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On the quantum theory of fields that are associated with some simple models of nonlinear wave equations

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Summary – Two models of nonlinear wave equations are studied. They give representations for scalar or vectorial particles with scalar or vectorial self-fields. Hamiltonian representations of classical field theories associated with the nonlinear wave equations are constructed. New quantum field theories are proposed for the introduced models. They are generalizations of the usual quantum field theory, but the masses are quantized in parallel with the amplitudes.

1. – Introduction.

In some work that I have pursued for several years ([1]), I have developed the study of models in which I have sought to obtain a representation for what one calls "corpuscles with spin" without dissociating the bare corpuscle from its field of interactions in its description.

In order to attempt to achieve this objective, I will start with what seems to me to be the simplest path: The description of a particle in uniform, rectilinear motion by a plane wave – i.e., one or more functions that take constant values on a four-dimensional planar manifold. This description is applied simply in the case of non-interacting particles in the usual theory of particles with spin. One sees easily in this case that the elements of the tensorial or spinorial functions are trigonometric functions of the argument $K_\mu x^\mu = \mu_0 n_\mu x^\mu = (1/\hbar)[Wt - (\mathbf{px})]$ that characterizes the plane wave. These functions depend upon one parameter $\mu_0 = m_0 c/\hbar$ that determines a frequency that is associated with the corpuscle in its proper system.

In order to extend this description, and to introduce some new parameters into the constitutive elements of the wave function that are attached to the proper fields of interaction of the particle that are capable of making them also appear like forms of the

([1]) G. PETIAU: Suppl. Nuovo Cimento **9** (1958), 548. Journ. Phys. **20** (1959), 784; Cahiers de Physique **13** (1959), 429; **14** (1960), 5; **15** (1961), 157; **16** (1962), 491; Compt. Rend. **259** (1964), 4525; **260** (1965), 1107.

corpuscle, I assume that the preceding frequency must be accompanied by some latent frequencies that are already presented in the description in the proper system.

The simplest way of attaining this result seems to me to be the consideration of plane waves that are no longer trigonometric functions, but *multi-periodic functions* that possess a real period, along with one or more pure imaginary periods. The simplest of these functions are elliptic or hyperelliptic functions.

Whereas the solutions of plane-wave type of the wave equations of particles with spin in the usual theories is always expressed by means of pairs of tensorial or spinorial functions that lead to a pair of trigonometric functions that are associated with differential equations of the type:

$$y_1' = y_2, \qquad y_2' = -y_1,$$

the theory of Jacobi elliptic functions couples *three* functions by nonlinear differential equations of the type:

$$y_1' = y_2 y_3, \qquad y_2' = -y_1 y_3, \quad y_3' = -k^2 y_1 y_2 .$$

One likewise defines systems of n hyperelliptic functions that are associated with systems of nonlinear differential equations of the type:

$$y_p' = \varepsilon_p\, y_1\, y_2 \dots y_{p-1}\, y_{p+1} \dots y_n, \qquad p = 1, 2, \dots, n.$$

This led me to construct systems of nonlinear, first-order, partial differential equations with structures that are close to those of the wave equations in the theory of particles with spin, but admit plane-wave solutions that are multi-periodic functions, elliptic or hyperelliptic, and of bounded amplitude.

I have, moreover, developed the study of various systems of this type that I have called corpuscle-field models. I would like to show here how these models also permit an extension to the quantum theory of fields, and notably lead to a quantization of the dynamic mass in parallel to the quantization of the amplitudes that one encounters in the usual quantum theory of fields.

2. – Study of the two models of the system of nonlinear wave equations.

Here, I would like to study two very simplified models that I interpret as capable of describing, the first one, a corpuscle of spin zero (i.e., a scalar or pseudo-scalar), and the second one, a corpuscle of spin zero with an associated proper field that has spin $\hbar$ of the vectorial type, or alternatively, a corpuscle of spin $\hbar$ with an associated proper field of spin 0.

2.1. *Corpuscle model of spin* 0. – *Associated field of spin* 0 (scalar or pseudo-scalar). – In this model, the wave function is the union of a vector $J^\mu(x^\nu)$ and two invariants $I(x^\mu)$, $K(x^\mu)$. These elements are determined by starting with the system of nonlinear, first-order partial differential equations:

$$(1) \qquad \begin{cases} \partial_\mu J^\mu = \kappa_1 KI, \\ \partial_\mu I = -\kappa_2 KJ_\mu, \\ \partial_\mu K = -\kappa_3 KJ_\mu. \end{cases}$$

κ_1, κ_2, κ_3 are three real, positive constants that are considered as given. One can, with no loss of generality, set $\kappa_1 = \kappa_2 = \kappa_3 = 1$. I will nevertheless preserve these constants because their introduction facilitates the study of the degeneracies of the system (1).

The choice of signs that was introduced before for κ_1, κ_2 κ_3 facilitates the calculations; in a more complete theory, they are left indeterminate. The conditions that I have introduced, moreover, – viz., the existence of plane waves of bounded amplitude – then leads to the determination of these signs according to the choice (+1, –1, –1) or (+1, +1, –1).

