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On some types of nonlinear wave equations and their solutions

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I propose to present some results here that I have obtained in the study of certain types of nonlinear wave equations that are introduced in wave mechanics.

Some of these equations have been encountered before, notably by SCHIFF, IVANENKO, FINKELSTEIN, and HEISENBERG. My starting point is very different from those authors, and I think that my method can complete and clarify some of their results.

I shall begin with some considerations on the hypotheses that can guide one in the search for nonlinear wave equations that are capable of generalizing the usual wave mechanics.

1. The types of solutions of the Klein-Gordon equation.

I shall begin by examining the solutions of the Klein-Gordon equation:

$$\square\Psi(x, y, z, t) + \mu_0^2\,\Psi = 0, \qquad \mu_0 = \frac{m_0 c}{\hbar} \tag{1}$$

that represent corpuscles without spin in ordinary wave mechanics.

According to the problem being studied, one can consider different types of waves Ψ that are solutions of that equation.

They are the solutions that one calls:

– plane waves
– invariant waves
– spherical waves
– proper field waves
– guided waves.

1. The *plane wave* solutions are obtained by starting with equation (1) and supposing that the functions Ψ (x, y, z, t) depend upon only one variable τ, which is a linear combination of x, y, z, t that takes the form:

(2) $$\tau = \frac{1}{\hbar}[Wt - (\mathbf{p}\,\mathbf{x})],$$

so:

(3) $$\Psi\,(x, y, z, t) = \Psi\,(t).$$

The parameters W, $\mathbf{p}$ are linked by the relation:

$$\frac{W^2}{c^2} = p^2 + m_0^2\,c^2 .$$

The function $\Psi\,(\tau)$ is determined by a *differential* equation:

(4) $$\frac{d^2\Psi(\tau)}{d\tau^2} + \Psi\,(\tau) = 0.$$

The general solution is then a linear combination of two types of solutions, namely, even and odd ones:

(5) $$\Psi_c = A\cos\tau, \quad \Psi_s = A'\sin\tau.$$

The different values of the functions $\Psi\,(\tau)$ can be considered to be deduced from the solution in the proper system:

$$\Psi = \Psi\,(\tau)$$

(so $\tau = 2\pi/\hbar\,m_0\,c^2 t = \mu_0\,ct$) by a Lorentz transformation.

We can already note that the solutions (5) are the solutions of (1) *that depend upon just one variable that is periodic, uniform, and has finite amplitude.*

2. The solutions that are called *invariant* are obtained by starting with (1) while considering functions $\Psi\,(x, y, z, t)$ that depend upon x, y, z, t only by the intermediary of *a single auxiliary variable* that is *a relativistic invariant.*

One generally takes:

$$u = \pm\sqrt{c^2t^2 - r^2}\,,$$
$$u^2 = c^2\,t^2 - (x^2 + y^2 + z^2).$$

One easily sees that:

$$\Box = \frac{d^2}{du^2} + \frac{3}{u}\frac{d}{du}.$$

Equation (1) is then written:

$$\left[\frac{d^2}{du^2} + \frac{3}{u}\frac{d}{du} + \mu_0^2\right]\Psi(u) = 0.$$

This is once more a *differential* equation whose general solution is expressed by means of Bessel functions:

$$\begin{aligned}\Psi\,(u) &= \frac{A}{u}J_1(\mu_0\,u)+\frac{A'}{u}N_1(\mu_0\,u)\\ &= \frac{B}{u}H_1^{(1)}(\mu_0\,u)+\frac{B'}{u}H_1^{(2)}(\mu_0\,u)\,.\end{aligned}$$

These functions might possibly possess only one critical point: namely, the point $u = 0$ that corresponds to the light cone.

3. For the introduction of spherical waves and guided waves, we shall suppose that there exists a privileged frame R_0 that is attached to the corpuscle, and in that frame the wave functions $\Psi\,(x, y, z, t)$ are expressed in the form of a product of a function of t – namely, $\Psi_1\,(t)$ – and a function of the spatial variables $\Psi_2\,(x, y, z)$ or $\Psi_2\,(r, \theta, \varphi)$:

$$\Psi\,(x, y, z, t) = \Psi_1\,(t)\,\Psi_2\,(x, y, z) = \Psi_1\,(t)\,\Psi_2\,(r, \theta, \varphi)$$

so one will then have:

$$\Psi_2\frac{d^2\Psi_1}{dt^2}-\Psi_1\Delta\Psi_2+\mu_0^2\Psi_1\Psi_2 = 0.$$

If one introduces two coupling constants λ_1, λ_2 such that:

$$\frac{d^2\Psi_1}{dt^2}+\lambda_1\,\Psi_1\,(t) = 0,$$

$$\Delta\Psi_2\,(x, y, z) + \lambda_2\,\Psi_2\,(x, y, z) = 0.$$

We suppose that λ_1 and λ_2 are real. [λ_1 will > 0, in order to avoid the possibility that the solutions $\Psi_1\,(t)$ are not of vanishing type; those solutions will be considered in a more general study that I shall not enter into here.] Hence:

$$\Psi_1\,(t) = C_1\,e^{i\sqrt{\lambda_1}\,t}+C_2\,e^{-i\sqrt{\lambda_1}\,t} = C_1'\cos\sqrt{\lambda_1}\,t+C_2'\sin\sqrt{\lambda_1}\,t\,.$$

There are two cases to consider for the functions Ψ_2:

Indeed, one has $\lambda_2 = \lambda_1 - \mu_0^2$, so:

a. $$\lambda_1 > \mu_0^2, \qquad \lambda_2 > 0.$$

$$\Delta\Psi_2 + \lambda_2\,\Psi_2 = 0$$

will admit the general solution:

$$\Psi_2\,(r, \theta, \varphi) = \frac{1}{\sqrt{r'}}[A\,J_{l+\frac{1}{2}}(\sqrt{\lambda_2}\,r)+B\,N_{l+\frac{1}{2}}(\sqrt{\lambda_2}\,r)]\,Y_l^n(\theta,\varphi)\,.$$

b. $\lambda_1 < \mu_0^2, \qquad \lambda_2 < 0.$

$$\Delta\Psi_2 - |\lambda_2|\Psi_2 = 0$$

will admit a solution that is bounded when $r \to \infty$:

$$\Psi_2(r, \theta, \varphi) = \frac{A}{\sqrt{r'}} K_{l+\frac{1}{2}}(\sqrt{\lambda_2}\, r) Y_l^n(\theta, \varphi).$$

