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On a nonlinear generalization of wave mechanics and the properties of the corresponding wave functions

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Abstract. –

1. – Introduction.

Here, I propose to present some results that I have obtained in the search for and study of solutions of several types of nonlinear wave equations that are likely to generalize the equations of wave mechanics.

Numerous authors have already sought to introduce nonlinear wave equations by starting with a phenomenological study of interactions by looking for a nonlinear theory whose quantum field theory is an approximation.

I have adopted another viewpoint by looking for whether some very general considerations might otherwise give them exactly, at least when restricted to classes of nonlinear wave equations that might be introduced. Starting from an analysis of these types of solutions of the Klein-Gordon equation, I was led [**16-18**] to examine the acceptable generalizations. Conversely, these generalizations satisfy wave equations that are generalizations of the Klein-Gordon equation. In the case of plane waves, I was also led [**16**] to discover a type of nonlinear equation that was already encountered by R. FINKELSTEIN, R. LE LEVIER, M. RUDERMANN [**6**], L. SCHIFF [**20**], and N. ROSEN and H. B. ROSENSTOCK [**19**].

2. – The principal solutions of the Klein-Gordon equation.

The usual wave mechanics represents corpuscles without spin by wave functions $\psi(x, y, z, t)$ that are solutions of the Klein-Gordon equation:

$$(1) \qquad \begin{cases} \square\psi + \mu_0^2\psi = 0, \\ \square = \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} - \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}\right), \ \mu_0 = \dfrac{m_0 c}{\hbar}, \ \hbar = \dfrac{h}{2\pi}. \end{cases}$$

Whereas general theorems show the equivalence of all the complete systems of solutions of wave equation, the applications of wave mechanics show that it is necessary for the problems that are examined to use basic systems that possess, for example, special symmetries. In a nonlinear generalization of wave mechanics that seems desirable in various regards since it is possible that only certain basic systems will be considered, so the equivalence between systems might result only from a degeneracy that is associated with the linear approximation.

Depending upon the problem that is being studied, the principal types of solutions of the Klein-Gordon equation are:

a) Solutions of plane-wave type,
b) 〃 〃 invariant wave type,
c) 〃 〃 spherical wave type,
d) 〃 〃 guided wave type.

a) The solutions of "plane-wave" type are obtained by starting with equation (1) upon supposing that the functions $\psi(x, y, z, t)$ depend upon only one variable – namely, τ – which is a linear combination of x, y, z, t:

$$\tau = \frac{1}{\hbar}[Wt - (\mathbf{px})] = Kct - (\mathbf{Kx}),$$

by the intermediary of four constants (W, p_1, p_2, p_3) or (K, K_1, K_2, K_3), which are such that:

$$W^2 = c^2p^2 + m_0^2c^4 \qquad \text{or} \qquad K^2 = \mathbf{K}^2 + \mu_0^2 .$$

The function $\psi(t)$ that is a solution of (1) is then a solution of the differential equation:

$$(2) \qquad \frac{d^2\psi(\tau)}{d\tau^2} + \psi(\tau) = 0.$$

The general solution of (2) is a combination of two types of solutions, one of them ψ_c being even and the other one ψ_s being odd:

$$(3) \qquad \psi_c = A\cos\tau, \qquad \psi_s = B\sin\tau.$$

The functions $\psi(t) = \psi(Kct - (\mathbf{Kx}))$ can be considered to be the result of a Lorentz transformation that is applied to a particular solution $\psi(t)$ of the proper system, which is a solution of (1) that is independent of x, y, z. One then has:

$$\tau = \mu_0 \, ct = \frac{2\pi}{h} m_0 \, c^2 t.$$

The plane waves (3) that are of the form $\psi(\tau)$ define a complete system of solutions of (1) that are functions of τ that are uniform, periodic, and have bounded amplitude.

b) The invariant solutions are obtained by starting with (1) upon considering the particular solutions of that equation that depend upon only one variable, which is a relativistic invariant.

One generally takes this variable to be:

(4) $$u = \sqrt{c^2 t^2 - r^2},$$

or

$$u^2 = c^2 t^2 - (x^2 + y^2 + z^2).$$

One easily sees that:

(5) $$\Box = \frac{d^2}{du^2} + \frac{3}{u} \frac{d}{du}.$$

Equation (1) then determines $\psi(u)$ by way of:

(6) $$\left[\frac{d^2}{du^2} + \frac{3}{u} \frac{d}{du} + \mu_0^2\right] \psi(u) = 0.$$

This is, moreover, a differential equation whose general solution is expressed by means of Bessel functions of order one:

(7) $$\psi(u) = \frac{A}{u} J_1(\mu_0 \, u) + N_1(\mu_0 \, u) = \frac{C_1}{u} H_1^{(1)}(\mu_0 u) + \frac{C_2}{u} H_1^{(2)}(\mu_0 u),$$

(J_1, N_1 are Bessel functions of the first and second kind, while $H_1^{(1)}$ and $H_1^{(2)}$ are the corresponding Hankel functions of order 1.)

c) and *d*) In order to introduce spherical waves and guided waves [**18**], we shall now assume that there exists a privileged frame R_0, in which the wave functions $\psi(x, y, z, t)$ are expressed in the form of a product of a function $\psi_1(t)$ of t and a function $\psi_2(x, y, z)$ or $\psi_2(r, \theta, \varphi)$:

(8) $$\psi(\mathbf{x}, t) = \psi_1(t) \, \psi_2(x, y, z) = \psi_1(t) \, \psi_2(r, \theta, \varphi).$$

We then have:

$$\psi_2(\mathbf{x})\frac{1}{c^2}\frac{d^2\psi_1}{dt^2}(t)-\psi_1(t)\Delta\psi_2(\mathbf{x})+\mu_0^2\psi_1\psi_2 = 0.$$

We introduce two constants λ_1, λ_2, which are such that:

$$\lambda_2 - \lambda_1 = \mu_0^2,$$

in which $\psi_1(t)$ and $\psi_2(\mathbf{x})$ satisfy the equations:

$$\text{(9)} \qquad \begin{cases} \dfrac{1}{c^2}\dfrac{d^2\psi_1}{dt^2}(t)+\lambda_1\psi_1(t)=0, \\ \Delta\psi_2(x,y,z)+\lambda_2\psi_2(x,y,z)=0. \end{cases}$$

(We assume that λ_1 and λ_2 are real and restrict ourselves here to the case in which $\lambda_1 > 0$, in order to not introduce solutions of a type that vanishes with t. These solutions that are damped in the course of time must not be discarded in a general study, which we shall not make here.)

