“Sur l’extension des idées de Galois à la théorie des équations différentielles. (Extrait d’une lettre addressée à Mr. Klein),” Math. Ann. **47** (1896), 161-162.

On the extension of Galois’s ideas to the theory of differential equations

(Extract from a latter addressed to Mr. Klein)

By

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Translated by D. H. Delphenich

This year, while giving my course on the study of *groups of transformations* of linear differential equations, on the subject of which I recently sent an article to your journal (Math. Annalen, v. 46), I realized that an auxiliary equation that plays an essential role in that study was defined in a too-specialized manner in that article, which might lead one to restrict the notion of a group of transformations.

If you would like to indeed refer to the cited location then at a certain point in it, I considered the algebraic differential equation:

$$f\left(x, V, \frac{dV}{dx}, \ldots, \frac{d^pV}{dx^p}\right) = 0 \qquad (p < m^2) \tag{4}$$

to have a common solution σ with equation (2) that does not belong to equation (3):

$$\varphi\left(x, V, \ldots, \frac{d^kV}{dx^k}\right) = 0 . \tag{3}$$

I then supposed that equation (4) is irreducible. That is not necessary, since it would suffice to take from among all algebraic equations such as (4) the one that has *lowest order* (or one of them, if there are several). Moreover, one can assume that equation (4) is algebraically irreducible with respect to $\frac{d^pV}{dx^p}$. It is then clear that any solution of f that does not belong to (3) will belong to equation (2), because otherwise the solution σ would satisfy an equation of order lower than p.

All of the reasoning that was made under the more specialized hypothesis that had been adopted will persist intact, and that is why one is led to the notion of the *group of transformations* of a linear differential equation in the most satisfying manner.

Permit me to add a remark *on the extension of Galois's ideas to the theory of nonlinear equations*. Let an algebraic differential equation be:

$$P\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^m y}{dx^m}\right) = 0 .$$

I consider the expression:

$$V = R\,(y_1, y_2, \ldots, y_m, x) ,$$

in which R is an arbitrarily-chosen rational function of m arbitrary integrals $y_1, y_2, \ldots, y_m$ of the preceding equation and the variable x. One can form the equation of order $m\,\mu$ that V must satisfy, which is an equation that we shall denote by E. Moreover, we will have rational functions of V and its derivatives for the y.

If the equation F is arbitrary then equation E will have no common integrals with an algebraic equation of lower order, since that would be true for only certain equations that are easy to form and are produced by the supposition that two or more integrals y in V are identical. We shall denote the set of those equations by φ.

If one abandons the general case then it can happen that E has a common integral that does not belong to φ for an algebraic differential equation of lower order. Let:

$$f\left(x, V, \frac{dV}{dx}, \ldots, \frac{d^p V}{dx^p}\right) = 0$$

be one such equation. Among the equations of that sort, consider the one that has *lowest order* (or one of them if there are several). That equation will lead to a theory that is entirely similar to the one that we had developed for linear equations. One will then have an algebraic relation between the integrals $y_1, y_2, \ldots, y_m$ and their derivatives, so *the group that is involved here will be the group of operations that replace that system of integrals with another one in that relation.*

You see that the indeterminacy in the number μ gives that theory a completely different character to what happens in the case of linear equations, where one can confine oneself to taking $\mu = m$.

Paris, 27 November 1895.