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On the statics of planar frameworks

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Translated by D. H. Delphenich

The following presentation is closely linked with the treatise of Herrn Geheimrat Klein ([1]) that recently appeared with the title "Über Selbstspannungen ebener Diagramme." The analytic-geometric considerations of the aforementioned paper might be placed alongside the graphical representation here.

It will be shown how appealing to the spatial polyhedron that belongs to certain planar diagrams can be managed conveniently for the construction of reciprocal diagrams, and therefore also for the graphical determination of the self-stresses in a planar framework, and in connection with the aforementioned treatise, the circumstances under which it appears for framework diagrams that are to be regarded as the projections of *one-sided*, closed polyhedra will be considered especially.

1. It will suffice here to assume that a spatial, *closed* polyhedron that is composed of planar polygons and can be regarded as a stress surface will give the *self-stresses* in that planar framework that can be regarded as its orthogonal projection ([2]).

We will employ, to our advantage, the representation of the planar bounding surfaces of polyhedra with the help of *altitude lines* and *gradients*, which is a method that Runge cared to use in his lectures on graphical statics.

A plane cuts the horizontal plane (i.e., the reference plane) along the line 0. We now choose a unit segment along a line in the plane to be represented that is perpendicular to the altitude line 0 until the perpendicular elevation over the horizontal plane is equal to the unit segment. We draw the parallel to 0 through the projection of the point thus arrived at onto the horizontal plane, and thus get the altitude line 1. The altitude lines 2, 3, … are parallels at equal distances, while the parallels – 1, – 2, … give the altitude lines for the rising part of the plane.

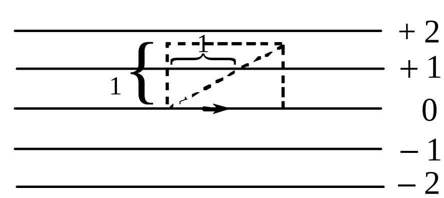

Figure 1.

The elevation of the plane that corresponds to a point whose projection has a perpendicular distance of 1 from the null line in the reference plane gives the magnitude

([1]) Mathematische Annalen **67** (1909).

([2]) F. Klein and K. Wieghardt, "Über Spannungsflächen und reziproke Diagramme, mit besonderer Berücksichtigung der Maxwellschen Arbeiten," Archiv der Mathematik und Physik. Ser. 3, Vol. VIII, Issues 1 and 2. (1904).

of the gradients of the plane. *We would like to agree that these gradients enter in as directed segments along an altitude line (ordinarily, the null line) and an arrow direction can go through it that gives the direction and magnitude of the vector of inclination of the plane (when seen from above the reference plane), after it has been rotated through* 90° *counter-clockwise.*

If the equation of the plane to be represented is:

$$z = ax + by + c$$

in a rectangular coordinate system whose *xy*-plane coincides with the reference plane then:

$$ax + by + c = 0$$

will be the equation of the altitude line 0, and the length of the gradient will be $\sqrt{a^2 + b^2}$.

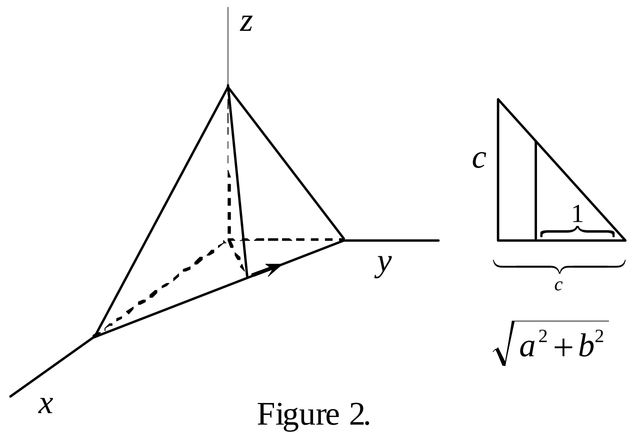

Figure 2.