We then obtain for the function $\psi_1(t)$:

$$\text{(10)} \qquad \psi_1(t) = c_1 \exp\,[i\sqrt{\lambda_1}ct] + c_2 \exp\,[-i\sqrt{\lambda_1}ct] = c_1'\cos(\sqrt{\lambda_1}ct)+c_2'\sin(\sqrt{\lambda_1}ct).$$

For $\psi(x, y, z)$, there are two cases to consider:

1) $\lambda_1 > \mu_0^2$, $\lambda_2 > 0$.

$$\Delta\psi_2 + \lambda_2\psi_2 = 0,$$

which then admits for its acceptable solutions:

$$\text{(11)} \qquad \psi_2(r,\,\theta,\,\varphi) = \frac{1}{\sqrt{r}}[AJ_{l+\frac{1}{2}}(\sqrt{\lambda_2}r)+RN_{l+\frac{1}{2}}(\sqrt{\lambda_2}r)]\,y_l^m(\theta,\varphi).$$

2) $\lambda_1 < \mu_0^2$, $\lambda_2 < 0$.

$$\Delta\psi_2 - |\lambda_2|\,\psi_2 = 0,$$

which has for its solutions that remain bounded when $r \to \infty$:

$$\text{(12)} \qquad \psi_2(r,\,\theta,\,\varphi) = \frac{A}{\sqrt{r}}K_{l+\frac{1}{2}}(\sqrt{|\lambda_2|}r)\,y_l^m(\theta,\varphi).$$

If we restrict ourselves to the case of $l = 0$ then we are no longer considering spherical functions $y_i^m(\theta,\varphi)$, and what remains is:

(13) $$\psi_2(r,\ \theta,\ \varphi) = \psi_2(r).$$

One thus has, in the case above:

(14) $$\psi_2(r) = A'\frac{\sin(\sqrt{\lambda_2}\,r)}{r} + B'\frac{\cos(\sqrt{\lambda_2}\,r)}{r},$$

(15) $$\psi_2(r) = \frac{A''}{r}\exp[-\sqrt{|\lambda_2|}\,r].$$

The solutions that are called "spherical waves" in orthodox wave mechanics are obtained by starting with these expressions upon setting:

(16) $$\lambda_1 = \frac{W^2}{\hbar^2 c^2} = K^2, \qquad \lambda_2 = \frac{p^2}{\hbar^2} = |\,\mathbf{K}\,|^2,$$

so

$$\lambda_2 - \lambda_1 = K^2 - |\,\mathbf{K}\,|^2 = \mu_0^2 .$$

One then necessarily has $\lambda_1 > \mu_0^2$, and for $l = 0$ the general spherical wave solution is written:

(17) $$\psi_{\text{sph.}} = \psi_1(t)\ \psi_2(r) = [c_1'\cos Kct + c_2'\sin Kct]\,A'\left[\frac{\sin|\,\mathbf{K}\,|\,r}{r} + A''\frac{\cos|\,\mathbf{K}\,|\,r}{r}\right]$$
$$= c_1''\frac{\sin(Kct \mp |\,\mathbf{K}\,|\,r)}{r} + c_2''\frac{\cos(Kct \mp |\,\mathbf{K}\,|\,r)}{r}.$$

Orthodox wave mechanics likewise considers the particular case of the solutions above for which one has:

(18) $$\lambda_1 = 0, \qquad \psi = \psi_2(r).$$

One is then dealing with the case 2) above. $\lambda_1 = 0$ entails that $|\,\lambda_2\,| = \mu_0^2$, and:

(19) $$\psi = \psi(r) = \psi_2(r) = \frac{C_0}{r}\exp[-\mu_0\, r].$$

Upon fixing the value of the constant C_0, this solution is considered to represent the field $\psi(r)$ that is created by a source C_0 that is localized to the point $r = 0$ in the proper system of the corpuscle (here, the frame R_0).

One passes from the solutions $\psi = \psi_1(t)\,\psi_2(x, y, z)$, with $\psi_1(t)$ and $\psi_2(x, y, z)$ given by (10), (11), and (12), or (10), (14), and (15), to the solutions of "guided wave" type, for which the corpuscle is localized and describes a trajectory (which is rectilinear and uniform in the absence of an external field), by performing a Lorentz transformation on the functions ψ that depends upon time explicitly.

For a corpuscle that displaces along the OZ axis with velocity v, we set:

$$ct = \cosh \gamma ct' - \sinh \gamma z', \qquad z = \cosh \gamma z' - \sinh \gamma ct',$$

$$x = x', \qquad y = y', \qquad \tanh \gamma = v,$$

so

$$r^2 = x'^2 + y'^2 + \cosh^2 \gamma (z' - \tanh \gamma ct')^2,$$

$$\sqrt{\lambda_1} t = \sqrt{\lambda_1} (\cosh \gamma ct' - \sinh \gamma z').$$

If we further write:

$$\sqrt{\lambda_1} = \mu_1, \qquad K_1 = \mu_1 \cosh \gamma, \qquad \mathbf{K}_1 = \mu_1 \sinh \gamma,$$

$$\sqrt{\lambda_1} t = K_1 \, ct' - |\, \mathbf{K}_1 \,|\, z',$$

$$r^2 = x'^2 + y'^2 + \left(\frac{K_1}{\mu_1}\right)^2 [z' - vt']^2 = \rho^2$$

then we obtain the expression for the "guided wave":

(20) $$\psi'(x', y', z', t') =$$

$$= \begin{Bmatrix} c_1' \cos \\ c_1'' \sin \end{Bmatrix} (K_1 ct' - |\, \mathbf{K}_1 \,|\, z') \cdot \left\{ A' \frac{\sin(\sqrt{\lambda_2}\,\rho')}{\rho'} + A'' \frac{\cos(\sqrt{\lambda_2}\,\rho')}{\rho'} \right\}.$$

The term in A'' introduces a polar singularity that displaces with a velocity v ($\rho' = 0$ for $x' = y' = 0$, $z' = vt'$). Likewise, if one restricts oneself to the regular part then a structure that is defined in R_0 by a combination of solutions of type (10), (11), and (12), or (10), (14), and (15) will generate a solution that is a combination of the $\psi'(x', y', z', t')$ above that displaces with the velocity v.

The particular solution:

$$\psi = \psi(r) = \frac{c_0 \exp[-\mu_0 r]}{r}$$

leads to the guided solution:

(21) $$\psi'(x', y', z', t') = \frac{c_0 \exp[-\mu_0 (x'^2 + y'^2 + \cosh^2 \gamma (z' - vt')^2)^{1/2}]}{[x'^2 + y'^2 + \cosh^2 \gamma (z' - vt')^2]^{1/2}},$$

which is ordinarily interpreted as a Yukawa field with a source that moves rectilinearly and uniformly with velocity $\tanh \gamma = v$.

Likewise, the solutions of plane-wave type remain within the scheme of guided waves when we set $\lambda_2 = 0$, $\lambda_2 = 0$, $\psi = \psi_1(t)$.

3. – Generalized plane waves that are deduced form Jacobi's elliptic functions.

In an extension of wave mechanics that is based in the Klein-Gordon equation, we must generalize either the set of solution types that we just considered or only certain other ones that, for physical reasons, lead us to consider them as attached more directly to the representation of matter.

If we first consider the solutions of plane-wave type then we have seen that they be considered as the result of a transformation of the Lorentz group that is applied to the particular solutions of the proper system:

(22) $$\psi_s = A' \sin \tau_0 , \qquad \psi_c = A'' \sin \tau_0 ,$$

with

(23) $$\tau_0 = \mu_0\, ct = \frac{2\pi}{h} m_0 c^2 t = 2\pi \nu_0\, t.$$

This form of solution exhibits a fundamental character of the representation of corpuscles in wave mechanics that L. de Broglie has often insisted upon: In the proper system of the corpuscle, the wave function is associated with a "clock;" i.e., a periodic function of proper time with a period $T = h/m_0\, c^2$ (or a frequency $\nu_0 = m_0\, c^2/h$).