For $\kappa_3 = 0$, the system (1) gives $K = K_0$ = const. What then remains is:

$$\partial_\mu J^\mu = (\kappa_1 K_0)\, I, \qquad \partial_\mu I = -(\kappa_2 K_0)\, J^\mu,$$

hence:

$$[\Box + \kappa_1 \kappa_2 (K_0)^2]\, I(x^\mu) = 0.$$

The function $I(x^\mu)$ then verifies the Klein-Gordon equation, with:

$$\mu_0^2 = \kappa_1 \kappa_2 K_0^2 .$$

For $\kappa_2 = 0$, one has I = const. I_0, and in this case $K(x^\mu)$ verifies the Klein-Gordon equation:

$$[\Box + \kappa_1 \kappa_3 (I_0)^2]\, K(x^\mu) = 0.$$

I will use the terms "plane waves" or "plane-wave functions" to refer to the solutions of (1) that are functions of just one variable u that is defined by starting with a given timelike quadri-vector that characterizes a direction of propagation, and will be defined by:

$$u = n_\mu x^\mu, \qquad n_\mu n^\mu = 1, \qquad |\, n^4 \,| \geq +1.$$

For propagation along the OZ axis:

$$n^\mu = (\pm \cosh \gamma, 0, 0, \sinh \gamma).$$

For plane-wave solutions, $I(x^\mu)$, $K(x^\mu)$, and J^μ reduce to functions of u and (1) is written:

$$n_\mu \frac{d}{du} J^\mu(u) = \kappa_1 KI,$$

$$n_\mu \frac{d}{du} I(u) = -\kappa_2\, KJ_\mu\,,$$
$$n_\mu \frac{d}{du} J^\mu(u) = -\ \kappa_3\, I\, J_\mu\,.$$

This leads us to set $J^\mu = n^\mu J_0(u)$, with $J_\mu J^\mu = (J_0(u))^2$.

The functions of u, $J_0(u)$, $I(u)$, $K(u)$ are then solutions of the differential system:

(2)
$$\begin{cases} J_0'(u) = \ \ \kappa_1 KI, \\ I'(u) = -\kappa_2 KJ_0, \\ K'(u) = -\kappa_3 IJ_0. \end{cases}$$

Introducing the functions $y_1(u)$, $y_2(u)$, $y_3(u)$ such that:

$$J_0(u) = \frac{1}{\sqrt{\kappa_2\kappa_3}}\, y_1(u), \qquad I(u) = \frac{1}{\sqrt{\kappa_1\kappa_3}}\, y_2(u), \qquad K(u) = \frac{1}{\sqrt{\kappa_1\kappa_2}}\, y_3(u),$$

(2) is further written:

(3)
$$y_1' = y_2\, y_3\,, \qquad y_2' = -\, y_1\, y_3\,, \qquad y_3' = -\, y_1\, y_2\,.$$

From this, we deduce the relations:

$$y_1 y_1' = -\, y_2 y_2' = -\, y_3 y_3' = y_1\, y_2\, y_3\,,$$

and the first integrals:

(4)
$$\begin{cases} y_1^2 + y_2^2 = \lambda_1^2 = \lambda^2\kappa_3, \\ y_1^2 + y_3^2 = \mu_1^2 = \mu^2\kappa_2. \end{cases}$$

As a result, we have the first integrals for the system (2):

(5)
$$\begin{cases} \kappa_2\kappa_3 J_0^2 + \kappa_1\kappa_3 I^2 = \lambda_1^2 = \lambda^2\kappa_3, \\ \kappa_2\kappa_3 J_0^2 + \kappa_1\kappa_2 K^2 = \mu_1^2 = \mu^2\kappa_3, \end{cases}$$

or furthermore:

(6)
$$\begin{cases} \kappa_2 J_0^2 + \kappa_1 I^2 = \lambda^2, \\ \kappa_3 J_0^2 + \kappa_1 K^2 = \mu^2. \end{cases}$$

The integration of the differential system (3), with (4) – or of (2), with (5) – is immediate: $y_1(u)$, $y_2(u)$ can be expressed by means of three Jacobi elliptic functions, namely, $\mathrm{sn}(u, k)$, $\mathrm{cn}(u, k)$, $\mathrm{dn}(u, k)$. This results immediately from the relations between Jacobi functions:

$$\mathrm{sn}^2\, u = \mathrm{cn}\, u\ \mathrm{dn}\, u, \qquad \mathrm{cn}^2\, u = -\,\mathrm{sn}\, u\ \mathrm{dn}\, u, \qquad \mathrm{dn}^2\, u = -\,k^2\ \mathrm{sn}\, u\ \mathrm{cn}\, u,$$

$$\mathrm{sn}^2\, u + \mathrm{cn}^2\, u = 1, \qquad k^2\,\mathrm{sn}^2\, u + \mathrm{dn}^2\, u = 1, \qquad \mathrm{sn}^2\, u - k^2\,\mathrm{sn}^2\, u = 1 - k^2 \qquad (0 \leq k \leq 1).$$

As a result, if we have:

$$\lambda_1^2 < \lambda_2^2$$

in (4) or (5) then the general solution of (4) is:

$$\text{(7)} \qquad \begin{cases} y_1(u) = \lambda_1 \mathrm{sn}(\mu_1 u + \varphi, \lambda_1 / \mu_1), \\ y_2(u) = \beta_1 \mathrm{cn}(\mu_1 u + \varphi, \lambda_1 / \mu_1), \\ y_3(u) = \mu_1 \mathrm{dn}(\mu_1 u + \varphi, \lambda_1 / \mu_1). \end{cases}$$

With $\lambda_1 = \lambda\sqrt{\kappa_3}$, $\mu_1 = \mu\sqrt{\kappa_2}$, the general solution of (2) is written:

$$\text{(8)} \qquad \begin{cases} J_0(u) = \dfrac{\lambda}{\sqrt{\kappa_2}}\,\mathrm{sn}(\mu_1 u + \varphi, \lambda_1 / \mu_1), \\[2ex] I(u) = \dfrac{\lambda}{\sqrt{\kappa_1}}\,\mathrm{cn}(\mu_1 u + \varphi, \lambda_1 / \mu_1), \\[2ex] K(u) = \dfrac{\lambda}{\sqrt{\kappa_1}}\,\mathrm{dn}(\mu_1 u + \varphi, \lambda_1 / \mu_1). \end{cases}$$

