^2 - Y_1^2)(\lambda_2^2 - Y_1^2)} , \\ \frac{dY_2(u)}{du} &= -\sqrt{(\lambda_1^2 - Y_2^2)(\lambda_3^2 + Y_2^2)} , \\ \frac{dY_3(u)}{du} &= -\sqrt{(\lambda_2^2 - Y_3^2)(Y_3^2 - \lambda_3^2)} . \end{aligned}$$

As a result, $Y_1(u)$, $Y_2(u)$, $Y_3(u)$ *will be determined by the inversion of the elliptic integrals of the first kind:*

$$u = \int_{Y_1^0}^{Y_1} \frac{dy}{\sqrt{(\lambda_1^2 - y^2)(\lambda_2^2 - y^2)}} ,$$

$$(15) \qquad u = \int_{Y_2}^{Y_2^0} \frac{dy}{\sqrt{(\lambda_1^2 - y^2)(\lambda_3^2 + y^2)}} ,$$

$$u = \int_{Y_3}^{Y_3^0} \frac{dy}{\sqrt{(\lambda_2^2 - y^2)(y^2 - \lambda_3^2)}}\,.$$

These *are made to correspond* with the *elliptic integrals that define the Jacobi functions.*

These functions, namely:

$$\begin{gathered} y_1 = \mathrm{sn}(u,k),\ \ y_2 = \mathrm{cn}(u,k),\ \ y_3 = d\mathrm{n}(u,k), \\ 0 \le k \le 1, \quad 1 - k^2 = k'^2, \quad 0 \le k' \le 1, \end{gathered} \tag{16}$$

are defined by the inversion of the integrals:

$$\begin{aligned} u &= \int_0^{y_1} \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}\,, \\ u &= \int_{y_2}^{1} \frac{dy}{\sqrt{(1-y^2)(k'^2+k^2y^2)}}\,, \\ u &= \int_{y_3}^{1} \frac{dy}{\sqrt{(1-y^2)(y^2-k'^2)}}\,. \end{aligned} \tag{17}$$

These functions are associated with each other by way of the relations:

$$\begin{aligned} &\mathrm{sn}^2(u, k) + \mathrm{cn}^2(u, k) = 1, \\ &k^2\, \mathrm{sn}^2(u, k) + \mathrm{dn}^2(u, k) = 1, \\ &\mathrm{dn}^2(u, k) - k^2\, \mathrm{cn}^2(u, k) = k'^2, \end{aligned} \tag{18}$$

and the differential relations:

$$\begin{aligned} &\frac{d}{du}\mathrm{sn}(u, k) = \mathrm{cn}\, u\ \mathrm{dn}\, u, \\ &\frac{d}{du}\mathrm{cn}(u, k) = -\,\mathrm{sn}\, u\ \mathrm{dn}\, u, \\ &\frac{d}{du}\mathrm{dn}(u, k) = -\,k^2\, \mathrm{sn}\, u\ \mathrm{cn}\, u, \end{aligned} \tag{19}$$

$$\mathrm{sn}(0, k) = 0, \quad \mathrm{cn}(0, k) = 1, \quad \mathrm{dn}(0, k) = 1.$$

If we write the integrals that define the functions Y_1, Y_2, Y_3 in the form:

(20)
$$\lambda_2\, u = \int_{Y_1^0/\lambda_1}^{Y_1/\lambda_1} \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{\lambda_1^2}{\lambda_2^2}y^2\right)}},$$

with:

(21)
$$\lambda_1^2 = (Y_1^0)^2 + (Y_2^0)^2, \quad \lambda_2^2 = (Y_3^0)^2 + (Y_1^0)^2,$$
$$\lambda_3^2 = (Y_2^0)^2 - (Y_3^0)^2, \quad \frac{\lambda_1^2}{\lambda_2^2} = \frac{(Y_1^0)^2 + (Y_2^0)^2}{(Y_3^0)^2 + (Y_1^0)^2} < 1,$$

then we have *for the particular initial values:*

$$Y_1^0 = 0, \qquad Y_2^0,\ Y_3^0, \qquad \lambda_1^2 = (Y_2^0)^2, \quad \lambda_2^2 = (Y_3^0)^2, \quad \lambda_3^2 = (Y_2^0)^2 - (Y_3^0)^2,$$

and, in turn:

(22)
$$Y_1(u; 0, Y_2^0, Y_3^0) = \lambda_1 \mathrm{sn}\left(\lambda_2 u, \frac{\lambda_1}{\lambda_2}\right) = Y_2^0 \mathrm{sn}\left(Y_3^0 u, \frac{Y_2^0}{Y_3^0}\right),$$
$$Y_2(u; 0, Y_2^0, Y_3^0) = \lambda_1 \mathrm{cn}\left(\lambda_2 u, \frac{\lambda_1}{\lambda_2}\right) = Y_2^0 \mathrm{cn}\left(Y_3^0 u, \frac{Y_2^0}{Y_3^0}\right),$$
$$Y_3(u; 0, Y_2^0, Y_3^0) = \lambda_2 \mathrm{dn}\left(\lambda_2 u, \frac{\lambda_1}{\lambda_2}\right) = Y_2^0 \mathrm{dn}\left(Y_3^0 u, \frac{Y_2^0}{Y_3^0}\right).$$

In the case of three positive initial values Y_1^0, Y_2^0, Y_3^0, the general solution is further expressed as a Jacobi function by associating the three values Y_1^0, Y_2^0, Y_3^0 *with a unique initial argument* u_0 that is defined by:

(23)
$$Y_1^0 = \lambda_1 \mathrm{sn}\left(\lambda_2 u_0, \frac{\lambda_1}{\lambda_2}\right), \qquad Y_2^0 = \lambda_1 \mathrm{cn}\left(\lambda_2 u_0, \frac{\lambda_1}{\lambda_2}\right), \qquad Y_3^0 = \lambda_2 \mathrm{dn}\left(\lambda_2 u_0, \frac{\lambda_1}{\lambda_2}\right).$$

From this, one deduces the general expressions:

(24)
$$Y_1(u; 0, Y_2^0, Y_3^0) = \lambda_1\ \mathrm{sn}\left[\lambda_2 (u + u_0), \frac{\lambda_1}{\lambda_2}\right],$$
$$Y_2(u; 0, Y_2^0, Y_3^0) = \lambda_1\ \mathrm{cn}\left[\lambda_2 (u + u_0), \frac{\lambda_1}{\lambda_2}\right],$$
$$Y_3(u; 0, Y_2^0, Y_3^0) = \lambda_1\ \mathrm{dn}\left[\lambda_2 (u + u_0), \frac{\lambda_1}{\lambda_2}\right].$$

The application of the addition formulas leads to the *general solution* of the system (Y):

$$Y_1(u;0,Y_2^0,Y_3^0) = \frac{\lambda_2\left[Y_2^0Y_3^0\,\mathrm{sn}\left(\lambda_2 u,\frac{\lambda_1}{\lambda_2}\right)+\lambda_2Y_1^0\,\mathrm{cn}(\lambda_2 u)\,\mathrm{dn}(\lambda_2 u)\right]}{\lambda_2^2-(Y_1^0)^2\,\mathrm{sn}^2\left(\lambda_2 u,\frac{\lambda_1}{\lambda_2}\right)},$$

(25)
$$Y_2(u;0,Y_2^0,Y_3^0) = \frac{\lambda_2\left[\lambda_2Y_2^0\,\mathrm{cn}\left(\lambda_2 u,\frac{\lambda_1}{\lambda_2}\right)-Y_1^0Y_3^0\,\mathrm{sn}(\lambda_2 u)\,\mathrm{dn}(\lambda_2 u)\right]}{\lambda_2^2-(Y_1^0)^2\,\mathrm{sn}^2\left(\lambda_2 u,\frac{\lambda_1}{\lambda_2}\right)},$$

$$Y_3(u;0,Y_2^0,Y_3^0) = \frac{\lambda_2\left[\lambda_2Y_3^0\,\mathrm{dn}\left(\lambda_2 u,\frac{\lambda_1}{\lambda_2}\right)-Y_1^0Y_2^0\,\mathrm{sn}(\lambda_2 u)\,\mathrm{cn}(\lambda_2 u)\right]}{\lambda_2^2-(Y_1^0)^2\,\mathrm{sn}^2\left(\lambda_2 u,\frac{\lambda_1}{\lambda_2}\right)},$$

In these expressions, the denominator can take on the equivalent forms:

$$\lambda_2^2-(Y_1^0)^2\,\mathrm{sn}^2(\lambda_2 u,k) = (Y_1^0)^2+(Y_2^0)^2-(Y_3^0)^2\,\mathrm{sn}^2(\lambda_2 u,k)$$

(26)
$$= (Y_1^0)^2\,\mathrm{cn}^2(\lambda_2 u,k)+(Y_3^0)^2 = \frac{\lambda_2^2}{\lambda_1^2}[(Y_1^0)^2\,\mathrm{dn}^2(\lambda_2 u,k)+(Y_2^0)^2],$$

$$\left(k=\frac{\lambda_1}{\lambda_2}=\sqrt{\frac{(Y_1^0)^2+(Y_2^0)^2}{(Y_1^0)^2+(Y_3^0)^2}}\right).$$

We have presented these expressions explicitly here in order to *exhibit the intervention of the combinations of the initial values of the three functions* in each of the functions, and to neatly show the *impossibility of approaching them by the perturbation methods* that are often utilized.

The elliptic functions $y_1(u) = \mathrm{sn}(u, k)$, $y_2(u) = \mathrm{cn}(u, k)$ have the real period $4K(k)$ and $y_3(u) = \mathrm{dn}(u, k)$ has the real period $2K(k)$, $K(k)$ being defined by the integral:

$$F(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = {}_2F_1\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2}\ ;k^2\\ 1\end{matrix}\right].$$

$\left({}_2F_1\left[\begin{matrix}\alpha,\ \beta \\ \gamma\end{matrix};z\right] \text{is the Gauss hypergeometric function.} \right)$.

The representation (25) of the functions $Y(u)$ introduces the functions:

$$\mathrm{sn}\left(\lambda_2 u, \frac{\lambda_1}{\lambda_2}\right),\quad \mathrm{cn}\left(\lambda_2 u, \frac{\lambda_1}{\lambda_2}\right),\quad \mathrm{dn}\left(\lambda_2 u, \frac{\lambda_1}{\lambda_2}\right),$$

which are periodic functions whose period for the variables u and Y_1, Y_2 is:

$$4K\left(\frac{\lambda_1}{\lambda_2}\right) = {}_2F_1\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2} \\ 1\end{matrix};\frac{\lambda_1^2}{\lambda_2^2}\right]. \tag{27}$$

The plane-wave solutions of the system J, (S_2) can be put into a simple correspondence with those of the system (Y).

If we write the plane-wave solutions of the system (J) as:

$$J_1(x^\lambda) \to J_1(u) = \frac{1}{\sqrt{\kappa_0\kappa_2}} L_1(u),$$

$$J_2(u) = \frac{1}{\sqrt{\kappa_0\kappa_2}} L_2(u), \qquad S^\mu(u) = \frac{n^\mu}{\sqrt{\kappa_1\kappa_2}} L_3(u)$$

then the functions $L(u)$ are solutions of the system:

$$\frac{dL_1}{du} = L_2\, L_3, \qquad \frac{dL_2}{du} = L_1\, L_3, \qquad \frac{dL_1}{du} = -L_1\, L_2. \tag{L}$$

This system admits the first integrals:

$$L_2^2 + L_3^2 = (l_2^0)^2 + (l_3^0)^2 = \mu_1^2, \qquad L_1^2 + L_3^2 = (l_1^0)^2 + (l_3^0)^2 = \mu_2^2,$$

and with $\mu_2^2 > \mu_1^2$ (for a convenient indexing):

$$(L_1(u))^2 - (L_1(u))^2 = \mu_2^2 - \mu_1^2 = \mu_3^2 > 0.$$

The system (L) is put into correspondence with the system Y by means of the relations:

$$\begin{array}{lll} L_3 \to -Y_1, & L_1 \to -Y_3, & L_2 \to -Y_2, \\ \mu_1^2 = \lambda_1^2, & \mu_2^2 = \lambda_2^2, & \mu_3^2 = \lambda_3^2, \\ L_3(0) = l_3^0 = -Y_1^0, & l_1^0 = -Y_3^0, & l_2^0 = -Y_2^0. \end{array} \tag{28}$$

Here, the particular values:

$$l_3^0 = 0, \qquad l_2^0 = \mu_1, \qquad l_1^0 = \mu_2$$

lead to the solution:

$$
\begin{aligned}
&L_1(u;0,l_1^0,l_2^0) = -\mu_2 \ \mathrm{dn}\left(\mu_2 u;\frac{\mu_1}{\mu_2}\right) = -\ l_1^0 \ \mathrm{dn}\left(l_1^0 u;\frac{l_2^0}{l_1^0}\right),\\
(29)\qquad &L_2(u;0,l_1^0,l_2^0) = -\mu_1 \ \mathrm{cn}\left(\mu_2 u;\frac{\mu_1}{\mu_2}\right),\\
&L_3(u;0,l_1^0,l_2^0) = -\mu_1 \ \mathrm{sn}\left(\mu_2 u;\frac{\mu_1}{\mu_2}\right).
\end{aligned}
$$