If we would like to generalize this concept, while attempting to enrich the notion of corpuscle, by no longer introducing just the one intrinsic constant $\nu_0 = m_0\, c^2/h$, but two or more constants then the most immediate generalization consists of taking the wave functions that represent the particle in its proper system to be certain Jacobi elliptic functions that possess a real period and a pure imaginary period, instead of the circular functions cos τ or sin τ. The definition of these functions introduces a real parameter k that is found between 0 and 1. For $k = 0$, these functions reduce to sin τ and cos τ. The given of k is equivalent to the introduction of an intrinsic supplementary parameter.

The theory of Jacobi functions introduces three principal functions:

$\mathrm{sn}(u, k)$	of period	$4K$ and $4iK'$,
$\mathrm{cn}(u, k)$	″	$4K$ and $4iK'$,
$\mathrm{dn}(u, k)$	″	$2K$ and $2iK'$.

The periods $K(k)$ and $K'(k)$ are defined by the integral:

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2\theta}} ,$$

and by

$$K'(k) = K(k'), \qquad \text{with } k'^2 = 1 - k^2.$$

Starting with these three functions, one constructs a system of 12 elliptic functions by adding to sn u, cn u, dn u, their inverse and quotients:

$$\mathrm{ns}\, u = \frac{1}{\mathrm{sn}\, u}, \qquad \mathrm{nc}\, u = \frac{1}{\mathrm{cn}\, u}, \qquad \mathrm{nd}\, u = \frac{1}{\mathrm{dn}\, u},$$

$$\mathrm{sc}\, u = \mathrm{tn}\, u = \frac{\mathrm{sn}\, u}{\mathrm{cn}\, u}, \qquad \mathrm{sd}\, u = \frac{\mathrm{sn}\, u}{\mathrm{dn}\, u},$$

$$\mathrm{cs}\, u = \frac{\mathrm{cn}\, u}{\mathrm{sn}\, u}, \qquad \mathrm{cd}\, u = \frac{\mathrm{cn}\, u}{\mathrm{dn}\, u},$$

$$\mathrm{ds}\, u = \frac{\mathrm{dn}\, u}{\mathrm{sn}\, u}, \qquad \mathrm{dc}\, u = \frac{\mathrm{dn}\, u}{\mathrm{cn}\, u},$$

and one has, notably, the following relations between these functions:

$$\mathrm{sn}(u + K, k) = \quad \mathrm{cd}(u, k),$$

$$\mathrm{cn}(u + K, k) = - k' \,\mathrm{sd}(u, k),$$

$$\mathrm{dn}(u + K, k) = \quad k' \,\mathrm{nd}(u, k),$$

$$\mathrm{sn}(u, 0) = \sin u, \qquad \mathrm{sn}(u, 1) = \tanh u,$$

$$\mathrm{cn}(u, 0) = \cos u, \qquad \mathrm{sn}(u, 1) = \frac{1}{\cosh u},$$

$$\mathrm{dn}(u, 0) = 1, \qquad \mathrm{dn}(u, 1) = \frac{1}{\cosh u}.$$

One will find the study of the properties of these functions in numerous books on applied mathematics. In the name of indicating some, I will cite only the works of APPEL and LACOUR [**1**], GREENHILL [**9**], TRICOMI [**22**], and the excellent little monograph of BOWMAN [**2**].

The generalization of the wave functions that are plane-wave solutions of the Klein-Gordon equation leads us to set $u = \tau$, so:

$$\tau = 4K(k)\,\nu_0 t = 4K(k)\frac{m_0 c^2}{\hbar}\, t = \frac{m_0 c^2}{\hbar}\, t = \mu_0\, ct. \tag{24}$$

Here, $4K$ is the analogue of the factor 2π in the trigonometric case, and this leads us to introduce a new reduced Planck constant:

$$\hbar' = \frac{h}{4K(k)}, \tag{25}$$

which replaces the usual constant $\hbar = h/2\pi$, while μ_0 will be coupled to the dynamical mass m_0 by the intermediary of $\hbar$:

$$\mu_0 = \frac{m_0 c}{\hbar}. \tag{26}$$

In the proper system, we have the possibility of defining even and odd doubly-periodic wave functions that reduce to sin τ and cos τ, respectively, for $k = 0$, according to the choices:

a) Either sn(τ, k) and cd(τ, k)
b) Or cs(τ, k) and sd(τ, k).

In addition, we may define a doubly-periodic wave function that reduces to a constant for $k = 0$ by considering the functions:

c) dn(τ, k) and nd(τ, k).

We shall now examine the second-order differential equations that the choice of these functions leads to by adopting, as a generalization, the equation:

$$\frac{d^2}{d\tau^2}\psi(\tau) + \psi(\tau) = 0.$$

For this, we examine the second-order differential equations whose solutions are Jacobi elliptic equations.

A) The equation:

$$y'^2 + (1 - 2k^2)\, y^2 + k^2\, y^4 - k'^2 = 0$$

has the solutions:

$$y = \text{cn}\, u \quad \text{if} \quad y(0) = 1,$$

$$y = k' \text{cn}\, u \quad \text{if} \quad y(0) = 0.$$

As a result:

$$y'' + (1 - 2k^2)\, y + 2k^2 y^2 = 0$$

has the solutions:

$$y = \text{cn}\, u \quad \text{if} \quad y(0) = 1, \quad y'(0) = 0,$$

$$y = k' \text{sd}\, u \quad \text{if} \quad y(0) = 0, \quad y'(0) = k'.$$

B) The equation:

$$y'^2 + (1 + 4k^2)\, y^2 - k^2\, y^4 - 1 = 0$$

has the solutions:

$$y = \text{sn}\, u \quad \text{if} \quad y(0) = 0,$$

$$y = \text{cd}\, u \quad \text{if} \quad y(0) = 1.$$

As a result:

$$y'' + (1 + k^2)\, y - 2k^2 y^2 = 0$$

has the solutions:

$$y = \operatorname{cn} u \qquad \text{if} \qquad y(0) = 0, \qquad y'(0) = 1,$$

$$y = \operatorname{cd} u \qquad \text{if} \qquad y(0) = 1, \qquad y'(0) = 0.$$

C) The equation

$$y'^2 - (1 + k'^2)\, y^2 + y^4 + k'^2 = 0$$

has the solutions:

$$y = \operatorname{dn} u \qquad \text{if} \qquad y(0) = 0,$$

$$y = k' \operatorname{nd} u \qquad \text{if} \qquad y(0) = k'.$$

As a result:

$$y'' - (1 + k^2)\, y + 2y^3 = 0$$

has the solutions:

$$y = \operatorname{dn} u \qquad \text{if} \qquad y(0) = 1, \qquad y'(0) = 0,$$

$$y = k' \operatorname{nd} u \qquad \text{if} \qquad y(0) = k', \qquad y'(0) = 0.$$

Returning from the differential equations that are verified by the functions $\psi(t)$ to the partial differential equations that determine the functions $\psi(x, y, z, t)$, one immediately sees by correspondence that:

A) (27)

$$\begin{cases} \psi_c = \lambda \quad \operatorname{cn}[(Kct - (\mathbf{Kx})), k], \\ \psi_s = \lambda k' \operatorname{sd}[(Kct - (\mathbf{Kx})), k] \end{cases}$$

are particular solutions of:

(28)

$$\Box\psi + (1 - 2k^2)\mu_0^2 \psi + \frac{2k^2 \mu_0^2}{\lambda^2} \psi^3 = 0.$$

B) (29)

$$\begin{cases} \psi_s = \lambda \operatorname{sn}[(Kct - (\mathbf{Kx})), k], \\ \psi_c = \lambda \operatorname{cd}[(Kct - (\mathbf{Kx})), k] \end{cases}$$

are particular solutions of:

(30)

$$\Box\psi + (1 + k^2)\mu_0^2 \psi - \frac{2k^2 \mu_0^2}{\lambda^2} \psi^3 = 0.$$

C) (31)

$$\begin{cases} \psi_{\mathrm{dn}} = \lambda \quad \operatorname{dn}[(Kct - (\mathbf{Kx})), k], \\ \psi_{\mathrm{nd}} = \lambda k' \operatorname{nd}[(Kct - (\mathbf{Kx})), k] \end{cases}$$

are particular solutions of:

(32)

$$\Box\psi - (1 + k'^2)\mu_0^2 \psi + \frac{2\mu_0^2}{\lambda^2} \psi^2 = 0.$$

Equations (28), (30), (32) have already been encountered by numerous authors, notably, L. SCHIFF [**20**], N. ROSEN and H. B. ROSENSTOCK [**19**], R. FINKELSTEIN, R. LE LEVIER and M. RUDERMAN [**6**], B. J. MALENKA [**13**], and D. IVANENKO [**11**].