The constant μ_1 is associated with the variable u in order to give the propagation term:

$$\mu_1\, u = \mu_1\, u_\mu\, x^\mu = K_\mu\, x^\mu = (1/\hbar)\,(Wt - \mathbf{px}).$$

As a consequence, μ_1 plays the role of reduced proper mass ($\mu_1 \approx m_0 c\ /\ \hbar$). In the limiting case where $\mu_3 = 0$, this gives us $K(u) = K_1 = \text{const.}$, $\kappa_1\,\kappa_2(K_0)^2 = \mu_1^2$. One then has:

$$k^2 = \frac{\lambda_1^2}{\mu_1^2} = \frac{\lambda^2\kappa_3}{\mu^2\kappa_2} = 0,$$

and the Jacobi functions reduce to trigonometric functions:

$$\mathrm{sn}(u, 0) = \sin u, \qquad \mathrm{cn}(u, 0) = \cos u, \qquad \mathrm{dn}(u, 0) = 1.$$

The nonlinear model considered here no longer introduces the dynamic mass as given a priori, *but as the first integral of a differential system, and in parallel to the amplitude* λ.

We have shown this already, and later on, we shall show that a quantization, which will be a restriction on the possible classical values of certain first integrals, will lead to the simultaneous quantization of the amplitudes and proper masses.

The wave function (8) is defined of the association of three Jacobi functions that are doubly-periodic functions. The functions $J_0(u)$ and $I(u)$ oscillate between $\pm\lambda/\sqrt{\kappa_2}$, $\pm\lambda/\sqrt{\kappa_1}$, with all of the character of trigonometric functions. The same is not true for $K(u)$, whose amplitude oscillates between two positive values:

$$\frac{\mu}{\sqrt{\kappa_1}} \qquad \text{and} \qquad \frac{\mu}{\sqrt{\kappa_1}}\sqrt{1-k^2} = \sqrt{\frac{\mu_1^2-\lambda_1^2}{\kappa_1\kappa_2}}.$$

If $k = \lambda_1 / \mu_1$ is small then $K(u)$ is presented as a "modulation" of the *dynamical* mass. These properties have led me to consider two functions $I(x^\mu)$ and $J_\mu(x^\mu)$ as constituting a more specifically *corpuscular* part of the global wave function here, and $K(u)$ as describing, more particularly, a "proper field" that is associated with the corpuscle. The structure of the system (1) closely couples these three functions, and, in particular, their attribute of having the same propagation.

The solution (7) or (8) corresponds to the hypothesis $\lambda_1^2 < \lambda_2^2$. In the case $\lambda_1^2 > \lambda_2^2$, the respective roles of the functions $I(u)$ and $K(u)$ are found to be switched.

The general solution of the system (3), (4) is then:

$$\text{(9)} \qquad \begin{cases} y_1(u) = \mu_1 \mathrm{sn}(\lambda_1 u + \varphi, \mu_1/\lambda_1), \\ y_2(u) = \lambda_1 \mathrm{dn}(\lambda_1 u + \varphi, \mu_1/\lambda_1), \\ y_3(u) = \mu_1 \mathrm{cn}(\lambda_1 u + \varphi, \mu_1/\lambda_1), \end{cases}$$

so

$$\text{(10)} \qquad \begin{cases} J_0(u) = \dfrac{\mu}{\sqrt{\kappa_3}} \mathrm{sn}(\lambda_1 u + \varphi, \mu_1/\lambda_1), \\ K(u) = \dfrac{\mu}{\sqrt{\kappa_1}} \mathrm{cn}(\lambda_1 u + \varphi, \mu_1/\lambda_1), \\ I(u) = \dfrac{\lambda}{\sqrt{\kappa_1}} \mathrm{dn}(\lambda_1 u + \varphi, \mu_1/\lambda_1). \end{cases}$$

This exchange of functions is well-known in the theory of elliptic functions; it is the transformation of the "reciprocal modulus":

$$\mathrm{sn}\left(ku, \frac{1}{k}\right) = k\,\mathrm{sn}(u, k),$$

$$\mathrm{cn}\left(ku, \frac{1}{k}\right) = \mathrm{dn}(u, k),$$

$$\mathrm{dn}\left(ku, \frac{1}{k}\right) = \mathrm{cn}(u, k).$$

2.2. **Corpuscle-field model of scalar and vectorial type**. I will now briefly examine a second model that describes a scalar corpuscle that is associated with a vectorial field, or conversely, a vectorial corpuscle that is associated with a scalar field.

Here, the global wave function is the union of a vector $J_\mu(x^\rho)$, an anti-symmetric tensor of second order $F_{\mu\nu}(x^\rho)$, and an invariant $K(x^\rho)$.

These tensors are coupled by the system of first-order nonlinear partial differential equations:

$$\left\{\begin{aligned} \partial_\mu J_\nu - \partial_\nu J_\mu &= \kappa_1 K F_{\mu\nu}, \\ \partial_\lambda F^{\mu\lambda} &= \kappa_2 K J^\mu, \\ \partial_\mu K &= \kappa_3 F_{\mu\lambda} J^\lambda, \end{aligned}\right. \tag{11}$$

κ_1, κ_2, κ_3 again being three positive constants.