(The solution in $I_{l+\frac{1}{2}}$ will diverge when $r \to \infty$.) If we confine ourselves to the case of l = 0 then:

$$\Psi_2(r, \theta, \varphi) = \Psi_2(r),$$

so one will have the cases:

a. $$\Psi_2(r) = A' \frac{\sin\sqrt{\lambda_2}\, r}{r} + B' \frac{\cos\sqrt{\lambda_2}\, r}{r}$$

b. $$\Psi_2(r) = \frac{A''}{r} e^{-\sqrt{|\lambda_2|}\, r}.$$

The solutions that are called "spherical" are obtained by starting from these expressions and setting:

$$\lambda_1 = \frac{W^2}{\hbar^2 c^2} = K^2, \qquad \lambda_2 = \frac{p^2}{\hbar^2} = |\mathbf{K}|^2,$$

$$\lambda_1 - \lambda_2 = \mu_0^2. \quad \text{Here} \quad \lambda_1 > \mu_0^2.$$

$$\begin{aligned}\Psi_{\text{sph}} &= \Psi_1(t)\,\Psi_2(r) \\ &= (C_1' \cos Kct + C_2' \sin Kct)\left(A' \frac{\sin|\mathbf{K}|r}{r} + A'' \frac{\cos|\mathbf{K}|r}{r}\right) \\ &= C_1'' \frac{\sin(Kct \mp |\mathbf{K}|r)}{r} + C_2'' \frac{\cos(Kct \mp |\mathbf{K}|r)}{r}.\end{aligned}$$

These waves are the spherical waves of ordinary wave mechanics (in the case l = 0).

In addition, one considers the particular case that one calls the *proper field of the particle.* Those waves correspond to the case in which:

$\lambda_1 = 0$ so $\Psi(x, y, z, t) = \Psi_2(r) = \Psi(r)$,

$\lambda_1 = 0$ implies that $|\lambda_2| = \mu_0^2$,

$$\Psi(r) = \frac{C_0}{r} e^{-\mu_0 r}.$$

When one fixes the value of the constant C_0 , that solution will be considered to be the field Ψ that is created by a source C_0 that is localized at the point $r = 0$ in the proper system of the source.

One passes from these general spherical wave solutions to the *guided-wave* solutions by considering the solutions (*a*) and (*b*) to be complete and performing a general Lorentz transformation on the frame R_0 .

Let:

$$ct = \cosh\gamma ct' - \sinh\gamma z', \qquad x = x',$$

$$z = \sinh\gamma ct' - \cosh\gamma z', \qquad y = y',$$

$$r^2 \to x'^2 + y'^2 + \cosh^2\gamma[z' - \tanh\gamma ct']^2,$$

$$\sqrt{\lambda_1}\, t \to [\cosh\gamma ct' - \sinh\gamma z'].$$

Setting $\sqrt{\lambda_1} = \mu_1$:

$$K_1 = \mu_1 \cosh\gamma, \qquad |\mathbf{K}_1| = \sinh\gamma, \qquad \tanh\gamma = v_1,$$

$$\sqrt{\lambda_1}\, t \to K_1\, ct' - |\mathbf{K}_1|\, z',$$

$$r^2 \to x'^2 + y'^2 + (z' - v_1 t')^2 = \rho'^2,$$

$$\Psi(x, y, z, t) = [C_1' \cos\sqrt{\lambda_1}t + C_1'' \sin\sqrt{\lambda_1}t]\left[A'\frac{\sin\sqrt{\lambda_2}t}{r} + A''\frac{\cos\sqrt{\lambda_2}t}{r}\right]$$

$$\to \Psi(x', y', z', t')$$

$$= [C_1' \cos(K_1 ct' - |\mathbf{K}_1|\, z') + C_1'' \sin(K_1 ct' - |\mathbf{K}_1|\, z')]\left[A'\frac{\sin\sqrt{\lambda_2}\rho'}{\rho'} + B'\frac{\cos\sqrt{\lambda_2}\rho'}{\rho'}\right].$$

Similarly, the field solution Ψ: $\Psi(r) = C_0\, e^{-\mu_0 r}/r$ will give the particular guided solution:

$$\Psi(x', y', z', t') = \frac{C_0 \exp\left(-\mu_0\sqrt{x'^2 + y'^2 + \cosh^2\gamma(z' - \tanh\gamma ct')^2}\right)}{\sqrt{x'^2 + y'^2 + \cosh^2\gamma(z' - \tanh\gamma ct')^2}},$$

which corresponds to the Yukawa field with a source in uniform, rectilinear motion.

Whereas the solutions of the invariant wave type are solutions of (1) with *fixed critical points*, the guided waves are solutions with *moving critical points* (i.e., ones that depend upon integration constants).

2. The plane waves associated with Jacobi's elliptic functions.

In order to extend wave mechanics, we must generalize either the set of those type of solutions (which will be equivalent to a linear schema when one confines oneself to the regular solutions) or just some of them that we are led to consider by physical reasons to be attached more directly to the representation of matter.

If we consider the solutions of *plane-wave* type then we have seen that they can be considered to be the result of applying transformations from the Lorentz group to the particular solutions in the proper system:

$$\Psi_0 = \begin{cases} A\sin\tau = A\sin\dfrac{2\pi}{h}m_0c^2t = A\sin 2\nu_0 t \\ A\cos\tau = A\cos\dfrac{2\pi}{h}m_0c^2t. \end{cases}$$

That form of the solution exhibits a fundamental character of the representation of corpuscles in wave mechanics:

In the proper system of the corpuscle, the wave function associates a "clock" with it; i.e., a periodic function of proper time with a period of 2π and frequency of $\nu_0 = m_0 c^2 / h$.

If we would like to generalize that concept while attempting to enrich the notion of corpuscle by introducing, not just the single intrinsic constant $\nu_0 = m_0 c^2 / h$, but two or more constants, then the most immediate generalization would consist of taking the wave function that represents the corpuscle in its proper system to be certain Jacobi elliptic functions that possess one real period and one imaginary period instead of the circular functions cos τ or sin τ, and they are defined by means of a number k that is found between 0 and 1 ($0 \leq k \leq 1$) by the integrals:

$$K(k) = \int_0^{2\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \qquad K' = K(k'),$$

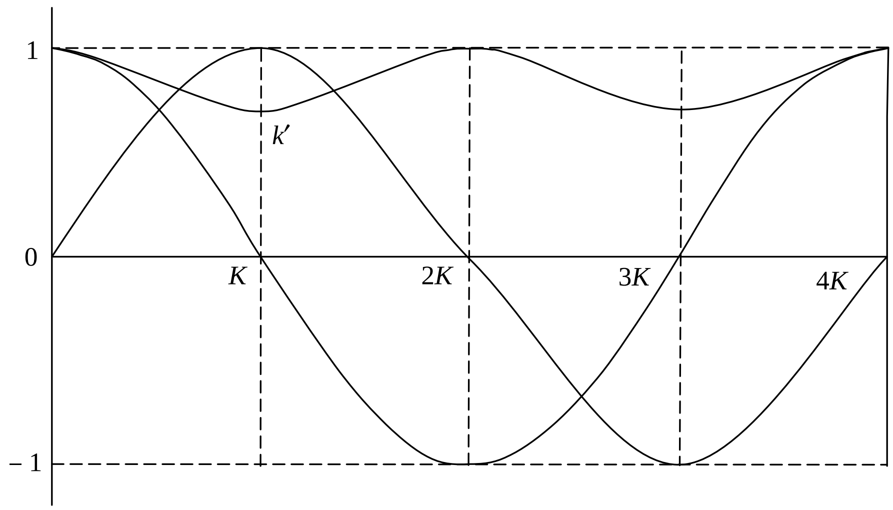
with $k'^2 = 1 - k^2$, $0 \leq k' \leq 1$.