These equations may be written in a general fashion:

(33) $$\Box\psi + \alpha\psi + \gamma\psi^3 = 0,$$

in which α and γ denote two constants.

4. – The plane-wave solutions of the nonlinear wave equation $\Box\psi + \alpha\psi + \gamma\psi^3 = 0$.

Conversely, we shall use the results above to characterize the plane-wave solutions of equations (33), which we divide into four types:

(34) $$\begin{cases} (A)\ \ \Box\psi + \mu_1^2\psi + \mu_2^2\psi^2 = 0, \\ (B)\ \ \Box\psi + \mu_1^2\psi - \mu_2^2\psi^3 = 0, \\ (C)\ \ \Box\psi - \mu_1^2\psi + \mu_2^2\psi^3 = 0, \\ (D)\ \ \Box\psi - \mu_1^2\psi + \mu_2^2\psi^3 = 0, \end{cases}$$

More precisely, we shall determine when there exist solutions of these equations of bounded-amplitude plane-wave type, under conditions that we shall specify.

A) Equation (*A*) admits for its plane-wave solutions:

(35) $$\psi = \lambda\,\mathrm{cn}[(Kct - (\mathbf{Kx})) + \xi_0, k],$$

with

$$K^2 - |\,\mathbf{K}\,|^2 = \mu_0^2,$$

$$\mu_0 = \frac{m_0 c}{\hbar'},$$

upon determining μ_0 and k by:

$$(1 - 2k^2)\,\mu_0^2 = \mu_1^2, \qquad \frac{2k^2\mu_0^2}{\lambda^2} = \mu_2^2,$$

so

(36) $$\begin{cases} \mu_0^2 = \mu_1^2 + \mu_2^2\lambda^2, \\ k^2 = \dfrac{\mu_2^2\lambda^2}{2(\mu_1^2 + \mu_2^2\lambda^2)}. \end{cases}$$

Here, one always $0 \leq k^2 \leq 1/2$. A plane wave is never spherical. The reduced dynamical mass μ_0 is always greater than μ_1, while the true dynamic mass m_0 has the value:

$$m_0 = \frac{\hbar\mu_0}{4cK(k)}.$$

If $\mu_0 > \mu_1$ is fixed then λ and k are determined by:

$$\lambda^2 = \frac{\mu_0^2 - \mu_1^2}{\mu_2^2}, \qquad k^2 = \frac{\mu_0^2 - \mu_1^2}{2\mu_0^2}. \tag{37}$$

If m_0 is given instead of μ_0 then the determination of k is more complex: In this case, one must solve the transcendental equation:

$$(1 - 2k^2)\, K^2(k) = \frac{\hbar^2\mu_1^2}{16m_0^2c^2}. \tag{38}$$

If one is given three constants μ_1, μ_2, and k ($0 \leq k^2 \leq 1/2$) then:

$$\begin{cases} \lambda^2 = \dfrac{2k^2\mu_1^2}{\mu_2^2(1-2k^2)}, \\ \mu_0^2 = \dfrac{\mu_1^2}{1-2k^2} \qquad \text{and} \qquad m_0^2c^2 = \dfrac{\mu_1^2}{16K^2(1-2k^2)}. \end{cases} \tag{39}$$

Plane waves (35) are solutions of (A) for any initial conditions $\psi(0)$ and $\psi'(0)$. For $\xi_0 = 0$, $\psi = \lambda \operatorname{cn} \tau$, and for $\xi_0 = K$, $\psi = \lambda k' \operatorname{sd} \tau$.

B) Equations of the form:

$$\Box\psi + \mu_1^2\psi - \mu_2^2\psi^3 = 0 \tag{34B}$$

admit for bounded-amplitude plane-wave solutions:

$$\psi = \lambda \operatorname{sn}[(Kct - (\mathbf{Kx})) + \xi_0, k], \qquad K^2 - |\,\mathbf{K}\,|^2 = \mu_0^2, \tag{40}$$

with

$$\begin{cases} \mu_0^2 = \mu_1^2 - \dfrac{\mu_2^2\lambda^2}{2}, \\ k^2 = \dfrac{\mu_2^2\lambda^2}{2\mu_1^2 - \mu_2^2\lambda^2}. \end{cases} \tag{41}$$

The condition $0 \leq k^2 < 1$ entails that:

$$0 \leq \lambda^2 \leq \frac{\mu_1^2}{\mu_2^2}.$$

This corresponds to restrictions on the initial data.

Indeed, the solution above exists only if the initial conditions satisfy the conditions:

$$(\psi'(0))^2 \leq \frac{\mu_1^2}{2\mu_2^2}$$

and

$$(\psi'(0))^2 \leq \frac{1}{\mu_2^2}[\mu_2^2 - \sqrt{2\mu_2^2(\psi'(0))^2}\,].$$

If these conditions are not satisfied then the plane-wave solutions of (B) are Jacobi elliptic functions that become unbounded periodic and it does not seem that such functions are likely to represent a physically realizable corpuscular structure.

Conversely, being given μ_0^2, μ_1^2, μ_2^2 determines λ^2 and k^2 by way of:

$$\text{(42)} \qquad \begin{cases} \lambda^2 = \dfrac{2(\mu_0^2 - \mu_1^2)}{\mu_0^2}, \\ k^2 = \dfrac{\mu_1^2 - \mu_0^2}{\mu_0^2}. \end{cases}$$

This solution becomes aperiodic for $\lambda^2 = \mu_1^2 / \mu_2^2$. Then $\mu_0^2 = \mu_1^2 / 2$ and:

$$\text{(43)} \qquad \psi_s = \frac{\mu_1}{\mu_2}\tanh(Kct - (\mathbf{Kx})), \qquad \psi_c = \frac{\mu_1}{\mu_2}.$$

The relation $m_0c = h\mu_0/ 4K$ then shows that if $\mu_0 = \mu_1 / \sqrt{2}$ remains finite as $K(1) \to \infty$ then the proper dynamical mass m_0 tends to zero.

C) Equations of the type (C):

$$\text{(34C)} \qquad \Box\psi - \mu_1^2\psi + \mu_2^2\psi^3 = 0$$

admit plane-wave solutions of either the type λ cn τ or the type λ dn τ for any initial conditions.