A first linear degeneracy, with $\kappa_3 = 0$, gives $K = K_0$ = const., and reduces (11) to the vector meson equations.

A second linear degeneracy, with $\kappa_2 = 0$, gives, upon setting $F_{\mu\lambda} J^\lambda = C$, with $\partial_\lambda F^{\mu\lambda} = 0$, the "scalar" system:

$$\begin{aligned} \partial_\lambda K &= \kappa_3 C, \\ \partial_\mu C^\mu &= \kappa_1 \tfrac{1}{2}(F_{\mu\nu} F^{\mu\nu}) K, \end{aligned}$$

so

$$\Box K = \kappa_1 \kappa_3 \tfrac{1}{2}(F_{\mu\nu} F^{\mu\nu}) K = 0.$$

We recover the Klein-Gordon equation under the condition that:

$$F_{\mu\nu} F^{\mu\nu} < 0.$$

We further describe the plane-wave states by a system of functions of the variable $u = n_\mu x^\mu$ ($n_\mu n^\mu = 1$, $| n^4 | \geq +1$), namely, $J_\mu(u)$, $F_{\mu\nu}(u)$, $K(u)$.

These functions will then be constructed from n^μ and a second spacelike vector n'^μ that is orthogonal to n^μ:

$$n'_\mu n'^\mu = -1, \qquad n_\mu n'^\mu = 0.$$

The structure of the system (11) leads us to introduce the functions $J_0(u)$, $F_0(u)$, $K(u)$, with:

$$\begin{aligned} J_\mu &= n'^\mu J_0(u), \\ F_{\mu\nu}(u) &= (n_\mu n'_\nu - n_\nu n'_\mu) F_0(u). \end{aligned}$$

One then has:

$$F_{\mu\lambda} J^\lambda = - n_\mu F_0 J_0 .$$

(11) now leads to the differential system:

(12)
$$\begin{cases} J'_0 = \kappa_1 F_0 K, \\ F'_0 = -\kappa_2 K J_0, \\ K' = -\kappa_3 F_0 J_0. \end{cases}$$

If we further set:

$$J_0(u) = \frac{1}{\sqrt{\kappa_2\kappa_3}}\, y_1(u), \qquad F_0(u) = \frac{1}{\sqrt{\kappa_1\kappa_3}}\, y_2(u), \qquad K(u) = \frac{1}{\sqrt{\kappa_1\kappa_2}}\, y_3(u)$$

then we come down to the system (2), with the first integrals (4), which gives us here:

(13)
$$\begin{cases} \kappa_1\kappa_3 J_0^2 + \kappa_1\kappa_3 F_0^2 = \lambda_1^2 = \lambda^2\kappa_3, \\ \kappa_2\kappa_3 J_0^2 + \kappa_1\kappa_2 F_0^2 = \mu_1^2 = \mu^2\kappa_2, \\ J_0^2 = -J_\mu J^\mu, \qquad F_0^2 = -\tfrac{1}{2} F_{\mu\nu} F^{\mu\nu}. \end{cases}$$

According to the relative values of λ_1^2 and μ_1^2, we have two further solutions:

1) If $\lambda_1^2 < \mu_1^2$:

(14)
$$\begin{cases} J_0(u) = \dfrac{\lambda}{\sqrt{\kappa_2}} \operatorname{sn}(\mu_1 u + \varphi, \lambda_1 / \mu_1), \\ F_0(u) = \dfrac{\lambda}{\sqrt{\kappa_1}} \operatorname{cn}(\mu_1 u + \varphi, \lambda_1 / \mu_1), \\ K(u) = \dfrac{\lambda}{\sqrt{\kappa_1}} \operatorname{dn}(\mu_1 u + \varphi, \lambda_1 / \mu_1). \end{cases}$$

The "corpuscular" functions are J^μ and $F^{\mu\nu}$ (or $J_0(u)$, $F_0(u)$), so the field function is $K(u)$. The linear limit $\kappa_3 = 0$ preserves the existence of the plane waves that describe a "vector meson":

$$J^\mu = n'^\mu \frac{\lambda}{\sqrt{\kappa_2}} \sin(\mu_1 u + \varphi),$$

$$F_{\mu\nu} = (n_\mu n'_\nu - n_\nu n'_\mu) \frac{\lambda}{\sqrt{\kappa_2}} \cos(\mu_1 u + \varphi),$$

with

$$\mu_1 = \mu_1^0 = \sqrt{\kappa_1\kappa_2}\, K_0 .$$

2) If $\mu_1^2 < \lambda_1^2$ then the system (12), (13) has the general solution:

(15)
$$\begin{cases} J_0(u) = \dfrac{\mu}{\sqrt{\kappa_2}}\,\mathrm{sn}(\lambda_1 u + \varphi, \lambda_1 / \mu_1), \\ F_0(u) = \dfrac{\mu}{\sqrt{\kappa_1}}\,\mathrm{cn}(\lambda_1 u + \varphi, \lambda_1 / \mu_1), \\ K(u) = \dfrac{\lambda}{\sqrt{\kappa_1}}\,\mathrm{dn}(\lambda_1 u + \varphi, \lambda_1 / \mu_1). \end{cases}$$

The "corpuscular" functions are the functions J^μ and K. The proper field function is the function $F_{\mu\nu}$.

Here, the linear degeneracy $\kappa_2 = 0$ is:

$$K(u) = \frac{\mu}{\sqrt{\kappa_1}}\cos(\lambda_1 u + \varphi),$$

$$C_\mu(u) = F_{\mu\lambda}\,J^\lambda = \frac{-\lambda\mu}{\sqrt{\kappa_1\kappa_3}}\sin(\lambda_1 u + \varphi),$$

and describes a scalar corpuscle.