We have pointed out the *important degeneracy of the system* (*J*), which admits the first integral:

$$\frac{J_1^2}{\kappa_1} - \frac{J_2^2}{\kappa_2} = \lambda_0^2,$$

and which leads, for $\lambda_0 = 0$, and upon writing ($\varepsilon = \pm 1$):

$$\frac{J_1}{\sqrt{\kappa_1}} = \varepsilon \frac{J_1}{\sqrt{\kappa_1}} = J_0(x^\lambda),$$

to the reduced system:

$$(30)\qquad \partial_\mu J_0 = \varepsilon \sqrt{\kappa_1\kappa_2}\ J_0\, S_\mu, \qquad \partial_\mu S^\mu = -\ \varepsilon\kappa_0 \sqrt{\kappa_1\kappa_2}\ J_0^2,$$

and upon denoting:

$$
\begin{aligned}
&J_0^2 = K_0, \qquad K^\mu = 2\varepsilon \sqrt{\kappa_1\kappa_2}\ S^\mu,\\
(K)\qquad &\partial_\mu K_0(x^\lambda) = K_0 K_\mu(x^\lambda), \quad \partial_\mu K^\mu(x^\lambda) = -\,2\ \kappa_0\,\kappa_1\,\kappa_2\ K_0(x^\lambda).
\end{aligned}
$$

This system, which is equivalent to the Liouville equation, takes a simple form in the case of plane waves:

$$K_0(x^\lambda) \to K_0(u), \qquad K^\mu = n^\mu\, K_1(u).$$

and leads to the system:

$$(31)\qquad \frac{d}{du}K_0(u) = K_0\, K_1, \qquad \frac{d}{du}K_1(u) = -\,K_0^2.$$

Since the initial values $K_0(u_0)$, $K_1(u_0)$ are associated with the first integral:

$$K_0^2(u) + K_1^2(u) = \lambda_1^2 = K_0^2(u_0) + K_1^2(u_0),$$

the general integral is written:

(32) $$K_0(u) = \frac{\lambda_1}{\cosh[\lambda_1(u-u_0)]}, \qquad K_0(u) = -\lambda_1 \tanh[\lambda_1(u-u_0)],$$

with

$$K_0(0) = \frac{\lambda_1}{\cosh \lambda_1 u_0}, \qquad K_0(u) = -\lambda_1 \tanh \lambda_1 u_0,$$

In the general case, this solution corresponds to the *degeneracy of the Jacobi elliptic function* for the value of the modulus $k = 1$:

(33) $$\mathrm{sn}(u;1) = \tanh u, \qquad \mathrm{cn}(u;1) = \mathrm{dn}(u;1) = \frac{1}{\cosh u}.$$

The *general Liouville solution in the case of two dimensions* (t, z) is recovered easily for the system (K_0, K_μ).

For the *Liouville solution* considered, according to d'Alembert, the variables u, v that are associated with t, z by way of:

$$u = \frac{1}{2}(t+z), \qquad v = \frac{1}{2}(t-z)$$

lead to:

$$\partial_\mu \partial^\mu \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial u\, \partial v}.$$

More generally, in order to exhibit the *correspondence between the variables u and v and the spinors* $x^{\dot{1}1}$, $x^{\dot{2}2}$, we introduce *two isotropic vectors* n'^μ, n''^μ, with $n'_\mu n'^\mu = 0$, $n''_\mu n''^\mu = 1/2$.

Setting:

$$u_1 = n'_\mu x^\mu, \qquad u_2 = n''_\mu x^\mu,$$

$$\frac{\partial}{\partial x^\mu} = n'_\mu \frac{\partial}{\partial u_1} + n''_\mu \frac{\partial}{\partial u_2}, \qquad \frac{\partial}{\partial u_1} = 2n''^\mu \frac{\partial}{\partial x^\mu}, \qquad \frac{\partial}{\partial u_2} = 2n'^\mu \frac{\partial}{\partial x^\mu},$$

we are lead to associate the vector K^μ with two functions $K_1(u_1, u_2)$, $K_2(u_1, u_2)$ by way of $K^\mu(x^\lambda) = K_1 n'^\mu + K_2 n''^\mu$. The system ($K$) is written, with $K_0(u_1, u_2)$:

(34) $$\frac{\partial K_0}{\partial u_1} = K_0 K_1, \qquad \frac{\partial K_0}{\partial u_2} = K_0 K_2, \qquad \frac{\partial K_2}{\partial u_1} + \frac{\partial K_1}{\partial u_2} = -4\kappa_0\, \kappa_1\, \kappa_2\, K_0(u_1, u_2).$$

The *general Liouville solution* is recovered upon writing, *a priori:*

(35) $$K_0(u_1, u_2) = \frac{\lambda A_1(u_1)A_2(u_2)}{[C_1(u_1)+C_2(u_2)]^{\alpha}}$$

(λ, α are to be determined).

From this, one deduces:

$$\frac{1}{K_0}\frac{\partial K_0}{\partial u_1} = \frac{A_1'}{A_1} - \frac{\alpha C_1'}{C_1+C_2} = K_1(u_1, u_2),$$

$$\frac{1}{K_0}\frac{\partial K_0}{\partial u_2} = \frac{A_1'}{A_2} - \frac{\alpha C_2'}{C_1+C_2} = K_2(u_1, u_2),$$

and from the second equation:

$$\alpha = 2, \qquad \lambda = -\frac{1}{\kappa_0\kappa_1\kappa_2},$$

$$K_0(u_1, u_2) = -\frac{C_1'(u_1)C_2'(u_2)}{\kappa_0\kappa_1\kappa_2(C_1+C_2)^2},$$

(36) $$K_1(u_1, u_2) = \frac{C_1''(u_1)}{C_1'(u_1)} - \frac{2C_1'(u_1)}{C_1(u_1)+C_2(u_2)},$$

$$K_2(u_1, u_2) = \frac{C_2''(u_1)}{C_2'(u_1)} - \frac{2C_2'(u_1)}{C_1(u_1)+C_2(u_2)}.$$

A model that is close to the one that was studied up to now is comprised of the association of an invariant $I(x^{\lambda})$ and some *field functions* that are associated with the description of the *electromagnetic* or *vector mesonic fields*, namely, *a vector* A^{μ} and a *second-order self-dual anti-symmetric* tensor $F^{\mu\nu}$.

With an adapted choice of signs, these three quantities are associated with each other by the three equations:

(37) $$\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} = \kappa_1 I F_{\mu\nu}, \qquad \partial_{\lambda} F^{\mu\lambda} = \kappa_1 I A^{\mu}, \qquad \partial_{\mu} I = \varepsilon_0\, \kappa_0\, F_{\mu\lambda} A^{\mu}.$$

(κ_0, κ_1, κ_2 are real *positive* constants ($\varepsilon_0 = \pm 1$)).

This system entails that $\partial_{\mu}(I A^{\mu}) = 0$, and, in turn, $\partial_{\mu} A^{\mu} = 0$.

The *particular solutions of plane-wave type* are further expressed by means of *Jacobi elliptic functions.*

Setting:

$$A^{\mu}(u) = \frac{1}{\sqrt{\kappa_2\kappa_0}}\, a^{\mu}\, Y_1(u), \qquad F^{\mu\nu} = \frac{1}{\sqrt{\kappa_1\kappa_0}}\, f^{\mu\nu}\, Y_2(u),$$

$$I(u) = \frac{1}{\sqrt{\kappa_1 \kappa_2}} Y_2(u), \qquad (u = n_\mu x^\mu, \qquad n_\mu n^\mu = 1),$$

the functions $Y(u)$ satisfy the relations:

$$(n_\mu a^\nu - n_\nu a^\mu) \frac{dY_1}{du} = f_{\mu\nu} Y_2 Y_3, \qquad n_\lambda f^{\mu\lambda} \frac{dY_2}{du} = - a^\mu Y_1 Y_3,$$

$$n_\mu \frac{dY_3}{du} = \varepsilon_0 f_{\mu\nu} a^\lambda Y_1 Y_2, \qquad \text{and} \qquad n_\mu a^\mu = 0.$$

If one determines $f_{\mu\nu}$ by starting with a_μ by means of $f_{\mu\nu} = n_\mu a_\nu - n_\nu a_\mu$, and conversely:

$$n_\lambda f^{\mu\lambda} = a^\mu,$$

then this definition entails that:

$$f_{\mu\lambda} a^\lambda = n_\mu (a_\lambda a^\lambda).$$

The functions $Y_j(u)$ are then associated by means of the system:

$$\text{(38)} \qquad \frac{dY_1}{du} = Y_2 Y_3, \qquad \frac{dY_2}{du} = - Y_1 Y_3, \qquad \frac{dY_3}{du} = \varepsilon_0 (a_\lambda a^\lambda) Y_1 Y_2.$$

Here, the *spacelike or timelike character of the vector* a^μ, and in turn, of A^μ, *plays an important role.*

In the theory of electromagnetic waves, the vector potential A^μ is of spacelike type ($A_\mu A^\mu < 0$), which leads to the *transversality of the electromagnetic field.*

Here, $F^{\mu\nu}(u)$ will be of *transversal* type if we choose, *a priori:*

$$a_\lambda a^\lambda = - 1.$$

In this case $\varepsilon_0 = + 1$, and we recall the expressions that we previously obtained for $Y_1(u)$, $Y_2(u)$, $Y_3(u)$ in terms of *Jacobi elliptic functions.* With:

$$Y_1^2 + Y_2^2 = \lambda_1^2, \qquad Y_1^2 + Y_3^2 = \lambda_2^2,$$

$$Y_1(u) = \lambda_1 \operatorname{sn}\left(\lambda_2 (u + u_0), \frac{\lambda_1}{\lambda_2}\right),$$

$$Y_2(u) = \lambda_1 \operatorname{cn}\left(\lambda_2 (u + u_0), \frac{\lambda_1}{\lambda_2}\right),$$

$$Y_3(u) = \lambda_1 \operatorname{dn}\left(\lambda_2 (u + u_0), \frac{\lambda_1}{\lambda_2}\right),$$

λ_1, λ_2, u_0 are expressed by starting with their initial values.

However, if the vector A^μ is of spacelike type, while preserving $\varepsilon_0 = +1$*:*

$$a_\lambda\, a^\lambda = +1$$

then the system reduces to the equations:

$$Y_1' = Y_2\, Y_3\,, \qquad Y_2' = -\, Y_1\, Y_3\,, \qquad Y_3' = +\, Y_1\, Y_2\,.$$

The condition:

$$A_\mu\, A^\mu > 0, \qquad \text{hence, } a_\mu\, a^\mu = +1,$$

is encountered in *quantum electrodynamics* with *vector mesons* for the *interactions that are associated with the absorption-re-emission of longitudinal waves* that are *necessary* to serve as the intermediaries of the *interactions that are called static.*

For this case, the equations above come down to the expressions in Jacobi elliptic functions that we encountered for the system (J). The *representative elliptic functions* of A^μ and $F^{\mu\nu}$ *are switched.*

In the two cases, one recovers the degenerate solutions for plane waves that we encountered in the preceding cases.

I have introduced and studied numerous extensions of the models that were presented up to now that are capable of describing more complex micro-physical elements *that are defined by the association of some system of fields* that are represented by groups of tensors or spinor-tensors.

The simplest of these structures leads to functions that are defined by *the inversion of hyperelliptic integrals* for the associated plane waves. A model of this structure type considers a field that is described by the association of n invariants (for $L_+^\uparrow$) $I_1(x^\lambda)$, …, $I_n(x^\lambda)$, and *a vector* $S^\mu(x^\lambda)$ by the intermediary of the nonlinear system of equations:

$$(39)\qquad \begin{aligned} \partial_\mu I_1 &= \kappa_1\, I_2\, I_3 \ldots I_n\, S_\mu\,, \\ \partial_\mu I_2 &= -\,\kappa_2\, I_1\, I_3 \ldots I_n\, S_\mu\,, \\ &\vdots \\ \partial_\mu I_n &= -\,\kappa_n\, I_1\, I_1 \ldots I_{n-1}\, S_\mu\,, \\ \partial_\mu S^\mu &= -\,\kappa_0\, I_1\, I_2 \ldots I_n\,, \end{aligned}$$

(κ_1, κ_2, …, κ_n, κ_0 are real, positive constants).