If one draws a vector through the origin of the coordinate system that is parallel to the vector that is carried by the altitude line (for the sake of brevity, we would like to refer to this vector along the altitude line as the *gradient* in what follows) then the coordinates of its endpoint will have the values:

$$x = b, \quad y = -a.$$

If we then (in a coordinate system whose axes are parallel to the *x* and *y* axes) shrink the gradients of the various planes of the polyhedron that belongs to a given diagram as the stress surface towards the origin, which is the *pole* to which the gradient of the horizontal plane shrinks, then the endpoints of those gradients will be precisely the points $x = b$, $y = -a$ of the reciprocal diagram that correspond to the polyhedral planes:

$$z = ax + by + c,$$

from formula (5) of the paper cited in rem. 1 on page 1.

In the application of a gradient to a point, we then have an exceptionally convenient means of defining the points of the reciprocal diagram.

If G_k is the gradient that is applied to the plane (*k*) of the polyhedron, and G_l is that of the plane (*l*) then, from formula (4) of the aforementioned paper, the geometric difference $G_l - G_k$ will be the magnitude of the stress along the projection of the edge of intersection (*kl*) of the planes (*k*) and (*l*), and therefore the rod stress in the rod of the framework that is the projection of the polyhedral edge in question.

2. We can now determine uniquely whether the stress that exists in the rod, which we know up to absolute value, is a *tension* or a *pressure*.

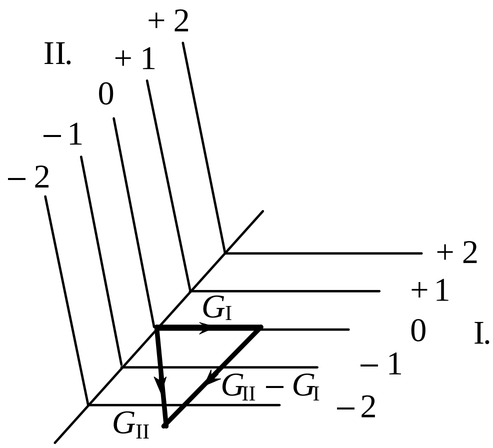

Figure 3.

We would next like to imagine, not a closed polyhedron, but a faceted surface with attached polyhedral zones that cover the plane only simply, as one would have for the stress surface of a framework that is in equilibrium under the action of a system of external forces. The projections of the two planar facets that come together at an edge of the stress surface then lie on different sides of the line of projection of the edge of the diagram.

We define the gradients G_{I} and G_{II} of the two planes according to what we established and obtain the indicated vector that is parallel to the projection of the edge of intersection of the planes I and II as the geometric difference of the gradients $G_{\mathrm{II}} - G_{\mathrm{I}}$ when we go from plane I to plane II. We will now get the correct sign for the stress when we establish that: *The vector gives us the effect of the force on the piece of the rod that is to the left of an observer that lies beyond the rod in the prescribed direction.* (That agrees with the general discussion on page 3 of the treatise that was cited in rem. 2 on page 1.)

In the figure, we will then get a compression in the rod when we go from I to II. When we go from II to I, the vector $G_{\mathrm{I}} - G_{\mathrm{II}}$ will have precisely the opposite arrow direction as $G_{\mathrm{II}} - G_{\mathrm{I}}$ did before, so we will once more get a compression in the part of the rod that is to be regarded as to the left of the direction in which one goes beyond it.

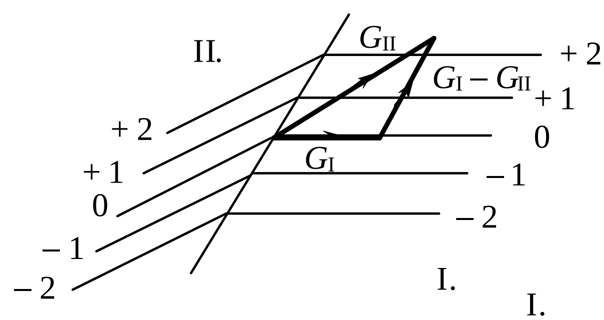

Figure 4.

The accompanying Figure 4 corresponds to a tension in the rod.

It obeys the rule for the simple covering of the xy-plane by the diagram.

