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THEORY

OF

VORTICES

Lectures taught during the second semester 1891-92.

BY

H. POINCARÉ, MEMBER OF THE INSTITUTE

Edited by **M. LAMOTTE**, licensee in science

Translated by D. H. Delphenich

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INTRODUCTION

The theory of vortex motions rests upon a theorem that is due to Helmholtz, and which constitutes the greatest advance that has been made up to now in hydrodynamical theories.

In full rigor, that theorem applies to only the motions of fluids in which there exists no friction, and which present a uniform temperature or depend upon only the pressure. One can still apply the theorem when these conditions are not satisfied exactly, but only real conditions that differ from them very slightly, by considering the results that are obtained as being not rigorously exact, but only a first approximation.

Vortex motions seem to play a considerable role in meteorological phenomena, which is a role that Helmholtz attempted to specify.

One is also tempted to find the mechanical explanation for the universe in the existence of such vortex motions. Instead of representing the space that is occupied by atoms, which are separated by distances that are immense in comparison to their own dimensions, Sir William Thomson assumed that matter is continuous, but that certain portions of it are animated with vortex motions that must preserve their individuality, from Helmholtz's theorem.

Finally, the equations that serve to define vortex motions present a certain formal analogy with the equations of electrodynamics. That will naturally bring the two theories closer together, and in certain cases, it will permit one to deduce the solution to a problem that is posed in one of the theories from a problem that was solved in the other one. Moreover, a certain number of attempts have been made to establish a closer link between them.

After recalling the equations of hydrodynamics, I will prove Helmholtz's theorem, and I will then develop its consequences that relate to the motion of fluids by comparing the results with those of electrodynamics.

FIRST CHAPTER

HELMHOLTZ'S THEOREM

1. Equations of hydrodynamics. – Let x_0, y_0, z_0 be the coordinates of a fluid molecule at the time origin $t = 0$; x, y, z are its coordinates at time t. u, v, w are the components of its velocity, ρ is the density of the fluid, and p is its pressure.

One can take the variables to be x_0, y_0, z_0 , which is the Lagrange system, or x, y, z, t, which is the Euler system. I will adopt the following notions in what follows: I let:

$$\frac{du}{dt}, \quad \frac{du}{dx_0}, \quad \frac{du}{dy_0}, \quad \frac{du}{dz_0}$$

denote the derivatives with respect to the Lagrange variables, and I let:

$$\frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial z}$$

be the derivatives that are taken with respect to the Euler variables.

In the Lagrange system, x, y, z are functions of x_0, y_0, z_0 :

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{dz}{dt} = w$$

are the components of the velocity, while:

$$\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}$$

are those of acceleration.

In order to pass from one system of variables to another, it will suffice to apply the ordinary rules of differentiation, and to write:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

or

(1) $$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},$$

and similarly:

(2) $$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}.$$

Let $d\tau$ be a volume element; the mass of liquid that it contains will be $\rho\, d\tau$. I will call the projections onto the axes of the resultant of all of the forces that act upon that element $\rho\, X\, d\tau$, $\rho\, Y\, d\tau$, $\rho\, Z\, d\tau$. The equations of hydrostatics, which express the idea that the element is in equilibrium, are the following ones:

$$\frac{\partial p}{\partial x} = \rho\, X, \qquad \frac{\partial p}{\partial y} = \rho\, Y, \qquad \frac{\partial p}{\partial z} = \rho\, Z. \tag{3}$$

In order to pass from these equations to those of hydrodynamics, one must add some fictitious inertial forces to the real forces (d'Alembert's principle). The components of these inertial forces are equal to the components of the acceleration, multiplied by the mass and with the sign changed: namely:

$$-\rho\, d\tau \cdot \frac{du}{dt}, \qquad -\rho\, d\tau \cdot \frac{dv}{dt}, \qquad -\rho\, d\tau \cdot \frac{dw}{dt}.$$

The equations of hydrodynamics will then be:

1. In the Lagrange system:

$$\frac{\partial p}{\rho\, \partial x} = X - \frac{du}{dt}, \qquad \frac{\partial p}{\rho\, \partial y} = Y - \frac{dv}{dt}, \qquad \frac{\partial p}{\rho\, \partial z} = Z - \frac{dw}{dt}. \tag{4}$$

2. In the Euler system:

$$\frac{\partial p}{\rho\, \partial x} = X - \frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z}, \quad \text{etc.} \tag{5}$$

2. In all of what follows, I will suppose that x, y, z are continuous functions of x_0, y_0, z_0; that condition is not always satisfied. Indeed, let two reservoirs be filled with liquid that are separated by wall that is pierced with an opening, and suppose that a high pressure prevails in one of them. That will produce a jet of liquid, outside of which the liquid will remain immobile, while inside of which it will take on a uniform motion. Suppose that the jet has the form of a cylinder that is parallel to the x-axis.

Outside of the cylinder, we will have:

$$x = x_0, \qquad y = y_0, \qquad z = z_0 ,$$

and inside of it, we will have:

$$x = x_0 + vt, \qquad y = y_0, \qquad z = z_0 .$$

x is thus a discontinuous function of x_0, y_0, z_0.

If x, y, z are continuous functions of x_0, y_0, z_0 then the liquid molecules that form a continuous curve or surface at the time $t = 0$ will again form a continuous curve or surface at an arbitrary epoch t; if the curve is closed at time 0 then it will again be closed at time t.

Indeed, suppose that the molecules form a certain arc of a curve in their initial positions. The equations of that arc of a curve can be put into the form:

$$x_0 = f_0(\alpha), \qquad y_0 = f_0'(\alpha), \qquad z_0 = f_0''(\alpha),$$

in which f_0, f_0', f_0'' are continuous functions of the parameter α.

The coordinates of the molecules become x, y, z at time t, which will be, by hypothesis, continuous functions of x_0, y_0, z_0. Consequently, they will be continuous functions of α:

$$x = f(\alpha), \qquad y = f'(\alpha), \qquad z = f''(\alpha).$$

These equations will again represent a continuous arc of the curve.

If the initial curve C_0 is closed then x_0, y_0, z_0 will be periodic functions of α. Since x, y, z are uniform functions of x_0, y_0, z_0 they will also be periodic functions of α, and the curve C that the molecules occupied at the epoch t will be likewise closed.

If the molecules occupy a continuous surface S_0 at the time origin then their coordinates can be expressed by:

$$x_0 = f_0(\alpha, \beta), \qquad y_0 = f_0'(\alpha, \beta), \qquad z_0 = f_0''(\alpha, \beta),$$

in which f_0, f_0', f_0'' are continuous functions of the parameters (α, β). The coordinates will become x, y, z at the epoch t, and they will be continuous functions of x_0, y_0, z_0, and consequently, of (α, β). Therefore:

$$x = f(\alpha, \beta), \qquad y = f'(\alpha, \beta), \qquad z = f''(\alpha, \beta),$$

in which f, f', f'' are continuous functions; consequently, these equations will again represent a continuous surface S.

3. Equations of continuity. – Consider a surface element $d\omega$ and seek to evaluate the quantity of fluid that crosses that element during the time dt. The molecules that traverse the element $d\omega$ at the epoch t will occupy a surface element $d\omega'$ at the time $t + dt$ that is infinitely close to $d\omega$. In particular, the ones that are found at the center of gravity G of $d\omega$ will go to G'; the ones that cross $d\omega$ at the epoch $t + dt$ will occupy that element itself. Finally, the ones that cross $d\omega$ between the two epochs t and $t + dt$ will be found in intermediate positions.

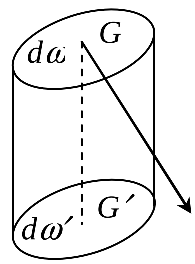

Figure 1.

In summary, all of the molecules that passed through $d\omega$ during the time dt will be found in a volume at the instant $t + dt$ that amounts to a cylinder that has the element $d\omega$ for its base and whose generators are parallel to GG' (Fig. 1). Moreover, $GG' = V\,dt$, where V is the velocity of the fluid at the instant considered. The height of that cylinder will be the projection of GG' onto the normal to $d\omega$, namely:

$$V\,dt \cdot \cos(GG'N) = V_n\,dt.$$

The quantity of fluid that crosses $d\omega$ during the time dt is then:

$$\rho\,V_n\,d\omega \cdot dt.$$

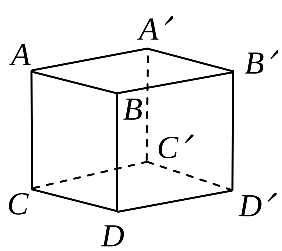

Figure 2.

Now, consider a rectangular parallelepiped whose edges are parallel to the coordinate axes, and are equal to dx, dy, dz, respectively (Fig. 2). From the preceding, the mass of fluid that traverses the face $ABCD$ that is perpendicular to OX during the time dt will be equal to:

$$\rho\,u\,dy\,dz\,dt,$$

and the mass that crosses the opposite face will be:

$$\left(\rho u + \frac{\partial \rho u}{\partial x}\right) dy\, dz\, dt.$$

Therefore, there is a mass of fluid between these two faces of the parallelepiped that is equal to:

$$-\frac{\partial \rho u}{\partial x}\, dx\, dy\, dz\, dt.$$

Upon performing the same calculation for the other two pairs of faces, one will find that the total fluid mass that passes between them in the parallelepiped during the time *dt* will be equal to:

$$-\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right) dx\, dy\, dz\, dt.$$

On the other hand, the increase in the mass $\rho\, dx\, dy\, dz$ of the fluid that is contained in the parallelepiped during the time *dt* is:

$$\frac{\partial \rho}{\partial t}\, dx\, dy\, dz\, dt,$$

since $\partial\rho / \partial t$ represents the increase in the density ρ during the time *dt*.

It is therefore necessary that:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0. \tag{6}$$

This is the equation of continuity in the Euler system. It can be written:

$$\frac{\partial \rho}{\partial t} + \sum \rho \frac{\partial u}{\partial x} + \sum u \frac{\partial \rho}{\partial x} = 0,$$

or, upon taking the relation (2) into account:

$$\frac{d\rho}{dt} + \rho \sum \frac{\partial u}{\partial x} = 0,$$

except that in the latter form, it refers to the two types of derivatives.

4. Simplification of the Lagrange equations. – The Lagrange equations are susceptible to being simplified when one makes the hypotheses that are necessary for the application of Helmholtz's principle, as we shall see.

In that case, the forces *X*, *Y*, *Z* will admit a potential *V*:

$$X = \frac{\partial V}{\partial x}, \qquad Y = \frac{\partial V}{\partial y}, \qquad Z = \frac{\partial V}{\partial z}.$$

No matter what the fluid, the density ρ, the pressure p, and the temperature T will be coupled by a relation such as:

$$\rho = f(p, T).$$

In order for the theorem to be applicable, it is necessary that ρ must be a function of only p. This will be true if one is dealing with a single liquid or a gas that obeys Mariotte's law and whose temperature is uniform ($\rho = p$), or a gas that is subject to an adiabatic transformation ($\rho = p^{\gamma}$). If the temperature is non-uniform then it will be necessary that the surfaces of equal pressure must coincide with the surfaces of equal temperature. If there are two superposed liquids then it will be necessary that the pressure must be constant on the separation surface in order for one to have the right to apply the theorem.

When ρ is a function of p, dp / ρ will be an exact differential, and:

$$\int \frac{dp}{\rho}$$

will be a function of p. If one sets:

$$V - \int \frac{dp}{\rho} = \psi$$

and differentiates with respect to x then it will become:

$$\frac{\partial \psi}{\partial x} = \frac{\partial V}{\partial x} - \frac{\partial p}{\rho\, \partial x}.$$

If one replaces $\frac{\partial p}{\rho\, \partial x}$ with its value that is inferred from that relation in the first Lagrange equation then, upon remarking that $\frac{\partial V}{\partial x} = X$, that system will become:

$$\frac{du}{dt} = \frac{\partial \psi}{\partial x}, \qquad \frac{dv}{dt} = \frac{\partial \psi}{\partial y}, \qquad \frac{dw}{dt} = \frac{\partial \psi}{\partial z}. \tag{7}$$

5. Helmholtz's theorem. – Consider an infinitude of fluid molecules that form a closed curve C_0 at the instant $t = 0$; they form another closed curve C [no. **2**] at the instant t. The integral:

$$J = \int (u\,dx + v\,dy + w\,dz) \tag{8}$$

is constant when it is taken along the curve C.

Helmholtz did not give his theorem in that form, as we shall see later on.

That theorem contains Lagrange's theorem as a special case: If there exist a velocity potential at the time origin then there will exist one at an arbitrary epoch.

Indeed, in that case, one will have:

$$u\,dx + v\,dy + w\,dz = d\varphi,$$

and the integral J will be zero at the time origin. If it is constant then it will always be zero, and the expression under the $\int$ sign will always be an exact differential.

6. Proof of the theorem. – Let the equations of the closed curve C_0 be:

$$x_0 = f_0(\alpha), \qquad y_0 = f_0'(\alpha), \qquad z_0 = f_0''(\alpha),$$

in which $f_0, \ldots$ are continuous periodic functions of α. Similarly, one will have:

$$x = f(\alpha), \qquad y = f'(\alpha), \qquad z = f''(\alpha)$$

for the curve C.