Those functions will tend to sin τ and cos τ when $k \to 0$.

The theory of Jacobi's elliptic functions defines three fundamental functions:

sn (u, k)	with periods $4K$ and $4iK'$,
cn (u, k)	with periods $4K$ and $4iK'$,
dn (u, k)	with periods $2K$ and $2iK'$.

Starting with them, one constructs a system of 12 fundamental functions by appending the inverses and quotients of the three principal functions.



In particular, we have the relations:

$$\operatorname{sn}(u + K, k) = \operatorname{cd}(u, k) = \frac{\operatorname{cn} u}{\operatorname{dn} u},$$

$$\operatorname{sn}(u, 0) = \sin u, \qquad \operatorname{cd}(u, 0) = \cos u,$$

$$\operatorname{sn}(u, 1) = \tanh u, \qquad \operatorname{cd}(u, 1) = 1,$$

$$\operatorname{cn}(u + K, k) = -k' \operatorname{sd}(u, k) = -k' \frac{\operatorname{sn} u}{\operatorname{dn} u},$$

$$\operatorname{cn}(u, 0) = \cos u, \qquad \operatorname{sd}(u, 0) = \sin u,$$

$$\operatorname{cn}(u, 1) = \frac{1}{\cosh u},$$

$$\operatorname{dn}(u + K, k) = -k' \quad \operatorname{nd}(u, k) = -\frac{k'}{\operatorname{dn} u},$$

$$\operatorname{dn}(u, 0) = 1, \qquad \operatorname{dn}(u, 1) = \frac{1}{\cosh u}.$$

Here, we are then led to set: $u = \tau$, so:

$$4K(k)\, v_0\, t = 4K(k) \frac{m_0 c^2}{h} t = \frac{m_0 c^2}{\hbar'} t = \mu_0\, ct.$$

Here, $4K$ is the analogue of the factor 2π in the trigonometric case.

$$\hbar' = \frac{h}{4K(k)} \qquad \text{replaces} \qquad \hbar = \frac{h}{2\pi}.$$

μ_0 will be defined by $m_0\, c^2 / \hbar'$. (We remark that $\hbar' < \hbar$).

In the proper system, we will be led to consider three possible systems:

1. A sn (τ, k) and A' cd (τ, k),

2. A cn (τ, k) and A' sd (τ, k).

These two types of functions are either even or odd and reduce to the functions sin τ and cos τ for $k = 0$.

3. A dn (τ, k), A' nd (τ, k).

The even functions reduce to constants for $k = 0$ [dn (0) = 1]. One knows that the functions sn u, cn u, dn u satisfy the following differential equations:

1. $$y'^2 + (1 - 2k^2)\, y^2 + k^2\, y^4 - k'^2 = 0,$$

which has the solutions:

$$y = \text{cn}\, u \qquad \text{for} \qquad y(0) = 1,$$

$$y = k'\, \text{sd}\, u \qquad \text{for} \qquad y(0) = 1.$$

As a result:

$$y'' + (1 - 2k^2)\, y + 2k^2\, y^3 = 0$$

will have the solutions:

$$y = \text{cn}\, u \qquad \text{if} \qquad y(0) = 1, \qquad y'(0) = 0,$$

$$y = k'\, \text{sd}\, u \qquad \text{if} \qquad y(0) = 0, \qquad y'(0) = k'.$$

2. $$y'^2 + (1 + k^2)\, y^2 - k^2\, y^4 - 1 = 0$$

will have the bounded solutions:

$$y = \text{sn}\, u \qquad \text{with} \qquad y(0) = 0,$$

$$y = \text{cd}\, u \qquad \text{with} \qquad y(0) = 1.$$

As a result:

$$y'' + (1 + k^2)\, y - 2k^2\, y^3 = 0$$

will have the solutions:

$$y = \text{sn}\, u \qquad \text{for} \qquad y(0) = 0, \qquad y'(0) = 1,$$

$$y = \text{cd}\, u \qquad \text{for} \qquad y(0) = 1, \qquad y'(0) = 0.$$

3. $$y'^2 - (1 + k'^2)\, y^2 + y^4 + k'^2 = 0$$

will have the solutions:

$$y = \mathrm{dn}\, u \qquad \text{with} \quad y\,(0) = 1,$$

$$y = k' \,\mathrm{nd}\, u \qquad \text{with} \quad y\,(0) = k'.$$

As a result:

$$y'' - (1 + k^2)\, y + 2y^3 = 0$$

will have the solutions:

$$y = \mathrm{dn}\, u \qquad \text{for} \quad y\,(0) = 1, \qquad y'(0) = 0,$$

$$y = k' \,\mathrm{dn}\, u \qquad \text{for} \quad y\,(0) = k', \qquad y'(0) = 0.$$

If we return from the differential equations that are satisfied by the functions $\Psi\ (\tau)$ to the partial differential equations that are satisfied by the functions $\Psi\ (x, y, z, t)$ then we will see that:

1.

$$\Psi_{\mathrm{c}} = \lambda\ \mathrm{sn}\, [\frac{1}{\hbar'}(Wt - px)],$$

$$\Psi_{\mathrm{s}} = \lambda\, k'\ \mathrm{sd}\, [\frac{1}{\hbar'}(Wt - px)],$$

are particular *plane-wave* solutions of:

$$\Box\Psi + (1 - 2k^2)\mu_0^2\,\Psi + \frac{2k^2\mu_0^2}{\lambda^2}\Psi^3 = 0.$$

2.

$$\Psi_{\mathrm{s}} = \lambda\ \mathrm{sn}\, [\frac{1}{\hbar'}(Wt - px)],$$

$$\Psi_{\mathrm{c}} = \lambda\ \mathrm{cd}\, [\frac{1}{\hbar'}(Wt - px)]$$

are particular solutions of:

$$\Box\Psi + (1 - 2k^2)\mu_0^2\,\Psi + \frac{2k^2\mu_0^2}{\lambda^2}\Psi^3 = 0.$$