2. – Hamiltonian formulation.

The theory that was developed up to here is a simple extension of the description of particles with spin by the wave equations of the first quantization.

I would now like to show how one can make the preceding models correspond to a theory of classical fields, and then to a quantum theory of fields.

I will consider the systems (1) and (11) in parallel, when considered to be field equations.

A plane-wave state of the field is characterized by n^μ, which is associated with the variable u. In the case of the first model, I can consider the state variable of the field to be one of the three functions $I(u)$, $K(u)$, and $J_0(u)$.

I would first like to consider the choice $J_0(u) = X_1 = X_1(u)$, where X_1 is the coordinate of the field. I will associate it with a conjugate momentum:

$$P_1 = P_1(u) = l_1\,\kappa_1\,KI,$$

l_1 being an arbitrary constant.

The system (2), (6) then gives:

$$\frac{dX_1}{du} = \frac{1}{l_1}P_1,$$

$$\frac{dP_1}{du} = -\,l_1\,[(\lambda_1^2 + \mu_1^2)X_1 - 2\kappa_2\kappa_3 X_1^3],$$

and leads us to introduce the Hamilton function:

(16) $$H_1(X_1, P_1, u) = \frac{1}{2l_1}P_1^2 + \frac{l_1}{2}[(\lambda_1^2 + \mu_1^2)X_1^2 - \kappa_2\kappa_3 X_1^4].$$

Taking (6) into account, one has:

(17) $$\begin{cases} H_1 = \dfrac{l_1}{2}\lambda^2\mu^2, \\ H_1 = \dfrac{1}{2l_1}P_1^2 + \dfrac{l_1}{2}[\lambda^2\mu^2 - (\lambda^2 - \kappa_2 X_1^2)(\mu^2 - \kappa_3 X_1^2)]. \end{cases}$$

The functions H_1 describes the evolution by taking into account the particular values λ_1^2, μ_1^2 of the two first integrals.

We can free ourselves from this condition by introducing a second momentum P_2 that is associated with a *cyclic variable* X_2, and writing:

(18) $$H_1 = \frac{1}{2l_1}P_1^2 + \frac{l_1}{2}[X_1^2 P_2^2 - \kappa_2\kappa_3 X_1^4].$$

The equations of motion (2) are then recovered easily upon starting with (18) by means of the two first integrals:

$$H_1 = \alpha_1, \qquad P_2 = \alpha_2,$$

if one associates them with the two constants λ and μ such that:

$$\lambda^2\mu^2 = \alpha_1, \qquad \lambda^2\kappa_3 + \mu^2\kappa_2 = \alpha_2^2 .$$

The Hamiltonian function H_1 corresponds to the Lagrange function:

(19) $$L_{(1)} = \frac{l_1}{2}\left[X_1'^2 + \frac{X_2'^2}{l_1^2 X_1^2} + \kappa_2\kappa_3 X_1^4\right].$$

Here, I have chosen the field variable to be $J_0(u)$. It is more natural to take the coordinate to be the function $I(u)$ or the function $K(u)$.

Let:

$$I(u) = Y_1(u).$$

We associate $Y_2(u)$ with the conjugate momentum:

$$H_1(u) = -\, l_2\, \kappa_2\, KJ_2 ,$$

l_2 again being an arbitrary constant.

Then:

$$Y_1' = \frac{1}{l_2}\Pi_1,$$

$$\frac{d\Pi_1}{du} = l_2[(2\lambda_1^2 - \mu_1^2)Y_1 - 2\kappa_1\kappa_3 Y_1^3].$$

These equations are the equations of motion that are deduced from the Hamilton function:

(20) $$H_2(Y_1, \Pi_1, u) = \frac{1}{2l_2}\Pi_1^2 + \frac{l_2}{2}[(\mu_1^2 - 2\lambda_1^2)Y_1^2 + \kappa_1\kappa_3 Y_1^4].$$

Here again, one sees that with the first integrals (5):

$$H_2 = \frac{l_2}{2}\frac{\lambda_1^2(\mu_1^2 - \lambda_1^2)}{\kappa_1\kappa_3}.$$

We can replace the constant $\mu_1^2 - 2\lambda_1^2$ in H_1 by the conjugate momentum of a cyclic variable Y_2, namely, Π_2. We then write:

a) If $\mu_1^2 - 2\lambda_1^2 > 0$ then:

(21) $$H_2 = \frac{1}{2l_2}\Pi_1^2 + \frac{l_2}{2}[\Pi_2^2 Y_1^2 + \kappa_1\kappa_3 Y_1^4];$$

b) If $\mu_1^2 - 2\lambda_1^2 < 0$ then:

(22) $$H_2 = \frac{1}{2l_2}\Pi_1^2 + \frac{l_2}{2}[-\Pi_2^2 Y_1^2 + \kappa_1\kappa_3 Y_1^4].$$

The corresponding Lagrangians are then deduced easily from these expressions.

A third choice of field variable is the function $K(u)$. The symmetry in the introduction of $I(u)$ and $K(u)$ then shows us that if we consider $K(u)$ to be a field variable – namely, $Z(u)$ – then the corresponding Hamiltonian function $H_3(Z_1, Z_2, u)$ will be deduced from H_2 with the exchange of constants $\lambda_1^2 \to \mu_1^2$ and κ_2, κ_3.