If one lets α denote the period of A then:

$$J = \int_0^A \left(u \frac{dx}{d\alpha} + v \frac{dy}{d\alpha} + w \frac{dz}{d\alpha} \right) d\alpha .$$

One basically needs to prove that:

$$\frac{dJ}{dt} = 0.$$

Now:

$$J = \int_0^A \sum u \frac{dx}{d\alpha}\, d\alpha,$$

$$\frac{dJ}{dt} = \int_0^A \sum \frac{du}{dx} \frac{dx}{d\alpha} d\alpha + \int_0^A \sum u \frac{d^2 x}{d\alpha\, dt}\, d\alpha.$$

I say that each of the sums Σ is an exact differential.

In fact:

$$\sum \frac{du}{dx}\frac{dx}{d\alpha}d\alpha = \sum \frac{\partial \psi}{\partial x}\frac{dx}{d\alpha}d\alpha,$$

$$\sum \frac{\partial \psi}{\partial x}\frac{dx}{d\alpha} = \frac{\partial \psi}{\partial x}\frac{dx}{d\alpha} + \frac{\partial \psi}{\partial y}\frac{dy}{d\alpha} + \frac{\partial \psi}{\partial z}\frac{dz}{d\alpha} = \frac{d\psi}{d\alpha},$$

so

$$\sum \frac{du}{dt}\frac{dx}{d\alpha}d\alpha = d\psi.$$

On the other hand:

$$u\frac{d^2x}{d\alpha\, dt}d\alpha = u\,\frac{du}{d\alpha}d\alpha = \frac{1}{2}\frac{du^2}{d\alpha}d\alpha = du^2, \text{ etc.}$$

Consequently:

$$\frac{dJ}{dt} = \int \left[d\psi + \tfrac{1}{2}d(u^2 + v^2 + w^2)\right].$$

Since the expression under the $\int$ sign is an exact differential, the integral along a closed curve will be zero, and:

(9) $$\frac{dJ}{dt} = 0.$$

7. Remark. – This theorem is true, on the condition that $d\psi$ is an exact differential – in other words, that ρ must be a function p and that the external forces must admit a potential; i.e., there is no friction. One sometimes expresses the last condition by saying that the theorem is true when there are no instantaneous forces, but that statement is imprecise.

8. Stokes's theorem. – In order to transform Helmholtz's theorem in the manner that I just demonstrated, I will make use of a theorem that is due to Stokes, which I would like to recall.

Let C be a closed curve, and pass from that curve to an arbitrary surface; the curve C bounds a certain area A on that surface. Let $d\omega$ be an element of that area, and let l, m, n be the direction cosines of the normal to $d\omega$. From Stokes's theorem, one has:

(10) $$\int_C (u\,dx + v\,dy + w\,dz) = \int d\omega\left[l\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + m\left(\frac{du}{dz} - \frac{dw}{dx}\right) + n\left(\frac{dv}{dx} - \frac{du}{dy}\right)\right],$$

in which the first integral is taken over all of the elements of C, while the second one is taken over all of the elements $d\omega$ of the area A.

1. First, suppose that the area A is planar and situated in the plane (x, y), for example. In that case:

$$l = m = 0, \qquad n = A, \qquad dz = 0, \qquad d\omega = dx\,dy,$$

and what remains will be:

$$\int_C u\,dx + v\,dy = \int dx\,dy\left(\frac{dv}{dx} - \frac{du}{dy}\right);$$

this is the statement of a well-known theorem in analysis. The same thing will be true for the other coordinate planes.

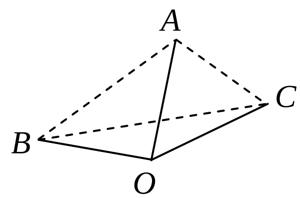

Figure 3.

2. The area A is planar, but situated in an arbitrary plane.

Let there be three infinitely-small lengths OA, OB, OC (Fig. 3) that are parallel to the axes. Join them into the triangle ABC; the triangle ABC is planar and infinitely small. I say that the theorem is true for that triangle. One obviously has:

$$\int_{ABCA} = \int_{ABOA} + \int_{BCOB} + \int_{CAOC} \;.$$

Indeed, the edges OA, OB, OC are traversed twice in the opposite senses, and all that will remain in the right-hand side will be the $\int$ that is taken around AB, BC, CA, as in the left-hand side. Since the triangles AOB, etc. are infinitely small, I can write:

$$\int (u\,dx + v\,dy + w\,dz) = \overline{AOB}\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + \overline{AOC}\left(\frac{du}{dz} - \frac{dw}{dx}\right) + \overline{BOC}\left(\frac{dv}{dx} - \frac{du}{dy}\right),$$

upon applying the equality (1) to each of the triangles that are situated in the coordinate planes. Now, these triangles are nothing but the projections of ABC onto these planes; therefore, if:

$$ABC = d\omega,$$

$$AOB = l\,d\omega, \quad AOC = m\,d\omega, \quad BOC = n\,d\omega$$

then one will indeed have:

$$\int u\,dx + v\,dy + w\,dz = \int l\,d\omega\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + \ldots$$

The theorem is then true in a general fashion, since an arbitrary area can be decomposed into triangles that are small enough to be treated as planar triangles such as *ABC*.

Maxwell made frequent use of that theorem. (See his treatise.)

9. Helmholtz's notations. Definition of vorticity. – Helmholtz set:

$$\frac{dw}{dy} - \frac{dv}{dz} = 2\xi,$$
$$\frac{du}{dz} - \frac{dw}{dx} = 2\eta, \tag{11}$$
$$\frac{dv}{dz} - \frac{du}{dy} = 2\zeta.$$

From Stokes's formula, one will then have:

$$\int (u\,dx + v\,dy + w\,dz) = 2\int d\omega\,(l\xi + m\eta + n\zeta).$$

From what we established [no. **6**], when this integral it taken over the area *A*, it will remain constant during the motion of that area.

The vector whose components are (ζ, η, ξ) is what Helmholtz called the *vorticity*. That name demands some explanation.

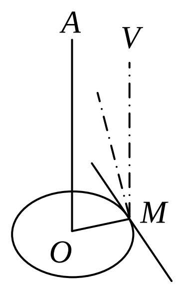

Figure 4.

Suppose that the curve *C* is a circumference (Fig. 4). Draw the vector *MV* that represents the velocity through the point *M* on the curve; its components are (*u*, *v*, *w*). The expression:

$$u\,dx + v\,dy + w\,dz$$

represents the product of the element of the curve MM' with the projection of its velocity onto the direction MM'. That product will represent the work that is done by a force that is numerically equal to the velocity when its point of application is displaced from *M* to M'. The integral *J* will then be equal to the work that is done by that force if the point *M* describes the entire circumference.

Decompose the vector *MV* into three other ones, one of which is parallel to the *OA* axis that is perpendicular to the plane of the circle, the second of which is directed along the tangent to the circle at M, and finally, the third of which is along the radius vector OM. Only the tangential component will do any work. Let R denote the radius of the circle; we represent that component by φR, where φ is an angular velocity. Set:

$$x = R \cos \omega, \quad y = R \sin \omega,$$

and it will become:

$$J = \int_0^{2\pi} \varphi R^2 d\omega .$$

Let φ_0 be the mean angular velocity along the circle, which is defined by the relation:

$$\int_0^{2\pi} \varphi \, d\omega = 2\pi \varphi_0 ,$$

so we will have:

$$J = 2\pi \varphi_0 R^2.$$

On the other hand, we know that in order to obtain the element of the integral J, one must multiply the element $d\omega$ of the area A by two times the projection $l\xi + m\eta + n\zeta$ of the vector (ξ, η, ζ) onto the normal. That projection is the normal component of the vorticity. If our circle is very small then we can take its area to be the element $d\omega$, and we will have:

$$J = 2\pi R^2 \cdot (l\xi + m\eta + n\zeta),$$

and consequently:

$$\varphi_0 = l\xi + m\eta + n\zeta .$$

φ_0 is the normal component of the vorticity.

10. Streamlines. – We have two vectors at each point: namely, the velocity, whose components are u, v, w, and the vorticity, whose components are ξ, η, ζ.

One can consider the lines whose tangent at each point represents the velocity, and whose differential equations are consequently given by:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} . \tag{12}$$

They are *streamlines*. Those lines are not necessarily the trajectories of molecules. That will be true only in the case of a permanent flow regime, for which the velocity is constant.

11. Vortex lines. – One can also consider the lines that are tangent to the vorticity vector at each of their points. Their differential equations are:

(13)
$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}.$$

They are the *vortex lines*. For example, suppose that the velocity is independent of x and parallel to the plane xy: $w = 0$, and the derivatives of u and v with respect to x are zero. From the defining equation (11) [no. **9**], one will then have:

$$\xi = \eta = 0,$$
$$\zeta = -\frac{du}{dy}.$$

The equations of the vortex lines will then become:

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{\zeta}$$

or

$$dx = dy = 0.$$

The vortex lines will then be lines that are parallel to Oz.

12. Vortex surfaces. – A *vortex surface* is a surface that is generated by vortex lines; in other words, it is a surface whose tangent plane at each point passes through the vorticity. The condition that expresses the idea that a surface $f(x, y, z) = 0$ is a vortex surface will then be:

(14)
$$\xi \frac{df}{dx} + \eta \frac{df}{dy} + \zeta \frac{df}{dz} = 0.$$

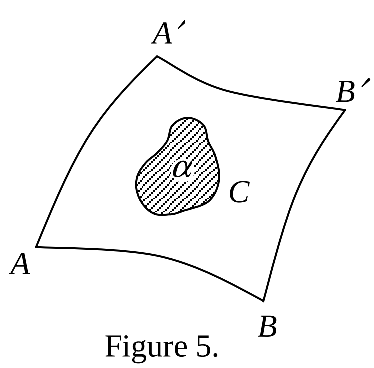

Figure 5.

Consider an arc of an arbitrary curve AB (Fig. 5), and draw vortex lines through the various points of that curve. The set of all of them will generate a vortex surface, which will be simply connected if the curve AB is not closed.

Trace a closed curve C on that surface that bounds a certain area α. The integral J, which is taken along C, is zero; indeed:

$$J_C = \int_\alpha 2\, d\omega\,(l\xi + m\eta + n\zeta).$$

In order for that integral to be zero for any area α, it is necessary that:

$$l\xi + m\eta + n\zeta = 0$$

for all elements of the surface; i.e., the normal component of the vorticity must be zero. The surface will then be a vortex surface.

14. Suppose that we have a certain number of molecules that occupy a vortex surface at a certain epoch $t = 0$. I say that these molecules will again occupy a vortex surface at an arbitrary epoch t.

Indeed, from the preceding, at the epoch $t = 0$, the integral J is zero for the surface that is occupied by these molecules. From Helmholtz's theorem, J will remain constant. That integral will then be zero at an arbitrary epoch, and the surface that is occupied by the molecules at that epoch will again be a vortex surface.

The intersection of two vortex surfaces will be a vortex line, and conversely, one can always pass two vortex surfaces through any vortex line.

Therefore, consider a string of molecules that occupies a vortex line C_0 at time 0. That line will take a certain position C at the time t; I say that C is also a vortex line.

Indeed, pass two vortex surfaces S_0 and S_0' through C_0. These two surfaces will become S and S' at the time t, which will have C for their intersection. From what we just saw, S and S' will remain vortex surfaces. Their intersection C will then be a vortex line again.

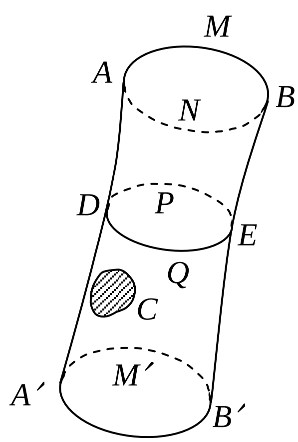

Figure 6.

15. Vortex tubes. – One calls the surface that is obtained by drawing vortex lines through the various points of a closed curve a *vortex tube* (Fig. 6). One can describe two kinds of closed curves on such a surface.

The curves C of the first kind bound a portion of an area on the surface by themselves.

The curves of the second kind, such as *DPEQ*, divide the surface into two regions, each of which is bounded by only the curve *DPEQ*. When one develops the surface, only the curves of the first kind will develop along closed curves.

16. Moment of a vortex tube. – When the integral J is taken along a closed curve of the first kind, it will be zero [no. **12**]. However, the argument no longer applies to the curves of the second kind, and the statement of theorem must be modified in the following manner:

When the integral J is taken along a closed curve of the second kind, it will have the same value for any such curve.

Indeed, let *DPEQ* and *D′P′E′Q′* be two closed contours: One can show that:

$$J_{DPEQ} = J_{D'P'E'Q'} .$$

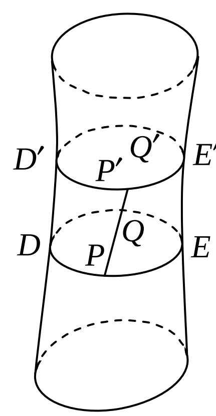

Figure 7.

Take a point P in the first curve and a point P' on the second one, and join them with PP' (Fig. 7). The contour:

$$PEQDP - PP' - P'D'Q'E'P' - P'P$$

can be regarded as a closed contour of the first kind. Therefore, J will be zero along that contour, or:

$$J_{PEQDP} + J_{PP'} + J_{P'D'Q'E'P'} + J_{P'P} = 0.$$

The second and fourth integrals vanish, since PP' is traversed twice in the opposite sense. What remains is then:

$$J_{PEQDP} + J_{P'D'Q'E'P'} = 0,$$

or

$$J_{PEQDP} = J_{P'E'Q'D'P'} .$$

The integral J, thus-determined, is called the *moment* of the vortex tube.