For a double covering, we next consider a plane I of the first sheet and a plane II of the second sheet, which are both connected along an edge of the contour polygon. If we attach the gradients of both planes according to our prescription then rule that was present will be obeyed for the transition from I to II. However, if we go from II to I then we will go beyond the rod in the same sense as above, so according to our rule, we would now obtain precisely the opposite stress to the previous one. Under the transition from the plane of the second sheet to that of the first sheet, we must then take the rod piece that lies to the *right* of the part that was exceeded as being definitive of the stress in the rod that is produced by the given force through the difference of gradients. The same

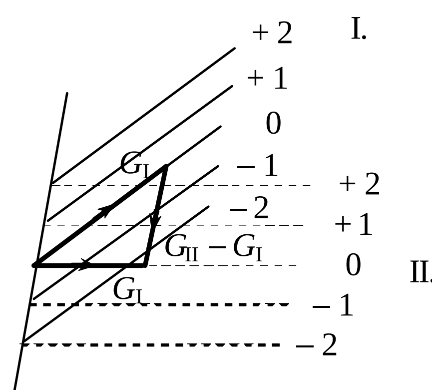

Figure 5.

thing will be true when we go from one plane of the second sheet to another plane of the second sheet.

We can then summarize this as: *If we move along the edge of intersection in a plane of the first sheet then the resulting vector will give the effect of the force on the left-hand rod segment, while if we move along the edge of intersect in a plane of the second sheet then the resulting vector will exhibit the effect of the force on the right-hand rod segment.*

3. We now apply these theorems to the case of a simple, closed polyhedron. Let the closed polyhedron be *two-sided* and belong to the diagram that is drawn as the stress surface in Fig. 9. The diagram has 7 nodes and 15 rods, so it will have $15 - (2 \cdot 7 - 3) = 4$ linearly-independent self-stresses. One can easily give four such independent self-stresses when one thinks of the rods as being endowed with stresses only in diagrams that contain the following figures:

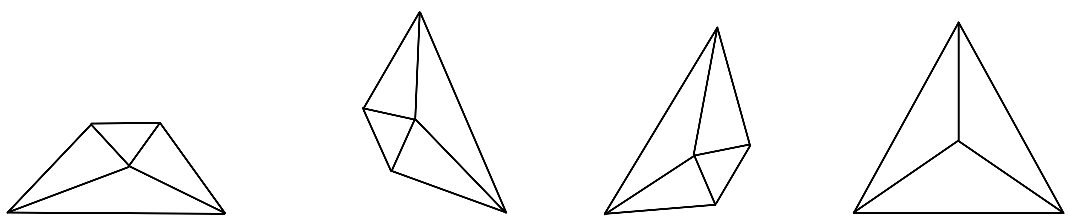

Figure 6.

We choose a completely well-defined polyhedron for our construction and assign the values to the *z*-ordinates that are indicated in the diagram, and thus obtain a well-defined self-stress in our diagram. (The planes of the polyhedron are denoted by *a*, …, *k* in Fig. 9, while the symbols of the hidden planes are enclosed in parentheses. Fig. 11 gives a representation of the spatial polyhedron that belongs to the diagram in skew parallel projection.)

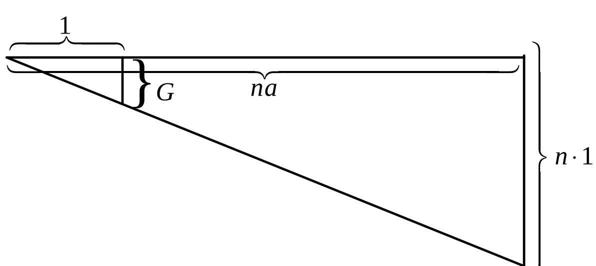

Figure 7.

The construction of the reciprocal diagram follows from the foregoing discussion with no further assumptions: One draws the gradients from an arbitrary point *O* using the given prescription and links the endpoints of the gradients that correspond to planes of the polyhedron that come together. In order to ascertain the gradient *G*, one needs only the direction of the altitude line in order to find its direction. Its length is obtained most simply when one draws the unit segment 1, which is chosen once, as the altitude of a right triangle and the distance *a* of two successive altitude lines as a hypotenuse section, while the other section of the hypotenuse is the length of the gradient. However, being more accurate about the signs will give an accordingly simpler construction.

