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On the groups of transformations of linear differential equations (*).

By

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On other occasions (*Comptes rendus*, 1883, and *Annales de la Faculté des Sciences de Toulouse*, 1887), I have shown how one can extend Galois's celebrated theory that relates to algebraic equations to linear differential equations. I drew attention to the notion of the *group of transformations* of a linear equation. The fundamental proposition in that subject consists of a theorem and its converse, which was stated in my article with a pointless restriction. Those questions have been recently dealt with more deeply by Vessiot in an extremely-remarkable thesis. However, Vessiot adopted a viewpoint in his work that is very different from my own, and the path that I have followed in order to lay the foundations for that theory, which is a path that closely approximates that of Galois for algebraic equations seems preferable to me in various regards. I thus believe that it would be useful to recall the question completely while filling in the slight gap that I allowed to persist in the converse of the fundamental theorem.

1. We first place ourselves in the case that is simplest and undoubtedly the most interesting for the applications, namely, that of a linear equation with rational coefficients. Therefore, let:

$$\frac{d^m y}{dx^m} + p_1 \frac{d^{m-1} y}{dx^{m-1}} + \cdots + p_m\, y = 0 \tag{1}$$

be one such equation, in which we suppose that the coefficients are rational functions of x, and let $y_1, y_2, \ldots, y_m$ be a fundamental system of integrals.

I imagine the following expression:

$$V = A_{11}\, y_1 + \cdots + A_{1m}\, y_m + A_{21} \frac{dy_1}{dx} + \cdots + A_{2m} \frac{dy_m}{dx} + \cdots + A_{mm} \frac{d^{m-1} y_m}{dx^{m-1}},$$

which is, as one sees, a linear and homogeneous expression with respect to the y and their derivatives of order up to $m - 1$. The coefficients A are rational functions of x that are chosen

(*) From the Comptes rendus de séances de l'Académie des Sciences, t. 119, Session on 8 October 1894.

arbitrarily. That function V satisfies a linear equation of order m^2 that is easy to form: Denote it by:

(2) $$\frac{d^{m^2}V}{dx^{m^2}} + P_1 \frac{d^{m^2-1}V}{dx^{m^2-1}} + \cdots + P_{m^2} V = 0 .$$

Moreover, upon differentiating V a number of times that equals $m^2 - 1$, one will have m^2 equations of first degree in the y and their derivatives, which will give:

$$y_1 = \alpha_1 V + \alpha_2 \frac{dV}{dx} + \cdots + \alpha_{m^2} \frac{d^{m^2-1}V}{dx^{m^2-1}} ,$$
$$y_2 = \beta_1 V + \beta_2 \frac{dV}{dx} + \cdots + \beta_{m^2} \frac{d^{m^2-1}V}{dx^{m^2-1}} ,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y_n = \lambda_1 V + \lambda_2 \frac{dV}{dx} + \cdots + \lambda_{m^2} \frac{d^{m^2-1}V}{dx^{m^2-1}} ,$$

in which the α, β, …, λ are rational in x.

Any integral of equation (2) corresponds to a system of integrals y_1, y_2, …, y_m of equation (1). That system cannot be fundamental. That will happen if the determinant of the y and their derivatives up to order $m - 1$ is zero. Upon writing that out, one will obtain a certain equation in V:

(3) $$\varphi\left(x, V, \frac{dy}{dx}, \ldots, \frac{d^k y}{dx^k}\right) = 0 ,$$

in which k is equal to at most $m^2 - 1$.

One will then have a fundamental system y_1, y_2, …, y_m if one takes V to be an integral of equation (2) that does not satisfy equation (3).

Having said that, it can happen, in general, i.e., if equation (1) is taken arbitrarily, that equations (2) will have no common solution with a (linear or nonlinear) differential equation with rational coefficients of order less than m^2, if one ignores the solutions that satisfy equation (3). However, things can be different in certain cases. Therefore, suppose that the differential equation of order p :

(4) $$f\left(x, V, \frac{dy}{dx}, \ldots, \frac{d^p y}{dx^p}\right) = 0$$

(f being a polynomial) fulfills that condition. Moreover, I suppose that the preceding equation is *irreducible*, i.e., it has no common solution with an equation of the same form and lower order. Under those conditions, *all* functions V that satisfy equation (4) will satisfy equation (2), and furthermore, equation (4) will have *no* common solution with equation (3). As a result, each solution to $f = 0$ will correspond to a fundamental system of integrals for the proposed linear equation.

Therefore, let $y_1, y_2, \ldots, y_m$ be the fundamental system that corresponds to a certain solution of the equation $f = 0$, and let $Y_1, \ldots, Y_m$ be the system that corresponds to an arbitrary solution of the same equation. One will have:

$$
(S) \qquad \begin{cases} Y_1 = a_{11}\, y_1 + \cdots + a_{1m}\, y_m, \\ Y_2 = a_{21}\, y_1 + \cdots + a_{2m}\, y_m, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ Y_m = a_{m1}\, y_1 + \cdots + a_{mm}\, y_m. \end{cases}
$$

The coefficients a depend upon only p arbitrary parameters, and one sees very easily that one can consider them to be algebraic functions of those parameters. Moreover, those substitutions form a *group*. I shall let G denote the continuous algebraic group of linear transformations that are defined by equations (S) and call it the *group of transformations* relative to the linear equation (1).