This system admits solutions of plane-wave type, namely:

$$I_j(x^\lambda) \to Y_j(u), \qquad S^\mu(x^\lambda) \to n^\mu\, Y_0(u),$$
$$(u = n_\mu\, x^\mu, \qquad n_\mu\, n^\mu = 1, \qquad j = 1, 2, \ldots, n).$$

These functions are associated with each other by means of the differential system:

$$(40)\qquad Y_1 Y_1' = -Y_2 Y_2' = \ldots = -Y_n Y_n' = -Y_0 Y_0' = (Y_1 Y_2 \ldots Y_n) \cdot Y_0\,,$$

which admits the n first integrals (that are associated with the initial values of Y_j, Y_0)

$$Y_1^2 + Y_2^2 = \lambda_1^2, \qquad Y_1^2 + Y_j^2 = \lambda_{j-1}^2,$$
$$Y_1^2 + Y_n^2 = \lambda_{n-1}^2, \qquad Y_1^2 + Y_0^2 = \lambda_0^2.$$

The functions $Y_j(u)$ and $Y_0(u)$ are thus determined by *starting with the differential equation:*

(41)
$$(Y_1')^2 = (\lambda_0^2 - Y_1^2)\left[\prod_{j=1}^{n-1}(\lambda_j^2 - Y_1^2)\right].$$

With a *choice of indexing* that puts the values of λ_j^2 in order:

$$\lambda_0^2 > \lambda_{n-1}^2 > \ldots > \lambda_j^2 > \ldots > \lambda_1^2,$$

the function $Y_1(u)$ is defined by *the inversion of the hyperelliptic integral:*

(42)
$$u = \int_{Y_1^0}^{Y_1} \frac{dy}{\sqrt{(\lambda_0^2 - y^2)(\lambda_1^2 - y^2)\cdots(\lambda_{n-1}^2 - y^2)}}.$$

From this, one deduces the set of functions Y_1, Y_2, Y_n, Y_0, *except for a uniformization that is associated with a more complete characterization of the physical objects that are represented.* The *non-uniformity* of a representation *constitutes a degeneracy of the representation that might be lifted by the consideration of a quality of the physical object* that was not taken into consideration in the original model.

Nevertheless, in the case of a system of four functions $Y_1(u)$, $Y_2(u)$, $Y_3(u)$, $Y_0(u)$, *the representation of the functions can once more be effected completely by means of Jacobi elliptic functions.*

I would like to specify the *elements of this correspondence that will be essential* in the second part of this paper.

We start with four functions $Y_1(u)$, $Y_2(u)$, $Y_3(u)$, $Y_0(u)$ that are associated with the differential system:

(43)
$$\frac{dY_1}{du} = Y_2Y_3Y_0, \qquad \frac{dY_2}{du} = -Y_1Y_3Y_0,$$
$$\frac{dY_3}{du} = -Y_1Y_2Y_0, \qquad \frac{dY_0}{du} = -Y_1Y_2Y_3,$$

with the first integrals:

$$Y_1^2 + Y_2^2 = \lambda_1^2, \qquad Y_1^2 + Y_3^2 = \lambda_2^2, \qquad Y_1^2 + Y_0^2 = \lambda_0^2,$$
$$\lambda_0^2 > \lambda_2^2 > \lambda_1^2.$$

Setting:

$$Z_0 = \frac{1}{Y_0^2}, \qquad Z_1 = \frac{Y_1}{Y_0}, \qquad Z_2 = \frac{Y_2}{Y_0}, \qquad Z_3 = \frac{Y_3}{Y_0},$$

the functions Z are associated with each other by means of the system:

(44)
$$Z_1' = \lambda_0^2 Z_2 Z_3, \qquad Z_2' = -(\lambda_0^2 - \lambda_1^2) Z_1 Z_3, \qquad Z_3' = -(\lambda_0^2 - \lambda_2^2) Z_1 Z_2,$$
$$Z_0' = 2Z_1\, Z_2\, Z_3 = \frac{2}{\lambda_0^2} Z_1 Z_1'.$$

The preceding first integrals are written:

$$Z_0 = \frac{1}{\lambda_0}(1 + Z_1^2) = \frac{1}{\lambda_0^2 - \lambda_1^2}(1 - Z_2^2) = \frac{1}{\lambda_0^2 - \lambda_2^2}(1 - Z_3^2),$$

and, in turn:

$$\frac{Z_1^2}{\lambda_0^2} + \frac{Z_2^2}{\lambda_0^2 - \lambda_1^2} = \frac{\lambda_1^2}{\lambda_0^2(\lambda_0^2 - \lambda_1^2)} = \nu_1^2,$$
$$\frac{Z_1^2}{\lambda_0^2} + \frac{Z_3^2}{\lambda_0^2 - \lambda_2^2} = \frac{\lambda_2^2}{\lambda_0^2(\lambda_0^2 - \lambda_2^2)} = \nu_2^2.$$

The preceding analysis leads us to the expressions:

$$Z_1(u) = \lambda_0\, \nu_1 \,\text{sn}\left[\nu_2(u + u_0); \frac{\nu_1}{\nu_2}\right],$$
$$Z_2(u) = \sqrt{\lambda_0^2 - \lambda_1^2}\; \nu_1 \,\text{cn}\left[\nu_2(u + u_0); \frac{\nu_1}{\nu_2}\right],$$
$$Z_3(u) = \sqrt{\lambda_0^2 - \lambda_1^2}\; \nu_2 \,\text{dn}\left[\nu_2(u + u_0); \frac{\nu_1}{\nu_2}\right],$$

and

$$Z_0 = \frac{1}{\lambda_0^2} + \nu_1^2 \text{sn}^2\left[\nu_2(u + u_0); \frac{\nu_1}{\nu_2}\right].$$

Here, the *modulus of the elliptic function* is defined by:

$$k_1 = \frac{\nu_1}{\nu_2} = \frac{\lambda_1}{\lambda_2}\sqrt{\frac{\lambda_0^2 - \lambda_2^2}{\lambda_0^2 - \lambda_1^2}}.$$

The definition $Z_0 = 1/Y_0^2$ gives, successively, for the functions Y:

(45)
$$Y_0(u) = \frac{1}{\sqrt{Z_0}} = \frac{\lambda_0}{\sqrt{1+\lambda_0^2\nu_1^2\mathrm{sn}^2\left[\nu_2(u+u_0), \frac{\nu_1}{\nu_2}\right]}},$$

$$Y_1(u) = Z_1 Y_0 = \frac{Z_1}{\sqrt{Z_0}} = \frac{\lambda_0^2\nu_1\mathrm{sn}\left[\nu_2(u+u_0); k_1\right]}{\sqrt{1+\lambda_0^2\nu_1^2\mathrm{sn}^2\left[\nu_2(u+u_0), k_1\right]}},$$

$$Y_2(u) = Z_2 Y_0 = \frac{Z_2}{\sqrt{Z_0}} = \frac{\lambda_1\mathrm{cn}\left[\nu_2(u+u_0); k_1\right]}{\sqrt{1+\lambda_0^2\nu_1^2\mathrm{sn}^2\left[\nu_2(u+u_0), k_1\right]}},$$

$$Y_3(u) = Z_3 Y_0 = \frac{Z_3}{\sqrt{Z_0}} = \frac{\lambda_2\mathrm{dn}\left[\nu_2(u+u_0); k_1\right]}{\sqrt{1+\lambda_0^2\nu_1^2\mathrm{sn}^2\left[\nu_2(u+u_0), k_1\right]}}.$$

The constants λ_0^2, λ_1^2, λ_2^2, and u_0, and in turn, ν_1^2, ν_2^2, k_1 are defined by *the initial (or maximal) values* of the functions Y.

The *addition theorem for elliptic functions* will give the complete expressions for the functions Y_0, Y_1, Y_2, Y_3 by starting with these values for $u = 0$.

The *system that associates the four functions*, when expressed as Jacobi functions, admits *a particularly important degeneracy*.

For $k_1 = \nu_1 / \nu_2 = 1$, one will have $\lambda_1^2 = \lambda_2^2$, namely, $Y_2^2 = Y_3^2$.

For $Y_2 = Y_3$, the system (43) reduces to three equations:

(46)
$$Y_1' = Y_2^2 Y_0, \qquad Y_2' = -Y_1 Y_2 Y_0, \qquad Y_0' = -Y_1 Y_2^2$$

that are associated with the first integrals:

$$Y_1^2 + Y_2^2 = \lambda_1^2, \qquad Y_1^2 + Y_0^2 = \lambda_0^2.$$

We again assume that $\lambda_0^2 > \lambda_1^2$ (namely, $Y_0^2 > Y_2^2$).

This *system is easily integrated by hyperbolic functions:*

(47)
$$Y_0(u) = \frac{\lambda_0\sqrt{\lambda_0^2-\lambda_1^2}\cosh[\nu_1(u+u_0)]}{\sqrt{\lambda_0^2\cosh^2[\nu_1(u+u_0)]-\lambda_1^2}},$$

$$Y_1(u) = \frac{\lambda_0\lambda_1\sinh[\nu_1(u+u_0)]}{\sqrt{\lambda_0^2\cosh^2[\nu_1(u+u_0)]-\lambda_1^2}},$$

$$Y_2(u) = \frac{\lambda_0\sqrt{\lambda_0^2-\lambda_1^2}}{\sqrt{\lambda_0^2}\cosh^2[\nu_1(u+u_0)]-\lambda_1^2},$$

$$\nu_1 = \frac{\lambda_1}{\lambda_0\sqrt{\lambda_0^2-\lambda_1^2}}.$$

These *particular solutions constitute a "solitary wave"* that is associated with equations (39) that determine the field functions I_1, I_2, I_3, S^μ in the case $I_2 = I_3$, and is written in the general form:

(48) $$\partial_\mu I_1 = \kappa_1 I_2^2 S_\mu, \qquad \partial_\mu I_2 = -\kappa_1 I_1 I_2 S^\mu, \qquad \partial_\mu S^\mu = -\kappa_0 I_1 I_2^2.$$

The structural models for the associated field, whose simplest characteristics I have presented very briefly, *are comprised in such a fashion as to lead to plane waves that are determined completely* and *defined by the inversion of Legendre elliptic integrals of the second type* or their hyperelliptic extensions that are adapted to the representation by Jacobi elliptic functions.

In the study of more complex structures, I was led to characterize some *covariant systems* of first-order partial differential equations *that admit plane waves that are defined by the inversion of elliptic integrals of the third type.*

In the case of the association of fields that are defined by tensors, some simple structures lead to this type of plane-wave solutions.

The simplest of these models associates a vector $A^\mu(x^\lambda)$, an anti-symmetric tensor of second order $F^{\mu\nu}(x^\lambda)$ (a field of electromagnetic or vector meson type) with an invariant $I(x^\lambda)$, and a vector $S^\mu(x^\lambda)$ that constitutes a field of type "spin 0," which is the simplest.

The system that associates these tensors with each other is written:

(49) $$\begin{gathered}\partial_\mu A_\nu - \partial_\nu A_\mu = \kappa_1(S_\lambda S^\lambda)I\,F_{\mu\nu}, \qquad \partial_\lambda F^{\mu\lambda} = \kappa_2(S_\lambda S^\lambda)I\,A^\mu, \\ \partial_\mu I = \kappa_3(S_\lambda S^\lambda)F_{\mu\nu}A^\lambda, \qquad \partial_\lambda S^\lambda = -\frac{1}{2}\kappa_0 I\,F_{\rho\sigma}(A^\rho S^\sigma - A^\sigma S^\rho).\end{gathered}$$

κ_1, κ_2, κ_3, κ_0 are three constants that are *a priori* positive, and the signs that were introduced have been chosen in order to facilitate the classification and ulterior choice of the solutions that are capable of being associated with physical objects.

For the plane wave $u = n_\mu x^\mu$ (with $n_\mu n^\mu = 1$), I will write:

$$A^\mu = a^\mu Y_1(u), \qquad F_{\mu\nu} = f_{\mu\nu} Y_2(u), \qquad I(u) = Y_3(u), \qquad S^\mu = m^\mu Y_0(u).$$

Substitution leads to the system:

$$(n_\mu a_\nu - n_\nu a_\mu)\,Y_1' = \kappa_1(m_\lambda m^\lambda)\,f_{\mu\nu}\,Y_0^2\,Y_2\,Y_3, \qquad n_\lambda f^{\mu\lambda}\,Y_2' = \kappa_2(m_\lambda m^\lambda)\,a^\mu\,Y_0^2\,Y_1\,Y_3,$$

$$n_\mu \, Y_3' = \kappa_3 (m_\lambda \, m^\lambda) \, f_{\mu\lambda} \, a^\lambda \, Y_0^2 \, Y_1 \, Y_2,$$

$$(n_\lambda \, m^\lambda) \, Y_0' = -\frac{1}{2} \, \kappa_1 \, frs \, (a^\rho \, m^\sigma - a^\sigma \, m^\rho) \, Y_1 \, Y_2 \, Y_3 \, Y_0 \, .$$

These relations demand the orthogonality condition:

$$n_\mu \, a^\mu = 0$$

that associates $f_{\mu\nu}$ and a_μ by means of the constitutive relations:

$$f_{\mu\nu} = K(n_\mu \, a_\nu - n_\nu \, a_\mu),$$

so

$$n_\lambda \, f^{\mu\lambda} = -Ka^\mu, \qquad f_{\mu\lambda} \, a^\lambda = K \, n_\mu (a_\lambda \, a^\lambda),$$

$$f_{\rho\sigma} \, (a^\rho \, m^\sigma - a^\sigma \, m^\rho) = -2K(a_\lambda \, a^\lambda)(n_\mu \, m^\mu).$$

We obtain the differential system:

$$\text{(50)} \quad \begin{aligned} Y_1' &= \kappa_1 K(m_\lambda \, m^\lambda) Y_0^2 \, Y_2 \, Y_3 \, , \\ Y_2' &= -\frac{\kappa_2}{K} (m_\lambda \, m^\lambda) Y_0^2 \, Y_1 \, Y_3 \, , \\ Y_3' &= \kappa_3 \, K(m_\lambda \, m^\lambda) Y_0^2 \, Y_1 \, Y_2 \, , \\ Y_0' &= \kappa_0 \, K(a_\lambda \, a^\lambda) \, Y_1 \, Y_2 \, Y_3 \, Y_0 \, , \end{aligned}$$

under the condition $n_\lambda \, m^\lambda \neq 0$.