In order to now distinguish between tension and compression, we appeal to a manner of presentation that will be of value to us for the transition to the one-sided polyhedra and which was also suggested by Runge on occasion.

We would like to start with a certain face of our two-sided polyhedron and perhaps paint its outer side white and the inner side red. If we then travel over the entire polyhedron then we will once more come to a face that is already painted (inside as well as outside), and in fact, in *the same* color that we would now like to again bring to it. The coloring of the individual faces is thus single-valued; each surface is white on the outside and red on the inside, and no matter how often we paint the outer or inner side of a face of the polyhedron, even for an arbitrary path, one and the same side will always be white, while the other one will always be red. If we now consider the diagram that is the projection of this painted polyhedron then, when seen from above, the faces of the polyhedron above the spatial contour will project to white polygons, while the ones below the contour will projection a red polygons, and it will no longer be clear what we mean in this diagram by the terms "white" and "red" polygons, with no further assumptions.

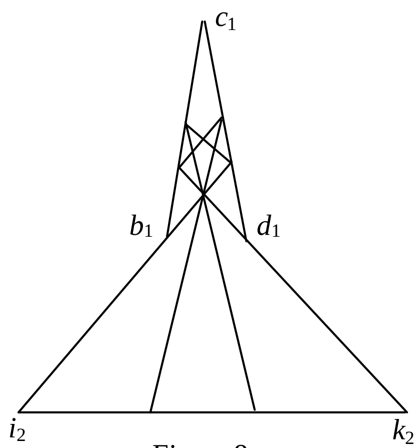

Figure 8.

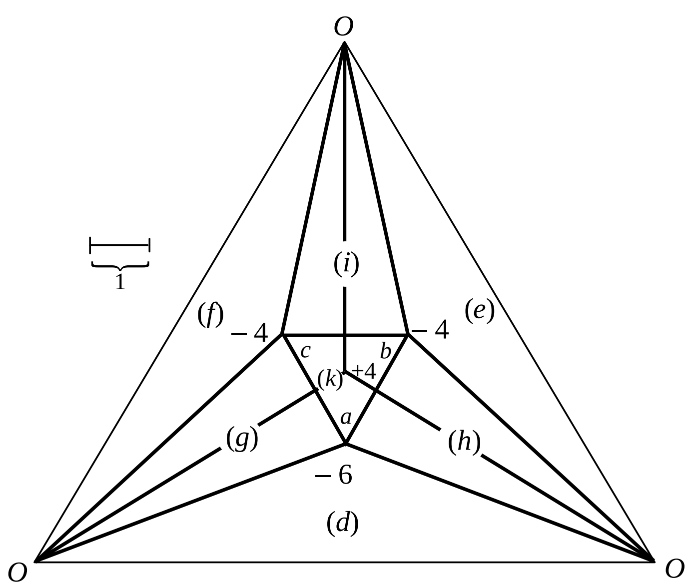

Figure 9.

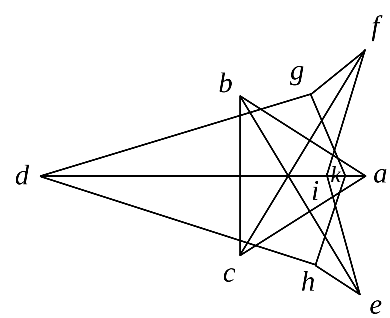

Figure 10.