– C_1 –

$$\text{(44)} \qquad \psi = \lambda\,\mathrm{dn}[(Kct - (\mathbf{Kx})) + \xi_0, k]$$

satisfies equations (C), where μ_0^2 and k^2 are determined by:

$$\text{(45)} \qquad \mu_0^2 = \frac{\mu_2^2\lambda^2}{2}, \quad k^2 = \frac{2(\mu_2^2\lambda^2 - \mu_1^2)}{\mu_2^2\lambda^2},$$

under the condition that:

$$\frac{\mu_1^2}{\mu_2^2} \le \lambda^2 \le \frac{2\mu_1^2}{\mu_2^2}.$$

For $\xi_0 = 0$, $\psi = \lambda \operatorname{dn} \tau$; and for $\xi_0 = K$, $\psi = \lambda k' \operatorname{nd} \tau$.

For $|\lambda| = \mu_1 / \mu_2$, $k = 0$, and ψ reduces to a constant: $\psi = \mu_1 / \mu_2$.

For $k^2 = 1$ – namely, $|\lambda| = \mu_1\sqrt{2}/\mu_2$ – ψ becomes aperiodic:

$$\left(\operatorname{dn}(u,1) = \frac{1}{\cosh u}\right),$$

but then $\mu_0^2 = \mu_1^2$, $\hbar' \to 0$. It is necessary that the proper dynamic mass m_0 tends to zero.

Conversely, if μ_0^2, μ_1^2, μ_2^2 are given then:

$$\lambda^2 = \frac{2\mu_0^2}{\mu_2^2}, \qquad k^2 = \frac{2\mu_0^2 - \mu_1^2}{\mu_0^2},$$

under the condition that $\mu_1^2 / 2 \le \mu_0^2 \le \mu_1^2$.

– C_2 –

(46) $$\psi = \lambda \operatorname{cn}[(Kct - (\mathbf{Kx})] + \xi_0, k],$$

with

$$\tfrac{1}{2} \le k^2 \le 1,$$

is a solution of (C).

(For $\xi_0 = 0$, $\psi_c = \lambda \operatorname{cn} \tau$, and for $\xi_0 = \mp K$, $\psi_c = \pm \lambda k' \operatorname{sd} \tau$).

μ_0^2 and k^2 are then determined by:

(47) $$\begin{cases} \mu_0^2 = \mu_2^2\lambda^2 - \mu_1^2, \\ k^2 = \dfrac{\mu_2^2\lambda^2}{2(\mu_2^2\lambda^2 - \mu_1^2)}, \end{cases}$$

under the condition that:

$$\lambda^2 \ge \frac{2\mu_1^2}{\mu_2^2}.$$

For $\lambda^2 = 2\mu_1^2 / \mu_2^2$, $k^2 = 1$, ψ becomes aperiodic:

(48) $$\psi_c = \frac{\mu_1\sqrt{2}}{\mu_2}\frac{1}{\cosh \tau}.$$

Conversely, if if μ_0^2, μ_1^2, μ_2^2 are given then:

(48) $$\lambda^2 = \frac{\mu_0^2 + \mu_1^2}{\mu_2^2}, \; k^2 = \frac{\mu_0^2 + \mu_1^2}{2\mu_0^2}, \qquad (\mu_0^2 \geq \mu_1^2).$$

D) Equations of the type:

(34D) $$\Box\psi - \mu_1^2\psi + \mu_2^2\psi^3 = 0$$

do not admit bounded-amplitude plane-wave solutions.

Indeed, depending upon the initial conditions, the solutions of the associated differential equation:

$$\frac{d^2\psi(\tau)}{d\tau^2} - \mu_1^2\psi - \mu_2^2\psi^3 = 0$$

have one of the forms:

$$\lambda \operatorname{tn} \tau, \qquad \lambda \frac{\operatorname{sn} \tau}{\operatorname{cd} \tau}, \qquad \lambda \operatorname{nc} \tau.$$

These doubly-periodic functions become unbounded periodic and are not physically acceptable. This leads us to discard the equations of the type (*D*).

In the three cases (*A*), (*B*), (*C*), we have obtained plane-wave solutions of stationary type.

Linear wave mechanics considers, above all, plane-waves of "progressive" type – i.e., ones of the form:

$$\psi = A \exp[i(Kct - (\mathbf{Kx})],$$

which correspond to solutions:

$$\psi = A \exp[\pm i\tau], \qquad (\tau = Kct - (\mathbf{Kx}))$$

of

$$\frac{d^2\psi}{d\tau^2} + \psi(\tau) = 0.$$

One may propose to determine waves of the same type for (34A), (34B), (34C) that reduce to the functions $A \exp[\pm i\tau]$.

If one considers the equation:

$$y''_{x^2} + \omega^2 y(x) = 0$$

then direct integration gives:

$$y'^2 + \omega^2 y^2 = \chi_0 = \text{const.}$$

$\chi_0 \neq 0$ leads to stationary waves $\sqrt{\chi_0 / \omega^2}\cos \omega x$ and $\sqrt{\chi_0 / \omega^2}\sin \omega x$, while $\chi_0 = 0$ leads to $y = A \exp[\pm i\omega x]$.

Here, the differential equation that is associated with equation (34A), for example, namely:

$$\psi''_{x^2} + \mu_1^2\psi + \mu_2^2\psi^3 = 0,$$

gives, by direct integration:

$$(\psi'_\tau)^2 + \mu_1^2\psi^2 + \frac{\mu_2^2}{2}\psi^4 = \chi_0 .$$

$\chi_0 \neq 0$ leads to the stationary real solutions that were considered previously.

$\chi_0 = 0$ leads to another type of solutions.

Setting $\psi = 1/\chi$, one easily sees that if $\chi_0 = 0$ then:

(50) $$\psi(\tau) = \frac{1}{C_1 \exp[i\mu_1\tau] - C_2 \exp[-i\mu_1\tau]},$$

in which C_1, C_2 denote two constants that are coupled by the relation:

(51) $$C_1C_2 = \frac{\mu_2^2}{8\mu_1^2}.$$

Setting $1/C_1 = \lambda_1$, $1/C_2 = -\lambda_2$, one further writes:

(52) $$\psi(\tau) = \frac{\lambda_1}{\exp[i\mu_1\tau] - (\mu_2^2/8\mu_1^2)\lambda_1^2 \exp[-i\mu_1\tau]} = \frac{\lambda_2}{\exp[-i\mu_1\tau] - (\mu_2^2/8\mu_1^2)\lambda_2^2 \exp[i\mu_1\tau]},$$

or further

(53) $$\psi(\tau) = \frac{\lambda_1}{\exp[i\mu_1\tau](1 + \mu_2^2\lambda_1^2/8\mu_1^2) - (\mu_2^2/4\mu_1^2)\lambda_1^2 \cos\mu_1\tau}$$

$$= \frac{\lambda_2}{[1 + (\mu_2^2/8\mu_1^2)\lambda_2^2]\exp[-i\mu_1\tau] - (\mu_2^2/4\mu_1^2)\lambda_2^2 \cos\mu_1\tau}.$$

These functions are simply periodic. One easily sees how the passage to the solutions of the Klein-Gordon case comes about from these solutions as $\mu_2 \to 0$.

The plane-wave $\psi(Kct - (\mathbf{Kx}))$, $(K^2 - |\,\mathbf{K}\,|^2 = \mu_1^2)$ is never purely progressive. A stationary term appears along with the progressive term. This may be further interpreted by saying that the plane waves of this type never have uniquely positive or uniquely negative energy. A beat term (*terme de battement*) always accompanies the principal progressive term with positive or negative energy.