Choosing m^λ and a^λ to be unitary vectors:

$$(m_\lambda \, m^\lambda) = \varepsilon_1 \, , \qquad (a_\lambda \, a^\lambda) = \varepsilon_0,$$

and setting $K = 1$ reduces this system to the form:

$$\text{(51)} \quad \begin{aligned} Y_1' &= \varepsilon_1 \kappa_1 Y_0^2 Y_2 Y_3, \\ Y_2' &= -\varepsilon_1 \kappa_2 Y_0^2 Y_1 Y_3, \\ Y_3' &= \varepsilon_0 \kappa_1 Y_0^2 Y_1 Y_2, \\ Y_0' &= \varepsilon_0 \kappa_0 Y_1 Y_2 Y_3 Y_0. \end{aligned}$$

For $a_\lambda \, a^\lambda = \varepsilon_0 = -1$ (and, with a simple change of scale), *a wave of transversal type* for this system is written:

$$Y_1' = \varepsilon_1 Y_0^2 Y_2 Y_3, \qquad Y_1' = -\varepsilon_1 Y_0^2 Y_1 Y_3,$$
$$Y_3' = -\varepsilon_1 Y_0^2 Y_1 Y_2, \qquad Y_0' = -Y_1 Y_2 Y_3 Y_0. \tag{52}$$

With $\varepsilon_1 = +1$ and S^μ time-like, this system admits the first integrals:

$$Y_1^2 + Y_2^2 = \lambda_1^2, \qquad Y_1^2 + Y_3^2 = \lambda_2^2, \qquad Y_0^2 + Y_1^2 = \lambda_0^2.$$

Assuming that $\lambda_0^2 > \lambda_2^2 > \lambda_1^2$, one obtains for $Y_1(u)$:

$$\left(\frac{dY_1}{du}\right)^2 = (\lambda_1^2 - Y_1^2)(\lambda_2^2 - Y_1^2)(\lambda_0^2 - Y_1^2)^2.$$

$Y_1(u)$ *will thus be defined by inversion of the elliptic integral of the third kind:*

$$u = \int_{Y_1^0}^{Y_1} \frac{dy}{(\lambda_0^2 - y^2)\sqrt{(\lambda_1^2 - y^2)(\lambda_2^2 - y^2)}} \tag{53}$$

(for $|Y_1| < \lambda_1$).

In the case $\varepsilon_0 = +1$ (A^μ, $F^{\mu\nu}$ of longitudinal type (static case)), the system reduces to:

$$Y_1' = \varepsilon_1 \kappa_1 Y_0^2 Y_2 Y_3, \qquad Y_2' = -\varepsilon_1 \kappa_2 Y_0^2 Y_1 Y_3,$$
$$Y_3' = \varepsilon_1 \kappa_3 Y_0^2 Y_1 Y_2, \qquad Y_0' = \kappa_0 Y_1 Y_2 Y_3 Y_0, \tag{54}$$

which again leads (for $\varepsilon_1 = +1$) to a solution of the same type, but with the functions Y_1 and Y_2 exchanged.

Numerous models for the associated field by the intermediary of their energies are described by functional structures of this kind.

An associated field structure that is very close to the one that was presented up to now *corresponds to numerous studies that were carried out by starting, a priori, with second-order partial differential equations.*

I will start by describing a field of "spin 0" that is defined by $I(x^\lambda)$ and $S^\mu(x^\lambda)$ *with self-coupling* and is associated with the *first-order equations:*

$$\partial_\mu I = [\alpha_1 + \beta_1 I^2]\, S_\mu, \qquad \partial_\mu S^\mu = -[\alpha_2 + \beta_2 (S_\lambda S^\lambda)]\, I. \tag{55}$$

α_1, α_2, β_1, α_2 are constants that are given *a priori*. With:

$$S_\mu = \frac{\partial_\mu I}{\alpha_1 + \beta_1 I^2},$$

I satisfies the second-order equation:

$$\partial_\mu \partial^\mu I = -\alpha_1(\alpha_2 + \beta_1 I^2) I + (2\beta_1 - \beta_2)(\alpha_1 + \beta_1 I^2)(S_\mu S^\mu) I,$$

or

(56) $$\partial_\mu \partial^\mu I = -\alpha_2(\alpha_1 + \beta_1 I^2)I + \frac{(2\beta_1 - \beta_2)}{\alpha_2 + \beta_1 I^2}(\partial_\mu I)(\partial^\mu I)\, I.$$

The condition between the constants:

(57) $$\beta_2 = 2\beta_1 ,$$

reduces the system to:

(58) $$\partial_\mu I = (\alpha_1 + \beta_1 I^2)\, S_\mu , \qquad \partial_\mu S^\mu = -[\alpha_2 + \beta_1(S_\lambda S^\lambda)]\, I,$$

and I satisfies the second-order equation:

(59) $$\partial_\mu \partial^\mu I + \alpha_1\alpha_2 I + \beta_1\alpha_2 I^3 = 0.$$

The plane-wave solutions of the system (55) are again obtained by writing:

$$u = n_\mu x^\mu, \qquad I = Y_1(u), \qquad S^\mu = n^\mu Y_2(u).$$

One then has:

(60) $$\frac{dY_1}{du} = (\alpha_1 + \beta_1 Y_1^2)\, Y_2 , \qquad \frac{dY_2}{du} = -(\alpha_2 + \beta_2 Y_2^2) Y_1 .$$

This system admits the first integral:

$$(\alpha_1 + \beta_1 Y_1^2)^{\beta_2/\beta_1}(\alpha_2 + \beta_2 Y_2^2) = \lambda_0^2 .$$

From this, we deduce the differential equation for $Y_1(u)$:

(61) $$\left(\frac{dY_1}{du}\right)^2 = \frac{1}{\beta_2}[\lambda_0^2(\alpha_1 + \beta_1 Y_1^2)^{2-\beta_2/\beta_1} - \alpha_2(\alpha_1 + \beta_1 Y_1^2)^2] .$$

We *recover an elliptic integral* that defines $Y_1(u)$ by a *Jacobi elliptic function* for $\beta_2 = 2\beta_1$ and for $\beta_2 = \beta_1$.

For $\beta_2 = 2\beta_1$, what remains is:

(62) $$\left(\frac{dY_1}{du}\right)^2 = \frac{1}{2\beta_1}[\lambda_0^2 - \alpha_1^2\alpha_2 - 2\alpha_1\alpha_2\beta_1 Y_2^2 - \alpha_2\beta_1^2 Y_1^4],$$

and for $\beta_2 = \beta_1$:

(63) $$\left(\frac{dY_1}{du}\right)^2 = \frac{1}{\beta_2}[(\alpha_1 + \beta_1 Y_1^2)(\lambda_0^2 - \alpha_1\alpha_2 - \alpha_2\beta_1 Y_1^2) .$$

According to the values of the constants and the initial values that are associated with λ_0^2, $Y_1(u)$, and $Y_2(u)$ (and, in turn, $S^\mu(u)$), they are represented by starting with the functions $\mathrm{sn}(\omega u, k)$, $\mathrm{cn}(\omega u, k)$, $\mathrm{dn}(\omega u, k)$, and *possibly their degeneracies.*

I would like to extend the preceding results by introducing, and determining completely, the solutions of "plane-wave" type for *a model of a "corpuscle-field*" in which *the physical object is recognized and described by the association of the field that belongs to types that linear quantum mechanics associates with particles with spin* $\hbar/2$ (states of "electronic" type), *to particles of spin* $\hbar$ (electromagnetic or "vector meson" states), *and particles of spin* $2\hbar$ (gravitational states).

The systems of partial fields whose association defines the physical object considered will be represented in space-time (coordinates x_0, $\mathbf{x}$) by *Van der Waerden tensor-spinors* and written in terms of *nonlinear first-order systems of partial differential equations* that extend the wave equations of the general theory of particles with spin and *associate the set of constitutive fields*, without possible separation, except for degeneracy.

In the model that we will study here, *the physical object is defined by three systems of partial fields*, so the representation will be effected by three groups of tensor-spinors that are symmetric with respect to the indices of each of the two types.

These systems of spinors will be associated with each other, and their evolution will be effected globally. Each of the first-order partial differential equations that relates to one of the types of partial fields will contain just one term: *an invariant spinor* that associates the set of spinors that describes the fields of the other two types.

The first type of field (states of spin $\hbar/2$) introduces *two spinors of rank one:* $\xi^m(x^\mu)$, $\eta^{\dot{i}}(x^\mu)$ that are associated with each other in the linear case by means of the Dirac equations, when written in the spinorial formalism.

For the *Van der Waerden spinors*, I will utilize the usual spinorial connection, which is denoted here by:

$$\xi_m = \varepsilon_{mr}\,\xi^r = -\,\varepsilon_{rm}\,\xi^r, \qquad \xi^m = \varepsilon^{mr}\,\xi_r\,,$$

$$\eta_{\dot{i}} = \varepsilon_{\dot{i}\dot{s}}\eta^{\dot{s}}, \qquad (\xi_1 = -\,\xi^2, (\xi_2 = -\,\xi^1),$$

$$(64)\qquad \varepsilon_{m_1m_2} = -\varepsilon_{m_2m_1} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}, \qquad \varepsilon_{\dot{m}_1\dot{m}_2} = -\varepsilon_{\dot{m}_2\dot{m}_1} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix},$$

$$\varepsilon^{mr}\,\varepsilon_{rs} = \delta^m{}_s\,, \qquad \varepsilon^m{}_m = 2, \qquad \varepsilon_m{}^m = -\,2.$$

The association between the vector A^μ (in the restricted sense of the group $L_+^\uparrow$) and the spinor of second rank $A^{\dot{l}m}$ will be effected by the reciprocal correspondence:

(65)
$$A^{\dot{l}m} = (g_\mu)^{\dot{l}m} A^\mu, \qquad A^\mu = \frac{1}{2}(g^\mu)_{\dot{l}m} A^{\dot{l}m} = \frac{1}{2}(g^\mu)^{\dot{l}m} A_{\dot{l}m},$$

$$[(g^0)^{\dot{l}m} \equiv \| \sigma_0 \|, \qquad (g^p)^{\dot{l}m} \equiv \| \sigma_p \|,$$

$$\sigma_p\, \sigma_q = i\sigma_r, \quad \sigma_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \qquad \sigma_2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix},$$

$$\sigma_3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \quad \sigma_0 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\ ].$$

The operators $(g^\mu)^{\dot{l}m}$ are associated with each other by the relations:

(66)
$$(g^\mu)^{\dot{r}}{}_l (g^\nu)_{\dot{r}m} + (g^\nu)^{\dot{r}}{}_l (g^\mu)_{\dot{r}m} = 2g^{\mu\nu}\,\varepsilon_{lm},$$
$$(g^\mu)_{\dot{l}}{}^r (g^\nu)_{\dot{m}r} + (g^\nu)_{\dot{l}}{}^r (g^\mu)_{\dot{m}r} = 2g^{\mu\nu}\,\varepsilon_{\dot{l}\dot{m}},$$

$$(g^\mu)_{\dot{l}r}(g_\varpi)_{\dot{m}s} = 2\varepsilon_{\dot{l}\dot{m}}\varepsilon_{rs}.$$

These relations entail the following relations for the "vector" $(A^\mu,\ A^{\dot{l}m})$:

(67)
$$A^{\dot{r}}{}_l A_{\dot{r}m} = (A_\mu A^\mu)\ \varepsilon_{lm}, \quad A_{\dot{l}}{}^{r} A_{\dot{m}r} = (A_\mu A^\mu)\,\varepsilon_{\dot{l}\dot{m}}, \quad A_{\dot{r}l} A^{\dot{r}l} = 2(A_\mu A^\mu),$$

and for two vectors $(A^\mu,\ A^{\dot{l}m})$, $(B^\mu,\ B^{\dot{l}m})$, the relations:

$$A^{\dot{r}}{}_l B_{\dot{r}m} - A^{\dot{r}}{}_m B_{\dot{r}l} = 2(A^\mu B_\mu)\ \varepsilon_{lm}, \qquad A_{\dot{r}l} B^{\dot{r}l} = 2(A_\mu B^\mu).$$

The vector x^μ will be associated with the spinor $x^{\dot{l}m}$ by way of:

(68)
$$x^{\dot{l}m} = (g_\mu)^{\dot{l}m} x^\mu, \qquad x^\mu = \frac{1}{2}(g^\mu)_{\dot{l}m} x^{\dot{l}m}.$$

The operator $\partial_\mu \equiv \partial / \partial x^\mu$ will be associated with the operator that is denoted $\partial_{\dot{l}m}$:

(69)
$$\partial_{\dot{l}m} = (g^\mu)_{\dot{l}m} \frac{\partial}{\partial x^\mu} = 2\frac{\partial}{\partial x^{\dot{l}m}}, \qquad \frac{\partial}{\partial x^\mu} = \frac{1}{2}(g_\mu)^{\dot{l}m}\partial_{\dot{l}m}.$$

The d'Alembertian operator $\Box \equiv \partial_\mu\, \partial^\mu$ will be represented by the combination:

$$\partial^{\dot{r}}{}_l \partial_{\dot{r}m} = \varepsilon_{lm}\, \partial^\mu\, \partial_\mu = \varepsilon_{lm}\ \Box, \qquad \partial^{\dot{r}l}\partial_{\dot{r}l} = 2\Box.$$

The first subsystem of partial fields (in the linear theory, they are the isolated states of spin $\hbar/2$) will be comprised by starting with *two spinors of rank one* $\xi^m(x^\lambda)$, $\eta^i(x^\lambda)$ *that are associated with a system that extends the Dirac equations* and is written:

$$(E_1) \qquad \partial^i{}_r \xi^r = \Omega_1 \eta^i, \qquad \partial^m{}_{\dot r} \xi^{\dot r} = \Omega_1 \xi^m .$$

Ω_1 will be *a spinorial invariant that is defined by starting with the set of spinors that represent the other two systems of fields*, and whose structure I will specify later on.