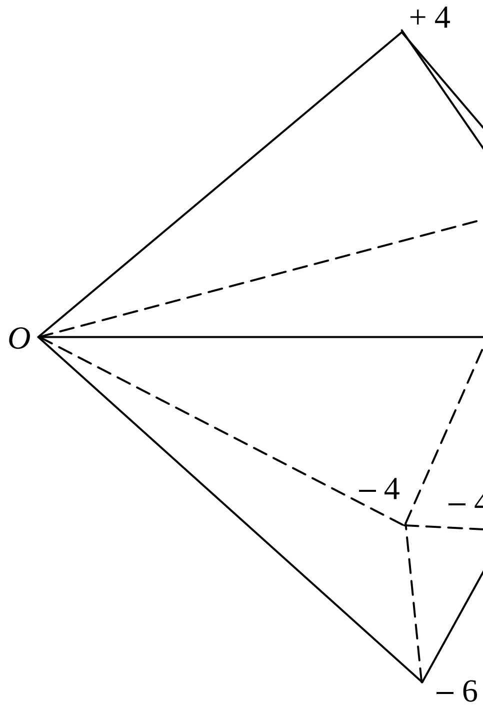

Figure 11.

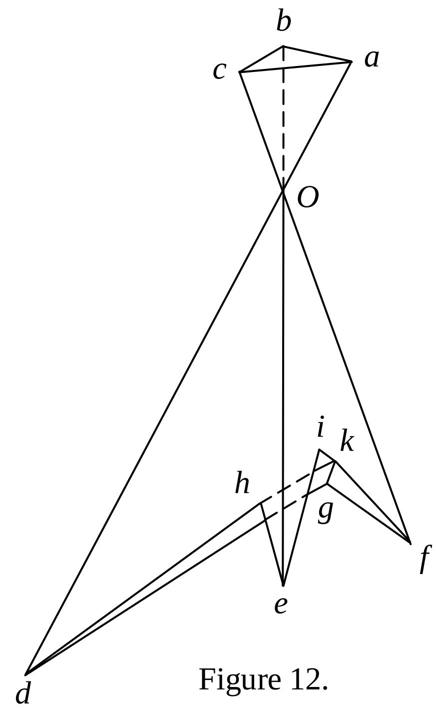

Figure 12.

We can express our sign rule for rod stresses thus: *If we come to a rod in a white polygon in the diagram then the vector of the reciprocal force plane will give us the force that acts upon the left part of the rod, while a red polygon will give us the one that acts upon the right part.* One can now easily find the tensions and compressions in our framework diagram. As usual, the compressed rods are characterized by thicker lines in Fig. 9.

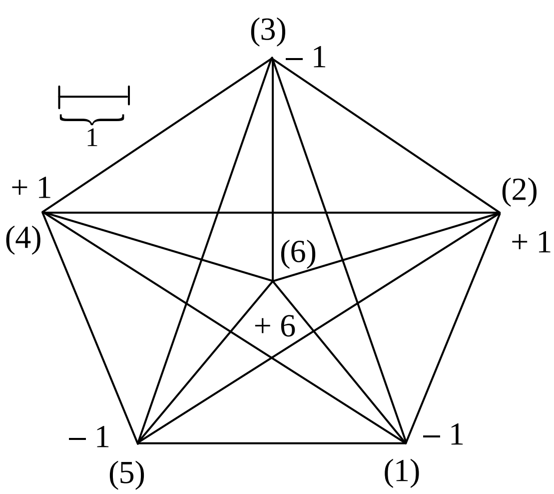

Figure 13.

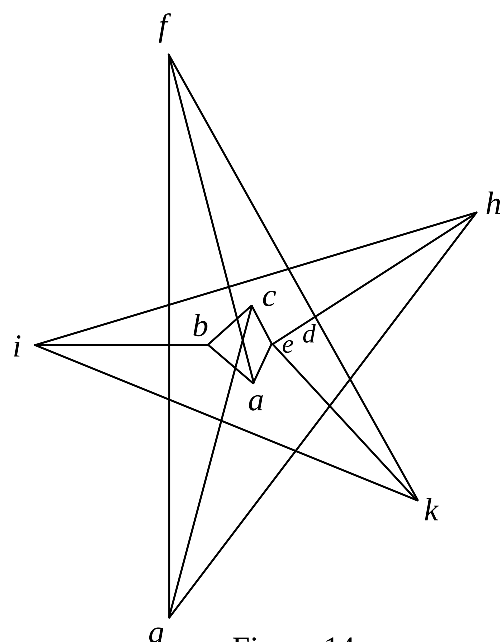

Figure 14.

