Pre-metric Electromagnetism
and
Complex relativity

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Common ground

- Representation of $SO_0(3, 1)$ by its isomorphism with $SO(3; \mathbb{C})$.

- Action of $SO(3; \mathbb{C})$ on the bundle of oriented complex orthonormal 3-frames on the bundle $\Lambda^2(M)$ when it is given an almost-complex structure.

Bigger picture

- Looking at geometry of spacetime in terms of tangent 2-planes, instead of tangent vectors.

- Plücker-Klein embedding represents 2-planes in $\mathbb{R}^4$ as decomposable bivectors or 2-forms.

- Shift of emphasis from metric geometry of M in terms of $T(M)$ to projective geometry in terms of $\Lambda^2(M)$.
    - $\mathbb{C}P^2$ plays a key role in wave motion.
- Formulation of wave mechanics using $SO(3; \mathbb{C})$ in place of $SL(2; \mathbb{C})$ would unify formalisms of EM, GR, and QM into a single methodology.
    - This might also suggest a geometric interpretation of wave mechanics to replace the statistical one.
    - Since the basic geometric objects described by (decomposable) 2-forms are tangent 2-planes, not tangent vectors, one needs to rethink the concept of geodesic motion in terms of surfaces, not curves.
        - Or possibly complex lines and curves.

Pre-metric electromagnetism

- Formulation of electromagnetism using an electromagnetic constitutive law:

$$\kappa: \Lambda^2(M) \to \Lambda_2(M)$$

  to replace the Hodge $* = 1/\lambda\ \kappa$, where λ is a scalar function on M.

- Maxwell equations take the form:

$$dF = 0, \qquad \delta\mathfrak{h} = \mathbf{J}, \qquad \mathfrak{h} = \kappa(F)$$

- $\delta : \Lambda_2(M) \to \Lambda_1(M)$ is defined by Poincaré duality # isomorphism using a volume element V on $T(M)$:

$$\delta = \#^{-1} d\#$$

$$\#: \Lambda_k(M) \to \Lambda^{4-k}(M), \quad \mathbf{b} \mapsto i_{\mathbf{b}}V$$

- If one assumes that $*^2 = -I$ (Lorentzian case) then $*$ defines an almost-complex structure on $\Lambda^2(M)$.

  - Complex scalar multiplication follows from:

    $$iF = *F$$

    In this case, we are dealing with a defining identity, not an equation.

  - $*$ implies a conformal structure on $T(M)$.

  - Also implies a complex orthogonal structure on $\Lambda^2(M)$.

    $<F, G> = (F \wedge G)(\mathbf{V})$

    $(F, G) = <F, *G>$

    $<F, G>_{\mathbf{C}} = (F, G) + i<F, G>$

  - Structure group of $\Lambda^2(M)$ reduces to:

    $$SO(3; \mathbb{C}) \cong SO_0(3; 1)$$

Complex relativity

- Also based in isomorphism of $SO_0(3, 1)$ and $SO(3; \mathbb{C})$.

- Usually, to represent $SO(3; \mathbb{C})$ in $\Lambda^2(M)$ one complexifies $\Lambda^2(M)$ and then decomposes $\Lambda^2_{\mathbb{C}}(M)$ into self-dual and anti-self-dual 2-forms.

  - $*F = \pm iF$ becomes an eigenvalue equation now, not an identity.

  - Representation of $SO(3; \mathbb{C})$ is then in the bundle of self-dual complex 2-forms.

  - Not actually necessary to complexify $\Lambda^2(M)$ if it is already given an almost-complex structure.

- Self-duality is only used in the Debever-Penrose decomposition of the curvature tensor.

- In the case of 2-forms with values in complex vector spaces, the operator $i*$ can be defined naturally.

- Lorentzian geometry follows from associating the $SO(3;\ \mathbb{C})$–principal bundle of oriented, complex orthonormal 3-frames in $\Lambda^2(M)$ with $\Lambda^2(M)$.

  - This bundle is given a canonical $\mathbb{C}^3$–valued 2-form Z^i, instead of an $\mathbb{R}^4$–valued 1-form.

  - Lorentzian connection can be defined by an $\mathfrak{so}(3;\ \mathbb{C})$–valued 1-form σ^i_j on bundle.

  - Connection describes infinitesimal parallel translation of complex 3-frames, 2-forms, and tangent 2-planes along curves.

- Torsion 3-form and curvature 2-form are then defined by Cartan structure equations:

$$\Psi^i = dZ^i + \sigma^i_j \wedge Z^i$$

$$\Sigma^i_j = d\sigma^i_j + \sigma^i_k \wedge \sigma^k_j$$

- Bianchi identities take the form:

$$d\Psi^i = \Sigma^i_j \wedge Z^j, \qquad d\Sigma^i_j = -\sigma^i_k \wedge \Sigma^k_j$$

- Levi-Civita connection is the unique $\mathfrak{so}$(3; ℂ)-valued connection 1-form with vanishing torsion:

$$dZ^i = -\sigma^i_j \wedge Z^j, \qquad \sigma_{ij} + \sigma_{ji} = 0$$

(Last equation only valid for orthonormal frame.)

- Vacuum Einstein equations follow from vanishing of trace-free Ricci tensor that is obtained from the self-dual part of the curvature 2-form.

- Geodesic equations of motion not as intrinsic to this type of geometry.