That moment will remain constant when the molecules that are situated on the tube are displaced, since we know that J remains constant.

17. Applications. Infinitely-thin vortex tubes. – Consider a vortex tube that is infinitely thin (Fig. 8). Draw a cross-section of that tube; it will be a closed curve of the second type. One can then calculate the moment of the tube by taking the integral J around that section. Now:

$$J = 2 \int d\omega \xi_n ,$$

as we have seen [no. **9**], where ξ_n is the normal component of the vorticity. In the present case:

$$J = 2\, d\omega \cdot \xi,$$

since we have only one element, to which the vorticity is normal, $d\omega$ is the cross-section of the tube (which is a cross-section that one can always assume to be parallel to the yz-plane), and ξ is the vorticity itself. One concludes that:

The product of the cross-section of an infinitely-thin vortex tube with the vorticity will be constant along the tube.

That product will also remain constant in time.

These two propositions result immediately from the fact that J is constant under the same conditions [no. **6**]; they apply to liquids and gases whenever there exists the function that we have called ψ.

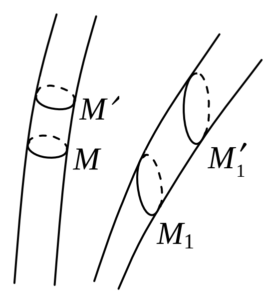

Figure 8.

18. Theorems that relate to only liquids. – Let a force tube be infinitely thin, and let two cross-sections of that tube be made infinitely-close to each other. The volume of the tube that is contained between these two sections (Fig. 8) can be equated to a cylinder that has one of them $d\omega$ for its base and a height of MM', where M and M' are the points of the two sections that are situated along the same vortex line. The volume of the cylinder will be:

$$MM' d\omega.$$

The molecules that constitute this volume will define another vortex tube at another epoch. The molecules that occupy the two cross-sections will occupy other sections that are infinitely close and will not necessarily be cross-sections. However, the volume that is found between them, which amounts to a cylinder, will be:

$$M_1 M_1' \, d\omega_1,$$

in which $M_1 M_1'$ is the length of a generator, and $d\omega_1$ is the cross-section.

If one is dealing with a liquid then its volume will remain constant, and:

$$MM' d\omega = M_1 M_1' \, d\omega_1.$$

Moreover, we have seen that $d\omega$ will vary for the opposite reason as the vorticity. The distance MM' between two molecules will thus vary in proportion to the vorticity. MM' thus represents the vorticity multiplied by a certain constant ε in magnitude, direction, and sense.

If one lets x, y, z be the coordinates of M then those of M' will be $x + \varepsilon\,\xi$, $y + \varepsilon\,\eta$, $z + \varepsilon\,\zeta$.

After an infinitely-small time dt, the coordinates of M will become:

$$x + u\,dt, \qquad y + v\,dt, \qquad z + w\,dt.$$

Those of M' will be:

$$x + \varepsilon\,\xi + u_1\,dt, \; \ldots, \qquad \text{etc.}$$

Now:

$$u_1 = u + \frac{\partial u}{\partial \xi}\varepsilon\xi + \frac{\partial u}{\partial \eta}\varepsilon\eta + \frac{\partial u}{\partial \zeta}\varepsilon\zeta\,.$$

The coordinates of M' will then become:

$$x + \varepsilon\,\xi\,u\,dt + \varepsilon\,dt\left(\xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z}\right), \qquad \text{etc.}$$

On the other hand, since the projections of MM' are equal to $\varepsilon\xi$, $\varepsilon\eta$, $\varepsilon\zeta$, those coordinates will be:

$$x + u\,dt + \varepsilon\left(\xi + \frac{d\xi}{dt}dt\right).$$

Upon equating these two expressions, one will have:

$$\frac{d\xi}{dt} = \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z}. \tag{15}$$

This relation is identical to the following one:

$$\frac{d\xi}{dt} = \xi\frac{\partial u}{\partial x} + \eta\frac{\partial v}{\partial x} + \zeta\frac{\partial w}{\partial x}. \tag{16}$$

Indeed, if we subtract corresponding sides of these equations then we will get:

$$0 = \eta\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \zeta\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right),$$

or

$$0 = -2\eta\zeta + 2\eta\zeta = 0.$$

The relation (15) and the other two that are deduced from it by permutation express Helmholtz's theorem, but in the case of liquids, exclusively.

19. Other proofs of Helmholtz's theorem.

1. *Helmholtz's proof.* – Helmholtz sought to obtain the equations in the latter form (15) that we just gave them by starting with Euler's equations.

We wrote [no. **4**] the Lagrange equations:

$$\frac{du}{dt} = \frac{\partial \psi}{\partial x}, \qquad \text{etc.}$$

in a form that contains both the Lagrange variables and those of Euler. In order to leave only the latter ones, we make the transformation:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}, \text{ etc.,}$$

and we will obtain:

$$(17) \qquad \begin{cases} (1) \quad \dfrac{\partial u}{\partial t} = \dfrac{\partial \psi}{\partial x} - u\dfrac{\partial u}{\partial x} - v\dfrac{\partial u}{\partial y} - w\dfrac{\partial u}{\partial z}, \\[2ex] (2) \quad \dfrac{\partial v}{\partial t} = \dfrac{\partial \psi}{\partial y} - u\dfrac{\partial v}{\partial x} - v\dfrac{\partial v}{\partial y} - w\dfrac{\partial v}{\partial z}, \\[2ex] (3) \quad \dfrac{\partial w}{\partial t} = \dfrac{\partial \psi}{\partial z} - u\dfrac{\partial w}{\partial x} - v\dfrac{\partial w}{\partial y} - w\dfrac{\partial w}{\partial z}. \end{cases}$$

Differentiate equation (3) with respect to y and equation (2) with respect to z, and subtract them; upon recalling the definitions of ξ, η, ζ [no. **9**], one will get:

$$(18) \qquad 2\,\frac{\partial \xi}{\partial t} = -\,2u\frac{\partial \xi}{\partial x} - 2v\frac{\partial \xi}{\partial y} - 2w\frac{\partial \xi}{\partial z}$$
$$-\,\frac{\partial u}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial y} - \frac{\partial w}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial z}.$$

On the other hand:

$$\frac{d\xi}{dt} = \frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} + w\frac{\partial \xi}{\partial z},$$

and the equation of continuity for liquids will reduce to:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

Upon taking this relation into account, one easily puts equation (4) into the form:

$$\frac{d\xi}{dt}=\xi\frac{\partial u}{\partial x}+\eta\frac{\partial v}{\partial y}+\zeta\frac{\partial w}{\partial z}.$$

In fact, we recover equation (16). However, this proof of Helmholtz's theorem applies to only liquids.

20. Kirchhoff's proof. – Kirchhoff took his starting point to be the Lagrange equations:

$$\frac{du}{dt}=\frac{\partial\psi}{\partial x}, \qquad \frac{dv}{dt}=\frac{\partial\psi}{\partial y}, \qquad \frac{dw}{dt}=\frac{\partial\psi}{\partial z},$$

when they were transformed in such a manner that they depended upon only the Lagrange variables.

We have:

$$\frac{d\psi}{dx_0}=\frac{\partial\psi}{\partial x}\frac{dx}{dx_0}+\frac{\partial\psi}{\partial y}\frac{dy}{dx_0}+\frac{\partial\psi}{\partial z}\frac{dz}{dx_0}.$$

Multiply the first Lagrange equation by dx / dx_0, the second one by dy / dx_0, the third one by dz / dx_0, and add them. We will get:

$$\frac{du}{dt}\frac{dx}{dx_0}+\frac{dv}{dt}\frac{dy}{dx_0}+\frac{dw}{dt}\frac{dz}{dx_0}=\frac{d\psi}{dx_0}$$

and two other analogous equations that are obtained by symmetry.

21. One can, moreover, give these equations a more general form by substituting three other variables a, b, c for x_0, y_0, z_0, which are defined by three arbitrary relations:

$$\begin{aligned} x_0 &= \varphi_0(a, b, c), \\ y_0 &= \varphi_1(a, b, c), \\ z_0 &= \varphi_2(a, b, c), \end{aligned}$$

in which a, b, c do not depend upon t. The derivatives with respect to t will be the same in the two systems of variables.

We perform the same operation as before, and find:

$$\frac{du}{dt}\frac{dx}{da}+\frac{dv}{dt}\frac{dy}{da}+\frac{dw}{dt}\frac{dz}{da}=\frac{d\psi}{da},$$

and two other equations that are obtained by changing a into b and then into c.

We will finally have the system:

$$(19)\qquad \begin{cases} (1) & \sum\frac{du}{dt}\frac{dx}{da}=\frac{d\psi}{da}, \\ (2) & \sum\frac{du}{dt}\frac{dx}{db}=\frac{d\psi}{db}, \\ (3) & \sum\frac{du}{dt}\frac{dx}{dc}=\frac{d\psi}{dc}. \end{cases}$$

Differentiate (1) with respect to b, (2) with respect to a, and then subtract them:

$$\sum\left(\frac{d^2u}{dt\,db}\frac{dx}{da}-\frac{d^2u}{dt\,da}\frac{dx}{db}\right)=0,$$

or, as is easy to verify:

$$\frac{d}{dt}\sum\left(\frac{du}{db}\frac{dx}{da}-\frac{du}{da}\frac{dx}{db}\right)=0,$$

and finally:

$$(20)\qquad \begin{aligned} &\frac{du}{db}\frac{dx}{da}-\frac{du}{da}\frac{dx}{db} \\ +\ &\frac{dv}{db}\frac{dy}{da}-\frac{dv}{da}\frac{dy}{db} \\ +\ &\frac{dw}{db}\frac{dz}{da}-\frac{dw}{da}\frac{dz}{db}=\text{const.} \end{aligned}$$

One obtains two other analogous equations by permuting a, b, c circularly and changing the value of the constant.

21. (cont.). As we shall show, these Kirchhoff equations are equivalent to the ones that we gave at the beginning:

$$J=\int u\,dx+v\,dy+w\,dz=\text{const.}$$

Indeed, consider a point M whose coordinates are (x, y, z) in the Euler system or a, b, c, t in that of Kirchhoff. x, y, z vary with t, but a, b, c are independent of t and depend upon only x_0, y_0, z_0. The point M belongs to a certain curve C. I choose a, b, c in such a manner that $c = 0$ for all points of that curve. If that condition is satisfied at the instant $t = 0$ then it will still be true at any other epoch.

If we regard a, b as the rectangular coordinates of a point in a plane, for the moment, then each point M of C will correspond to a point M' of the plane, and when M describes the curve C, M' will describe a certain curve C' that will be closed if C is closed, except that the curve C' is fixed, while the curve C is mobile. Take the integral:

$$\int_C u\,dx$$

around the curve C:

$$\int_C u\,dx = \int_{C'} \left(u\frac{dx}{da}da + u\frac{dx}{db}db \right),$$

and the second integral around C'. Transform that integral by the Stokes formula [no. **8**]:

$$\int_{C'} \left(u\frac{dx}{da}da + u\frac{dx}{db}db \right) = \iint \left[\frac{d}{da}\left(u\frac{dx}{db} \right) - \frac{d}{db}\left(u\frac{dx}{da} \right) \right],$$

in which $\iint$ is taken over the area A' that is bounded by the curve C'.

Perform the indicated differentiations, and after reductions, one will have:

$$\int_C u\,dx = \iint \left(\frac{du}{da}\frac{dx}{db} - \frac{du}{db}\frac{dx}{da} \right) da\,db.$$

Upon performing the same transformation on $\int_C v\,dy$ and $\int_C w\,dz$ and then adding, we will find that:

(21) $$J = \int_C u\,dx + v\,dy + w\,dz = \int_{A'} \left(\frac{du}{da}\frac{dx}{db} - \frac{du}{db}\frac{dx}{da} \right) da\,db.$$

The area A' does not vary, since C' is fixed; the Σ that is placed under the $\int$ sign is constant by virtue of the assertion of Kirchhoff. Therefore, J = const.

CHAPTER II

CONSEQUENCES OF HELMHOLTZ'S THEOREM

22. Case of permanent motion. – A motion is *permanent* when the functions that we have defined – viz., u, v, w, ψ – do not depend upon t, but only upon the Euler variables x, y, z. Consequently, in the case of permanent motions:

$$\frac{\partial u}{\partial t} = 0, \;\ldots,\; \frac{\partial \psi}{\partial t} = 0, \text{ etc.},$$

and [no. **1**]:

$$\frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},$$
$$\frac{d\psi}{dt} = u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y} + w\frac{\partial \psi}{\partial z},$$

which is a relation that applies to an arbitrary function, moreover.

Under these conditions, one can deduce a certain number of consequences of the fundamental theorem of Helmholtz.

23. Theorem. – If the motion is permanent then there will exist an infinitude of surfaces on which one can trace an infinitude of streamlines and an infinitude of vortex lines.

Here is the significance of that statement:

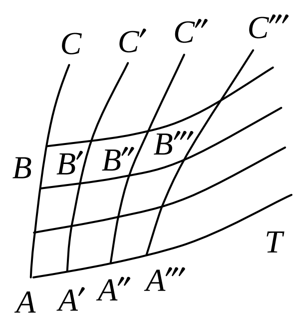

Figure 9.