3.

$$\Psi_{\mathrm{dn}} = \lambda\ \mathrm{dn}\, [\frac{1}{\hbar'}(Wt - px)],$$

$$\Psi_{\mathrm{nd}} = \lambda\, k'\ \mathrm{dn}\, [\frac{1}{\hbar'}(Wt - px)]$$

are particular solutions of:

$$\Box\Psi - (1 + k'^2)\mu_0^2\,\Psi + \frac{2\mu_0^2}{\lambda^2}\Psi^3 = 0.$$

3. The plane-wave solutions of wave equations with nonlinear terms in Ψ^3.

Conversely, we can utilize those results to characterize the plane-wave solutions to the following four types of equations:

(α) $$\Box\Psi + \mu_1^2\,\Psi + \mu_2^2\Psi^3 = 0,$$

(β) $$\Box\Psi + \mu_1^2\,\Psi - \mu_2^2\Psi^3 = 0,$$

(γ) $$\Box\Psi - \mu_1^2\,\Psi + \mu_2^2\Psi^3 = 0,$$

(δ) $$\Box\Psi - \mu_1^2\,\Psi - \mu_2^2\Psi^3 = 0.$$

More precisely, we shall seek to determine when they exist under some conditions that specify the solutions to those equations that are *plane waves of bounded amplitude.*

α) When equation (α) is written:

$$\Box\Psi + \mu_1^2\,\Psi + \mu_2^2\Psi^3 = 0,$$

it will have plane-wave solutions:

$$\Psi = \lambda\,\mathrm{cn}\,[\frac{1}{\hbar'}\,(Wt - px) + \xi_0\,,\,k]$$

when one determines μ_0 and k by:

$$(1 - 2k^2)\,\mu_0^2 = \mu_1^2\,, \qquad \frac{2k^2\mu_0^2}{\lambda^2} = \mu_2^2\,,$$

$$(\mu_0 = \frac{m_0\,c^2}{\hbar'}).$$

One then deduces:

$$\mu_0^2 = \mu_1^2 + \mu_2^2\lambda^2\,, \qquad k^2 = \frac{\mu_2^2\,\lambda^2}{2(\mu_1^2 + \mu_2^2\,\lambda^2)}\,.$$

One verifies that one will always have $0 \le k^2 \le 1/2$ here. Therefore, the plane wave Ψ is never aperiodic. The dynamic mass μ_0 is always greater than μ_1 .

Conversely, if one fixes $\mu_0 > \mu_1$ then λ and k will be fixed. One will then have:

$$\lambda^2 = \frac{\mu_0^2 - \mu_1^2}{\mu_2^2}\,, \qquad k^2 = \frac{\mu_0^2 - \mu_1^2}{2\mu_0^2}\,.$$

Only λ will depend upon μ_2, while k will depend upon only μ_0^2 and μ_1. The solutions above are the only plane-wave solutions to equation (α) with no restrictions on the initial conditions.

β) Equation (β):

$$\Box\Psi + \mu_1^2\,\Psi - \mu_2^2\Psi^3 = 0$$

will admit the functions:

$$\Psi = \lambda\,\mathrm{sn}\,[\frac{1}{\hbar'}\,(Wt - px) + \xi_0\,,\,k)]$$

for bounded periodic functions. Here, one has:

$$\mu_0^2 = \mu_1^2 - \frac{\mu_2^2\,\lambda^2}{2}, \qquad k^2 = \frac{\mu_2^2\,\lambda^2}{2\mu_1^2 - \mu_2^2\,\lambda^2}\,.$$

μ_0 is less than μ_1.

$0 \leq k^2 \leq 1$ implies the condition that $\lambda^2 < \mu_1^2/\mu_2^2$. That corresponds to some restrictions on the initial conditions Ψ (0), Ψ' (0) that are necessary for the solutions to (β) to have bounded amplitude.

Indeed, the general study of the solutions to:

$$y''_{x^2} + \alpha\,y - |\,\gamma|\,y^3 = 0$$

shows that according to the values of the initial conditions y_0 and y'_0, those equations will admit the following solutions:

1. $$Y = \lambda\,\mathrm{sn}\,[\omega(x - x_0) + \xi_0]$$

$$\text{if}\quad 0 \leq y_0^2 \leq \frac{1}{|\,\gamma|}(\alpha - \sqrt{2\,|\,\gamma|\,y_0'^2}) \text{ and } y_0'^2 \leq \frac{\alpha^2}{2\,|\,\gamma|}\,.$$

2. $$Y = \lambda\,\mathrm{scd}\,[\omega(x - x_0) + \xi_0]$$

$$\text{if}\quad \left.\begin{matrix}\alpha - \sqrt{2\,|\,\gamma|\,Y_0'^2} \\ 0\end{matrix}\right\} \leq |\,\gamma|\,Y_0^2 \leq \alpha + \sqrt{2\,|\,\gamma|\,Y_0'^2}\,.$$

3. $$Y = \frac{\lambda}{k}\,\mathrm{ns}\,[\omega(x - x_0) + \xi_0]$$

$$\text{if}\quad \frac{1}{|\,\gamma|}(\alpha + \sqrt{2\,|\,\gamma|\,y_0'^2}) \leq Y_0^2 \leq \frac{1}{|\,\gamma|}\sqrt{\alpha + 2\,|\,\gamma|\,y_0'^2}\,.$$

4. $$Y = \lambda \,\mathrm{nc}\,[\omega(x - x_0) + \xi_0]$$

$$\text{if} \quad \frac{1}{|\gamma|}[\alpha + \sqrt{\alpha^2 + 2|\gamma|\,y_0'^2}\,] \leq Y_0^2 .$$

Only the solution $Y = \mathrm{sn}\,[\omega(x - x_0) + \xi_0]$ will have bounded amplitude. Equation (β) will then have periodic solutions of an acceptable type only if the initial functions satisfy the conditions:

$$[\Psi'(0)]^2 \leq \frac{\mu_1^2}{2\mu_2^2},$$

$$[\Psi(0)]^2 \leq \frac{1}{\mu_2^2}(\mu_1^2 - \sqrt{2\mu_2^2(\Psi_0')^2}\,].$$

Conversely, being given that $\mu_0 < \mu_1$ will determine λ^2 and k^2:

$$\lambda^2 = \frac{2(\mu_0^2 - \mu_1^2)}{\mu_2^2}, \qquad k^2 = \frac{\mu_0^2 - \mu_1^2}{\mu_0^2}.$$