5. – The composition of wave functions in nonlinear theories.

Equations (34A), (34B), (34C) are not linear and the sum of two solutions is not a solution. Nevertheless – and this is a point upon which we shall now insist – there exists an addition theorem – or, if one prefers, a composition theorem – for the plane-wave solutions of these equations.

This results immediately from the theorem on the addition of elliptic functions.

Consider the cn u. We have seen that:

$$\operatorname{cn}(u+K) = -k' \operatorname{sd} u, \qquad \operatorname{sd}(u+K) = \frac{1}{k'}\operatorname{cn} u.$$

One may show that the addition theorems for elliptic functions that are given in the classical treatments likewise take the following forms:

$$\text{(54)} \qquad \begin{cases} \operatorname{cn}(u\pm v) = \dfrac{\operatorname{cn} u \operatorname{cn} v \mp k'^2 \operatorname{sd} u \operatorname{sd} v}{1 \pm k^2 \operatorname{cn} u \operatorname{sd} u \operatorname{cn} v \operatorname{sd} v}, \\ \operatorname{sd}(u\pm v) = \dfrac{\operatorname{sd} u \operatorname{cn} v \pm \operatorname{sd} v \operatorname{cn} u}{1 \mp k^2 \operatorname{sd} u \operatorname{cn} u \operatorname{sd} v \operatorname{cn} v}, \\ \operatorname{sn}(u\pm v) = \dfrac{\operatorname{sn} u \operatorname{cd} v \pm \operatorname{cd} v \operatorname{sn} u}{1 \mp k^2 \operatorname{sn} u \operatorname{cd} u \operatorname{sn} v \operatorname{cd} v}, \\ \operatorname{cd}(u\pm v) = \dfrac{\operatorname{cd} u \operatorname{cd} v \mp \operatorname{sn} u \operatorname{sn} v}{1 \mp k^2 \operatorname{sn} u \operatorname{cd} v \operatorname{sn} v \operatorname{cd} v}. \end{cases}$$

If, for a corpuscle that is represented by equation (34A), one considers states τ_1 and τ_2 that correspond to the wave functions:

$$\text{(55)} \qquad \begin{cases} \psi_c^{(1)} = \lambda \operatorname{cn} \tau_1, \qquad \psi_s^{(1)} = \lambda k' \operatorname{sd} \tau_1, \\ \psi_c^{(2)} = \lambda \operatorname{cn} \tau_2, \qquad \psi_s^{(1)} = \lambda k' \operatorname{sd} \tau_2 \end{cases}$$

then the state function (1) + (2), or $\psi(\tau_1 + \tau_2)$ corresponds to the functions:

$$\text{(56)} \qquad \begin{cases} \psi_c^{(1)+(2)} = \lambda \quad \operatorname{cn}(\tau_1 + \tau_2, k), \\ \psi_s^{(1)+(2)} = \lambda k' \operatorname{sd}(\tau_1 + \tau_2, k). \end{cases}$$

The addition theorem then gives:

(57)
$$\begin{cases} \psi_c^{(1)+(2)} = \dfrac{\lambda^3[\psi_c^{(1)}\psi_c^{(2)} - \psi_s^{(1)}\psi_s^{(2)}]}{\lambda^4 + (k^2/k'^2)\psi_s^{(1)}\psi_s^{(2)}\psi_c^{(1)}\psi_c^{(2)}}, \\ \\ \psi_s^{(1)+(2)} = \dfrac{\lambda^3[\psi_c^{(1)}\psi_c^{(2)} + \psi_s^{(1)}\psi_s^{(2)}]}{\lambda^4 - (k^2/k'^2)\psi_s^{(1)}\psi_s^{(2)}\psi_c^{(1)}\psi_c^{(2)}}. \end{cases}$$

The possibility of constructing state functions for two corpuscles by starting with state functions for one corpuscle makes possible the construction of a state space that is necessary for the introduction of a second quantization.

Second quantization has been generally considered as necessitating a linear theory. It seems to me that this is not necessary, but that second quantization is essentially attached to the possibility of constructing states for 2, 3, … particles by starting with states of one particle. For this, it suffices that in the theory considered there exists a theorem of the addition or composition of states; i.e., that by starting with functions that represent a state with *n* particles and a state with one particle one can construct a state of *n* + 1 particles.

The acceptable wave functions will then be the ones that admit an addition theorem. This condition, which is necessary but not sufficient, seems *a priori* very large. Nevertheless, we shall see that one may ascribe a particularly remarkable solution to the determination of these functions.

Indeed, WEIERSTRASS has proved a remarkable theorem (see, for example, the treatise on elliptic functions of HANCKOCK [**10**]) that answers our question.

WEIERSTRASS called an algebraic relation that links the functions $\Phi(u)$, $\Phi(v)$, $\Phi(u + v)$ an *algebraic addition theorem*, and here is his theorem:

Any function for which there exists an algebraic addition theorem is an elliptic theorem or one of its degenerate cases.

The application of this theorem to the plane wave solutions leads us to the wave equations that were considered above in a limiting fashion.

Nevertheless, the algebraic nature of an addition theorem for wave functions is not imposed from the standpoint of physical interpretation, and nothing leads us to think that nature obeys rules that translate into algebraic laws.

I would, moreover, now like to consider a simple example of wave equations that generalize the preceding equations and for which there will exist a non-algebraic addition theorem for plane waves.

For this, I consider the nonlinear wave equations:

(58)
$$\begin{cases} (\alpha) \qquad \Box\psi + \mu_1^2 \sin\psi \ = 0, \\ (\beta) \qquad \Box\psi + \mu_1^2 \sinh\psi = 0. \end{cases}$$

If one considers that these equations are "approached" by the equations that were obtained by replacing sin ψ and sinh ψ with the first terms in their series developments then these equations are the generalizations of:

(59)
$$\begin{cases} (\alpha') \qquad \Box\psi + \mu_1^2\psi - \dfrac{\mu_1^2}{6}\psi^3 = 0, \\ (\beta') \qquad \Box\psi + \mu_1^2\psi + \dfrac{\mu_1^2}{6}\psi^3 = 0. \end{cases}$$

If one then sets $\psi = \lambda\varphi$, $\mu_1^2\lambda^2 / 6 = \mu_2^2$ then one obtains the following equations for φ:

(60)
$$\begin{cases} (\alpha'') \qquad \Box\psi + \mu_1^2\varphi - \mu_1^2\varphi^3 = 0, \\ (\beta'') \qquad \Box\psi + \mu_1^2\varphi + \mu_1^2\varphi^3 = 0. \end{cases}$$

One recovers the preceding equations of types (34A) and (34B).

The solutions of "plane-wave" type of equations (58α) and (58β) may be obtained without difficulty.

If one sets:

$$\tau = Kct - (\mathbf{Kx}), \qquad \text{with} \quad K^2 - |\,\mathbf{K}\,|^2 = \mu_0^2$$

then the "plane-wave" solutions of (58α) and (58β) will be of the form:

$$\psi(\mathbf{x}, t) = \psi(\tau),$$

$\psi(\tau)$ being a solution of the *differential* equations:

(61)
$$\begin{cases} \dfrac{d^2\psi(\tau)}{d\tau^2} + \dfrac{\mu_1^2}{\mu_0^2}\sin\psi(\tau) = 0, \\ \\ \dfrac{d^2\psi(\tau)}{d\tau^2} + \dfrac{\mu_1^2}{\mu_0^2}\sinh\psi(\tau) = 0, \end{cases}$$

or

(62)
$$\begin{cases} (\alpha) \ \ \psi''_{\tau^2} + \chi_1 \sin\psi(\tau) = 0, \\ (\beta) \ \ \psi''_{\tau^2} + \chi_1 \sinh\psi(\tau) = 0. \end{cases}$$

We shall examine the solutions of (62α), while those of (62β) are obtained by a parallel analysis.