4. How do the *one-sided,* closed polyhedra behave as stress surfaces? Like Klein, in his aforementioned Annals paper, we take the polyhedron to be a pyramid that is erected over the free edges of a spatial Möbius band that consists of five triangles. Its projection is the diagram that is depicted in Fig. 13. The numbers that are set next to the six nodes give the *z*-coordinates of the vertices of the spatial polyhedron, while the ones in parentheses define the numbering of the nodes. The planes of the polyhedron are:

a : 132	*f* : 136
b : 243	*g* : 356
c : 354	*h* : 526
d : 415	*i* : 246
e : 521	*k* : 416.

The diagram has six nodes and fifteen rods, so there will be $15 - (2 \cdot 6 - 3) = 6$ systems of self-stresses. For a certain choice of the ordinates, we will once more select a certain self-stress. Fig. 15 exhibits the spatial polyhedron, and in it – as in the remaining spatial figures, as well – the actual polyhedral edges that correspond to framework rods, as long as they are hidden ones, are indicated by dashed lines, while the hidden piercing edges are not indicated.

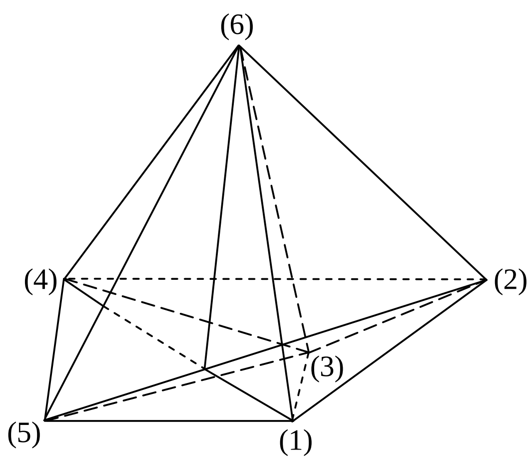

Figure 15.

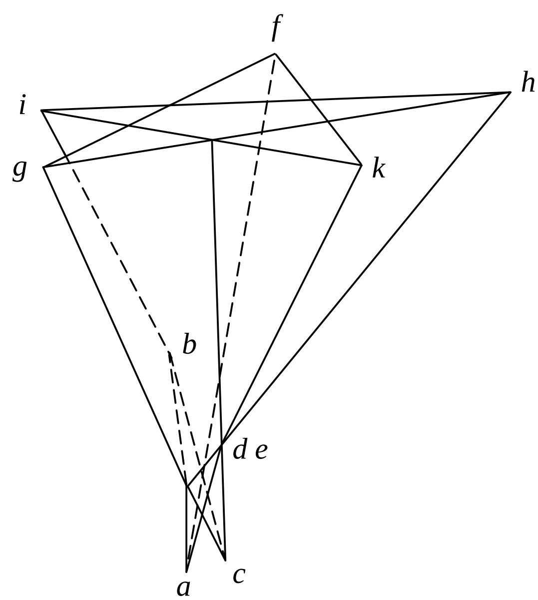

Figure 16.

The reciprocal plane – viz., Fig. 14 – can be constructed in precisely the same way as for a two-sided polygon, and a deviation from the usual situation first comes about when one determines the signs of the stresses.

If we once again begin to paint the polyhedron by starting with a face of it – say, we paint the upper side of the starting face white and its lower side, red. We can then choose the manner by which the entire polyhedron is ultimately painted so that each face will be white on one side and red on the other. We thus also get completely determined colors for the projections of the polygons, and can then distinguish tensions and compressions by our rules. However, if we now wander further over our spatial polyhedron in a suitable way then that will show that we must paint surface pieces that we previously painted in one color with precisely the other color. Naturally, that carries over to the polygons in the diagram, and our rules would now yield precisely the opposite sign for the stresses as they did before; i.e., we do not get a uniquely-determined sign for the stress.