One can draw a streamline AC and a vortex line AT through a point A (Fig. 9). If we draw vortex lines through the various points of AC then they will generate a certain surface. Similarly, if we draw streamlines through the various points of AT then they will generate another surface. The theorem implies that these two surfaces are identical.

Furthermore, if we draw streamlines AC, $A'C'$, $A''C''$, $A'''C'''$ through the points A, A', A'', A''' of AT then a vortex line that is drawn through an arbitrary point B of AC will meet AC, $A'C'$, …, etc.

That proposition is almost obvious. Indeed, when the motion is permanent, the streamlines will be the trajectories of the fluid molecules. Now, consider the molecules that are at A, A', A'', A''' at the epoch $t = 0$; they will be taken to B, B', B'', B''' at the epoch t. Since the vorticity is preserved by virtue of Helmholtz's theorem, the molecules B, B', B'', B''' will again be on the same streamline.

24. General equation of these surfaces. – We have set [no. **4**]:

$$\psi = V - \int \frac{dp}{\rho},$$

$$T = \tfrac{1}{2}(u^2 + v^2 + w^2).$$

I say that the equation for these surfaces that we just defined is:

$$\psi - T = \text{const.}$$

In order to prove that, it is sufficient to show that $\psi - T$ is constant along the streamlines, on the one hand, and along the vortex lines, on the other.

1. *Along the streamlines.* – These lines are the trajectories of the molecules. We follow a molecule along its motion. With the Lagrange equations, only t is variable, so:

$$d\psi = \frac{d\psi}{dt}dt,$$

$$dT = \frac{dT}{dt}dt,$$

$$dT = u\,du + v\,dv + w\,dw = u\frac{du}{dt} + v\frac{dv}{dt} + w\frac{dw}{dt},$$

$$\frac{dT}{dt} = u\,\frac{\partial\psi}{\partial x} + v\frac{\partial\psi}{\partial y} + w\frac{\partial\psi}{\partial z} = u\frac{du}{dt} + v\frac{dv}{dt} + w\frac{dw}{dt},$$

so, from the Lagrange equations [no. **4**]:

$$\frac{du}{dt} = \frac{\partial\psi}{\partial x}, \quad \text{etc.}$$

Therefore:

$$\frac{d\psi}{dt} = \frac{dT}{dt} \qquad \text{or} \qquad d\psi - dT = 0.$$

(1) $$\psi - T = \text{const.}$$

2. *Along the vortex lines*. – These lines have the equations:

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} = d\alpha,$$

or:

$$dx = \xi\, d\alpha, \qquad dy = \eta\, d\alpha, \qquad dz = \zeta\, d\alpha.$$

I say that:

$$\frac{d\psi}{d\alpha} = \frac{dT}{d\alpha}.$$

Indeed:

$$\frac{d\psi}{d\alpha} = \frac{\partial \psi}{\partial x}\xi + \frac{\partial \psi}{\partial y}\eta + \frac{\partial \psi}{\partial z}\zeta,$$

$$\frac{\partial \psi}{\partial x} = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}, \qquad \text{etc.}$$

Substitute:

$$\begin{aligned}\frac{d\psi}{d\alpha} = \;& u\left(\xi\frac{\partial u}{\partial x} + \eta\frac{\partial v}{\partial x} + \zeta\frac{\partial w}{\partial x}\right)\\ &+ v\left(\xi\frac{\partial u}{\partial y} + \eta\frac{\partial v}{\partial y} + \zeta\frac{\partial w}{\partial y}\right)\\ &+ w\left(\xi\frac{\partial u}{\partial z} + \eta\frac{\partial v}{\partial z} + \zeta\frac{\partial w}{\partial z}\right).\end{aligned}$$

On the other hand:

$$\begin{aligned}\frac{dT}{d\alpha} &= u\frac{du}{d\alpha} + v\frac{dv}{d\alpha} + w\frac{dw}{d\alpha}\\ &= u\left(\xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z}\right)\\ &\quad + v\left(\xi\frac{\partial v}{\partial x} + \eta\frac{\partial v}{\partial y} + \zeta\frac{\partial v}{\partial z}\right)\\ &\quad + w\left(\xi\frac{\partial w}{\partial x} + \eta\frac{\partial w}{\partial y} + \zeta\frac{\partial w}{\partial z}\right).\end{aligned}$$

However, we have seen [no. **18**] that:

$$\xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} = \xi\frac{\partial u}{\partial x} + \eta\frac{\partial v}{\partial x} + \zeta\frac{\partial w}{\partial x},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \text{ etc.}$$

Consequently:

$$\frac{d\psi}{d\alpha} = \frac{dT}{d\alpha} \qquad \text{and} \qquad \psi - T = \text{const.}$$

25. Bernoulli's theorem. – The direction of the vorticity will be indeterminate in the case where the vorticity is zero – i.e., when there exists a velocity function:

$$\xi = \eta = \zeta = 0.$$

An arbitrary line can then be regarded as a vortex line, and $\psi - T$ will be constant in all of space; that is *Bernoulli's theorem*.

26. Determination of the velocity as a function of the vorticity. – We propose to determine the components of the velocity u, v, w when we are given the components ξ, η, ζ of the vorticity.

Since vorticity is conserved, if we can solve that problem then we will know the velocity with which it displaces, and consequently, its direction and magnitude at each epoch $t + dt$ that is infinitely different from the first one, and then at an arbitrary epoch, by integration.

We first remark that this problem is indeterminate, in general, except in two cases: viz., when one is dealing with a homogeneous liquid that occupies an indefinite space or a homogeneous liquid that fills the vessel that contains it completely.

27. Simply-connected and multiply-connected volumes. – Before beginning the study of the question that was posed, it is indispensible to define what we call "simply-connected" and "multiply-connected" volumes. These are notions that we will have to make constant use of.

A volume with simple connectivity – or a *simply-connected* volume – is a volume in which no holes are present: e.g., the sphere, ellipsoid, and cube.

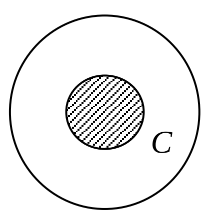

Figure 10.

Any closed curve that is traced inside of such a volume can be reduced to a point by deforming it in a continuous manner without leaving the volume (Fig. 10), so it will then sweep out a certain area A that is bounded by the curve exclusively.

We then agree to say that a volume is simply-connected when any closed curve inside of that volume can be regarded as the contour of a planar area that is situated completely inside of the volume. If one adopts that definition then the volume that is contained between two concentric spheres will again be simply-connected.

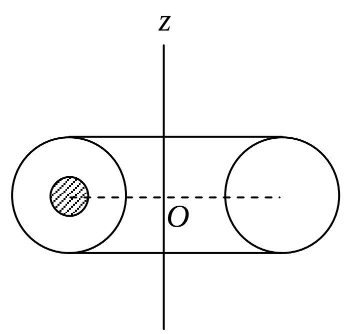

Figure 11.

28. A *multiply-connected volume* is a volume in which one or more holes are present. The number of holes defines the *order of multiplicity;* e.g., a torus (Fig. 11).

One can trace closed curves of two kinds in multiply-connected volumes: As we defined them in the preceding paragraph, the curves of the first kind can be reduced to a point without leaving the volume. For the torus, these will be the circumferences that are traced in a meridian plane and concentric to one of the meridian circumferences.

The curves of the second kind cannot be reduced to a point by a continuous deformation without leaving the volume. For example, for a torus, they would be the circumferences that are traced in the plane perpendicular to the axis and have their centers along that axis.

29. Having said that, suppose that the vorticity is zero:

$$\xi = \eta = \zeta = 0$$

and consider the integral:

$$J = \int (u\,dx + v\,dy + w\,dz)\,.$$

That integral will be zero when it is taken along a closed curve of the first kind. Indeed, [no. **9**]:

$$J = \int 2\xi_n\,d\omega\,,$$

where ξ_n is the normal component of the vorticity, and $d\omega$ is an area element that is bounded by the curve. Since, by hypothesis:

$$\xi_n = 0, \qquad \text{one will have} \qquad J = 0.$$

That proposition is no longer true for curves of the second kind. Indeed, suppose that the volume is that of a torus, and that:

$$u = \frac{y}{x^2 + y^2}, \qquad v = -\frac{x}{x^2 + y^2}, \qquad w = 0.$$

The streamlines are the circles that have centers situated on the axis and are traced in planes that are perpendicular to that axis:

$$u\,dx + v\,dy = \frac{y\,dx - x\,dy}{x^2 + y^2} = d\,\arctan\frac{y}{x}.$$

The velocity function arctan y / x is not uniform, but it is it susceptible to an infinitude of determinations that differ by π. If we take the integral J along a curve of the second kind then that integral will not be equal to 0, but to π or a multiple of π, so one will arrive at another determination of the function upon returning to the starting point.

30. Cuts. – If a volume is multiply-connected then it will be possible to render it simply connected by making cuts in it. In particular, if the volume is doubly-connected then it will suffice to make only one cut. For example, a torus can be rendered simply connected by cutting along one of its meridian circles.

The curves that do not cross the cut will be of the first kind; the curves that do cross the cut will be of the second kind.

The velocity function will remain uniform when one does not cross the cut, and the integral J will be zero when one takes it along a curve that does not cross the cut.

On the contrary, consider two infinitely-close points on one and the other curve that are infinitely close to the cut, but situated on one side and the other of the cut. The velocity function will present a discontinuity between these points. The difference in the values that it takes at the two points will be finite and equal to the value of the integral J, when taken along a curve of the second kind that links the two points.

31. Theorem. – That difference is constant; in other words, the value of the integral J will be the same for all of the integration curves that cross the cut once.

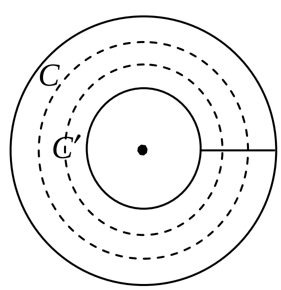

Figure 12.

Indeed, suppose that the curve C is deformed in a continuous manner without leaving the volume until it becomes, for example C' (Fig. 12). During that transformation, the C will sweep out a certain area that is situated completely inside of the volume.

The integral J will be zero when it is taken along the complete contour of that area CC'.

The two curves C and C' are traversed in the opposite sense, so:

$$J_C - J_{C'} = 0$$

or

$$J_C = J_{C'} .$$

The value of the discontinuity in the velocity function φ on one side and the other of the cut is then the same at all points of that cut; let A be that value.

If the curve of the second kind along which one integrates crosses the cut two times then the discontinuity of the function φ will be $2A$, etc. In a general manner, if the integration contour crosses the cut n times in the direct sense and n' times in the opposite sense then the value of the integral will be $(n - n')\,A$.

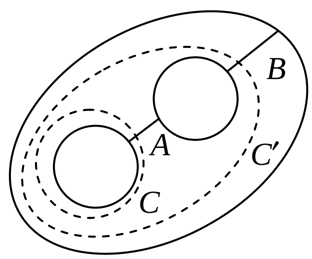

Figure 13.

32. If the volume is triply-connected (Fig. 13) then one must make two cuts in order to render it simply-connected. The velocity function φ will then be determined completely. However, it will present a discontinuity across each of these cuts. That discontinuity will have a constant value A along the first cut, and a constant value B along the second one that is generally different from A.

If the integration contour meets the first cut twice then:

$$J_C = A.$$

If the contour C' meets the second cut twice without crossing the first one then:

$$J_{C'} = B.$$

Finally, if the integration contour crosses the first cut, in a general manner, n times in the direct sense and n' times in the opposite sense, and crosses the second cut p times in the direct sense and p' times in the opposite sense then one will have:

$$J = (n - n')\,A + (p - p')\,B.$$

CHAPTER III

DETERMINING THE VELOCITY COMPONENTS AS FUNCTIONS OF THE VORTICITY COMPONENTS. SPECIAL CASE OF LIQUIDS

33. We established [no. **3**] the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{1}$$

in the general case, but when one is treating a liquid, the density ρ will be constant, and that equation will reduce to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2}$$

Suppose that the vorticity is zero everywhere; in other words, that the expression:

$$u\,dx + v\,dy + w\,dz$$

is an exact differential $d\varphi$; φ will be the velocity function.

The continuity equations will then be written:

$$\Delta\varphi = 0,$$

upon setting:

$$\Delta\varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2},$$

as usual.

34. THEOREM. – There are two cases in which these conditions cannot be satisfied without the liquid being at rest:

1. When the liquid that fills the indefinite space is found to be at rest at infinity.
2. When the liquid fills a closed, simply-connected, solid vessel completely.

We shall prove these two propositions by appealing to Green's theorem, which is expressed by the equation:

$$\int \varphi \frac{d\varphi}{dn} d\omega = \int \varphi \Delta\varphi \, d\tau + \int \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \right] d\tau. \tag{3}$$

The integral on the left-hand side is taken over all elements $d\omega$ of a closed surface; the other two are taken over all of the elements of the volume that is bounded by the surface. $d\varphi / dn$ is the derivative of φ when it is estimated along the normal to the surface at the center of gravity of the element $d\omega$. Here, it is the projection of the velocity onto that normal. The function φ must be uniform in the interior to the volume τ.