2. – One can establish the following proposition in regard to that group, which recalls Galois's fundamental theorem in the theory of algebraic equations:

Any rational function of x and of $y_1, y_2, \ldots, y_m$ and their derivatives, which are expressed rationally as functions of x will remain invariable when one performs the substitutions of the group G on $y_1, y_2, \ldots, y_m$.

Indeed, consider one such function. Upon replacing $y_1, y_2, \ldots, y_m$ in y with their values as functions of V and equating that to a rational function, one will have:

$$
F\left(x, V, \frac{dV}{dx}, \ldots, \frac{d^pV}{dx^p}\right) = R\,(x)\,,
$$

in which F and R are rational. Now, that equation is found to be verified for a certain solution V of $f = 0$. As a result, it will be verified for all solutions, from the irreducibility of the latter equation. That amounts to saying that the rational function considered will not change when one performs the substitution S on $y_1, y_2, \ldots, y_m$.

3. – The foregoing reproduces what was said before. We now arrive at the converse theorem. We shall show that:

Any rational function of $x, y_1, y_2, \ldots, y_m$, and their derivatives that remains invariable under the substitutions of the group G is a rational function of x.

I have proved that (*Annales de Toulouse*, t. I, pp. 5) only when the proposed linear equation has an integral that is rational in the neighborhood of each singular point (for the other cases, I said *uniform* function of x, instead of *rational* function of x). That restriction is pointless, and we shall easily prove the theorem. Let Φ ($x, y_1, y_2, \ldots, y_m$) be a function that satisfies the stated conditions. We must show that if one replaces $y_1, y_2, \ldots, y_m$ with a certain fundamental system then the function Φ will be a rational function of x. Now, replace the y and their derivatives with their values as functions of V; we will have:

$$\Phi = F\left(x, V, \frac{dV}{dx}, \ldots, \frac{d^pV}{dx^p}\right).$$

I say that if one takes V to be an arbitrary function of (4) then that expression will be a rational function of x. I first remark that from the hypothesis that was made in regard to Φ, the function $F(x, V, \ldots)$ will represent the same function of x, no matter what the function V that satisfies equation (4). Now, let μ be the degree in $\frac{d^pV}{dx^p}$ of the latter equation. For arbitrarily-given values of:

$$x,\ V,\ \frac{dV}{dx},\ \ldots,\ \frac{d^{p-1}V}{dx^{p-1}},$$

the equation $f = 0$ will have μ distinct roots. Upon appealing to equation (4), one can suppose that the derivative of order p of V figures in F only to a degree that is at most $\mu - 1$. Having made that substitution, F will become a function:

$$F_1\left(x, V, \frac{dV}{dx}, \ldots, \frac{d^pV}{dx^p}\right)$$

that is rational with respect to the letters that it depends upon and will contain $\frac{d^pV}{dx^p}$ to the power at most $\mu - 1$ in its numerator and its denominator. Since, for a given value of x, the function F_1 will take the same value for all values of x, V, $\frac{dV}{dx}$, …, $\frac{d^pV}{dx^p}$ that satisfy the relation:

$$f\left(x, V, \frac{dV}{dx}, \ldots, \frac{d^pV}{dx^p}\right) = 0,$$

which is irreducible and of degree μ in $\frac{d^pV}{dx^p}$, it is necessary that F_1 should depend upon only x. *The function* Φ *is then a rational function of* x, which is what we wished to establish.

One notes that in the preceding proofs, one did not consider the rational functions of x, y_1, y_2, …, y_m and their derivatives to contain the undetermined functions y_1, y_2, …, y_m, but one must also intend that y_1, y_2, …, y_m should represent a certain fundamental system of the proposed linear equation.

4. – We just addressed the simplest case. In order to have the theory in full generality, we can suppose that the coefficients p_1, …, p_m of the linear equation are rational functions of x and a certain number of *adjoint* functions:

$$A(x),\ B(x),\ \dots, L(x),$$

and their derivatives up to an arbitrary order. No modification of the preceding is necessary, except that the coefficients of the equation $f = 0$ will not necessarily be rational functions of x, but rational functions of x, and of A, B, …, L, and their derivatives, and those are the functions to which the fundamental theorem and its converse apply. They relate to functions that are expressed rationally with the aid of x, and of A, …, L, and their derivatives.

5. – In order to finish specifying the notion of a group of transformations for a linear equation, one must further prove that the double property that the substitutions of that group enjoy in regard to the proposed equation belongs to them exclusively. To that end, place oneself in the same case as in no. **1** and consider the left-hand side:

$$f\left(x, V, \frac{dV}{dx}, \dots, \frac{d^pV}{dx^p}\right) = 0$$

of equation (4), in which one first supposes that V is an arbitrary solution of (2).

Upon replacing V with its value in terms of y_1, y_2, …, y_m, and their derivatives, the expression f will become a function:

$$\Phi(x, y_1, y_2, \dots, y_m, \dots).$$

That function will be zero when one takes y_1, y_2, …, y_m to be a fundamental system that corresponds to a solution V to equation (4). A substitution Σ' that is performed on y_1, y_2, …, y_m and which does not belong to a group G cannot leave the value of Φ invariable, because such a substitution amounts, in a general manner, to the replacement of one solution of equation (2) with another one – say V with V' – so one will have:

$$f\left(x, V', \frac{dV'}{dx}, \dots, \frac{d^pV'}{dx^p}\right) = 0,$$

and as a result, since V' satisfies (4), the substitution Σ' will belong to the group G, whose characteristic properties are indeed thereby exhibited.