Pre-metric Electromagnetism and Complex relativity

David Delphenich

Physics Department
Bethany College
Lindsborg, KS 67456

delphenichd@bethanylb.edu
Common ground

- Representation of $SO_0(3, 1)$ by its isomorphism with $SO(3; \mathbb{C})$.

- Action of $SO(3; \mathbb{C})$ on the bundle of oriented complex orthonormal 3-frames on the bundle $\Lambda^2(M)$ when it is given an almost-complex structure.

Bigger picture

- Looking at geometry of spacetime in terms of tangent 2-planes, instead of tangent vectors.

- Plücker-Klein embedding represents 2-planes in $\mathbb{R}^4$ as decomposable bivectors or 2-forms.
• Shift of emphasis from metric geometry of $M$ in terms of $T(M)$ to projective geometry in terms of $\Lambda^2(M)$.

• $\mathbb{C}P^2$ plays a key role in wave motion.

• Formulation of wave mechanics using $SO(3; \mathbb{C})$ in place of $SL(2; \mathbb{C})$ would unify formalisms of EM, GR, and QM into a single methodology.

• This might also suggest a geometric interpretation of wave mechanics to replace the statistical one.

• Since the basic geometric objects described by (decomposable) 2-forms are tangent 2-planes, not tangent vectors, one needs to rethink the concept of geodesic motion in terms of surfaces, not curves.

• Or possibly complex lines and curves.
Pre-metric electromagnetism

- Formulation of electromagnetism using an electromagnetic constitutive law:
  \[ \kappa: \Lambda^2(M) \to \Lambda_2(M) \]
  to replace the Hodge \( * = 1/\lambda \kappa \), where \( \lambda \) is a scalar function on \( M \).

- Maxwell equations take the form:
  \[ dF = 0, \quad \delta \mathbf{J} = \mathbf{J}, \quad \mathbf{J} = \kappa(F) \]

- \( \delta: \Lambda_2(M) \to \Lambda_1(M) \) is defined by Poincaré duality \( \# \) isomorphism using a volume element \( V \) on \( T(M) \):
  \[ \delta = \#^{-1} d\# \]

  \( \#: \Lambda_k(M) \to \Lambda^{4-k}(M), \quad b \mapsto i_b V \)
• If one assumes that \( \ast^2 = -I \) (Lorentzian case) then \( \ast \) defines an almost-complex structure on \( \Lambda^2(M) \).

• Complex scalar multiplication follows from:

\[
iF = \ast F
\]

In this case, we are dealing with a defining identity, not an equation.

• \( \ast \) implies a conformal structure on \( T(M) \).

• Also implies a complex orthogonal structure on \( \Lambda^2(M) \).

\[
<F, G> = (F \wedge G)(\mathbf{V})
\]

\[
(F, G) = <F, \ast G>
\]

\[
<F, G>_{\mathbb{C}} = (F, G) + i<F, G>
\]

• Structure group of \( \Lambda^2(M) \) reduces to:

\[
SO(3; \mathbb{C}) \cong SO_0(3; 1)
\]
Complex relativity

- Also based in isomorphism of $SO_0(3, 1)$ and $SO(3; \mathbb{C})$.

- Usually, to represent $SO(3; \mathbb{C})$ in $\Lambda^2(M)$ one complexifies $\Lambda^2(M)$ and then decomposes $\Lambda^2_{\mathbb{C}}(M)$ into self-dual and anti-self-dual 2-forms.

- $*F = \pm iF$ becomes an eigenvalue equation now, not an identity.

- Representation of $SO(3; \mathbb{C})$ is then in the bundle of self-dual complex 2-forms.

- Not actually necessary to complexify $\Lambda^2(M)$ if it is already given an almost-complex structure.
• Self-duality is only used in the Debever-Penrose decomposition of the curvature tensor.

• In the case of 2-forms with values in complex vector spaces, the operator $i^*$ can be defined naturally.

• Lorentzian geometry follows from associating the $SO(3; \mathbb{C})$–principal bundle of oriented, complex orthonormal 3-frames in $\Lambda^2(M)$ with $\Lambda^2(M)$.

• This bundle is given a canonical $\mathbb{C}^3$–valued 2-form $Z^i$, instead of an $\mathbb{R}^4$–valued 1-form.

• Lorentzian connection can be defined by an $\mathfrak{so}(3; \mathbb{C})$–valued 1-form $\sigma^i_j$ on bundle.

• Connection describes infinitesimal parallel translation of complex 3-frames, 2-forms, and tangent 2-planes along curves.
• Torsion 3-form and curvature 2-form are then defined by Cartan structure equations:

\[ \Psi^i = dZ^i + \sigma^i_j \wedge Z^j \]

\[ \Sigma^i_j = d\sigma^i_j + \sigma^i_k \wedge \sigma^k_j \]

• Bianchi identities take the form:

\[ d\Psi^i = \Sigma^i_j \wedge Z^j, \quad d\Sigma^i_j = -\sigma^i_k \wedge \Sigma^k_j \]

• Levi-Civita connection is the unique \( \mathfrak{so}(3; \mathbb{C}) \)-valued connection 1-form with vanishing torsion:

\[ dZ^i = -\sigma^i_j \wedge Z^j, \quad \sigma_{ij} + \sigma_{ji} = 0 \]

(Last equation only valid for orthonormal frame.)

• Vacuum Einstein equations follow from vanishing of trace-free Ricci tensor that is obtained from the self-dual part of the curvature 2-form.

• Geodesic equations of motion not as intrinsic to this type of geometry.