35. Liquid that occupies an indefinite space. – We apply Green's theorem to a sphere of very large radius.

Since we suppose that the liquid is at rest at infinity, $d\varphi / dn$ will be zero on all of the surface of that sphere; the integral on the left-hand side will be zero. The first integral on the right-hand side will also be zero, since $\Delta\varphi = 0$; consequently, the same thing will be true for the latter:

$$\int \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 + \left(\frac{\partial \varphi}{\partial z} \right)^2 \right] d\tau = 0.$$

The differential element is essentially positive, since it is a sum of squares; that equality will then imply the following one:

$$\frac{\partial \varphi}{\partial x} = 0, \qquad \frac{\partial \varphi}{\partial y} = 0, \qquad \frac{\partial \varphi}{\partial z} = 0,$$

or

$$u = 0, \quad v = 0, \quad w = 0.$$

The velocity will then be zero.

36. Liquid that fills a fixed vessel completely.

1. *Simply-connected vessel.* Once more, apply Green's theorem, upon choosing the wall of the vessel to be the integration surface and the volume of the vessel to be the volume. Since the wall is immobile, the velocity of the liquid at a point of that wall can only be tangential, and the normal components $d\varphi / dn$ will be zero; therefore:

$$\int \varphi \frac{d\varphi}{dn} d\omega = 0,$$

since:

$$\int \varphi \Delta \varphi \, d\tau = 0,$$

and consequently:

$$\int \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 + \left(\frac{\partial \varphi}{\partial z} \right)^2 \right] d\tau = 0.$$

One then deduces, as above, that:

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial z} = 0.$$

The velocity is then zero at all points.

37. The preceding argument is valid only for a simply-connected volume. If the vessel is multiply-connected then the function φ will no longer be uniform, and Green's theorem will cease to be applicable.

38.

2. *Doubly-connected vessel.* – Suppose that the vessel is doubly-connected, and that it has the form of a torus, for example. Make a cut along a meridian circle; the closed contours of the second kind will meet that cut. The function of the velocity φ will be uniform as long as one does not cross that cut; however, the function φ will present a discontinuity from one side to the other that is constant over all of the cut surface.

I say that if one is given that constant – in other words, the value of J along a curve of the second kind – then the motion of the liquid will be determined entirely.

Indeed, suppose that there are two possible solutions, and let φ' and φ'' be the two velocity functions that correspond to these solutions; let φ_1' and φ_2' be the values of φ' on both sides of the cut, and let φ_1'' and φ_2'' be the values of φ''. We will have:

$$\varphi_1' - \varphi_2' = J_0,$$
$$\varphi_1'' - \varphi_2'' = J_0,$$

in which J_0 is the given constant; φ' and φ'' are uniform as long as one does not cross the cut, moreover. If one subtracts the respective sides of the two equations above then one will get:

$$\varphi_1' - \varphi_1'' = \varphi_2' - \varphi_2''.$$

The function φ' and φ'' will then have the same value on both sides of the cut, so it will be uniform and continuous in the whole volume, and one can apply Green's theorem; one then deduces that:

$$\frac{\partial(\varphi' - \varphi'')}{\partial x} = 0, \qquad \text{etc.}$$

or

$$\frac{\partial \varphi'}{\partial x} = \frac{\partial \varphi''}{\partial x}, \quad \frac{\partial \varphi'}{\partial y} = \frac{\partial \varphi''}{\partial y}, \quad \frac{\partial \varphi'}{\partial z} = \frac{\partial \varphi''}{\partial z}.$$

The components of the velocity are the same in both cases; there is then only one possible motion.

39.

3. *Triply-connected vessel.* In this case, one must make two cuts in order to render the volume simply-connected.

The motion will be given when one is given:

$$\varphi_1 - \varphi_2 = J_0, \quad \varphi_3 - \varphi_4 = J_1,$$

in which $\varphi_1 - \varphi_2$ is the difference between the values of φ on the two sides of the first cut, and $\varphi_3 - \varphi_4$ is that difference relative to the second cut.

As before, upon assuming that there exist two solutions φ' and φ'', one will find that:

$$\varphi_1' - \varphi_2' = \varphi_1'' - \varphi_2'',$$
$$\varphi_3' - \varphi_4' = \varphi_3'' - \varphi_4''.$$

Since the function $\varphi' - \varphi''$ is uniform and continuous inside of the volume, one will deduce from Green's theorem that:

$$\varphi' - \varphi'' = \text{const.}$$

or

$$\frac{\partial \varphi'}{\partial x} = \frac{\partial \varphi''}{\partial x}, \quad \text{etc.}$$

40. Non-zero vorticity. – In the case for which the vorticity is non-zero, the Helmholtz problem will be determinate, and if one solution exists then only one will exist.

Indeed, what is the nature of that problem? It amounts to determining u, v, w from the equations:

$$(4) \qquad \begin{cases} 2\xi = \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}, \\ 2\eta = \dfrac{\partial u}{\partial z} - \dfrac{\partial v}{\partial x}, \\ 2\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \end{cases}$$

when one is given the components ξ, η, ζ of the vorticity.

Suppose that we have found two solutions:

$$\begin{aligned} u &= u', & u &= u'', \\ v &= v', & v &= v'', \\ w &= w', & w &= w''. \end{aligned}$$

We will have:

$$2\xi = \frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z},$$
$$2\xi = \frac{\partial w''}{\partial y} - \frac{\partial v''}{\partial z},$$

so

$$0 = \frac{\partial(w' - w'')}{\partial y} - \frac{\partial(v' - v'')}{\partial z},$$

along with two other analogous equations. These three equations express the idea that the sum:

$$(u' - u'')\,dx + (v' - v'')\,dy + (w' - w'')\,dz$$

is an exact differential $d\varphi$. One can thus set:

$$u' - u'' = \frac{\partial \varphi}{\partial x}, \; \ldots, \qquad \text{etc.}$$

Write down that the continuity equation is satisfied for $u = u'$, …, etc., and for $u = u''$, …, etc., and that will give:

$$\sum \frac{\partial u'}{\partial x} = 0, \qquad \sum \frac{\partial u''}{\partial x} = 0\,.$$

Thus, upon subtracting corresponding sides:

$$\sum \frac{\partial(u' - u'')}{\partial x} = 0;$$

i.e.:

$$\Delta\varphi = 0.$$

If the vessel is filled completely then the normal component of the velocity must be zero at each point of the wall. If one lets l, m, p be the direction cosines of the normal at a point of the wall then the normal component of the velocity will be:

$$lu + mv + pw,$$

and if that component is zero:

$$lu' + mv' + pw' = 0,$$
$$lu'' + mv'' + pw'' = 0.$$

Consequently:

$$l\frac{\partial \varphi}{\partial x} + m\frac{\partial \varphi}{\partial y} + p\frac{\partial \varphi}{\partial z} = \frac{\partial \varphi}{\partial n} = 0.$$

If the vessel is simply-connected then we will find, upon reasoning as we did above [no. **36**], that:

$$\varphi = \text{const.},$$

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial z} = 0,$$

or:

$$u' = u'', \quad v' = v'', \quad w' = w''.$$

The problem thus admits only one solution.

41. Suppose that the vessel is multiply-connected, so the preceding argument will no longer be legitimate; one must introduce one or more conditions, as well.

For example, suppose that the volume is doubly-connected. Make a cut, and let J_0 be the value of the integral that is taken along a closed curve that crosses the cut once.

The problem will be determinate when one is given the value of J_0, in addition to the values of ξ, η, ζ.

Indeed, suppose that there can exist two solutions (u', v', w') and (u'', v'', w''), and we prove, as before [no. **38**], that:

$$u' - u'' = \frac{\partial\varphi}{\partial x}, \qquad v' - v'' = \frac{\partial\varphi}{\partial y}, \qquad w' - w'' = \frac{\partial\varphi}{\partial z},$$

$$\frac{\partial\varphi}{\partial n} = 0, \qquad \Delta\varphi = 0.$$

On the other hand:

$$\int u'dx + v'dy + w'dz = J_0,$$
$$\int u''dx + v''dy + w''dz = J_0,$$

and upon subtracting corresponding sides:

$$\int \left(\frac{\partial\varphi}{\partial x} dx + \frac{\partial\varphi}{\partial y} dy + \frac{\partial\varphi}{\partial z} dz \right) = 0$$

or

$$\int d\varphi = 0.$$

The function φ thus remains uniform, even when one suppresses the cut; it must then reduce to a constant. Consequently:

$$\frac{\partial\varphi}{\partial x} = u' - u'' = 0, \qquad \text{or} \qquad u' = u'', \qquad \text{etc.}$$

42. If the volume is triply-connected then one must make two cuts. In order to determine the problem, it is necessary to give the value J_0 of the integral J along a closed curve that meets only the first cut once and its value J_1 along a closed curve that meets only the second cut once, in addition to the values of ξ, η, ζ.

43. Analogy between Helmholtz's hydrodynamical equations and Maxwell's electrodynamical equations.

1. Suppose that the liquid considered occupies an indefinite space and is at rest.

In that case, the Helmholtz equations present the same form as the Maxwell system of equations that relate to a magnetic field.

Maxwell called the components of the current u, v, w, which signified that a surface element $d\omega$ that is normal to Ox is crossed by a quantity of electricity $u\,d\omega\,dt$, …, etc., during the time interval dt. α, β, γ are the components of the magnetic field that is produced by the current, and a, b, c are the components of the magnetic induction, which will reduce to α, β, γ when there is neither a permanent magnet nor soft iron present. The Maxwell equation:

$$\frac{\partial a}{\partial x}+\frac{\partial b}{\partial y}+\frac{\partial c}{\partial z}=0$$

will then reduce to the following one:

$$\frac{\partial \alpha}{\partial x}+\frac{\partial \beta}{\partial y}+\frac{\partial \gamma}{\partial z}=0.$$

If we compare the two systems then we will find that:

Maxwell ([1])	**Helmholtz**
$4\pi u=\frac{\partial \gamma}{\partial y}-\frac{\partial \beta}{\partial z}$	$2\xi=\frac{dw}{dy}-\frac{dv}{dz}$
$4\pi v=\frac{\partial \alpha}{\partial z}-\frac{\partial \gamma}{\partial x}$	$2\eta=\frac{du}{dz}-\frac{dw}{dx}$
$4\pi w=\frac{\partial \beta}{\partial x}-\frac{\partial \alpha}{\partial y}$	$2\zeta=\frac{dv}{dx}-\frac{du}{dy}$
$\frac{\partial \alpha}{\partial x}+\frac{\partial \beta}{\partial y}+\frac{\partial \gamma}{\partial z}=0.$	$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$

([1]) See *Électricité et optique*, I, § 102 and § 118.

One sees that in order to pass from the Helmholtz equations to those of Maxwell, it will suffice to change ζ, η, ξ, u, v, w into $2\pi u$, $2\pi v$, $2\pi w$, α, β, γ.

We have proved that if such a system admits one solution then that solution will be unique.

Now, suppose that we know the magnitude, direction, and sense of the vorticity vector. Divide that vector by 2π and assume that this vector, thus reduced, represents an electric current. The system of currents thus-obtained will produce a magnetic field, and the vector that represents that field will represent the velocity of the fluid molecule at the same point. The lines of magnetic force will be the streamlines of the hydrodynamical current.

44. Case in which only one vortex tube exists. – Suppose that there exists just one closed vortex tube that has an infinitely-small section, that the vorticity at each point has a very large value in order that the tube should be finite, and finally, that the vorticity is zero everywhere outside of the tube.

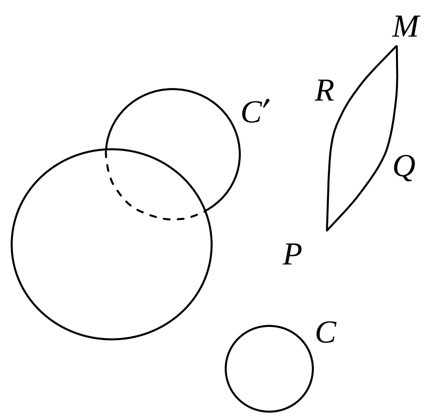

Figure 14.

By reason of the last hypothesis, there will be a velocity function φ outside of the tube. However, the volume external to the tube is doubly-connected, since one can trace two kinds of closed curve in it: The ones, such as C (Fig. 14), that do not enlace the tube, and the other ones C' that do enlace it, in the manner of the rings in a chain. When the integral J is taken along the former curves, it will be zero. However, when it is taken along the tubes of the second kind, it will no longer be equal to zero, but to the moment of the tube.

Since the section of the tube is infinitely small, that tube can be equated to a curve, which I will assume to be closed, and which I will call the *axis* of the tube. Indeed, we can pass a certain surface through the axis of the vortex tube and take the cut to be the area that is bounded by the axis of the tube on that surface. None of the closed curves that cross that area will be of the first kind; the ones that do cross it will be of the second kind [no. **30**].

45. Suppose that the function φ is zero at infinity, which is permissible, since that function is given only by its derivatives and is, consequently, determined only up to a constant. In order to define the value of φ at a given point P, we take the integral J along a curve that joins it to a point that is infinitely distant from the point P considered without crossing the cut. That definition will obviously be sufficient only if the function φ is uniform, and consequently, if the value thus-calculated does not depend upon the curve that is followed in order to take the point P to infinity. Now, that condition is fulfilled. Indeed, consider two arbitrary paths MQP, MRP that join a point M that is very distant to the point P. The integral J is zero along the closed contour $MQPRM$, which does not cross the cut; therefore:

$$\int_{MQP} d\varphi + \int_{PRM} d\varphi = 0$$

or

$$\int_{MQP} d\varphi = \int_{MRP} d\varphi = \varphi_P .$$

The value of φ that is calculated at a point M depends upon the surface that is chosen for the cut; the same thing is true for two cuts that do not include the point M, but things will be different when the point M is found between the two surfaces that are successively chosen for cuts.