Here again, only λ will depend upon μ_2, while k will depend upon only μ_0 and μ_1. The solution $\Psi = \lambda\,\mathrm{Sn}\,[\ldots]$ will become aperiodic for $\lambda^2 = \mu_1^2 / \mu_2^2$ or $\mu_0^2 = \mu_1^2 / 2$, sn $(u, 1) = \tanh u$, so:

$$\Psi = \frac{\mu_1}{\mu_2} \tanh\,[\frac{1}{\hbar}(Wt - px)].$$

γ) Equation (γ), namely:

$$\Box\Psi - \mu_1^2\,\Psi + \mu_2^2\Psi^3 = 0,$$

will admit solutions of type λ cn τ or λ dn τ according to the initial conditions.

a. $$\Psi = \lambda\,\mathrm{dn}\,[\frac{1}{\hbar}(Wt - px) + \xi_0, k]$$

will be a solution of (γ) if one has:

$$\mu_0^2 = \frac{\mu_2^2\,\lambda^2}{2}, \quad k^2 = \frac{2(\mu_2^2\,\lambda^2 - \mu_1^2)}{\mu_2^2\,\lambda^2}$$

under the condition that $\mu_1^2 / \mu_2^2 < \lambda^2 < 2\mu_1^2 / \mu_2^2$.

Ψ reduces to a constant for $|\lambda| = \mu_1 / \mu_2$, $k = 0$.

Ψ will become aperiodic for $k^2 = 1$ or $|\lambda| = \mu_1\sqrt{2} / \mu_2$.

[dn (u, 1) = 1 / cosh u]:

$$\Psi = \frac{\mu_1\sqrt{2}}{\mu_2}\frac{1}{\cosh[\frac{1}{\hbar}(Wt - px) + \xi_0]}\,.$$

Conversely, if μ_0 is fixed then:

$$\lambda^2 = \frac{2\mu_0^2}{\mu_2^2}, \qquad k^2 = \frac{2\mu_0^2 - \mu_1^2}{\mu_0^2},$$

in which μ_0 is less than μ_1 and is such that $\mu_1^2 / 2 < \mu_0^2 < \mu_1^2$. Here again, only λ will involve the constant μ_2 ; k depends upon only μ_0 and μ_1 .

b. $$\Psi = \lambda \operatorname{cn}[\frac{1}{\hbar}(Wt - px) + \xi_0 , k], \qquad \text{with} \quad \tfrac{1}{2} \leq k^2 \leq 1,$$

satisfies equation (γ) if:

$$\mu_0^2 = \mu_2^2\,\lambda^2 - \mu_1^2, \qquad k^2 = \frac{\mu_2^2\,\lambda^2}{2(\mu_0^2\,\lambda^2 - \mu_1^2)},$$

under the condition that $\lambda^2 \geq 2\mu_1^2 / \mu_2^2$.

Conversely, if μ_0^2 is given with the condition that $\mu_0^2 \geq \mu_1^2$ then:

$$\lambda^2 = \frac{\mu_0^2 + \mu_1^2}{\mu_2^2}, \qquad k^2 = \frac{\mu_0^2 + \mu_1^2}{2\mu_0^2}.$$

Here again, k is determined by μ_0 and μ_1 ; μ_2 will enter into only the determination of λ.

A solution of that form will become aperiodic for $k = 1$, $\lambda^2 = 2\mu_1^2 / \mu_2^2$, so cn ($u$, 1) = 1 / cosh u (or rather $\mu_0 = \mu_1$):

$$\Psi = \frac{\mu_1\sqrt{2}}{\mu_2}\frac{1}{\cosh[\frac{1}{\hbar}(Wt - px) + \xi_0]}\,.$$

δ) The equation:

$$\Box\Psi - \mu_1^2\,\Psi - \mu_2^2\Psi^3 = 0$$

never admits periodic plane-wave solutions of bounded amplitude. One will have solutions of one of the forms:

$$\lambda \operatorname{tn} \tau, \qquad \lambda \operatorname{scd} \tau, \qquad \lambda \operatorname{nc} \tau$$

according to the initial conditions. We have obtained *real*, bounded, periodic, plane-wave solutions in the three cases $(\alpha),(\beta),(\gamma)$. Those solutions will then have the character of stationary waves.

However, in a number of problems, wave mechanics considers "progressive" plane waves of the form:

$$\Psi = A e^{\pm\frac{i}{\hbar}(Wt-px)},$$

which correspond to solutions:

$$\Psi = A e^{\pm i\tau}$$

of the differential equation:

$$y''_{\tau^2} + y(\tau) = 0.$$

One can demand to know: Do there exist progressive solutions for the nonlinear equations that were considered here?

If one considers $y''_{t^2} + \mu_0^2 c^2 y = 0$ then they will be given by integration:

$$y'_{t^2} + \mu_0^2 c^2 y^2 = \kappa_0 \qquad (= \text{arbitrary constant} > 0).$$

If $\kappa_0 \neq 0$ then one will be led to stationary-wave solutions:

$$y = \frac{\sqrt{\kappa_0}}{\mu_0} \sin \mu_0 ct,$$

or

$$y = \frac{\sqrt{\kappa_0}}{\mu_0} \cos \mu_0 ct,$$

whereas the progressive solutions $y = A e^{\pm i\mu_0 ct}$ will correspond to $\kappa_0 = 0$, so $y' = \pm i \mu_0 cy$.

If one considers the equation:

$$y''_{\tau^2} + \mu_1^2 y + \mu_2^2 y^3 = 0$$

then direct integration will give:

$$y'^2_{\tau} + \mu_1^2 y^2 + \frac{\mu_2^2}{2} y^4 = \kappa_0 .$$

$\kappa_0 \neq 0$ will lead to the real stationary solutions in λ cn τ that were considered before.

$\kappa_0 = 0$ will lead to a new type of solution that is necessarily complex.

In order to obtain those solutions easily, it will suffice to remark that if one sets $y = 1/z$ and starts with:

$$y'^2_{\tau} + \mu_1^2 y^2 + \frac{\mu_2^2}{2} y^4 = 0$$

then one will immediately have:

$$z'^2 + \mu_1^2 z^2 + \frac{\mu_2^2}{2} = 0.$$

One easily sees that one will then have:

$$y = \frac{1}{C_1\, e^{i\mu_1\tau} - C_2 e^{-i\mu_1\tau}},$$

in which C_1, C_2 are two constants that are coupled by $C_1\, C_2 = \mu_2^2 / 8\mu_1^2$.