This is the contradiction that one comes to when one considers one-sided polyhedra as stress surfaces. As Herr Geheimrat Klein showed, one can avoid it when one extends the one-sided surface to a two-sided one by double covering, so, in particular, by adding an extension surface that has precisely the same vertices, edges, and faces as the starting surface. One must then regard the given diagram as the double projection of this two-sided surface, while the reciprocal diagram is the doubly-counted one-sided surface. Each polygon that projects to white as a plane of the starting surface will project to red as a plane of the extension surface, and conversely, and the determination of the sign of the tension in a rod by our rule will yield precisely the same compression in it, and thus, merely *null stresses.*

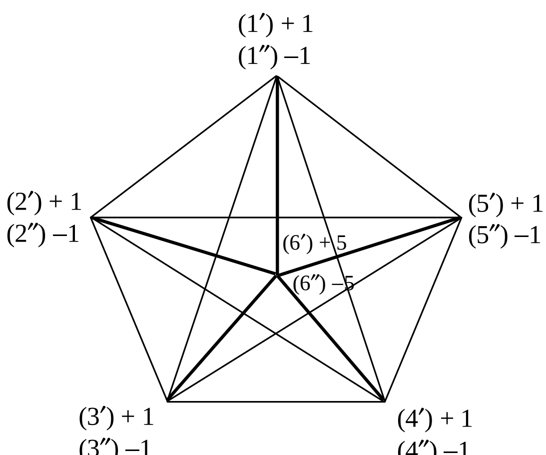

Figure 17.

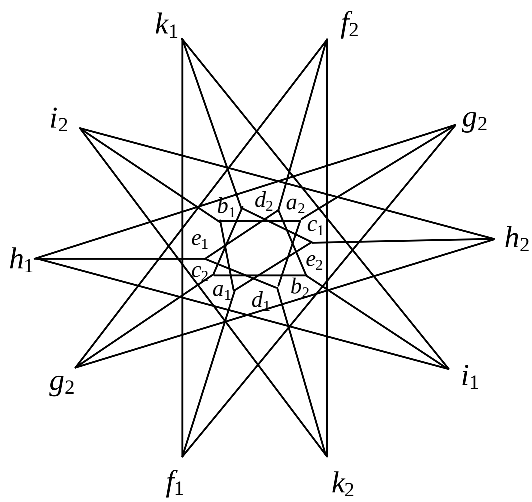

Figure 18.

5. We now come to the actual determination of the rod stresses that would correspond to certain self-stress state in our diagram. *From the discussion in the repeatedly-cited treatise, one achieves that by employing a two-sided polygon whose double projection is our diagram.* As was shown there, for any well-defined stress state, there is always a double polyhedron that has the *xy*-plane for its symmetry plane; we would like to choose a polyhedron whose vertices have the *z*-coordinates that are indicated for the nodes in the diagram of Fig. 17, moreover. (Once more, the numbers of the vertices of the polyhedron are in parentheses.) The polyhedral planes are:

a_1 : 1′ 2″ 3′	b_1 : 2″ 3′ 4″	c_1 : 3′ 4″ 5′	d_1 : 4″ 5′ 1′	e_1 : 5′ 1″ 2′
a_2 : 1″ 2′ 3″	b_2 : 2′ 3″ 4′	c_2 : 3″ 4′ 5″	d_2 : 4′ 5″ 1′	e_2 : 5″ 1′ 2″
f_1 : 6′ 1′ 3′	g_1 : 6′ 3′ 5′	h_1 : 6′ 5′ 2′	i_1 : 6′ 2′ 4′	k_1 : 6′ 4′ 1′
f_2 : 6″ 1″ 3″	g_2 : 6″ 3″ 5″	h_2 : 6″ 5″ 2″	i_2 : 6″ 2″ 4″	k_2 : 6″ 4″ 1″.

The symmetry of the stress surface relative to the *xy*-plane corresponds to the symmetry of the reciprocal diagram Fig. 18 relative to the pole.

We now obtain two parallel lines in the reciprocal diagram for each rod of the diagram, and those two parallels will be of equal length as a result of the symmetry behavior of our polyhedron. Our sign convention produces the same sign for the forces along these two parallels; *the rod stress will be obtained by adding the two partial stresses.* (The compressed rods in the diagram are once more characterized by dashed lines.)