In the case of the model *B*, the passage from the Hamiltonian formalism is effected in the same fashion. The choice of the field variable $J_0(u) = X_1(u)$ leads to the Hamilton function $H_1(X_1, X_2, u)$. The choice of $F_0(u) = Y_1$ leads to $H_2(Y_1, Y_2, u)$, and the choice $K(u) = Z_1$ leads to the function $H_3(Z_1, Z_2, u)$.

4. – The quantum theory of fields that is associated with the nonlinear models.

I would now like to start from the preceding Hamiltonian formalism and associate it with a quantum theory of fields for the models that I have introduced.

For this, I will assume that a quantum theory of fields is constructed:

a) By formally associating the Hamiltonian function for a theory of classical fields with a "Hamiltonian operator" that is obtained by a formal correspondence function $\to$ operator, for which I have adopted the usual choice:

$$P \to P_{\text{op}} = -\,i\hbar\frac{\partial}{\partial X}, \qquad\qquad X \to X_{\text{op}} = X_0 .$$

As a consequence, a Hamilton function corresponds to a second-order differential operator.

b) By constructing a "Schrödinger equation" from $(H)_{\text{op}}$.

By definition, here this will be the second-order partial differential equation:

$$i\hbar\frac{\partial}{\partial u}\Phi(X_1, X_2, u) = (H)_{\text{op}}\Phi(X_1, X_2, u). \tag{23}$$

c) By considering the general solution of that equation and isolating a "quantized manifold" among them that constitutes a complete system of orthogonal functions. These functions are determined by the possibility of associating the Schrödinger equation, after separating the variables, with *one or more* (regular or singular) Sturm-Liouville problems.

This possibility restricts the values of the first integrals of the Schrödinger equation to quantized values or proper values that are classified by "quantum numbers." They correspond to field functions $\Phi(X_1, X_2, u)$ that are orthogonal with respect to convenient measures.

For the preceding models, I would like to first consider the Hamiltonian H_1 that corresponds to the field variable $I(u)$. Here, $Y_1 = Y_1(u) = I(u)$ for the plane-wave states.

The Schrödinger equation that is associated with $(H)_{\text{op}}$ is written:

$$i\hbar\frac{\partial}{\partial u}\Phi(X_1, X_2, u) = (H_2)_{\text{op}}\Phi(Y_1, Y_2, u) =$$

$$= \frac{1}{2l_2}\left[-\hbar^2\frac{\partial^2}{\partial Y_1^2} + \frac{l_2}{2}\left[Y_1^2\left(-\hbar^2\frac{\partial^2}{\partial Y_2^2}\right) + \kappa_1\kappa_3 Y_1^4\right]\right]\Phi(Y_1, Y_2, u).$$

We can separate the variables by setting:

(24) $$\Phi(Y_1, Y_2, u) = \exp\left[-\frac{i}{\hbar}\varepsilon u\right]\Phi(Y_1, Y_2) = \exp\left[\frac{i}{\hbar}[\beta Y_2 - \varepsilon u]\right]\Phi(Y_1).$$

The function $\Phi(Y_1)$ is then determined by the differential equation:

(25) $$\left[\frac{\hbar^2}{2l_2}\frac{d^2}{dY_1^2} + \varepsilon\left[\beta^2 Y_1^2 + \kappa_1\kappa_3 Y_1^4\right]\right]\Phi(Y_1) = 0.$$

From the preceding field theory, we are led to introduce ε and β^2 by their expressions as functions of λ and μ, namely:

$$\varepsilon = \frac{l_2}{2}\lambda_1^2\frac{(\mu_1^2 - \lambda_1^2)}{\kappa_1\kappa_3} = \frac{l_2}{2}\frac{1}{\kappa_1}(\mu_1^2 - \lambda_1^2)\lambda^2, \qquad \beta^2 = \mu_1^2 - 2\lambda_1^2.$$

$\Phi(Y_1)$ is then a solution of a differential equation that we write as either:

(26) $$\left[\frac{d^2}{dY_1^2} + \frac{l_2^2}{\hbar^2}\left[\frac{\lambda^2}{\kappa_1}(\mu_1^2 - \lambda_1^2) - (\mu_1^2 - 2\lambda_1^2)Y_1^2 - \kappa_1\kappa_3 Y_1^4\right]\right]\Phi(Y_1) = 0$$

or

(27) $$\left[\frac{d^2}{dY_1^2} + \frac{l_2^2}{\hbar^2}\left[(\lambda^2 - \kappa_1 Y_1^2)\left(\frac{\mu_1^2 - \lambda_1^2}{\kappa_1} + \kappa_3 Y_1^2\right)\right]\right]\Phi(Y_1) = 0.$$

The equation of the harmonic oscillator, which is the basis for the quantum theory of fields, *is now replaced with the equation for an anharmonic oscillator.*

The limiting case $\kappa_3 = 0$ – so $\lambda_1^2 = 0$ – brings us down to the harmonic oscillator and the usual quantum theory of fields.

Up to now, the theory that we developed is not quantized. For the quantization, we must consider the general solutions of equation (26) or (27), and among them, one must restrict ε and β_1^2 (or λ_1^2 and μ_1^2) to particular values.

Consider eq. (26), which we further write:

(28) $$\left[\frac{d^2}{dY^2} + \beta_1 - \beta_2 Y^2 - \beta_3 Y^4\right]\Phi(Y_1) = 0$$

(with Y, instead of Y_1, now).

The regular general solution for bounded Y_1 of (26), (27), or (28) is written:

$$\Phi(Y, \varepsilon, \beta^2) = \Phi(Y, \lambda_1^2, \mu_1^2) = \Phi(Y, \beta_1, \beta_2).$$

We first suppose that $\beta_1 > 0$, $\beta_2 > 0$.