The second subsystem of partial fields is an extension of the wave equations of the linear theory of spin $\hbar$ particles (electromagnetic or vector meson field).

These wave equations, when written in spinorial formalism (tensors adapted to the group $L_+^\uparrow$), associate a spinor (vector potential) $A^{\dot l m}$ and two *symmetric* tensor-spinors of second order:

$$F^{\underline{m_1 m_2}}(x^\lambda) \equiv F^{\underline{m_2 m_1}}(x^\lambda), \qquad G^{\underline{\dot i_1 \dot i_2}} \equiv G^{\underline{\dot i_2 \dot i_1}}$$

(restricting to $L_+^\uparrow$ for A^μ and a *self-dual* $F^{\mu\nu} = -F^{\nu\mu}$).

In the most general linear theory, these tensor-spinors are associated with each other by the system of first order:

$$\partial_{\dot r m_1} A^{\dot r}{}_{m_2} + \partial_{\dot r m_2} A^{\dot r}{}_{m_1} = \kappa_1 F_{\underline{m_1 m_2}},$$

$$(70) \qquad \partial_{\dot l}{}^r F_{\underline{rm}} + \partial^{\dot r}{}_m G_{\underline{\dot r \dot l}} = -\kappa_1 A_{\underline{\dot l m}},$$

$$\partial_{\dot i_1 r} A_{\dot i_2}{}^r + \partial_{\dot i_2 r} A_{\dot i_1}{}^r = \kappa_1 G_{\underline{\dot i_1 \dot i_2}} .$$

This system reduces to two equations by the restriction (which is equivalent to the *reality* of the tensors A^μ, $F^{\mu\nu}$):

$$A_{\dot l m} = A_{m \dot l}, \qquad G_{\underline{\dot m_1 \dot m_2}} = F_{\underline{m_1 m_2}},$$

namely:

$$(71) \qquad \partial_{\dot r m_1} A^{\dot r}{}_{m_2} + \partial_{\dot r m_2} A^{\dot r}{}_{m_1} = \kappa_1 F_{\underline{m_1 m_2}}, \qquad \partial^{\dot i}{}_r F^{\underline{rm}} + \partial_r{}^m F^{\underline{\dot r \dot l}} = \kappa_1 A^{\dot l m}$$

(κ_1 constant).

This will be extended, and the field $(A^{\dot l m}, F^{\underline{m_1 m_2}})$ will be associated with other partial fields in a more general system that we write:

$$(E_2) \qquad \begin{aligned} &\partial_{\dot r m_1} A^{\dot r}{}_{m_2} + \partial_{\dot r m_2} A^{\dot r}{}_{m_1} = 2\Omega_2 F_{\underline{m_1 m_2}}, \\ &\partial^{\dot i}{}_r F^{\underline{rm}} + \partial_{\dot r}{}^m F^{\underline{\dot r \dot l}} = 2\Omega_2 A^{\dot l m}. \end{aligned}$$

Here, Ω_2 will be a *second spinorial invariant* that is constructed *symmetrically* from the *tensor-spinors of the other two subsystems.*

Equations (71) of the linear case entail the condition on $A^{\dot l m}$ that:

(72) $$\partial_{\dot r m_1} A^{\dot r}{}_{m_2} + \partial_{\dot r m_2} A^{\dot r}{}_{m_1} = 0,$$

so

$$\partial_{\dot r m}(\Omega_2 A^{\dot l m}) = 0.$$

The third subsystem that is associated with the notion of gravitational field – a field that is essentially comprised of all particle masses – will be written by starting with irreducible tensor-spinors of the *linear quantum theory of spin* $2\hbar$ *particles.*

The general theory defines this type of fields by a system of *five tensor-spinors:*

$$\Phi_{(0)}^{\underline{\dot l_1 \dot l_2},\underline{m_1 m_2}}, \qquad \Phi_{(1)}^{\dot l,\underline{m_1 m_2 m_3}}, \qquad \Phi_{(2)}^{|\underline{m_1 m_2 m_3 m_4}}, \qquad \Phi_{(3)}^{\underline{\dot l_1 \dot l_2 \dot l_3},m}, \qquad \Phi_{(4)}^{\underline{\dot l_1 \dot l_2 \dot l_3 \dot l_4}}.$$

(9, 8, 5, 8, 5 gives 35 complex components in all), and they are associated with each other by means of the partial differential equations:

(74)
$$\begin{aligned}
\partial_{\dot r}{}^{|\underline{m_1}}\Phi_{(1)}^{\dot r,\underline{m_1 m_2 m_3 m_4}|} &= \kappa_0 \Phi_{(2)}^{\underline{m_1 m_2 m_3 m_4}|}, \\
\partial^{|\underline{\dot l_1}}{}_{,r}\Phi_{(2)}^{\underline{r m_1 m_2 m_3 m_4}|} + \partial_{\dot r,}{}^{|\underline{m_1}}\Phi_{(0)}^{\underline{\dot r \dot l,m_2 m_3}|} &= \kappa_0 \Phi_{(1)}^{\dot l,\underline{m_1 m_2 m_3 m_4}|}, \\
\partial^{|\underline{\dot l_1}}{}_{,r}\Phi_{(1)}^{\underline{\dot l_2}|,\underline{r m_1 m_2}} + \partial_{\dot r,}{}^{|\underline{m_1}}\Phi_{(3)}^{\underline{\dot l_1 \dot l_2 \dot r},\underline{m_2}|} &= \kappa_0 \Phi_{(0)}^{\underline{\dot l_1 \dot l_2},\underline{m_1}|}, \\
\partial^{|\underline{\dot l_1}}{}_{,r}\Phi_{(0)}^{\underline{\dot l_2 \dot l_3}|,\underline{rm}} + \partial_{\dot r,}{}^{m}\Phi_{(4)}^{\underline{\dot l_1 \dot l_2 \dot l_3 \dot r},\underline{m_2}|} &= \kappa_0 \Phi_{(3)}^{\underline{\dot l_1 \dot l_2 \dot l_3},m}, \\
\partial^{|\underline{\dot l_1}}{}_{,r}\Phi_{(3)}^{\underline{\dot l_1 \dot l_2 \dot l_3}|,r} &= \kappa_0 \Phi_{(4)}^{\underline{\dot l_1 \dot l_2 \dot l_3 \dot l_4}}.
\end{aligned}$$

In these relations, I have denoted the *symmetry operations on the indices by underlining the associated indices,* and in *the case of symmetry of the partial groups,* I have denoted:

$$A^{|\underline{\dot L_1},M_1} B^{\underline{\dot L_2},M_2} \equiv A^{\dot L_1,M_1} B^{\dot L_2,M_2} + A^{\dot L_2,M_1} B^{\dot L_1,M_2},$$

$$A^{\dot L_1,|\underline{M_1}} B^{\dot L_2,\underline{M_1}|} \equiv A^{\dot L_1,M_1} B^{\dot L_2,M_2} + A^{\dot L_1,M_2} B^{\dot L_2,M_1}.$$

By the *association of this representation with a macroscopic theory* that is described by *real tensors,* I will assume the *reduction of the five tensors* to three by the correspondence:

(75)
$$\begin{aligned}
\Phi_{(1)}^{\dot l,\underline{m_1 m_2 m_3}} &\equiv \Phi_{(3)}^{\dot l,\underline{m_1 m_2 m_3}}, \\
\Phi_{(2)}^{\underline{m_1 m_2 m_3 m_4}} &\equiv \Phi_{(4)}^{\underline{m_1 m_2 m_3 m_4}}.
\end{aligned}$$

For the linear case, the system above reduces to three equations:

(76)
$$\begin{aligned}
\partial_{\dot r}{}^{|\underline{m_1}}\Phi_{(1)}^{\dot r,\underline{m_1 m_2 m_3 m_4}|} &= \kappa_0 \Phi_{(2)}^{\underline{m_1 m_2 m_3 m_4}|}, \\
\partial^{|\underline{\dot l_1}}{}_{,r}\Phi_{(2)}^{\underline{r m_1 m_2 m_3 m_4}|} + \partial_{\dot r,}{}^{|\underline{m_1}}\Phi_{(0)}^{\underline{\dot r \dot l,m_2 m_3}|} &= \kappa_0 \Phi_{(1)}^{\dot l,\underline{m_1 m_2 m_3 m_4}|}, \\
\partial^{|\underline{\dot l_1}}{}_{,r}\Phi_{(1)}^{\underline{\dot l_2}|,\underline{r m_1 m_2}} + \partial_{\dot r,}{}^{|\underline{m_1}}\Phi_{(1)}^{\underline{\dot l_1 \dot l_2 \dot r},\underline{m_2}|} &= \kappa_0 \Phi_{(0)}^{\underline{\dot l_1 \dot l_2},\underline{m_1 m_2}|}.
\end{aligned}$$

In the general corpuscular theory that we propose, this system will be extended by introducing, in place of the constant κ_0, *a spinorial invariant* that is constructed symmetrically from the set of tensor-spinors that are associated with the partial fields of spin $\hbar/2$ and $\hbar$.

The evolution of the components of the gravitational field that are associated with those of the other partial fields will thus be described by starting with the equations:

$$(E_3) \qquad \begin{aligned} \partial^{|\underline{\dot{l}_1}}{}_{,r}\,\Phi_{(1)}^{\underline{l_2}|,\underline{rm_1m_2}} + \partial_{\dot{r},}{}^{|\underline{m_1}}\,\Phi_{(1)}^{\underline{\dot{r}\dot{l}_1\dot{l}_2},\underline{m_2}|} &= \Omega_0\,\Phi_{(0)}^{\underline{\dot{l}_1\dot{l}_2},\underline{m_1m_2}}, \\ \partial_{,\dot{r}}{}^{|\underline{m_1}}\,\Phi_{(0)}^{\underline{\dot{r}l}|,\underline{m_2m_3}} + \partial^{\dot{l}}{}_{r,}\,\Phi_{(2)}^{\underline{rm_1m_2m_3}} &= \Omega_0\,\Phi_{(1)}^{\dot{l},\underline{m_1m_2m_3}}, \\ \partial_{\dot{r},}{}^{|\underline{m_1}}\,\Phi_{(1)}^{\dot{r},\underline{m_2m_3m_4}|} &= \Omega_0\,\Phi_{(2)}^{\underline{m_1m_2m_3m_4}}. \end{aligned}$$

This system leads to the conditions on $\Phi_{(0)}^{\underline{\dot{l}_1\dot{l}_2},\underline{m_1m_2}}$:

$$(77) \qquad \begin{gathered} \partial_{\dot{r},}{}^{m_i}\left(\Omega_0\,\Phi_{(0)}^{\underline{\dot{r}\dot{l}},\underline{m_jm_k}}\right) = \partial_{\dot{r}}{}^{m_j}\left(\Omega_0\,\Phi^{\underline{\dot{r}\dot{l}},\underline{m_im_k}}\right) \\ \partial^{\dot{l}_i}{}_{,r}\left(\Omega_0\,\Phi_{(0)}^{\underline{\dot{l}_j\dot{l}_k},\underline{rm}}\right) = \partial^{\dot{l}_j}{}_{,r}\left(\Omega_0\,\Phi^{\underline{\dot{l}_k\dot{l}_i},\underline{rm}}\right) \\ (i, j, k = 1, 2, 3). \end{gathered}$$

Ω_1, Ω_2, Ω_3 are *three spinorial invariants* (which are, a priori, real) that are constructed from three tensor-spinors (of vector type (for $L_+^{\uparrow}$)) that are associated with each of the principal fields, respectively:

$$(78) \qquad \begin{aligned} \Omega_1 &= E_{\dot{l}m}G^{\dot{l}m}, \\ \Omega_2 &= j_{\dot{l}m}G^{\dot{l}m}, \\ \Omega_3 &= j_{\dot{l}m}E^{\dot{l}m}. \end{aligned}$$

$j^{\dot{l}m}$ will be the homologue of the "current" vector in the linear theory of particles of spin $\hbar/2$:

$$(79) \qquad j^{\dot{l}m} = \xi^{\dot{l}}\xi^m + \eta^{\dot{l}}\eta^m,$$

and satisfies the equation of continuity (Ω_1 is real):

$$\partial_{\dot{l}m}\,j^{\dot{l}m} = 0.$$

$E^{\dot{l}m}$ will be defined by starting with the components $A^{\dot{l}m}$ and $F^{\underline{m_1m_2}}$ by forming the (real) Hermitian combination:

$$(80) \qquad E_{\dot{l}m} = F_{\underline{\dot{l}\dot{r}}}A^{\dot{r}}{}_m + F_{\underline{rm}}A_{\dot{l}}{}^r,$$

(which is the spinorial homologue of the vector $E_\mu = F_{\mu\lambda}A^\lambda$).