If one sets $1 / C_1 = \lambda_1$, $1 / C_2 = \lambda_2$ then one can further write that solution as:

$$y = \frac{\lambda_1}{e^{i\mu_1\tau} - \dfrac{\mu_2^2}{8\mu_1^2}\lambda_1^2\, e^{-i\mu_1\tau}} = \frac{\lambda_2}{e^{-i\mu_1\tau} - \dfrac{\mu_2^2}{8\mu_1^2}\lambda_2^2\, e^{i\mu_1\tau}},$$

or alternatively:

$$y = \frac{\lambda_1}{\left(1 + \dfrac{\mu_2^2}{8\mu_1^2}\lambda_1^2\right) e^{-i\mu_1\tau} - \dfrac{\mu_2^2}{4\mu_1^2}\lambda_1^2 \cos\mu_1\tau},$$

$$y = \frac{\lambda_2}{\left(1 + \dfrac{\mu_2^2}{8\mu_1^2}\lambda_2^2\right) e^{-i\mu_1\tau} - \dfrac{\mu_2^2}{4\mu_1^2}\lambda_2^2 \cos\mu_1\tau}.$$

One sees that the plane waves that correspond to those solutions are never purely progressive, but involve a stationary term that depends upon the nonlinearity factor μ_2, along with a progressive term.

Similarly, one will find the following progressive solutions for equations (β) and (γ):

(β)

$$\Psi(\tau) = \frac{\lambda_1}{e^{i\mu_1\tau}\left(1 - \dfrac{\mu_2^2}{8\mu_1^2}\lambda_1^2\right) + \dfrac{\mu_2^2\,\lambda_1^2}{4\mu_1^2}\cos\mu_1\tau}$$

$$= \frac{\lambda_2}{e^{-i\mu_1\tau}\left(1 - \dfrac{\mu_2^2}{8\mu_1^2}\lambda_2^2\right) + \dfrac{\lambda_1^2\mu_2^2}{4\mu_1^2}\cos\mu_1\tau},$$

(γ)

$$\Psi(\tau) = \frac{\lambda_1}{e^{\mu_1\tau}\left(1 - \dfrac{\mu_2^2\lambda_1^2}{8\mu_1^2}\right) + \dfrac{\mu_2^2\,\lambda_1^2}{4\mu_1^2}\cosh\mu_1\tau}$$

$$= \frac{\lambda_2}{e^{-\mu_1 \tau}\left(1 - \frac{\mu_2^2 \lambda_2^2}{8\mu_1^2}\right) + \frac{\lambda_1^2 \mu_2^2}{4\mu_1^2} \cosh \mu_1 \tau}.$$

The equations considered up to now are not linear, and the sum of the two solutions is not a solution.

Nevertheless (and this is a point upon which I must now insist), there exists an addition theorem for the plane-waves that were considered up to now, or if one prefers, a theorem on the composition of wave functions.

With a convenient composition, one can start from two solutions and construct a third one.

That addition theorem must result from the addition theorem for elliptic functions here,

I shall first recall the relations:

$$\mathrm{cn}\,(u + K) = -\,k\,\mathrm{sd}\,u,$$

$$\mathrm{sd}\,(u + K) = \frac{1}{k'}\,\mathrm{cn}\,u,$$

$$\mathrm{sn}\,(u + K) = \mathrm{cd}\,u.$$

One can show that the classical addition theorems of elliptic functions can be likewise written in the following form, which is more adapted to our problem:

$$\mathrm{cn}\,(u \pm v) = \frac{\mathrm{cn}\,u\,\mathrm{cn}\,v \mp k'^2 \mathrm{sd}\,u\,\mathrm{sd}\,v}{1 \mp k^2 \mathrm{cn}\,u\,\mathrm{sd}\,u\,\mathrm{cn}\,v\,\mathrm{sd}\,v},$$

$$\mathrm{sd}\,(u \pm v) = \frac{\mathrm{sd}\,u\,\mathrm{cn}\,v \pm \mathrm{sd}\,v\,\mathrm{cn}\,u}{1 \mp k^2 \mathrm{sd}\,u\,\mathrm{sd}\,u\,\mathrm{cn}\,v\,\mathrm{cn}\,v}.$$

If we consider two states that correspond to the values τ_1 and τ_2 that are characterized by the wave functions:

$$\Psi_c^{(1)} = \lambda\,\mathrm{cn}\,\tau_1, \qquad \Psi_s^{(1)} = \lambda\,K'\,\mathrm{sd}\,\tau_1,$$
$$\Psi_c^{(2)} = \lambda\,\mathrm{cn}\,\tau_2, \qquad \Psi_s^{(2)} = \lambda\,K'\,\mathrm{sd}\,\tau_2,$$

then the state function (1) + (2) – or $\Psi\,(\tau_1 + \tau_2)$ – will be characterized by the functions:

$$\Psi_c^{(1)+(2)} = \lambda\,\mathrm{cn}\,(\tau_1 + \tau_2\,,\,k),$$

$$\Psi_s^{(1)+(2)} = \lambda\,k'\,\mathrm{sd}\,(\tau_1 + \tau_2\,,\,k)\,.$$

The addition theorem will then give us:

$$\Psi_{\mathrm{c}}^{(1)+(2)} = \frac{\lambda^3(\Psi_{\mathrm{c}}^{(1)}\,\Psi_{\mathrm{c}}^{(2)} - \Psi_{\mathrm{s}}^{(1)}\,\Psi_{\mathrm{s}}^{(2)})}{\lambda^4 + \dfrac{k^2}{k'^2}\Psi_{\mathrm{s}}^{(1)}\,\Psi_{\mathrm{s}}^{(2)}\Psi_{\mathrm{c}}^{(1)}\,\Psi_{\mathrm{c}}^{(2)}},$$

$$\Psi_{\mathrm{s}}^{(1)+(2)} = \frac{\lambda^3(\Psi_{\mathrm{s}}^{(1)}\,\Psi_{\mathrm{c}}^{(2)} + \Psi_{\mathrm{s}}^{(1)}\,\Psi_{\mathrm{c}}^{(2)})}{\lambda^4 - \dfrac{k^2}{k'^2}\Psi_{\mathrm{s}}^{(1)}\,\Psi_{\mathrm{s}}^{(2)}\Psi_{\mathrm{c}}^{(1)}\,\Psi_{\mathrm{c}}^{(2)}}.$$

The possibility of starting with state functions for one corpuscle and constructing state functions for two or more corpuscles makes a second quantization of the theory possible.

4. More general nonlinear wave equations.

The generalization of wave functions that was considered up to now started with solutions of plane-wave type.

I would now like to examine the possibility of generalizing the waves of invariant type. Indeed, it seems reasonable that the nonlinear wave equations to be considered must present the relativistic invariance of linear wave mechanics.

We have seen that after introducing the variable $u = \sqrt{c^2t^2 - (x^2 + y^2 + z^2)}$ into the equation $\Box\Psi + \mu_0^2\Psi = 0$, we could determine invariant solutions $\Psi\ (u)$ by means of the *differential* equation $\left(\dfrac{d^2}{du^2} + \dfrac{3}{u}\dfrac{d}{du} + \mu_0^2\right)\Psi\ (u) = 0$, and I have indicated its solutions.