We must now determine that value of φ. Let μ be the moment of the vortex tube:

$$\mu = 2\sqrt{\xi^2 + \eta^2 + \zeta^2}\, d\omega,$$

in which $d\omega$ is a cross-section of the tube, which we have assumed to be infinitely small. If we replace the vorticity (ξ, η, ζ) with the current (u, v, w) then each component of the current will be equal to the corresponding component of the vorticity divided by 2π. The intensity of the current, when measured tangentially to the vortex tube, will be:

$$i = \sqrt{u^2 + v^2 + w^2}\, d\omega .$$

Consequently:

$$\mu = 4\pi i .$$

If one determines the value of i from that equality then the magnetic force and the velocity of a fluid molecule at a point will be represented by the same vector. The velocity function will be the magnetic potential of the current. As one knows, that potential will have:

$$i\,\sigma$$

for its expression at a given point, where σ is the solid angle in which one sees the contour of the current from that point. (Cf., *Électricité et optique*, t. I, page 107.) As a result:

$$\varphi = i\,\sigma = \frac{\mu\sigma}{4\pi} .$$

σ is the solid angle in which one sees the axis of the vortex tube from the point considered.

If there are several vortex tubes then the function φ that relates to the system will be the sum of the functions φ_1, φ_2, …, etc., that relate to each of them, and the same relation will persist.

46. Case of a rectilinear and indefinite vortex tube. – Let a vortex tube be rectilinear.

Apply the preceding rule. We must replace the tube with an indefinite, rectilinear current that possesses an intensity:

$$i = \frac{\mu}{4\pi}.$$

From the law of Biot and Savart, the action of this current on a magnetic pole M is perpendicular to the plane MPQ and inversely proportional to the distance r from the point M to the line PQ. The velocity of a fluid molecule M will then be perpendicular to the plane MPQ, and will vary inversely with its distance from the axis of the vortex tube.

47. That result can be obtained directly, moreover, without the intermediary of the electrodynamical comparison.

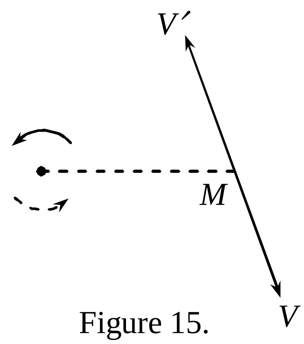

Figure 15.

By reason of symmetry, the velocity must be found in the plane R that is drawn through M perpendicularly to PQ. On the other hand, if we consider the plane PMQ then that will not be a symmetry plane, properly speaking. Indeed, take the plane of the figure to be the plane R (Fig. 15). The line $P\varphi$ projects onto that plane at N. MN is the trace of the plane PMQ. Suppose that the vorticity has the sense that is indicated by the arrow, and that the velocity is directed along MV.

Take the image of the figure with respect to MN. The moment of the vortex tube will keep the same value, but the vorticity will change sense; it will become MV'. Since MV' must be symmetric to MV with respect to MN, MV must be perpendicular to MN, and consequently, to the plane MPQ, since we know that MV is situated in the plane R.

48. In order to find the magnitude of the velocity, recall that:

$$J = \int (u\,dx + v\,dy + w\,dz) = \mu.$$

Choose the integration contour to be the circle in the plane that is perpendicular to *PQ* that is described by having its center at *N* and a radius of *MN*. Take *PQ* to be the *z*-axis, the point *N* to be the origin, and two rectangular diameters of the circle to be the *x* and *y* axes. In that system of axes:

$$x = \rho \cos \omega, \qquad y = \rho \sin \omega,$$

$$\begin{aligned} dx &= -\rho \sin \omega\, d\omega, & dy &= \rho \cos \omega\, d\omega, & dz &= 0, \\ u &= -V \sin w, & v &= V \cos w, & w &= 0. \end{aligned}$$

Thus:

$$\mu = \int_0^{2\pi} \rho V (\sin^2 \omega + \cos^2 \omega)\, d\omega = 2\pi\rho\, V$$

$$V = \frac{\mu}{2\pi\rho}.$$

That velocity is therefore inversely proportional to the distance $MN = \rho$, as we found by another method.

49. Direct proof. – It is not indispensible for one to obtain the expression for the function φ in order to take recourse to the comparison between the hydrodynamical equations and the electrodynamical ones, as we have done; that expression can be obtained directly, as I will show.

To abbreviate, we say that the function φ is *generated by a contour C* when it is due to a vortex tube whose contour *C* is the axis, and we agree to take a vortex tube whose moment is equal to 1. That choice of unit will not affect the generality of our proof. I shall first establish some theorems that we will need in order to find the expression for the function φ.

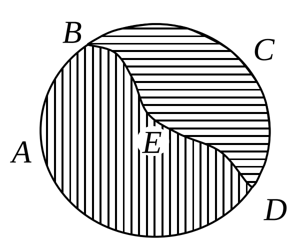

Figure 16.

50. Theorem I. – Consider a closed curve *ABCD* (Fig. 16); join two points of that curve *B*, *D* with an arbitrary path *BED*. Then, form two partial contours *ABED*, *BCDE*, and a total contour *ABCD*. Assume that these contours define the axes of three vortex tubes T', T'', and T. Each of these contours generates a function φ. Let φ', φ'', and φ be the functions that correspond to T', T'', and T, respectively. I say that:

$$\varphi = \varphi' + \varphi''.$$

Indeed, we can pass a certain surface through the three curves, which will determine two cuts. The function φ admits the two cuts, φ' admits only the cut (1), and φ'' admits the cut (2). In order to establish the theorem, it is sufficient that one have:

$$\varphi - \varphi' - \varphi'' = 0$$

identically. That function will verify the Laplace equation:

$$\Delta(\varphi - \varphi' - \varphi'') = 0,$$

since:

$$\Delta\varphi = \Delta\varphi' = \Delta\varphi'' = 0;$$

it is annulled at infinity, just like the partial functions φ, φ', φ''. It is permissible to apply Green's theorem [no. **34**] if it is uniform; i.e., if the integral:

$$\int d\varphi - \int d\varphi' - \int d\varphi'' = 0$$

along an arbitrary closed contour. Suppose that the integration curve is of the first kind; i.e., it does not meet any cut. The three partial integrals will then be zero. If the curve crosses only the cut (1) then $\int d\varphi$ will be equal to the moment of the tube T; i.e. to 1, by hypothesis. $\int d\varphi'$ is equal to the moment of the tube T', which is also 1. $\int d\varphi''$ is zero. The equation is still verified; one will establish it in the same manner if the integration curve meets only the cut (2).

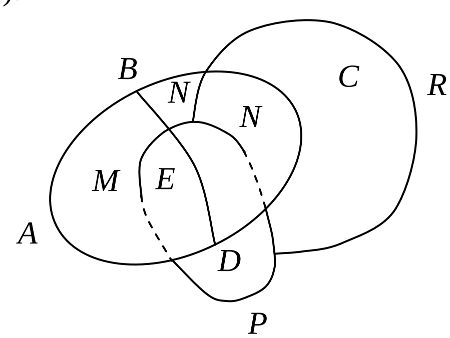

Figure 17.

Furthermore, an arbitrary closed contour can always be replaced with a series of contours, each of which meets only one cut (Fig. 17). Therefore, the contour *MNPQ*, which meets the two cuts, can be replaced with *MNRQM*, which meets only the first cut, and *NPQRN*, which meets only the second one. Indeed, traversing these two contours amounts to traversing the original contour in a well-defined sense, and the arc *NRQ*, once in one sense and once in another; that arc will then disappear in the result. Consequently, along an arbitrary contour:

$$\int d\varphi - \int d\varphi' - \int d\varphi'' = 0.$$

The function $\varphi - \varphi' - \varphi''$ is uniform and identically zero, as a consequence of Green's theorem.

51. THEOREM II. – The function φ that is generated by a planar contour C is zero at any point of the plane.

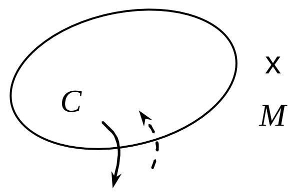

Figure 18.

Let C be the contour (Fig. 18). Represent the direction of the vorticity by an arrow, and take the figure to be symmetric with respect to the contour; φ must not change. The point M, which belongs to the symmetry plane, does not change. The moment of the tube keeps the same absolute value, but changes in sign, since the motion of the vortex does not change sense. The function φ must simultaneously not change and modify its origin; it can only be zero.

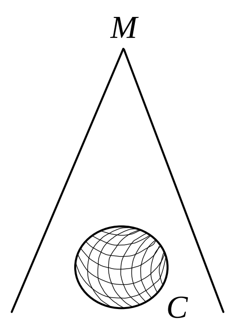

Figure 19

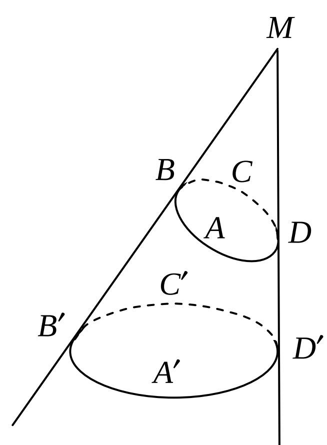

Figure 20.

52. THEOREM III. – Suppose that a contour C is traced in the surface of a cone that has its summit at M (Fig. 19). One can trace two types of curves on that surface: The one type bounds an area in which one does not find the summit; the other one makes a circuit around the cone that bounds an area that does contain the summit.

a. The function φ will be zero at the point M for the curves of the first type. Indeed, one can decompose C into infinitely-small contours, each of which can be associated with a planar element that is situated in the plane that is tangent to the cone. Since all of these tangent planes pass through the summit M of the cone, the function φ that is generated by each of them will be zero. Since the function φ that is generated by the entire contour C is the sum of elementary functions [no. **50**], it will also be zero.

b. Now, let there be two curves of the second kind $ABCD$ and $A'B'C'D'$ (Fig. 20).

I say that the functions φ that are generated by these two curves have the same value at the point M.

Indeed, join a point B of the first curve to a point B' of the second curve by the generator BB', for example; similarly, join D to D' with DD'. I can replace the contour $ABCD$ with the contours $A'B'C'D'$, $ABB'A'D'DA$, $CDD'C'B'BC$. In fact, upon successively describing these three contours in the sense that is indicated by the succession of letters, I will traverse each of the arcs two times in contrary senses, except for $ABCD$. Now, the functions φ that are generated by the two contours are zero, from the first part of the theorem (*a*). Therefore, the functions that are generated by the curves $ABCD$ and $ABCD$, which have the same perspective at the point M, will have the same value at that point.

53. Infinitely-small contour. Form of the function φ. – Suppose that the contour is infinitely-small. *A priori*, the function φ can depend upon the distance r from the point M to the elementary surface that is bounded by the contour, the angle ψ that the line that joins the point M to the center of gravity of the element makes with that element, the area of that element, and finally, its form. In other words, φ can depend upon r, ψ, and the solid angle and form of the cone that has the contour for its director and its summit at the point M.

I first say that φ cannot depend upon the form of that cone. Indeed, that area, which is infinitely-small to first order, can be decomposed into squares that will be infinitely-small of second order. All of these squares have the form, so the angle ψ will have the same value for each of them, up to higher-order infinitesimals. In addition, one can render their number very large in order that they will collectively differ from the area considered as little as one desires, no matter what its form. The value of φ that is generated by the total contour will be the sum of the functions φ that relate to each of the squares. However, since these functions are the same for each square – since r and ψ are the same, and the squares have the same form – the total function φ will be proportional to the number of squares – i.e., to the area that is bounded by the contour – and it will be independent of its form. Consequently, we can set:

$$\varphi = d\sigma\, f(r,\ \psi),$$

in which $d\sigma$ is the solid angle of the cone, and f is a function that we must determine.

$d\sigma$ has the same value all along the cone. On the other hand, two closed curves C and C' that are traced on the cone must generate the same function φ. However, r and ψ can be arbitrary for these two curves, so it will be necessary that:

$$f(r, \psi) = \text{const.} = A.$$

54. If one is dealing with a finite, closed curve then one can decompose it into elementary curves. φ will be proportional to the solid angle $d\sigma$ for each of them. One will have:

$$\varphi = A\,\sigma$$

for them collectively, in which σ is the total solid angle.