$G^{\dot{l}m}$ is constructed from the components of the *gravitational field*, and will be defined by the combination:

(81) $$G^{\dot{l}m} = \Phi_{(1)\dot{l}\underline{\dot{r}_1\dot{r}_2},}{}^{s}\Phi^{\underline{\dot{r}_1\dot{r}_2},}_{(0)\ \underline{sm}} + \Phi^{\dot{s}}_{(1),\underline{\dot{l}\dot{r}_1\dot{r}_2}}\Phi_{(0)\underline{\dot{s}l},}{}^{\underline{r_1 r_2}} .$$

A second possible form will be constructed by starting with:

$$\Phi^{\dot{l},\underline{m_1 m_2 m_3}}_{(1)} \qquad \text{and} \qquad \Phi^{\underline{m_1 m_2 m_3 m_4}}_{(2)} .$$

The analysis that follows will show the identity of these two choices.

Starting with the system (E_1), (E_2), (E_3) that determines *the evolution that is associated with the three systems of fields that comprise a physical object*, I would like to determine completely the *plane-wave solutions.*

Here, a component of the "plane-wave" field will be a component of the tensor-spinor that is a function of the unique variable u that is invariant with respect to the transformations of the spinorial group , namely, $u = n_\mu x^\mu$, n_μ being a *time-like* unitary vector of $n_\mu n^\mu = 1$.

The relations of association between spinors and tensors (in the restricted sense of $L_+^\uparrow$) lead us to take the homologue of a propagation vector n^μ to be a spinor $n^{\dot{l}m}$ (which is Hermitian: $n^{\dot{l}m} = n^{m\dot{l}}$) that is associated with n^μ by the relations:

$$n^{\dot{l}m} = (g_\mu)^{\dot{l}m} n^\mu , \qquad n^\mu = \frac{1}{2}(g^\mu)_{\dot{l}m} n^{\dot{l}m} ,$$
$$n^{\dot{r}}{}_l n_{\dot{l}m} = (n_\mu n^\mu)\, \varepsilon_{lm} = \varepsilon_{lm} ,$$
$$n_{\dot{l}}{}^r n_{\dot{m}l} = \varepsilon_{\dot{l}\dot{m}} .$$

With

$$x^{\dot{l}m} = (g_\mu)^{\dot{l}m} x^\mu , \qquad x^\mu = \frac{1}{2}(g^\mu)_{\dot{l}m} x^{\dot{l}m} ,$$

one deduces from this that:

$$\begin{aligned} n^{\dot{r}}{}_l (x)_{\dot{r}m} - n^{\dot{r}}{}_m (x)_{\dot{r}l} &= 2(n_\mu x^\mu)\, \varepsilon_{lm} , \\ &= 2u\varepsilon_{lm} , \end{aligned}$$

so

(82) $$u = \frac{1}{2} n_{\dot{l}m} x^{\dot{l}m} .$$

For a field function $\Phi_A(x^\mu)$ that reduces to a plane wave:

$$\Phi_A(x^\mu) \to \Phi_A(n_\mu x^\mu) \to \Phi_A(u),$$

(83) $$\begin{aligned} \partial_{\dot{l}m}\Phi_A(u) &= (g^\mu)_{\dot{l}m}\partial_\mu \Phi_A , \\ &= 2\frac{\partial}{\partial x^{\dot{l}m}}\Phi_A(u) = \eta_{\dot{l}m}\frac{\partial}{\partial u}\Phi_A(u) . \end{aligned}$$

In the system (E_1), (E_2), (E_3), the set of tensor-spinors will be taken to be functions of a single variable *u* and, in turn, to come down to a *system of differential equations*, while Ω_1, Ω_2, Ω_3 will be expressed by functions of *u* and *constitute the – essentially nonlinear – coupling terms between the three partial systems.*

The system (E_1) reduces to:

(84)
$$n^i{}_r \frac{d}{du}\xi^r(u) = \Omega_1(u)\eta^l ,$$
$$n_{\dot r}{}^m \frac{d}{du}\eta^{\dot r}(u) = \Omega_1(u)\xi^l(u).$$

This form leads us to associate $\xi^\mu(u)$, $\eta^i(u)$ *with two real functions* $Y_1(u)$, $Y_2(u)$ by the relations:

$$\xi^m(u) = \alpha^m Y_1(u), \qquad \eta^i(u) = \beta^i Y_2(u) .$$

Substitution leads to:

$$n^i{}_r \alpha^r \frac{dY_1}{du} = \Omega_1(u)\beta^i Y_2 , \qquad n_{\dot r}{}^m \beta^{\dot r} \frac{d}{du} Y_2 = \Omega_1 \alpha^m Y_1(u) ,$$

and upon introducing an arbitrary constant λ_0, by using:

$$\beta_{\dot l} = \lambda_0 n_{\dot l r}\alpha^r ,$$

the system reduces to the form:

(85)
$$\frac{dY_1}{du} = \lambda_0 \Omega_1(u) Y_2(u), \qquad \frac{dY_2}{du} = -\frac{1}{\lambda_0}\Omega_1(u) Y_1(u),$$

In the case of linear equations (Dirac particles of spin $\hbar/2$), Ω_1 is constant, $\Omega_1 = \mu_0$, and this system entails that:

$$Y_1'' = \lambda_0 \mu_0 Y_2' = -\mu_0^2 Y_1 .$$

We have associated the "current":

$$j^i = \xi^i \xi^m + \eta^i \eta^m$$

with the partial field ξ^m, η^i. Here, with:

$$\xi^m = \alpha^m Y_1(u), \qquad \eta^i = \lambda_0 n^i{}_r \alpha^r Y_2(u)$$

it is written:

(86)
$$j^{lm}(u) = \alpha^l \alpha^r (Y_1(u))^2 + \lambda_0^2 n^i{}_r n_{\dot s}{}^m \alpha^{\dot s} \alpha^r (Y_2(u))^2 .$$

The system ($Y_1(u)$, $Y_2(u)$) has, consequently:

$$Y_1 Y_1' = -\lambda_0^2 Y_2 Y_2'.$$

As a result, $Y_1 Y_1' + \lambda_0^2 Y_2 Y_2' = 0$. One thus has – *independently of the form of* Ω_1:

$$Y_1^2 + \lambda_0^2 Y_2^2 = \text{const.}$$

For the subsystem (E_2), we write the plane waves in the form:

$$A^{\dot{l}m} = a^{\dot{l}m} Z_1(u), \qquad F^{\underline{m_1 m_2}} = f^{\underline{m_1 m_2}} Z_2(u),$$

$Z_1(u)$, $Z_2(u)$ being real functions.

Substitution leads to the relations:

$$\text{(87)} \qquad \begin{aligned} (n_{\dot{r}}{}^{m_1} a^{\dot{r}m_2} + n_{\dot{r}}{}^{m_2} a^{\dot{r}m_1}) \frac{dZ_1}{du} &= 2\Omega_2(u) f^{\underline{m_1 m_2}} Z_2(u), \\ (n^{\dot{l}}{}_{\dot{r}} f^{\underline{rm}} + n_{\dot{r}}{}^{m_2} f^{\underline{\dot{r}\dot{l}}}) \frac{dZ_2}{du} &= 2\Omega_2(u) a^{\dot{l}m} Z_1(u), \end{aligned}$$

and with (73):

$$(n_{\dot{r}m_1} a^{\dot{r}}{}_{m_2} - n_{\dot{r}m_2} a^{\dot{r}}{}_{m_1}) \frac{d}{du}(\Omega_2 Z_1) = 0;$$

hence, one has the following structural relation for the coefficients $a^{\dot{l}m}$:

$$\text{(88)} \qquad n_{\dot{r}m_1} a^{\dot{r}}{}_{m_2} = n_{\dot{r}m_2} a^{\dot{r}}{}_{m_1}.$$

These relations lead us to *associate the amplitudes* $f^{\underline{m_1 m_2}}$, $a^{\dot{l}m}$ with each other by ways of the relations:

$$\begin{aligned} f^{\underline{m_1 m_2}} &= C_1 (n_{\dot{r}}{}^{m_1} a^{\dot{r}m_2} + n_{\dot{r}}{}^{m_2} a^{\dot{r}m_1}), \\ n^{\dot{l}}{}_r f^{\underline{rm}} &= -2C_1 a^{\dot{l}m}, \end{aligned}$$

If C_1 is an arbitrary real constant then $Z_1(u)$, $Z_2(u)$ are associated with the differential system:

$$\text{(89)} \qquad \begin{aligned} \frac{dZ_1}{du} &= 2C_1 \Omega_2(u) Z_2(u), \\ \frac{dZ_2}{du} &= -\frac{1}{2C_1} \Omega_2(u) Z_1(u). \end{aligned}$$

The coupling term $E^{\dot{l}m}$ is written for the plane wave:

$$\text{(90)} \qquad E_{\dot{l}m} = E_{\dot{l}m}(u) = (f_{\dot{l}\dot{r}} a^{\dot{r}}{}_m + f_{rm} a_{\dot{l}}{}^r) Z_1(u) Z_2(u) = 4C_1 n_{\dot{r}s} (a_{\dot{l}}{}^s a^{\dot{r}}{}_m) Z_1 Z_2.$$

This expression leads, notably, to the following expression for the product $n_{\dot{l}m}E^{\dot{l}m}(u)$:

$$n_{\dot{l}m}E^{\dot{l}m}(u) = -4C_1(a_{\dot{r}s}a^{\dot{r}s})\,Z_1(u)\,Z_2(u).$$

For the vector a^{μ} that is associated with $a^{\dot{l}m}$, we have $a_{\dot{l}m}a^{\dot{l}m} = 2(a_{\mu}\,a^{\mu})$, and as a consequence:

(91) $$n_{\dot{l}m}E^{\dot{l}m}(u) = -8C_1(a_{\mu}\,a^{\mu})Z_1(u)\,Z_2(u).$$

This *term thus depends upon the space-like or time-like character of the vector, a^{μ}.*

The relation $n_{\dot{r}}^{\;m_1}a^{\dot{r}m_2} - n_{\dot{r}}^{\;m_2}a^{\dot{r}m_1} = 0$ expresses the *orthogonality* of a_{μ} and a^{μ}, namely, $n_{\mu}\,a^{\mu} = 0$.

In the linear theory of electromagnetic type:

$$E^{\mu} = F^{\mu\lambda}A_{\lambda} \to f^{\mu\lambda}a_{\lambda}\,Z_1Z_2 = (n^{\mu}a^{\lambda} - n^{\lambda}a^{\mu})\,a_{\lambda}\,Z_1Z_2 = n^{\mu}(a_{\mu}\,a^{\mu})Z_1(u)\,Z_2(u).$$

$E^{\mu}(u)$ *depends upon the space-like or time-like character of the vector* $A^{\mu}(u)$ that is orthogonal to n^{μ}. As a consequence, the *space-like or time-like character of A^{μ} leads us to distinguish the transversal waves, which have $a_{\mu}\,a^{\mu} = -1$ and A^{μ} space-like, from the longitudinal waves, which have $a_{\mu}\,a^{\mu} = +1$ and A^{μ} time-like.*

In linear quantum mechanics, the interactions that happen by the exchange of "particles of spin $\hbar$" lead one to consider the interactions of transversal waves $a_{\mu}\,a^{\mu} < 0$, and to associate the *static interactions* (electromagnetic or nuclear static) with *interactions that happen by the exchange of longitudinal waves* $a_{\mu}\,a^{\mu} > 0$.

We have defined Ω_0 by $\Omega_0 = j_{\dot{l}m}E^{\dot{l}m}$.