Eq. (28) is of the type:

$$y'' + \varphi(x)\, y = 0.$$

One knows that if $\varphi(x) > 0$ then the general solution is of sinusoidal type (concave towards OX) and that if $\varphi(x) < 0$ then the general solution is of exponential type.

As a consequence, if $\mu_1^2 > \lambda_1^2$ (which was the first case considered, where $I(x^\mu)$ is a "corpuscular" function) then $\Phi(Y, \beta_1, \beta_2)$ will be of sinusoidal type for $Y^2 < \lambda^2 / \kappa_1$ and of exponential type for $Y^2 > \lambda^2 / \kappa_1$. Moreover, from the Polya-Szegö theorem, since the function $\varphi(x)$ is decreasing for $Y^2 < \lambda^2 / \kappa_1\, \kappa_3$, the sequence of relative maxima of $\Phi(Y, \beta_1, \beta_2)$ will be increasing.

The situation will thus be the same as in the case of the Schrödinger equation for the harmonic oscillator.

The quantization will be obtained by associating eq. (28) with *two simultaneous Sturm-Liouville problems.*

a) We may fix β_2 (or $\mu_1^2 - 2\lambda_1^2$) and consider the singular Sturm-Liouville problem that is obtained by subjecting $\Phi(Y, \beta_1, \beta_2)$ to being annulled as $Y^2 \to \infty$.

Since eq. (28) is written:

$$y'' - q(x)\, y + \lambda\, r(x)\, y = 0,$$

here we have $q(x) = \beta_1 Y^2 + \beta_2 Y^4$. The Titchmarsh theorems then show that there is a discrete spectrum of admissible values for β_1. We write them $\beta_1(n_1, \beta_2)$, where the integer n_1 classifies the corresponding solutions $\Phi(Y)$ by their number of nodes. These associated functions, namely, $\Phi(Y, \beta_1(n_1), \beta_2)$ are then orthogonal on $-\infty, +\infty$, with the weighting function $\rho(Y) = 1$:

$$\int_{-\infty}^{+\infty} \Phi(Y, \beta_1(n_1'), \beta_2)\, \Phi(Y, \beta_1(n_1''), \beta_2)\, dY = C\delta_{n_1', n_1''}.$$

b) We can fix β_1 (or $\lambda_1^2(\mu_1^2 - \lambda_1^2)$), and again consider a singular Sturm-Liouville problem on $-\infty < Y < +\infty$.

If we set $\Phi(Y) = (1/Y)\, \Psi(Y)$ then eq. (28) takes the form:

$$\left[\frac{d}{dY}\left(\frac{1}{Y^2}\Psi\right) + \left(\frac{2}{Y^4} + \frac{\beta_1}{Y^2} - \beta_2 Y^2 - \beta_2\right)\right]\Psi(Y) = 0.$$

The Titchmarsh theorem then shows that since β_3 is different from zero, there is a discrete spectrum of values for β_2 – namely, $\beta_2(n_1, \beta_2)$ – for which, on the one hand, $\Phi(Y)$ is annulled as $Y^2 \to \infty$, and, on the other hand, the solutions $\Phi(Y, \beta_1, \beta_2(n_2))$ are orthogonal on $(-\infty, +\infty)$, with the weighting function $\rho(Y) = Y^2$:

$$\int_{-\infty}^{+\infty} \Phi(Y, \beta_1, \beta_2(n_2')) \, \Phi(Y, \beta_1, \beta_2(n_2'')) \, Y^2 dY = C \delta_{n_2', n_2''} .$$

The *simultaneous* consideration of these two Sturm-Liouville problems leads to a system of functions:

$$\Phi(Y, \beta_1(n_1), \beta_2(n_2))$$

that are orthogonal on $(-\infty, +\infty)$, with the weighting function $\rho(Y) = 1$, for fixed β_2, and $\rho(Y) = Y^2$ for fixed β_1.

(The situation here is the same as in the case of the associated Legendre functions $P_\nu^\mu(\cos\theta)$, for which there are two orthogonality relations, either for fixed μ with $\rho(\theta) = -\sin\theta$ or for fixed ν with $\rho(\theta) = 1/\sin\theta$ that lead to the "quantized" values $\nu = n_1$, $\mu = n_2$).

We thus have a simultaneous quantization of β_1 and β_2 with a discrete spectrum, and, in turn, of the amplitude λ_1 and the reduced proper mass μ_1 by the expressions:

$$\lambda_1^2 = \tfrac{1}{2}[(\beta_2^2 + 4\kappa_1\kappa_3\beta_1)^{1/2} - \beta_2] ,$$
$$\mu_1^2 = (\beta_2^2 + 4\kappa_1\kappa_3\beta_1)^{1/2} .$$

For $\kappa_3 = 0$, one recovers the quantization conditions for the harmonic oscillator.

c) For the same eq. (28), but written in the form (27), we may pose another Sturm-Liouville problem. In order to do this, we write that equation as:

$$\left[\frac{d^2}{dY^2} + \frac{l_2^2}{\hbar^2}\lambda_1^2 \rho_3(Y) - \frac{l_2^2}{\hbar^2}[Y^2(\mu_1^2 - \lambda_1^2) + \kappa_1\kappa_3 Y^4\right]\Phi(Y) = 0,$$

with

$$\rho_3(Y) = \frac{\mu_1^2 - \lambda_1^2}{\kappa_1\kappa_3} + Y^2.$$

The preliminary condition $\mu_1^2 > \lambda_1^2$ gives us $\rho_3(Y) > 0$ for $-\infty < Y < +\infty$.