In order to determine A, suppose that the point M describes an arbitrary closed contour, so one will have:

$$\int d\varphi = A \int d\sigma,$$

and if one calls the moment of the vortex μ then:

$$\int d\varphi = \mu.$$

On the other hand:

$$\int d\sigma = 4\pi,$$

so:

$$\mu = 4\pi A$$

and

$$\varphi = \frac{\mu\sigma}{4\pi}.$$

55. Liquid that fills a simply-connected vessel completely. – We propose to determine u, v, w from the equations:

$$2\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},$$

$$\dots\dots\dots\dots\dots\dots$$

$$\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} = 0.$$

In the case of a liquid that fills a simply-connected vessel, it is necessary that the component of the velocity that is normal to the wall must be zero at any point of that wall. If one calls the direction cosines of the normal l, m, n then that condition will be written:

$$l\,n + m\,v + n\,w = 0.$$

In order to obtain the case that corresponds to that of electrodynamics, one must suppose that the currents prevail in the interior of the surface S of the wall, and that all of the exterior space is occupied by a perfect conductor. If one starts out at rest and then progressively increases the interior currents then induction currents will be produced in the exterior space. When the permanent regime is established, the electromotive force will disappear. However, the induction currents will persist when the exterior medium is a perfect conductor; i.e., when it presents zero resistance. Upon making that hypothesis, the problem of electrodynamics will coincide with that of Helmholtz.

Indeed, let there be a closed circuit, and let N be the flux of magnetic force that traverses it, so the electromotive induction is dN / dt, and from Ohm's law:

$$\frac{dN}{dt} = R\,i\,.$$

If we suppose that the conductor is perfect then $R = 0$, and as a result:

$$\frac{dN}{dt} = 0, \qquad N = \text{const.}$$

If we start at rest then $N = 0$ to begin with, and it will remain constantly zero so no line of force will traverse the surface S; i.e., the component of the magnetic force that is normal to the surface will be zero.

56. Special cases. – Let the velocity be parallel to the xy-plane and depend upon only x and y, so:

$$w = 0, \qquad \frac{du}{dz} = \frac{dv}{dz} = 0.$$

Upon supposing that these conditions are satisfied at the origin of time, they will always be true:

1. If the liquid is indefinite, because any plane that is parallel to the xy-plane will be a symmetry plane.

2. If the liquid fills up a cylinder that is parallel to Oz and is indefinite in the two senses.

The same thing will be true if the cylinder is bounded by two planes that are perpendicular to the z-axis. Indeed, when one introduces a partition into the liquid, one will generally impose one or more conditions on the motion, namely, that the component of the velocity that is normal to the partition must be zero on it. However, in the case that we are occupied with, that condition will be fulfilled before one creates the partition, and its existence will not modify the motion.

57. From the hypotheses that we have made:

$$(6) \qquad \begin{aligned} 2\xi &= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0, \\ 2\eta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \\ 2\zeta &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \end{aligned}$$

and the equation of continuity will reduce to:

$$(7) \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

All of the vortices are parallel to Oz, so all of the vortex tubes will be cylinders that have Oz for their axis.

Follow one of these tubes in its motion; I say that its cross-section will remain constant.

Indeed, consider a portion of the liquid that is bounded by the surface of a vortex tube and two cross-sections that are distant from h.

If we call the area of that cross-section ω then the volume of the liquid will be $h\,\omega$.

The liquid is incompressible, so that volume will remain constant. On the other hand, the vortex tube will be conserved, so the volume will remain cylindrical. A molecule that is situated on a cross-section at the origin of time will always remain so, since its velocity will be situated in that plane. The two sections that bound the cylinder will always remain at the same distance. Since $h\omega$ and h are constant, it will then follow that ω is constant.

In particular, if we consider a vortex tube whose section is infinitely-small $d\omega$ then its moment μ will be given by:

$$\mu = 2\, d\omega \cdot \zeta,$$

so $d\omega$ must be constant, as well as μ. Therefore, ζ is constant, and:

$$\frac{d\zeta}{dt} = 0.$$

(We use $d\zeta / dt$, and not $\partial\zeta / \partial t$, since we are following a molecule in its motion; i.e., we adopt the Lagrange variables.)

58. The case that we just treated is the one that Helmholtz called the case of a rectilinear vortex.

In particular, suppose that we have a vortex tube whose section by the xy-plane is a circle of radius R, and ζ = const. inside of that circle; $\zeta = 0$ outside of it, and there is a velocity function. Take the center of the circle to be the origin. Let M be an arbitrary point (Fig. 21). Set:

$$OM = \rho = \sqrt{x^2 + y^2}\,.$$

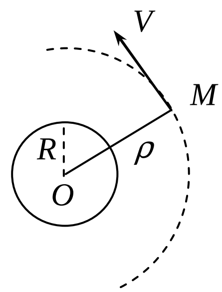

Figure 21.

The velocity V of the point M will be perpendicular to the radius vector OM, by reason of symmetry:

$$u = -V\,\frac{y}{\rho}, \qquad v = V\,\frac{x}{\rho},$$

and V will depend upon only ρ. Take the integral:

$$J = \int (u\,dx + v\,dy) = \int 2\zeta\,d\omega$$

along the circumference that is described by 0 as its center with the radius $OM = \rho$: $\int u\,dx + v\,dy$ represents the work that is done on a material point that describes the circumference by a force that is represented by the vector (u, v, w), which is the velocity; that vector has a constant magnitude and is directed along the tangent to the circumference at any points, so:

$$\int (u\,dx + v\,dy) = 2\pi\rho\,V.$$

We obtain another expression for J by means of integrals $\int 2\zeta\,d\omega$ that are extended over the entire surface of the circle OM.

One must distinguish two cases:

1. The point M can be inside of a circle of radius R ($\rho < R$). ζ will then be constant inside of the circle ρ, and:

$$J = \pi\zeta\rho^2 = 2\pi\rho\,V.$$

2. The point M is exterior to the circle R ($\rho > R$). ζ will then be constant inside of the circle R, and zero outside of it, so:

$$J = 2\pi\zeta R^2 = 2\pi\rho\, V.$$

We deduce from this that:

$$V = \zeta\,\rho$$

if ρ is smaller than R, and:

$$V = \zeta\,\frac{R^2}{\rho}$$

if ρ is greater than R.

In the latter case, we remark that moment of the vortex tube will be equal to:

$$2\pi\zeta R^2 = 2\pi\, m$$

upon setting:

$$m = \zeta R^2,$$

and V will take the form:

$$V = \frac{m}{\rho}.$$

That formula will persist if R becomes very small, but ζ becomes very large, in such a fashion that m remains finite.

59. This result can be compared with some other results of three different kinds:

1. *Electrodynamical comparison*. – We have seen that from the law of Biot and Laplace, the velocity of a molecule is represented by the same vector as the magnetic force that is produced by a current that traverses the vortex tube [no. **45**].

60.

2. *Analytical comparison*. – Outside of the tube $\zeta = 0$. From that condition and the equation of continuity (2) [no. **57**], one will have:

$$\text{(8)} \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}, \qquad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}.$$

These equations express the idea that $v + \sqrt{-1}\;u$ is a function of $x + y\sqrt{-1}$. This is easy to verify in the present case. Indeed:

$$v + \sqrt{-1}\;u = m\left(\frac{x}{x^2+y^2} - \sqrt{-1}\,\frac{x}{x^2+y^2}\right) = \frac{m}{x+\sqrt{-1}\,y}.$$

Set:

$$Z = x + \sqrt{-1}\; y,$$

and we can then write:

(9) $$v + \sqrt{-1}\; u = f(Z) = \frac{m}{Z}.$$

61. If there are several vortex tubes then the *xy*-plane will cut each of them along an infinitely-small circle that one can make agree with the points $a_1, a_2, \dots, a_n$ that coincide with their centers. Let $2\pi m_1, 2\pi m_2, \dots, 2\pi m_n$ be the moments of these tubes. If $a_1, a_2, \dots, a_n$ have the coordinates $a_1', a_1'', \dots, a_2', a_2'', \dots$ then these points will be the ones that are affixed to the imaginary quantities:

$$a_1 = a_1' + \sqrt{-1}\; a_1'',$$
$$a_2 = a_2' + \sqrt{-1}\; a_2''.$$

In order to obtain the value of $v + \sqrt{-1}\; u$ that corresponds to the first tube a_1, it will suffice to take formula (9) upon transporting the origin to the point $a_1, \dots$, and similarly for the other tubes, so the total value $v + \sqrt{-1}\; u$ will be the sum of the partial values that are thus obtained:

$$v + \sqrt{-1}\; u = \frac{m_1}{Z - a_1} + \frac{m_2}{Z - a_2} + \cdots + \frac{m_n}{Z - a_n} = \sum \frac{m_k}{Z - a_k}.$$

That expression is the derivative of the function:

$$\theta(Z) = \sum m_k \log(Z - a_k).$$

Let M be the point that is affixed to A, and let ρ_1 be the distance Ma_1 or the modulus of $Z - a_1$. Similarly, $\rho_2 = Ma_2$; …, $\rho_n = Ma_n$. Let ω_1 be the argument of $Z - a_1$; it is the angle that Ma_1 makes with Ox, …, etc.

$$\theta(Z) = \sum m_k \log \rho_k + \sqrt{-1} \sum m_k \omega_k,$$

or, upon setting:

(10) $$\psi = \sum m_k \log \rho_k,$$

(11) $$\varphi = \sum m_k \omega_k,$$

(12) $$\theta(Z) = \psi + \sqrt{-1}\; \varphi.$$

Differentiate that identity with respect to Z, upon remarking that:

$$\frac{\partial Z}{\partial x} = 1, \qquad \frac{\partial Z}{\partial y} = \sqrt{-1},$$

$$\theta'(Z) = v + \sqrt{-1}\; u = \frac{\partial \psi}{\partial x} + \sqrt{-1}\frac{\partial \varphi}{\partial x},$$

$$\theta''(Z) = \sqrt{-1}\; v - u = \frac{\partial \psi}{\partial y} + \sqrt{-1}\frac{\partial \varphi}{\partial y}.$$

Hence, upon defining:

(13) $$v = \frac{\partial \psi}{\partial x}, \qquad u = -\frac{\partial \psi}{\partial y},$$

(14) $$u = \frac{\partial \varphi}{\partial x}, \qquad v = \frac{\partial \varphi}{\partial y}.$$

From the relations (14), one sees that φ is the velocity function.

62.

3. *Electrostatic comparison.* – Suppose that the electricity is distributed uniformly over an indefinite line: The attraction of that electrified line to an exterior point will vary inversely with the distance.

Replace the vortex tube with a uniform distribution of electricity along its axis. The attraction at a point M will be directed along the normal that is drawn from M to the axis.

The velocity will be represented by the same vector that one will have rotated by 90 degrees. If there are several tubes then one will make the same transformation. One composes the partial attractions, and the resultant velocity will be represented by the resultant of these attractions when one rotates them through 90 degrees.

On will arrive at the same result by supposing that the points $a_1, a_2, \ldots, a_n$ act upon the point M in inverse proportion to the distance.

If we consider some vortex tubes that are distributed in an arbitrary manner and a point M that is very distant from them, so the distance is infinitely large of first order, then the attraction (or the velocity) will be infinitely small of first order.

63. The curves φ = const. – i.e., the ones along which the argument of $e^{\theta(Z)}$ is constant – are the lines that are normal to the velocity at each of their points. Indeed, for these curves:

$$d\varphi = u\, dx + v\, dy = 0.$$

The curves ψ = const. – i.e., the ones along which the real part of $\theta(Z)$ or the modulus of $e^{\theta(Z)}$ is constant – are the streamlines. Indeed, along these curves, one will have:

$$v\,dx - u\,dy = 0,$$

or

$$\frac{dx}{u} = \frac{dy}{v}.$$

In electrodynamics, the equations φ = const. represent the equipotential lines, and the equations ψ = const. represent the lines of force; the opposite is true in electrostatics.

64. Special case of two vortex tubes. – If there are only two vortex tubes a_1 and a_2 then $\theta(Z)$ will involve only two terms:

$$\theta(Z) = m_1 \log(Z - a_1) + m_2 \log(Z - a_2).$$

The equation of the streamlines will be:

$$m_1 \log \rho_1 + m_2 \log \rho_2 = \text{const.}$$

If $m_1 = m_2$ then the equation will become:

$$\rho_1\,\rho_2 = \text{const.}$$

The streamlines will then be Cassini ovals.

If $m_1 = - m_2$ then the streamlines that are represented by the equation:

$$\frac{\rho_1}{\rho_2} = \text{const.}$$

will be the circumferences, with respect to which the points a_1 and a_2 will be conjugate.

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CHAPTER IV

MOTION OF VORTEX TUBES

65. Theorem on the conservation of the center of gravity. – Suppose that we have n vortex tubes $a_1, a_2, \ldots, a_n$ that have the moments $2\pi m_1, 2\pi m_2, \ldots, 2\pi m_n$, resp.; assume that the tubes displace, but that their moments remain the same. If we regard $m_1, m_2, \ldots$, resp., as masses then we can construct their center of gravity G. I say that the point G will remain fixed under the displacement of the tube.

Let $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ be the coordinates of $a_1, a_2, \ldots, a_n$, resp., and let x_0, y_0 be those of G, so these coordinates will be coupled by the relations:

$$x_0 \sum m_K = \sum m_K x_K ,$$
$$y_0 \sum m_K = \sum m_K y_K .$$

If one treats tubes of finite dimensions, instead of infinitely-thin ones, then one will define x_0, y_0 in an analogous manner:

$$x_0 \int 2\zeta \, d\omega = \int 2x\zeta \, d\omega ,$$
……………………….., etc.,

in which the integrals are taken over all of the elements $d\omega$ of the sections of the different tubes.