For the plane-wave states:

(92) $$\Omega_0(u) = 4C_1 n_{\dot{r}s}a^{\dot{r}m}a^{\dot{l}s}\alpha_{\dot{l}}\alpha_m[Y_1^2 + \lambda_0^2 Y_2^2]Z_1Z_2.$$

We write this term:

$$\Omega_0 = 4C_1\omega_0(Y_1^2 + \lambda_0^2 Y_2^2)Z_1Z_2,$$

with

$$\omega_0 = n_{\dot{r}s}a^{\dot{r}m}a^{\dot{l}s}\alpha_{\dot{l}}\alpha_m.$$

We determine the plane-wave solutions of the system (E_3) by setting:

(93) $$\begin{aligned}\Phi_{(0)}^{\underline{\dot{l}_1\dot{l}_2},\underline{m_1m_2}}(u) &= g^{\underline{\dot{l}_1\dot{l}_2},\underline{m_1m_2}}H_1(u),\\ \Phi_{(1)}^{\dot{l},\underline{m_1m_2m_3}}(u) &= h^{\dot{l},\underline{m_1m_2}}H_2(u),\\ \Phi_{(2)}^{\underline{m_1m_2m_3m_4}}(u) &= k^{\underline{m_1m_2m_3m_4}}H_3(u).\end{aligned}$$

Substitution in (E_3) leads to the expressions:

$$(n^{|\underline{i_1}}{}_r h^{\underline{i_2}|, r\underline{m_1 m_2}} + n_{\dot r,}{}^{|\underline{m_1}} h^{\dot r \underline{i_1 i_2}, \underline{m_2}|}) \frac{dH_2}{du} = \Omega_0(u) g^{\underline{i_1 i_2}, \underline{m_1 m_2}} H_1(u),$$

$$n_{\dot r}{}^{|\underline{m_1}} g^{\dot r \underline{i_1}, \underline{m_1 m_2}} \frac{dH_1}{du} + n^{\underline{i_1}}{}_r k^{r\underline{m_1 m_2 m_3}} \frac{dH_3}{du} = \Omega_0 h^{\underline{i}, \underline{m_1 m_2 m_3}} H_2(u),$$

$$n_{\dot r}{}^{|\underline{m_1}} h^{\dot r, \underline{m_2 m_3 m_4}|} \frac{dH_2}{du} = \Omega_0 k^{\underline{m_1 m_2 m_3 m_4}} H_3(u),$$

and to:

$$n_{\dot r}{}^{m_i} g^{\dot r \underline{l}, \underline{m_j m_k}} = n_{\dot r}{}^{m_j} g^{\dot r \underline{l}, \underline{m_k m_i}} .$$

These expressions lead us *to associate the amplitudes with each other* by way of the relations:

(94)
$$g^{\underline{i_1 i_2}, \underline{m_1 m_2}} = C_0[n^{|\underline{i_1}}{}_r h^{\underline{i_2}|, r\underline{m_1 m_2}} + n_{\dot r}{}^{|\underline{m_1}} h^{\dot r \underline{i_1 i_2}, \underline{m_2}|}],$$
$$k^{\underline{m_1 m_2 m_3 m_4}} = C_0 n_{\dot r}{}^{|\underline{m_1}} h^{\dot r, \underline{m_2 m_3 m_4}|}.$$

These entail the relations between the functions:

(95)
$$\frac{dH_2}{du} = C_0\, \Omega_0\, H_1(u), \qquad \frac{dH_2}{du} = C_0\, \Omega_0\, H_3(u),$$

and consequently:

$$H_1(u) \equiv H_3(u).$$

The second relation then reduces to:

(96)
$$[n_{\dot r}{}^{|\underline{m_1}} g^{\dot r \underline{i_1}, \underline{m_2 m_3}|} + C_0 n^{\underline{i_1}}{}_r h^{\dot s, \underline{m_2 m_3 m_4}|}] \frac{dH_1}{du} = \Omega_0 h^{\underline{i_1}, \underline{m_1 m_2 m_3}} H_2 .$$

One deduces this upon taking into account the relation:

$$h^{\underline{i}, \underline{m_1 m_2 m_3}} = -\frac{1}{2C_0} n_{\dot r}{}^{|\underline{m_1}} g^{\dot r \underline{i}, \underline{m_2 m_3}|} .$$

The functions $H_1(u)$, $H_2(u)$ are then associated with each other by means of the system:

(97)
$$\frac{d}{du} H_1(u) = -\frac{1}{C_0} \Omega_0(u) H_2(u),$$
$$\frac{d}{du} H_2(u) = C_0 \Omega_0(u) H_1(u).$$

With the expressions above for the amplitudes, one evaluates the coupling factor:

$$G_{\dot l m}(u) = -\frac{1}{2C_0}[n_{\dot l r} g_{\underline{\dot r_1 \dot r_2}}{}^{\underline{rs}} g^{\underline{\dot r_1 \dot r_2}}{}_{, \underline{sm}} + n_{\dot r m} g^{\underline{\dot r \dot s}}{}_{\underline{r_1 r_2}} g_{, \underline{\dot s l}}{}^{\underline{r_1 r_2}}] H_1(u) H_2(u) .$$

I will write this expression as:

(98) $$G_{\underline{l}m}(u) = g_{\underline{l}m} H_1(u) H_2(u) .$$

The expressions for j_{lm}, $E_{\underline{l}m}$, $G_{\underline{l}m}$ permit us to evaluate the coupling terms $\Omega_1(u)$, $\Omega_2(u)$, $\Omega_0(u)$. With:

$$j^{lm} = \alpha^l \alpha^m Y_1^2(u) + \lambda_0^2 n_{\dot{s}}{}^m \alpha^{\dot{s}} \alpha^r Y_2^2(u) ,$$
$$E_{\underline{l}m} = 4C_1 n_{\dot{r}s} a_l^{\;s} a^{\dot{r}}{}_m Z_1(u) Z_2(u) ,$$

taking into account relations (78) and defining the invariants a_0 and g_0 by:

$$a_{\dot{r}l} a^{\dot{r}}{}_m = a_0\ \varepsilon_{lm} , \qquad a_0 = \frac{1}{2}(a_{\dot{r}}{}^s a^{\dot{r}}{}_s) = -(a_\mu\, a^\mu),$$

(99) $$g_0 = g_{\underline{i_1 i_2}, \underline{m_1 m_2}}\, g^{\underline{i_1 i_2}, \underline{m_1 m_2}} ,$$

one obtains:

(100) $$\Omega_1(u) = E_{\underline{l}m} G^{\underline{l}m} = -2\frac{C_1}{C_0} a_0\, g_0\, Z_1\, Z_2\, H_1\, H_2,$$

with

(102) $$\omega_2 = g^{\underline{\dot{r}_1 \dot{r}_2}, \underline{s_1 s_2}} [n_{l s_1} g_{\underline{\dot{r}_1 \dot{r}_2}, \underline{m s_2}} + n_{\dot{r}_1 m} g_{\underline{l \dot{r}_2}, s_1 s_2}] \alpha^i \alpha^m .$$

With these different expressions, the six functions $Y_1(u)$, $Y_2(u)$, $Y_3(u)$, $H_1(u)$, $H_2(u)$ are associated with each other by way of the differential system:

(103)
$$\frac{dY_1(u)}{du} = \lambda_0\, \Omega_1(u)\, Y_2(u), \qquad \frac{dY_2(u)}{du} = -\lambda_0\, \Omega_1(u)\, Y_1(u),$$
$$\frac{dZ_1(u)}{du} = 2C_1\, \Omega_2(u)\, Z_2(u), \qquad \frac{dZ_2(u)}{du} = -\frac{1}{2C_1}\, \Omega_2(u)\, Z_1(u),$$
$$\frac{dH_1(u)}{du} = -\frac{1}{C_0}\, \Omega_0(u)\, H_2(u), \qquad \frac{dH_2(u)}{du} = C_0\, \Omega_0(u)\, H_1(u).$$

with

(104)
$$\Omega_1 = -2\frac{C_1}{C_0} a_0\, g_0\, Z_1\, Z_2\, H_1\, H_2,$$
$$\Omega_2 = -\frac{\omega_2}{C_0}[Y_1^2 + \lambda_0^2 Y_2^2] H_1\, H_2,$$
$$\Omega_0 = 4C_1 \omega_0 (Y_1^2 + \lambda_0^2 Y_2^2)\, Z_1\, Z_2 .$$

The factors of proportionality λ_0, C_0, C_1 being arbitrary, we perform the simplifying choice:

$$\lambda_0 = 1, \qquad C_1 = \frac{1}{2}, \qquad C_0 = -1,$$

which gives:

$$\Omega_1 = a_0\, g_0\, Z_1\, Z_2\, H_1\, H_2,$$
(105)
$$\Omega_2 = \omega_2\,(Y_1^2 + \lambda_0^2 Y_2^2)\, H_1\, H_2,$$
$$\Omega_0 = 2\,\omega_0\,(Y_1^2 + \lambda_0^2 Y_2^2)\, Z_1\, Z_2,$$

and the differential system:

$$Y_1' = \Omega_1\, Y_2, \qquad Y_2' = -\,\Omega_1\, Y_1,$$
(106)
$$Z_1' = \Omega_2\, Z_2, \qquad Z_2' = -\,\Omega_2\, Z_1,$$
$$H_1' = \Omega_0\, H_2, \qquad H_2' = -\,\Omega_0\, H_1.$$

This system will be simplified upon remarking that the equations in (Y_1, Y_2) admit the first integral:

$$Y_1^2 + Y_2^2 \equiv (Y_1^0)^2 + (Y_2^0)^2 = v_0^2$$

(and, in turn, v_0 *bounds* | Y_1 | and | Y_2 |).

The preceding system is then decomposed into two subsystems that are associated with each other for constant v_0^2 by way of:

(107)
$$Y_1' = a_0 g_0 (Z_1 Z_2 H_1 H_2) Y_2,$$
$$Y_2' = -a_0 g_0 (Z_1 Z_2 H_1 H_2) Y_1,$$

$$Z_1' = \omega_2 v_0^2 H_1 H_2 Z_2,$$
$$Z_2' = -\omega_2 v_0^2 H_1 H_2 Z_1,$$
(108)
$$H_1' = 2\omega_0 v_0^2 Z_1 Z_2 H_2,$$
$$H_2' = -2\omega_0 v_0^2 Z_1 Z_2 H_2.$$

In this system, the constants a_0, g_0, ω_0, ω_2 are positive or negative, but a change in their signs will correspond uniquely to the exchange of indices 1 and 2 in each of the pairs of functions that are associated with the principal tensors.

We shall completely determine the general solution to this system by placing ourselves in the case $a_0\, g_0 > 0$, $\omega_2 > 0$, $\omega_0 > 0$, *a priori*.

The functions $Y_1(u)$, $Y_2(u)$ are determined completely by starting with the functions Z and H.

Setting:

(109) $$Y_1 = v_0 \sin \Phi(u), \qquad Y_2 = v_0 \cos \Phi(u),$$

$\Phi(u)$ is determined by:

(110) $$\Phi(u) = \frac{a_0 g_0}{\omega_2 \nu_0^2} Z_1^2 + \varphi_0 ,$$

φ_0 will be *a phase that is associated with the initial values of the set of functions.*

The integration of the system (Z, H) will be effected by introducing the new associated functions:

$$\varphi_1(u) = \nu_0 \sqrt{2\omega_0}\, Z_1(u), \qquad \varphi_2(u) = \nu_0 \sqrt{2\omega_0}\, Z_2(u),$$
$$\psi_1(u) = \nu_0 \sqrt{2\omega_0}\, H_1(u), \qquad \psi_2(u) = \nu_0 \sqrt{2\omega_0}\, H_2(u),$$

which leads to the system of differential equations:

(111) $$\varphi_1'(u) = (\psi_1\psi_2)\varphi_2(u), \quad \varphi_2'(u) = -(\psi_1\psi_2)\varphi_1(u),$$
$$\psi_1'(u) = (\varphi_1\varphi_2)\psi_2(u), \quad \psi_2'(u) = -(\varphi_1\varphi_2)\psi_1(u).$$

This system admits *three principal first integrals:*

(112) $$\varphi_1^2 + \varphi_2^2 = \lambda_1^2 = 2\omega_0\nu_0^2(Z_1^2 + Z_2^2),$$
$$\psi_1^2 + \psi_2^2 = \lambda_2^2 = \omega_0\nu_0^2(H_1^2 + H_2^2),$$
$$\varphi_1^2 + \psi_2^2 = \mu_1^2 = \nu_0^2(2\omega_0 Z_1^2 + \omega_2 H_2^2),$$

and the associated integrals:

(113) $$\psi_1^2 + \varphi_2^2 = \nu_0^2(\omega_2 H_1^2 + 2\omega_0 Z_2^2) = \lambda_1^2 + \lambda_2^2 - \mu_1^2 = \mu_2^2 > 0,$$
$$\psi_1^2 - \varphi_1^2 = \nu_0^2(Z_1^2 - H_1^2) = \lambda_2^2 - \mu_1^2 = \mu_0^2 .$$

Here, we have chosen $\lambda_2^2 - \mu_1^2 = \mu_0^2 > 0$, so:

$$\mu_2^2 = \mu_0^2 + \lambda_1^2 .$$

These relations permit us to simply represent the functions φ and ψ by means of *hyperelliptic integrals.*

$\varphi_1(u)$ and $\psi_1(u)$ are deduced from:

$$\left(\frac{d\varphi_1}{du}\right)^2 = (\lambda_1^2 - \varphi_1^2)(\mu_0^2 + \varphi_1^2)(\mu_1^2 - \varphi_1^2) ,$$

$$\left(\frac{d\psi_1}{du}\right)^2 = (\lambda_1^2 - \psi_1^2)(\psi_0^2 - \mu_1^2)(\mu_2^2 - \psi_1^2) .$$

In the first part, we have shown that this type of relation leads to functions that are expressed completely by Jacobi elliptic functions.