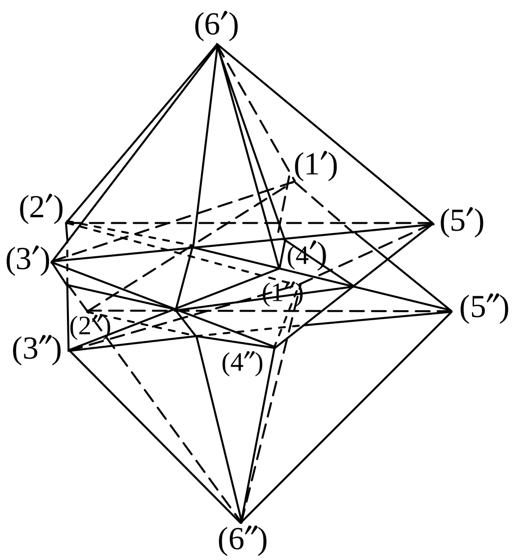

Figure 19.

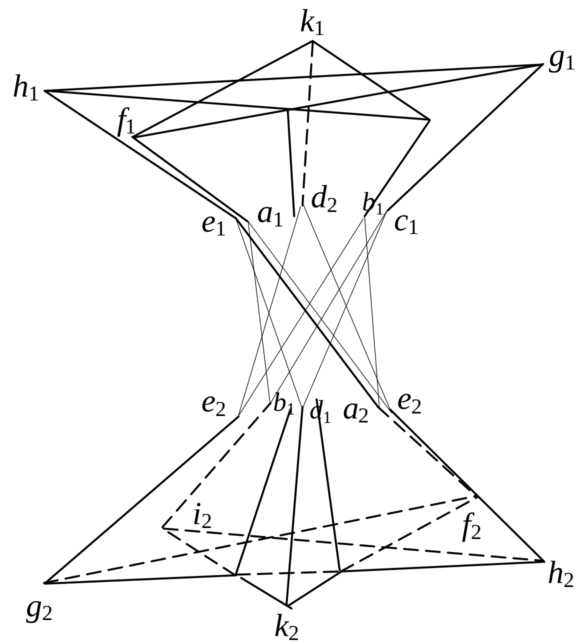

Figure 20.

6. Here is a final remark on the reciprocal spatial polyhedron itself. We have chosen the stress surface arbitrarily. Naturally, it is easy to construct the spatial polyhedron, as was done in Figs. 11, 15, 19. However, the points of the reciprocal polyhedron can also be obtained conveniently. From the present reciprocal affinity, the relationship between the coordinates x, y, z of the initial surface $z = z(x, y)$ and the coordinates ξ, η, ζ of the reciprocal surface is given by:

$$\xi = \frac{\partial z}{\partial y}, \qquad \eta = -\frac{\partial z}{\partial x}, \qquad \zeta = z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y},$$

so for the case of the plane $z = ax + by + c = 0$:

$$\xi = b, \qquad \eta = -a, \qquad \zeta = c.$$

The ζ-coordinates of the vertices of the reciprocal polyhedra are then simply the ordinates of the point of intersection of the planes in the starting surface that correspond to them with the z-axis.

In Figs. 12, 16, 20, these reciprocal polyhedra are represented in skew parallel projection. The planes in the starting polyhedron that are denoted by a, b, … always correspond to the points in the reciprocal polyhedron that have the same names. The connectivity of the reciprocal polyhedron is very easy to see. By contrast, the mutual intersections of the polyhedral surfaces are already rather complicated. This is especially the case in Fig. 20. On that basis, only the upper and lower parts of the polyhedron are indicated with the intersecting lines, while in the middle part, only the edges of the polyhedron whose projections are framework rods (with no concern for visibility or invisibility) are drawn. The construction of the mutual intersecting lines meets up with no major difficulties. One easily recognizes that each of the ten faces of the polyhedron – e.g., $i_2k_2d_1c_1b_1i_2$ – together with their intersecting lines with the other faces is represented in the accompanying figure.

Göttingen, 1 June 1909.