One of the characteristics of that equation is that it presents a critical point for its solutions only at the point $u = 0$; i.e., on the light cone.

It seems that one can demand that the wave functions (which are solutions of more general equations) should preserve that property, or at least, that they should possess only solutions with *fixed critical points* (i.e., ones that are independent of the integration constants).

If we seek a relativistically-invariant second-order partial differential equation then the most general form for one will be written:

$$\text{(E)} \qquad \Box\Psi_0 + F\left[\left(\frac{1}{c}\frac{\partial\Psi}{\partial t}\right)^2 - \sum\left(\frac{\partial\Psi}{\partial x}\right)^2, ct\frac{\partial\Psi}{\partial t} - x\frac{\partial\Psi}{\partial x}, \Psi, u\right] = 0.$$

The invariant solutions will be functions *of only u* that satisfy the differential equation:

$$\frac{d^2\Psi}{du^2} + \frac{3}{u}\frac{d\Psi}{du} + F\left[\left(\frac{d\Psi}{du}\right)^2, u\frac{d\Psi}{du}, u, \Psi(u)\right] = 0.$$

If equation (E) must likewise admit plane-wave solutions then upon setting:

$$m_0\, c\tau = [Wt - (\mathbf{px})]$$

it will be necessary that:

$$\frac{W^2}{c^2} - p^2 = m_0^2 c^2$$

and equation (E) will lead to a differential equation that depends upon only τ.

Now:

$$\Box\Psi \to \frac{d^2\Psi}{d\tau^2}, \qquad \left(\frac{1}{c}\frac{\partial\Psi}{\partial t}\right)^2 - \sum\left(\frac{\partial\Psi}{\partial x}\right)^2 \to \left(\frac{d\Psi}{d\tau}\right)^2, \qquad ct\frac{\partial\Psi}{\partial t} - x\frac{\partial\Psi}{\partial x} \to m_0\, c\tau\frac{d\Psi}{d\tau},$$

and as a result, (E) will give a differential equation in τ if u does not enter into the equation.

If that equation is realized then:

$$\text{(E}_1\text{)} \qquad \Box\Psi_0 + F\left[\left(\frac{1}{c}\frac{\partial\Psi}{\partial t}\right)^2 - \sum\left(\frac{\partial\Psi}{\partial x}\right)^2, t\frac{\partial\Psi}{\partial t} - x\frac{\partial\Psi}{\partial x}, \Psi\right] = 0$$

will give rise to two differential equations:

$$\text{(E}_1\text{)} \qquad \frac{d^2\Psi}{du^2} + \frac{3}{u}\frac{d\Psi}{du} + F\left[\left(\frac{d\Psi}{du}\right)^2, u\frac{d\Psi}{du}, \Psi(u)\right] = 0,$$

$$\text{(E}_2\text{)} \qquad \frac{d^2\Psi}{d\tau^2} + F\left[\left(\frac{d\Psi}{d\tau}\right)^2, \tau\frac{d\Psi}{d\tau}, \Psi(\tau)\right] = 0.$$

Generalizing the properties of the invariant plane wave functions of ordinary wave mechanics will lead us to postulate:

α) The solutions of E_2 must have fixed critical points.

β) The solutions of E_2 must be continuous, uniform functions that are bounded (or ones of finite amplitude).

One can obtain all of the second-order differential equations of the form:

$$y'' = R\,(y, y') \qquad \text{or} \qquad y'' = R\,(x, y, y'),$$

in which R is rational in y' and algebraic in x and y and its integrals are either uniform or have fixed critical points, by means of some results of a paper by P. PAINLEVÉ [**2**] that was completed by B. GAMBIER [**1**]. Those papers gave all of the forms that are allowable within the scope of the hypotheses above. The equation:

$$y'' + \alpha y + \beta y^3 = 0$$

is equation (4) of table (9) in Painlevé.

Among the remarkable nonlinear wave equations that one can deduce from Painlevé's tables, I would like to point out only the equation:

$$\text{(C)} \qquad \Box\Psi - \left(1 - \frac{1}{n}\right)\left[\left(\frac{1}{c}\frac{\partial\Psi}{\partial t}\right)^2 - \sum\left(\frac{\partial\Psi}{\partial x}\right)^2\right]\frac{1}{\Psi} + \mu_n^2\,\Psi = 0,$$

in which n is an integer.

One can easily find *all* of the solutions of that equation.

Indeed, let $\varphi(x, y, z, t)$ be such that:

$$(\mathrm{C}_1) \qquad \Box\varphi + \mu_1^2\,\varphi = 0,$$

and let $\Psi_1 = [\varphi(x, y, z, t)]^n$, so equation ($C_1$) will imply that:

$$\Box\Psi_1 - \left(1 - \frac{1}{n}\right)\frac{1}{\Psi_1}\left[\left(\frac{1}{c}\Psi'_t\right)^2 - \sum(\Psi'_x)^2\right] + n\,\mu_1^2\,\Psi_1 = 0.$$

One then sets:

$$\mu_n^2 = n\,\mu_1^2, \qquad \mu_n = \mu_1\sqrt{n}\,.$$

As a result, equation (C) will admit all of the solutions of (C_1) (plane waves, invariant waves, guided waves, etc.), with:

$$\Psi(x, y, z, t; \mu_n) = [\varphi(x, y, z, t; \frac{\mu_n}{\sqrt{n}})]^n$$

for a corpuscle of mass $\mu_1 = \mu_n / \sqrt{n}$ with $\Psi = \varphi^n$.

As a result, consider the equation:

$$\Box\Psi - \lambda\left[\frac{1}{c}\left(\frac{\partial\Psi}{\partial t}\right)^2 - \sum\left(\frac{\partial\Psi}{\partial x}\right)^2\right]\frac{1}{\Psi} + \mu^2\,\Psi = 0.$$

In order to require that $\lambda = 1 - 1/n$, with n an integer, it will suffice *to impose upon that equation* the demand that it must admit *plane-wave solutions* that are *uniform* functions of τ, and the states of mass $\mu_n = m / \sqrt{n}$ will result from that. Here, one sees how a *uniformity condition* that is imposed upon wave functions can lead to a quantization of mass.

I would now like to point out another viewpoint that perhaps richer in possible extensions.

Up to now, second quantization has been considered to be a linear attribute.

I think that this is not necessary, and it seems to me that second quantization is essentially attached to the possibility of constructing states with two particles, …, n particles upon starting with states with one particle.

In order for that to be true, it would suffice that there should exist *a theorem of addition or composition of states* in the theory considered; i.e., that when one starts with wave functions that represent a state with n particles and a state with one particle, one can construct the wave function that represents a state with n + 1 or n – 1 particles.

The acceptable wave functions will be the ones that admit a theorem of addition or composition. That condition seems very broad. Nevertheless, that problem will possess a partial solution.