Equations (62α) is well-known in physics: It is the equation of pendulum motion.

While the Klein-Gordon equation associates a corpuscle in its proper system with the motion of a sinusoidal oscillator, the nonlinear equations considered here are associated with a pendulum motion.

The solutions of:

(58α)
$$\Box\psi + \mu_1^2 \sin\psi = 0$$

are defined only up to a multiple of 2π. If $\psi_0(\tau)$ is a solution then the same will be true for:

$$\psi(\tau) = \psi_0(\tau) + 2n\pi.$$

Likewise, if one sets:

$$\psi_2(\tau) = \psi_0(\tau) \pm \frac{n\pi}{2}$$

then the functions $\psi_2(\tau)$ will satisfy:

$$(58g) \qquad \Box\psi_2(\tau) + \mu_1^2 \cos\psi_2(\tau) = 0$$

The solutions of (58α) thus permit us to write down those of (58β) and (58γ) immediately.

In order to obtain the plane-wave solutions of (58α), it suffices for us to consider the differential equation (62α), which gives, by direct integration:

$$(63) \qquad (\psi'_\tau)^2 - 2\chi_1 \cos \psi = \chi_0,$$

χ_0 being a constant such that:

$$\chi_0 = \psi_0'^2 - 2\chi_1 \cos \psi_0 .$$

We deduce from this that:

$$(64) \qquad (\psi'_\tau)^2 = (\psi_0 + \psi_1)\left[1 - \frac{4\chi_1}{\chi_0 + 2\chi_1}\sin^2\frac{\psi}{2}\right],$$

and this leads us to consider two cases:

$$1) \qquad \frac{4\chi_1}{\chi_0 + 2\chi_1} \leq 1, \quad \text{namely,} \qquad \chi_0 \geq 2\chi_1 .$$

Setting $k^2 = 4\chi_1 / (\chi_0 + 2\chi_1)$ gives:

$$(65) \qquad (\psi'_\tau)^2 = \frac{4\chi_1}{k^2}\left[1 - k^2 \sin^2\frac{\psi}{2}\right],$$

$$2) \qquad \frac{4\chi_1}{\chi_0 + 2\chi_1} = k_1^2 > 1,$$

so

$$(67) \qquad (\psi'_\tau)^2 = \frac{4\chi_1}{k_1^2}\left[1 - k_1^2 \sin^2\frac{\psi}{2}\right].$$

In the first case, one immediately has:

$$2\int_0^{\psi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \pm\frac{2\sqrt{\chi_1}}{k}\tau + 2\xi_0 ,$$

or

$$F\left(\frac{\psi}{2}, k\right) = \pm\frac{\sqrt{\chi_1}}{k}\tau + 2\xi_0 ,$$

where $F(\varphi, k)$ denotes the Legendre elliptic integral:

$$F(\varphi, k) = \int_0^{\varphi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} .$$

Introducing the function $\mathrm{am}(u, k) = \varphi$, which is such that:

$$\sin\varphi = \mathrm{sn}(u, k) \qquad \text{and} \qquad \cos\varphi = \mathrm{cn}(u, k),$$

one obtains:

(68) $$\frac{\psi}{2} = \mathrm{am}\left(\pm\frac{\sqrt{\chi_1}}{k}\tau + \xi_0, k\right),$$

(69) $$\begin{cases} \sin\dfrac{\psi}{2} = \mathrm{sn}\left(\dfrac{\sqrt{\chi_1}}{k}\tau + \xi_0, k\right), \\ \cos\dfrac{\psi}{2} = \mathrm{cn}\left(\dfrac{\sqrt{\chi_1}}{k}\tau + \xi_0, k\right). \end{cases}$$

$0 \leq k^2 \leq 1$ entails the condition that:

$$\cos^2\frac{\psi_0}{2} \leq \frac{\psi_0'^2}{4\chi_1} .$$

The solution of the second case, for which:

$$\cos^2\frac{\psi_0}{2} > \frac{\psi_0'^2}{4\chi_1} ,$$

which demands that:

$$\psi_0^2 < 4\chi_1,$$

is deduced from the solution to the first case by the relation:

$$F(\varphi, k_1) = k\,F(\varphi_1, k),$$

with

$$\varphi_1 = \arcsin(k_1 \sin\varphi),$$

or by the formula that is called the *reciprocal modulus formula:*

$$\mathrm{sn}(ku, k_1) = k\ \mathrm{sn}(u, k),$$

which gives:

(70) $$\sin\frac{\psi}{2} = k\ \mathrm{sn}(\sqrt{\chi_1}\,\tau + \xi_1, k).$$

In the case considered, we therefore have some simple expressions in terms of elliptic functions for the plane-wave solutions for the wave equation (58).

Here, there exists a further addition theorem for the wave functions that are solutions of plane-wave type.

Indeed, let:

$$\tau_1 = K_1ct - (\mathbf{K}_1\mathbf{x}), \qquad \tau_2 = K_2ct - (\mathbf{K}_2\mathbf{x}),$$

with

$$K_1^2 - (\mathbf{K}_1)^2 = K_2^2 - (\mathbf{K}_2)^2 = \mu_0^2,$$

in which $\psi(\tau_1)$ and $\psi(\tau_2)$ denote the preceding solutions, so $\psi(\tau_1 + \tau_2)$ is expressed in terms of $\psi(\tau_1)$ and $\psi(\tau_2)$.

Indeed, we have:

$$\psi(\tau) = 2\,\mathrm{am}\left[\frac{\sqrt{\chi_1}}{k}\tau + \xi_0\right].$$

The addition theorem for the functions am u gives us:

(71) $$\mathrm{am}(u_1 \pm u_2) = \arctan(\mathrm{tn}\,u_1\,\mathrm{dn}\,u_2) \pm \arctan(\mathrm{tn}\,u_2\,\mathrm{dn}\,u_1) =$$

$$= \arctan\left[\frac{\sin\varphi_1}{\cos\varphi_1}\sqrt{1-k^2\sin^2\varphi_2}\right] \pm \arctan\left[\frac{\sin\varphi_2}{\cos\varphi_2}\sqrt{1-k^2\sin^2\varphi_1}\right],$$

$$(\sin\varphi_2 = \mathrm{sn}\,u_1, \qquad \cos\varphi_2 = \mathrm{cn}\,u_1, \qquad \sin\varphi_2 = \mathrm{sn}\,u_2, \qquad \cos\varphi_2 = \mathrm{cn}\,u_2).$$

One immediately deduces the corresponding addition theorem for the functions $\psi(\tau_1)$, $\psi(\tau_2)$. It is not necessary to emphasize the non-algebraic character of this addition theorem.

It might be interesting to attach the plane-wave solutions of equations (34A), (34B), (34C) to the developments of the quantum theory of fields.