We shall adapt this general correspondence to the case that is encountered here.

We first remark that the first integrals that are associated with the initial values of the functions $\varphi_1(u)$, $\varphi_2(u)$, $\psi_1(u)$, $\psi_2(u)$ (or Z_1, Z_2, H_1, H_2) permit us to reduce them to *three independent functions*.

We set:

(114)
$$K_0(u) = \frac{1}{(\psi_1(u))^2} > 0, \qquad K_1(u) = \mu_2\lambda_2 \frac{\varphi_1(u)}{\psi_1(u)},$$
$$K_2(u) = \mu_0\lambda_2 \frac{\varphi_2(u)}{\psi_1(u)}, \qquad K_3(u) = \mu_0\mu_2 \frac{\psi_2(u)}{\psi_1(u)},$$

and conversely:

$$\varphi_1 = \frac{1}{\mu_2\lambda_2}\frac{K_1}{\sqrt{K_0}}, \qquad \varphi_2 = \frac{1}{\mu_0\lambda_2}\frac{K_2}{\sqrt{K_0}}, \qquad \psi_1 = \frac{1}{\mu_0\mu_2}\frac{K_3}{\sqrt{K_0}},$$

K_1, K_2, K_3 being associated with each other by way of:

$$K_1^2 + K_2^2 = \lambda_1^2\lambda_2^2, \qquad K_1^2 + K_3^2 = \mu_1^2\mu_2^2,$$

$K_0(u)$ is determined by starting with functions $K_1(u)$ or $K_2(u)$ or $K_3(u)$ by the expressions:

(115)
$$\mu_0^2\mu_2^2\lambda_2^2K_0(u) = \mu_2^2\lambda_2^2 - K_1^2 = \mu_0^2\lambda_2^2 + K_2^2 = \mu_0^2\mu_2^2 + K_3^2 .$$

The system (φ, ψ) leads to the differential system for $K_1(u)$, $K_2(u)$, $K_3(u)$:

(116)
$$\frac{dK_1}{du} = K_2\,K_3, \qquad K_2' = -K_1\,K_3, \qquad K_3' = -K_1\,K_2,$$
$$K_1^2 + K_2^2 = \lambda_1^2\lambda_2^2, \qquad K_1^2 + K_3^2 = \mu_1^2\mu_2^2 .$$

This determines the functions $K(u)$ by means of *Jacobi elliptic functions* of modulus k such that:

$$k^2 = \frac{\mu_1^2\mu_2^2}{\lambda_1^2\lambda_2^2}.$$

The necessary condition $0 \le k^2 \le 1$ shows us that *the choice of signs that has been made* imposes the following conditions here:

$$\mu_1^2 < \lambda_1^2 + \lambda_2^2, \qquad \lambda_1^2 < \mu_1^2,$$

namely:

$$0 < 2\omega_0 Z_2^2 + \omega_2 H_1^2, \qquad 2\omega_0 Z_2^2 < \omega_2 H_1^2 .$$

The preceding general study leads us to the expressions for the functions K_1, K_2, K_3, K_0 as Jacobi elliptic functions of the arguments:

$$\omega(u + u_0) = \mu_1\mu_2 (u + u_0)$$

and the modulus:

$$k = \frac{\lambda_1\lambda_2}{\mu_1\mu_2},$$

namely:

$$\text{(117)} \qquad \begin{aligned} K_1(u) &= \lambda_1\lambda_2 \mathrm{sn}[\omega(u+u_0), k], \\ K_2(u) &= \lambda_1\lambda_2 \mathrm{cn}[\omega(u+u_0), k], \\ K_3(u) &= \mu_1\mu_2 \mathrm{dn}[\omega(u+u_0), k], \end{aligned}$$

and

$$K_0(u).$$

The reciprocal formulas:

$$Z_1(u) = \frac{1}{\nu_0\sqrt{2\omega_0}}\varphi_1(u) = \frac{1}{\nu_0\sqrt{2\omega_0}\mu_2\lambda_2}\frac{K_1(u)}{\sqrt{K_0(u)}},$$

$$Z_2(u) = \frac{1}{\nu_0\sqrt{2\omega_0}}\varphi_2(u) = \frac{1}{\nu_0\sqrt{2\omega_0}\mu_0\lambda_2}\frac{K_2(u)}{\sqrt{K_0(u)}},$$

$$H_1(u) = \frac{1}{\nu_0\sqrt{\omega_2}}\psi_1 = \frac{1}{\nu_0\sqrt{\omega_2}}\frac{1}{K_0(u)},$$

$$H_2(u) = \frac{1}{\nu_0\sqrt{\omega_2}}\psi_2(u) = \frac{1}{\nu_0\sqrt{\omega_2}}\frac{1}{\mu_0\mu_2}\frac{K_3(u)}{\sqrt{K_0(u)}},$$

and

$$Y_1(u) = \nu_0 \sin \Phi(u), \quad Y_2(u) = \nu_0 \cos \Phi(u),$$

with

$$\Phi(u) = \frac{a_0 g_0}{\omega_2 \nu_0^2} Z_1^2 + \varphi_0,$$

give us the complete expressions for the set of plane-wave solutions that represent the associated field. They are represented by *Jacobi elliptic functions whose parameters are defined by the initial or maximal values of the associated fields.*

Nevertheless, for $k = 1$, *these functions become aperiodic.*

This conditions is expressed by:

$$\text{(118)} \qquad \lambda_1\lambda_2 = \mu_1\mu_2,$$

namely:

$$(\varphi_1^2 + \varphi_2^2)(\psi_1^2 + \psi_2^2) = (\varphi_1^2 + \varphi_2^2)(\psi_1^2 + \psi_2^2).$$

This condition is verified either for:

$$\psi_1^2 = \varphi_1^2, \qquad \lambda_2^2 = \mu_1^2,$$

and $\lambda_1^2 = \mu_2^2$, or for:

$$\psi_2^2 = \varphi_2^2, \qquad \text{or} \qquad \mu_1^2 = \lambda_1^2, \qquad \mu_2^2 = \lambda_2^2.$$

Considering the first case $\psi_1 = \varphi_1$ the system (111) is written:

(119) $$\varphi_1'(u) = (\varphi_1\ \varphi_2)\ \psi_2, \qquad \varphi_2' = -\ \varphi_1^2\psi_2, \qquad \psi_2' = -\ \varphi_1^2\varphi_2,$$

with

$$\varphi_1^2 + \varphi_2^2 = \lambda_1^2, \qquad \varphi_1^2 + \psi_2^2 = \lambda_2^2, \qquad \psi_2^2 - \varphi_2^2 = \lambda_2^2 - \lambda_1^2 \equiv \lambda_3^2,$$

and leads to the functions:

(120) $$\varphi_1(u) \equiv \psi_1(u) = \frac{\lambda_1\lambda_2}{\sqrt{\lambda_3^2\cosh^2(\lambda_1\lambda_2 u) + \lambda_1^2}},$$

$$\varphi_2(u) = -\frac{\lambda_1\lambda_3\sinh(\lambda_1\lambda_2 u)}{\sqrt{\lambda_3^2\cosh^2(\lambda_1\lambda_2 u) + \lambda_1^2}},$$

$$\psi_2(u) = \frac{\lambda_2\lambda_3\cosh(\lambda_1\lambda_2 u)}{\sqrt{\lambda_3^2\cosh^2(\lambda_1\lambda_2 u) + \lambda_1^2}}.$$

This aperiodic solution leads to the functions:

(121) $$Z_1(u) = \frac{1}{\nu_0\sqrt{2\omega_0}}\varphi_1(u), \qquad Z_2(u) = \frac{1}{\nu_0\sqrt{2\omega_0}}\varphi_2(u),$$

$$H_1(u) = \frac{1}{\nu_0\sqrt{\omega_2}}\varphi_1(u), \qquad H_2(u) = \frac{1}{\nu_0\sqrt{\omega_2}}\psi_2(u).$$

By these various expressions, we have thus obtained the set of field functions ($Y_1(u)$, $Y_2(u)$, $Z_1(u)$, $Z_2(u)$, $H_1(u)$, $H_2(u)$), and consequently, the tensors:

$$\xi^m, \qquad \eta^i, \qquad A^{lm}, \qquad F^{\underline{m_1m_2}}, \ \underset{(0)}{\Phi}{}^{\underline{l_1l_2},\underline{m_1m_2}}, \qquad \underset{(1)}{\Phi}{}^{l,\underline{m_1m_2m_3}}, \qquad \underset{(2)}{\Phi}{}^{\underline{m_1m_2m_3m_4}},$$

in the case of solutions of plane-wave type.

This neatly exhibit the classes of states of the fields that are associated with the classes of states of physical observables and are separate for the same type of corpuscle.

The representation of plane-waves obtained admits some laws of composition and association between systems of initial or maximal values. These laws are essentially deduced from the formulas that are called addition or composition formulas for the Jacobi elliptic functions and the transformations that associate Jacobi functions of different periods that constitute or simply extend the Gauss or Landen transformations of the general theory of elliptic functions.

These necessary transformations, which are very strongly non-linear, show the necessity of considering the representation of microphysical phenomena by the types of groups of transformations that are associated with nonlinear composition laws.

BIBLIOGRAPHY

a) On the nonlinear generalizations of the wave equations of quantum mechanics and the representation of corpuscular elements by the systems of associated fields, G. PETIAU.

1957 – C. R. Acad. Sci. Paris, t. **244**, 1 April 1957, pp. 1890-1893.
C. R. Acad. Sci. Paris, t. **244**, 20 May 1957, pp. 2580-2583.

1958 – Nuovo Cimento Supplemento, al. t. **9**, series X, 1958, pp. 542-569.
C. R. Acad. Sci. Paris, t. **246**, 5 May 1958, pp. 2581-2583.

1959 – J. de Physique, t. **20**, October 1959, pp. 784-796, "Sur certain types d'équations d'ondes non linéaires généralisant les équations de la théorie des particules à spin."
C. R. Acad. Sci. Paris, t. **248**, 23 February 1959, pp. 1129-1132.
C. R. Acad. Sci. Paris, t. **248**, 16 March 1959, pp. 1620-1622.
Cahiers de Physique, t. **13**, no. 111, November 1959, pp. 429-462. "Sur quelques modèles de corpuscles-champs en théorie non linéaires des particules à spin."

1960 – Cahiers de Physique, t. **14**, no. 113, January 1960, pp. 1-24. "Les généralisations nonlinéaires des équations d'ondes de la mécanique ondulatoire."

1961 – Cahiers de Physique, t. **14**, no. 128, April 1960, pp. 157-170.
Conférence Internationale d'Aix en Provence sur les particules élémentaires, t. **1**, pp. 171-181; "Étude d'un modéle susceptible de donner une représentation unifiée du système de particules élémentaires."

1962 – Cahiers du Physique, t. **16**, no. **16**, no. 148, November 1962, pp. 491-516.

1965 – Nuovo Cimento, series X, t. **40**, 1 November 1965, pp. 84-101, "Sur les théories quantiques des champs associées à des modèles simples d'équations d'ondes non linéares."

1965 – Annales Institut Henri Poincaré, **A**, t. **III**, no. 3, 1965, pp. 127-159. "Les équations d'ondes non linéaires d'un systèmes spin 0, spin 2*h* et les théories quantique des champs associée."
C. R. Acad. Sci. Paris, t. **260**, 25 January 1965, pp. 1107-1110.

1968 – C. R. Acad. Sci. Paris, t. **267 B**, 15 July 1968, pp.169-171.

1969 - Annales Institut Henri Poincaré, **A**, t. **X**, no. 4, 1969, pp. 381-417. "Sur une modèle général de théorie des particules fondamentales et la quantification des masses propres."

b) On the general spinorial formalism and its application in the representation of quantum electromagnetic and gravitational fields.

[1] G. PETIAU, "Le formalisme spinoriel et ses applications en mécanique quantique relativiste," Cours de 3rd Cycle, University of Paris VI, 1972-1977, multigraphed.

[2] G. PETIAU, "Sur certains aspects de l'emploi des spineurs dans la représentation des grandeurs et des équations de l'electromagnétisme," Symposia Mathematica Bologna, t. **XVIII**, 1976, pp. 369-388.

[3] G. PETIAU, "Formulation d'une théorie macroscopiques spinorielle des champs de gravitation généralisée prolongeant la théorie quantique microscopic des champs de spin 2*h*." Communication presented at the Conference on Gravitation and General Relativity, August 1977, University of Waterloo, Canada.

[4] G. PETIAU, "Sur la représentation, par des systèmes différentiels non linéaires du premier ordre, de modèles corpusculaires, association de champs microphysiques, électromagnétiques et gravitationnels," Communication presented at the 9th International Conference on General Relativity and Gravitation, July 1980, Friedrich Schiller University, Jena, D. D. R.

(Manuscript received on 3 July 1981)