We may then consider $\mu_1^2 - \lambda_1^2 = \nu_1^2$ to be fixed. The condition $\Phi(Y) \to 0$ for $|Y| \to \infty$ will then lead to quantized values for λ_1^2. The Titchmarsh theorem again shows us that the spectrum of λ_1^2 is discrete with fixed ν_1^2.

These values $\lambda_1^2(n_2, \nu_1^2)$ correspond to functions:

$$\Phi(Y, \lambda_1^2(n_3), \nu_1^2)$$

that are orthogonal on $(-\infty, +\infty)$, with the weighting function $\rho_3(Y)$:

$$\int_{-\infty}^{+\infty} \Phi(Y, \lambda_1^2(n_3'), \nu_1^2)\, \Phi(Y, \lambda_1^2(n_3''), \nu_1^2)\, \rho_3(Y)\, dY = C\delta_{n_3', n_3''}.$$

Up to now, the quantization of our model has been deduced from the Hamilton function H_2 that is associated with the field variable $Y_1(u) = I(u)$.

We may likewise construct a quantum theory of fields that are associated with our model in which the Hamiltonian H_1 corresponds to the field variable $J_0(u) = X_1(u)$.

Here, the Schrödinger equation is:

$$i\hbar \frac{\partial}{\partial u} \Phi(X_1, X_2, u) = \left[-\frac{\hbar^2}{2l_1} \frac{\partial^2}{\partial X_1^2} + \frac{l_1}{2} \left[X_1^2 \left(-\hbar^2 \frac{\partial^2}{\partial X_2^2} \right) - \kappa_2 \kappa_3 X_1^4 \right] \right] \Phi(X_1, X_2, u).$$

Separating the variables by writing:

$$\Phi(X_1, X_2, u) = \exp\left[\frac{i}{\hbar} (\alpha_2 X_2 - \varepsilon u) \right] \Phi(X_1),$$

with

$$\alpha_2^2 = \lambda_1^2 + \mu_1^2 \qquad \text{and} \qquad \varepsilon = \frac{l_1}{2} \lambda^2 \mu^2,$$

$\Phi(X_1)$ is determined by starting with the differential equation:

$$\left[\frac{d^2}{dX^2} + \frac{l_1^2}{\hbar^2} \left[\lambda^2 \mu^2 - (\lambda_1^2 + \mu_1^2) X^2 + \kappa_2 \kappa_3 X^4 \right] \right] \Phi(X_1) = 0,$$

which we also write as:

$$\left[\frac{d^2}{dX^2} + \frac{l_1^2}{\hbar^2} (\lambda^2 - \kappa_2 X^2)(\mu_1^2 - \kappa_3 X^2) \right] \Phi(X) = 0.$$

Here again, the equation of the harmonic oscillator of the usual quantum theory of fields is completed by an anharmonic term in X^4.

The general solution of this equation has sinusoidal behavior either for $0 < X^2 < \lambda_1^2 / \kappa_1\ \kappa_3$ or for $X^2 > \mu_1^2 / \kappa_2\ \kappa_3$.

The first case is the extension of the case of the harmonic oscillator that we recover for $\kappa_3 = 0$.

The simultaneous quantization of the parameters:

$$\alpha_1 = \lambda^2 \mu^2 = \lambda_1^2\ \mu_1^2 / \kappa_2\ \kappa_3 \qquad \text{and} \qquad \alpha_2 = \lambda_1^2 + \mu_1^2$$

will again be obtained from *two simultaneous Sturm-Liouville problems* upon imposing the condition on the functions $\Phi(X)$ that they go to zero as $|X| \to \infty$.

In the first case, one fixes $\alpha_2 = \lambda_1^2 + \mu_1^2$ and one will determine the quantized values of α_1 – namely, $\alpha_1(n_1)$ – for which the functions $\Phi(X, \alpha_1(n_1), \alpha_2)$ are orthogonal on $(-\infty, +\infty)$, with $\rho_1(X) = 1$:

$$\int_{-\infty}^{+\infty} \Phi(X, \alpha_1(n_1'), \alpha_2)\, \Phi(X, \alpha_1(n_1''), \alpha_2)\, dX = C\delta_{n_1', n_1''}.$$

The Titchmarsh theorem shows us that the spectrum $\alpha_1(n_1)$ is discrete.

The second Sturm-Liouville problem considers α_1 to be fixed and determines a spectrum (which is again discrete) $\alpha_2(n_2) = \lambda_1^2 + \mu_1^2$. The corresponding functions:

$$\Phi(X, \alpha_1, \alpha_2(n_2))$$

are orthogonal on $(-\infty, +\infty)$, with the weighting function X^2.

$$\int_{-\infty}^{+\infty} \Phi(X, \alpha_1, \alpha_2(n_2'))\, \Phi(X, \alpha_1, \alpha_2(n_2''))\, X^2 dX = C\delta_{n_2', n_2''}.$$

The consideration of these two simultaneous Sturm-Liouville problems thus again leads us to a *discrete spectrum* for the *amplitudes* λ_1 *and proper masses* μ_1 .

The quantum theory of fields that we have associated with our model thus extends the quantum theory of classical fields by simultaneously quantizing the amplitudes of the wave functions the proper masses.

The results that were reported here relate to a very simple model of corpuscle-field that is described by a system of nonlinear wave functions. However, these results extend without difficulty to the more complex models of corpuscle-field that we introduced, moreover. One then sees a much larger extension of the theory of classical fields in which an entire sequence of anharmonicity constants is quantized at the same time as the amplitudes of the multi-periodic functions that are associated with the various tensorial components of the global wave function.

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