This amounts to expressing the plane waves in the case (34A), for example, of the form:

(72) $$\psi(\tau) = \lambda\,\mathrm{cn}\,\tau = \lambda\,\mathrm{cn}[\mu_0\,ct, k],$$

in the proper system, by means of functions:

(73) $$A\cos\tau' = A\cos\mu_0'ct \qquad \text{or} \qquad A\sin\tau' = A\sin\mu_0'ct.$$

The theory of elliptic function immediately provides us with two developments of this type.

a) The development of elliptic functions into Fourier series gives us, for cn u:

(74) $$\mathrm{cn}\,u = \frac{2\pi}{Kk}\sum_{n=0}^{\infty}\frac{q^{n+\frac{1}{2}}}{1+q^{2n-1}}\cos\left[(2n+1)\frac{\pi u}{2K}\right],$$

with

$$q = \exp\left[-\pi\frac{K'}{K}\right].$$

From this, we deduce that:

(75) $$\psi(\tau) = \lambda\,\mathrm{cn}(\mu_0\,ct,\,k) = \lambda\frac{2\pi}{kK}\sum_{n=0}^{\infty}\frac{q^{n+\frac{1}{2}}}{1+q^{2n-1}}\cos(\mu_0'ct),$$

with

(76) $$\mu_n' = (2n+1)\,\frac{\pi}{2K}\,\mu_0\,.$$

The wave $\psi(\tau)$ can be considered to be the result of a particular series of plane-wave solutions of the Klein-Gordon equation with a sequence of reduced proper masses μ_n' that are odd multiples of the reduced proper mass:

(77) $$\mu_0' = \frac{\pi}{2K}\mu_0 < \mu_0\,.$$

b) The development of cn u into an infinite product of elliptic functions gives:

(78) $$\mathrm{cn}\,u = 2q^{1/4}k'^2k^{-1/2}\cos\frac{\pi u}{K}\cdot\prod_{n=1}^{\infty}\left[\frac{1+2q^{2n}\cos(\pi u/K)+q^{4n}}{1-2q^{2n-1}\cos(\pi u/K)+q^{4n-2}}\right].$$

This gives us:

(79) $$\psi(\tau) = \lambda\,\mathrm{cn}[\mu_0\,ct,\,k] = 2q^{1/4}k'^2k^{-1/2}\cos\mu_0'ct\prod_{n=1}^{\infty}\left[\frac{1+2q^{2n}\cos(\pi u/K)+q^{4n}}{1-2q^{2n-1}\cos(\pi u/K)+q^{4n-2}}\right],$$

with

(80) $$\mu_0' = \frac{\pi}{K}\mu_0,$$

here. Thus:

$$\mu_0' \geq \mu_0 \qquad \text{for } \pi/2 \leq K(k) < \pi,$$
$$\mu_0' < \mu_0 \qquad \text{for } K(k) > \pi.$$

The plane-wave $\psi(\tau)$ is therefore expressed in terms of an infinite product of combinations of plane-wave solutions of a Klein-Gordon equation for a corpuscle of reduced proper mass $\mu_0' = (\pi/K)\,\mu_0$.

6. – Invariant solutions and radial solutions of the preceding equations.

We conclude this study by briefly examining solutions of equations of the type:

$$\square\psi + \mu_1^2\psi \pm \mu_2^2\psi^2 = 0 \tag{34}$$

that have the "invariant wave" type – viz., $\psi(u)$, with $u^2 = c^2t^2 - (x^2 + y^2 + z^2)$ – and the solutions of the type $\psi = \psi(r)$, with $r^2 = x^2 + y^2 + z^2$.

These particular solutions are determined by the differential equations:

$$\left[\frac{d^2}{du^2} + \frac{3}{u}\frac{d}{du} + \mu_1^2\right]\psi(u) \pm \mu_2^2\psi^3 = 0, \tag{81}$$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \mu_1^2\right]\psi(r) \mp \mu_2^2\psi^3 = 0. \tag{82}$$

Equations of this type have been the object of numerous mathematical studies, notably, those of R. O. FORNAGUERA [**7**], M. CIMINO [**3**], and JAIČNICYM [**12**]. Their integration does not seem to be attached to the transcendentals that were characterized up to now.

Here, we point out only the results that relate to the case $\mu_1 = 0$.

Equations (81) and (82) then reduce to:

$$\left[\frac{d^2}{du^2} + \frac{3}{u}\frac{d}{du}\right]\psi(u) \pm \mu_2^2\psi^3 = 0, \tag{83}$$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right]\psi(r) \mp \mu_2^2\psi^3 = 0. \tag{84}$$

Equation (83) admits the remarkable particular solution:

$$\psi(u) = \frac{\sqrt{\pm 1}}{\mu_2 u},$$

from which, one deduces the invariant wave that is singular on the light cone:

$$\psi(x, y, z, t) = \frac{\sqrt{\pm 1}}{\mu_2\sqrt{c^2t^2 - (x^2 + y^2 + z^2)}}. \tag{85}$$

Equation (84), in the form:

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right]\psi(r) + \mu_2^2\psi^3(r) = 0,$$

reduces to the Emden equation [**4**] that was studied by E. A. MILNE [**14**], N. FAIRCLOUGH [**5**], and R. H. FOWLER [**8**], notably.

Indeed, if one sets:

$$\psi(r) = \frac{1}{\mu_2}\varphi(r)$$

then $\varphi(r)$ is determined by the equation:

$$\varphi''_{r^2} + \frac{2}{r}\varphi'_r + \varphi^3 = 0,$$

which is the canonical form of the Emden equation that was adopted by E. A. MILNE [**14**].

If $\varphi(r)$ is a solution then one sees immediately that:

$$\lambda\,\varphi(\lambda r)$$

is likewise a solution.

E. A. MILNE has studied the different forms for the solutions $\varphi(r)$ such that:

$$\varphi(r_0) = 0, \qquad \left(\frac{d\varphi}{dr}\right)_{r=r_0} = -\frac{1}{C^{1/2}}$$

for $r_0 = 1$ according to the various values for C. Notably, he showed that there exists only one positive integral such that $\varphi(1) = 0$, and which takes a value $\varphi(0)$ for $r = 0$ that remains finite. Conversely (Emden solution), if one considers a solution $\varphi(r)$ that takes a finite value for $r = 0$ (for which, one may set $\varphi(0) = 1$, with a convenient value for λ) and is such that $(d\varphi/dr)_{r=r_0} = 0$ then one finds the function that was tabulated by N. FAIRCLOUGH [5] that is annulled for $r = r_0 = 6.9011$, and at that point $\varphi'(r_0) = -0.40231$ and $r_0^2\varphi'(r_0) = -2.0150$.

Here, the general integral of:

$$\varphi''_{r^2} + \frac{2}{r}\varphi'_r + \varphi^3 = 0$$

depends upon the two constants λ and C. For any finite value of λ, there exists a value of $C = C_0$ for which there exists a solution for any given $\varphi(0)$. This solution is annulled for $r = r_0$, and the tangent of that solution for $r = r_0$ defines $C = C_0$. For the other values of $C \neq C_0$, there exist solutions $\varphi(r, C)$ such that $\varphi(r_0, C) = \varphi(r_0, C_0)$, but they are divergent for $r \to 0$, the one tending towards $+\infty$, the other towards $-\infty$. E. A. MILNE has shown the general allure of these functions by means of a diagram. Nevertheless, I do not believe that the analysis of Milne has been extended to the domain $r > r_0$, except in the general study of R. O. FORNAGUERA and in a note of JAIČNICYM, whose results do not seem to agree with those of Milne.

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