Indeed, WEIERSTRASS has proved a remarkable theorem that relates to our problem.

WEIERSTRASS gave the name of *algebraic addition theorem* to an algebraic relation that couples the functions $\Phi(u)$, $\Phi(v)$, $\Phi(u+v)$, and here is his theorem:

Any function for which there exists an algebraic addition theorem is an elliptic function or one of its degenerate cases.

That will then lead us to the nonlinear equations that we have considered.

Nonetheless, the mathematical hypotheses that are introduced are not justified from the physical viewpoint. Nothing leads us to assume that nature obeys rules that translate into algebraic laws.

I would like to conclude, moreover, by considering a simple example in which I shall recover all of the character that is admissible for a nonlinear wave equation without the hypothesis of algebra coupling or algebraic addition.

In order to do that, consider the nonlinear equation:

$$\text{(D)} \qquad \Box\Psi(x, y, z, t) + \mu_0^2 \sin \lambda\Psi = 0.$$

In the first approximation (?), $\lambda\Psi$ will be small, and that equation will be written:

$$\Box\Psi + \mu_0^2 \lambda\Psi = 0$$

or

$$\Box\Psi + \mu_1^2 \lambda\Psi = 0, \qquad \mu_1^2 = \mu_0^2\,\lambda\,.$$

In second approximation:

$$\Box\Psi + \mu_0^2 \lambda\Psi - \frac{\mu_0^2\,\lambda^3}{6}\Psi^3 = 0,$$

and with $\mu_2^2 = \mu_0^2\,\lambda^3/6$:

$$\Box\Psi + \mu_1^2 \lambda\Psi - \mu_2^2\Psi^3 = 0.$$

This is an equation of a nonlinear type that was considered before.

If we set $\lambda\,\Psi = \varphi$ then we will be reduced to:

$$(D_1) \qquad \Box\varphi + \mu_1^2 \sin\varphi = 0,$$

which depends upon just one parameter μ_1 .

If one seeks the plane-wave solutions to that equation then one will be reduced to the equation:

$$\frac{d^2\varphi(\tau)}{d\tau^2} + \mu_1^2 \sin\varphi(\tau) = 0.$$

That equation is well-known in physics: viz., it is the pendulum equation.

Whereas the Klein-Gordon equation associates a corpuscle in its proper system with the simplest oscillator, here we associate it with a pendulum motion.

The solutions of (D) or (D_1) are defined only up to a multiple of 2π: $\varphi_1(\tau) = \varphi_0(\tau) + 2\pi n$. Similarly, if one sets $\varphi' = \varphi \pm n\pi / 2$ then the functions φ' will satisfy:

$$\Box\varphi' \pm \mu_1^2 \cos\varphi' = 0.$$

The solutions to (D_1) simultaneously give solutions to:

$$\Box\varphi \pm \mu_1^2 \sin\varphi = 0,$$

$$\Box\varphi \pm \mu_1^2 \cos\varphi = 0.$$

We shall now determine the plane-wave solutions to (D_1). The equation $\varphi'' + \mu_1^2 \sin\varphi = 0$ gives:

$$\varphi'^2 - 2\mu_1^2 \cos\varphi = C_0, \qquad C_0 = \text{Const.}$$

$$C_0 = \varphi_0^2 - 2\mu_1^2 \cos\varphi_0 ,$$

and thus, the condition:

$$-2\mu_1^2 + \varphi_0'^2 \leq C_0 \leq 2\mu_1^2 + \varphi_0'^2 ,$$

$$\varphi'^2 = (C_0 + 2\mu_1^2)\left(1 - \frac{4\mu_1^2}{C_0 + 2\mu_1^2}\sin^2\frac{\varphi}{2}\right).$$

Then let:

$$\text{A)} \qquad \frac{4\mu_1^2}{C_0 + 2\mu_1^2} = k^2 < 1,$$

which demands that $C_0 > 2\mu_1^2$, so:

$$2\mu_1^2 \leq C_0 \leq 2\mu_1^2 + \varphi_0'^2 .$$

That gives:

$$\varphi'^2 = \frac{4\mu_1^2}{k^2}(1 - k^2 \sin^2 \frac{\varphi}{2});$$

$$\frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi / 2}} = \pm \frac{2\mu_1}{k} d\tau .$$

Let $\chi = \varphi / 2$:

$$\int_0^{\chi} \frac{d\kappa}{\sqrt{1 - k^2 \sin^2 \chi}} = \pm \frac{\mu_1}{k}\tau + \xi_0 .$$

On the left-hand side, one recognizes the Legendre integral:

$$F(\chi\, k) = F(\varphi / 2, k).$$

Now, if $F(\varphi, k) = u$, and conversely, $\varphi = \text{am } u$, then:

$$\sin \varphi = \text{sn } u, \quad \cos \varphi = \text{cn } u.$$

One will then have that:

$$\frac{\varphi}{2} = \text{am } (\frac{\mu_1}{k} \tau + \xi_0).$$

B) Let $\frac{4\mu_1^2}{C_0 + 2\mu_1^2} = k_1^2 > 1$, so $| k_1 \sin \varphi / 2 | < 1$.

Let $k_1 = 1 / k$ and utilize the relation $F(\varphi, k_1) = k\, F(\varphi_1, k)$ with $k_1 = 1 / k$, so one will get:

$$\varphi_1 = \text{arc sin } (k_1 \sin \varphi),$$

or the formula that is called the *reciprocal modulus formula:*

$$\text{sn } (ku, k_1) = k \text{ sn } (u, k),$$

which will give:

$$\sin \frac{\varphi}{2} = k \text{ sn } (\mu_1 \tau + \xi_1, k).$$

The determination of the plane-wave solutions will then be complete.

We shall show that the *corresponding plane-wave functions possess an addition theorem.*

Let $\varphi(u_1 + u_2) = 2 \text{ am } (u_1 + u_2)$, so:

$$\text{am } (u_1 \pm u_2) = \text{arc tan } (\tan u_1 \text{ dn } u_2) \pm \text{arc tan } (\tan u_2 \text{ dn } u_1)$$

$$= \arctan\left(\frac{\sin\varphi_1/2}{\cos\varphi_1/2}\sqrt{1-k^2\sin^2\frac{\varphi_2}{2}}\right) \pm \arctan\left(\frac{\sin\varphi_2/2}{\cos\varphi_2/2}\sqrt{1-k^2\sin^2\frac{\varphi_1}{2}}\right)$$

It is not necessary to emphasize the non-algebraic character of that addition theorem! Nonetheless, we see that it is possible to construct nonlinear wave equations that possess solutions whose character generalizes that of the wave functions of ordinary wave mechanics by starting with relatively simple physical models.

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