Differentiate with respect to t: Since we have constrained the moments of the tubes to remain constant, ζ and ω will not depend upon t [no. **57**], and:

$$\frac{dx_0}{dt} \int \zeta \, d\omega = \int \frac{dx}{dt} \zeta \, d\omega = \int u\zeta \, d\omega . \tag{1}$$

I would like to establish that the latter integral is zero. In order to do that, I consider the integral:

$$\int [(u^2 - v^2)\, dx + 2uv\, dy] ,$$

which is taken around a circle of very large radius. That integral will be zero. Indeed, for R very large, u and v will be infinitely-small of first order [no. **62**]; u^2, v^2 are second-order infinitesimals. The integration path is infinitely large, but only of first order, so the integral will be negligible.

On the other hand, transform that integral using formula (1) of § **8**:

$$\int [(u^2 - v^2)\, dx + 2uv\, dy] = \int d\omega \left[\frac{\partial (u^2 - v^2)}{\partial y} - 2 \frac{\partial (uv)}{\partial x} \right].$$

Perform the differentiations:

$$\frac{\partial(u^2 - v^2)}{\partial y} - 2\frac{\partial(uv)}{\partial x} = 2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y} - 2u\frac{\partial v}{\partial x} - 2v\frac{\partial u}{\partial x}$$
$$= -\ 2v\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - 2u\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

However, from the continuity equation, we will have:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

and, on the other hand, by definition:

$$\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 2\zeta.$$

Therefore:

$$\int [(u^2 - v^2)\,dx + 2uv\,dy] = -\,4 \int u\zeta\,d\omega.$$

The first integral is zero, as well as the second one; therefore:

$$\frac{dx_0}{dt} = 0.$$

One likewise proves that:

$$\frac{dy_0}{dt} = 0,$$

and as a result, the point G is fixed.

66. Motion of the center of gravity of a vortex tube. – I would now like to study the motion of the center of gravity of one of these vortex tubes. We have:

$$\frac{dx_0}{dt} \int \zeta\,d\omega = \int u\zeta\,d\omega.$$

Set:

$$u = u' + u'',$$

in which u' is the velocity of the tube considered if it alone exists, and u'' is the velocity of the other two:

$$\int u\zeta\,d\omega = \int u'\zeta\,d\omega + \int u''\zeta\,d\omega.$$

The integral $\int u'\zeta\,d\omega$ is zero, because if the first tube exists by itself then its center of gravity will be fixed.

Consequently, if we would like to determine the velocity of the center of gravity of one of the vortex tubes then it will suffice to take into account the velocity that is communicated by the other vortices.

67. Let $a_1, a_2, \ldots, a_n$ be the vortex tubes. Set:

$$\rho_{12} = \overline{a_1\,a_2}\,,$$
$$\rho_{13} = \overline{a_1\,a_3}\,,$$

and in a general manner:

$$\rho_{ik} = \overline{a_i\,a_k}\,.$$

Consider the function:

$$P = \sum m_i m_k \log \rho_{i,k}\,. \tag{2}$$

P is a function of the $2n$ coordinates $x_1, y_1, \ldots, x_n, y_n$.

One must determine dx_1 / dt and dy_1 / dt . We just saw that the velocity of the point x_1, y_1 will be the same if the tube a_1 is suppressed and only the other tubes persist. From equations (13) [no. **61**], we will then have:

$$\frac{dx_1}{dt} = -\frac{\partial \psi}{\partial y_1}\,,$$
$$\frac{dy_1}{dt} = \frac{\partial \psi}{\partial x_1}\,,$$

where:

$$\psi = \sum m_k \log \rho_{1,k}\,.$$

I say that these formulas are equivalent to the following ones:

$$m_1 \frac{dx_1}{dt} = -\frac{\partial P}{\partial y_1}\,,$$
$$m_1 \frac{dy_1}{dt} = \frac{\partial P}{\partial x_1}\,.$$

Indeed, P can be written as:

$$P = m_1 \sum m_k \log \rho_{1,k} + \sum m_i m_k \log \rho_{i,k}\,,$$

in which none of the indices i and k in the second term is equal to 1.

On the other hand, the ρ that are affected with the index 1 are the only ones that depend upon x_1 and y_1, so:

$$\frac{\partial P}{\partial x_1} = m_1 \frac{\partial(\sum m_k \log \rho_{1k})}{\partial x_1} = m_1 \frac{\partial \psi}{\partial x_1},$$

and similarly:

$$\frac{\partial P}{\partial y_1} = m_1 \frac{\partial \psi}{\partial y_1}.$$

68. In a general manner, we then obtain the following equations:

$$\text{(I)} \qquad m_k \frac{dx_k}{dt} = -\frac{\partial P}{\partial y_k}, \qquad m_k \frac{dy_k}{dt} = \frac{\partial P}{\partial x_k}.$$

In this form, one recognizes Hamilton's canonical equations, up to the factor m_k ; in order to get the canonical form exactly, it will suffice to take the variables to be:

$$x_1, x_2, \ldots, x_n \quad \text{and} \quad m_1 y_1, m_2 y_2, \ldots, m_n y_n .$$

69. Integration of the equations. – The integration of equations (I) is possible when there exist only three vortex tubes, as we shall show.

70. THEOREM. – We can first recover the theorem of the conservation of the center of gravity. Indeed, the function P depends upon only the distances ρ, and consequently upon only the differences $x_1 - x_2, \ldots, y_1 - y_2, \ldots$, etc. Therefore:

$$\frac{\partial P}{\partial y_1} + \frac{\partial P}{\partial y_2} + \ldots + \frac{\partial P}{\partial y_n} = 0,$$

namely:

$$\text{(3)} \qquad \sum \frac{\partial P}{\partial y_k} = 0 \quad \text{or:} \quad \sum m_k \frac{dx_k}{dt} = 0,$$

$$\sum m_k x_k = \text{const.}$$

Similarly:

$$\sum m_k y_k = \text{const.}$$

The center of gravity of the system then remains fixed.

71. The *vis viva* theorem. – Multiply the two sides of equations (I) by dx_k, dy_k, respectively. If one operates similarly on all of the analogous equations and adds them then one will have:

$$\sum \frac{dP}{dx_k} dx_k + \sum \frac{dP}{dy_k} dy_k = 0$$

or:

$$dP = 0.$$

Therefore:

$$P = \text{const.} \tag{4}$$

That relation expresses the theorem of *vis viva*. That is not obvious immediately, and we must eliminate several difficulties. Indeed, from our hypothesis, the *vis viva* will be infinite, for three reasons.

1. The liquid is indefinite in all of its senses. However, we have seen that the motion is not modified by introducing two solid planar partitions that are perpendicular to the Oz axis. We can thus limit ourselves to the consideration of the liquid that is contained between the two planes.

2. Even with that restriction, the *vis viva* will still be infinite, since the liquid extends indefinitely in the xy-plane. The components u and v of the velocity will tend to 0 when one goes indefinitely far along a circumference whose radius R is regarded as infinite of first order; u and v are first-order infinitesimals [no. **62**]. The elementary *vis viva* will be a second-order infinitesimal, but the surface of the circle will be a second-order infinitude, so the total *vis viva* will be infinite. Upon calling the sum of the moments of all of the vortex tubes $2\pi M$, the velocity on the circumference of radius R will have the value:

$$V = \frac{M}{R},$$

up to second-order infinitesimals.

Suppose that M is zero. The velocity (u, v) will then be a second-order infinitesimal. The elementary *vis viva* will be of fourth order, and the total *vis viva* will be finite.

3. This further supposes that the vortex tubes are not infinitely thin. Otherwise, the velocity in the neighborhood of these tubes would be a first-order infinitude, and the *vis viva* would be a second-order infinitude.

72. We assume that the *vis viva* is finite. As we just saw, in order for that to be true, it will suffice that:

1. The liquid is bounded by two parallel planes that are perpendicular to Oz.
2. The sum of the moments of all the tubes is zero.
3. The tubes have a finite section.

Consider two small surface elements $d\omega$ and $d\omega'$ that correspond to the values ζ and ζ' of the vorticity, resp. Letting $2\pi\, dm$, $2\pi\, dm'$ be the moments of the elementary tubes that are bounded by these elements, one will have:

$$2\zeta\, d\omega = 2\pi\, dm,$$
$$2\zeta'\, d\omega' = 2\pi\, dm'.$$

The term in P that corresponds to these elements will be:

$$dm\, dm' \log \rho,$$

if one calls the distance between the two elements ρ, and:

$$P = \iint dm\, dm' \log \rho$$

or

(5) $$P = \iint \frac{\zeta\zeta'\, d\omega\, d\omega' \log \rho}{\pi^2},$$

in which the integral is calculated by taking all of the pair-wise combinations of the elements $d\omega$ and $d\omega'$, each of which is taken once. Let x, y, and x', y' be the coordinates of the centers of gravity $d\omega$ and $d\omega'$, resp.; the value of ψ at the point (x, y) will be:

$$\psi = \int dm' \log \rho = \int \frac{\zeta' d\omega' \log \rho}{\pi}.$$

On the other hand:

$$2\pi^2 P = \iint \zeta\zeta'\, d\omega\, d\omega' \log \rho,$$

in which the integral is taken over *all* of the combinations $(d\omega, d\omega')$, so each of them will be used twice, and:

(6) $$2\pi^2 P = \pi \int \zeta\psi\, d\omega.$$

73. We will be able to write down that formula immediately when we refer to the electrostatic comparison [no. **62**].

Indeed, if we consider dm, dm' to be the electric masses that are spread over the elements $d\omega$, $d\omega'$, resp., then the function ψ will represent (up to a constant factor) the electrostatic potential, and P will represent the electrostatic energy. One knows that a relation of the form (6) exists between these two functions.

74. Replacing 2ζ by its value in the expression for P will give:

$$4\pi P = \int 2\zeta\psi\, d\omega = \int \left(\frac{du}{dx} - \frac{dv}{dy}\right) \psi\, d\omega.$$

Consider the integral:

(7) $$\int (v\, dy + u\, dx)\, \psi.$$

That integral will be zero when it is taken along a circle of very large radius, since we have supposed [no. **72**] that the algebraic sum of the moments of all the tubes is zero. It will then happen that u and v are of second order, since the length of the circumference is only a first-order infinitude. Upon transforming it by Stokes's theorem:

$$\int \left(\frac{d(\psi v)}{dx} - \frac{d(\psi u)}{dy}\right) d\omega = 0,$$

or, upon performing the differentiations:

$$\int \psi \left(\frac{dv}{dx} - \frac{du}{dy}\right) d\omega + \int \left(v \frac{d\psi}{dx} - u \frac{d\psi}{dy}\right) d\omega = 0.$$

Now:

$$2\zeta = \frac{dv}{dx} - \frac{du}{dy},$$

$$\frac{d\psi}{dx} = v, \qquad \frac{d\psi}{dy} = -u.$$

Therefore:

(8) $$4\pi P + \int (v^2 + u^2)\, d\omega = 0.$$

P then represents (up to a constant factor) the *vis viva* $\int (v^2 + u^2)\, d\omega$, and consequently, that *vis viva* will be constant.

75. THEOREM. – The moment of inertia of the masses m with respect to the Oz axis is constant.

Confer an infinitely-small rotation ε around the z-axis upon all of the system. Upon neglecting the second-order infinitesimals, the coordinates x_i and y_i will become:

$$x_i - y_i\, \varepsilon, \qquad y_i + x_i\, \varepsilon.$$

P, which depends upon only the distances ρ_{ik} [no. **71**], will not change.

Therefore, if one writes that $dP = 0$ then one will get:

$$\sum -\frac{dP_i}{dx_i} y_i \varepsilon + \sum \frac{dP}{dy_i} x_i \varepsilon = 0$$

or

$$\sum \left(x_i \frac{dP}{dy_i} - y_i \frac{dP_i}{dx_i} \right) = 0.$$

In the electrostatic comparison, that equation signifies that the sum of the moments of the attractions that are exerted on either of the electrified lines, when taken with respect to the *z*-axis, is zero. That is obvious, since the attractions are pair-wise equal and of opposite sign.

If we replace dP / dy_i and dP / dx_i with their values $m_i\, dx_i / dt$ and $m_i\, dy_i / dt$, resp., then we will find that:

$$\sum m_i \left(x_i \frac{dx_i}{dt} + y_i \frac{dy_i}{dt} \right) = 0$$

or, upon integrating:

(9) $$\sum m_i (x_i^2 + y_i^2) = \text{const.}$$

76. THEOREM. – The sum of the moments of the quantities of motion with respect to the *x*-axis is constant. If *f* is a homogeneous function of first order then Euler's theorem will give:

$$x \frac{df}{dx} + y \frac{df}{dy} + z \frac{df}{dz} = f$$

or:

$$\sum x \frac{df}{dx} = f$$

or finally, that:

$$\sum x \frac{d \log f}{dx} = 1.$$

If one applies this to the function $P \cdot \rho_{ik}$ then it will be a homogeneous function of first order in x_i, y_i, and the other coordinates x_k, y_k do not enter in. Consequently:

$$x_i \frac{d \log \rho_{i,k}}{dx_i} + y_i \frac{d \log \rho_{i,k}}{dy_i} + \ldots = 1$$

or, upon multiplying all of the terms by $m_i\, m_k$:

$$\sum \left(x_p \frac{d m_i m_k \log \rho_{i,k}}{dx_p} + y_p \frac{d m_i m_k \log \rho_{i,k}}{dy_p} \right) = m_i\, m_k ,$$

in which, the summation is extended over all values of p from 1 to n. One must then, in turn, take all of the possible combinations of i and k, and take the sum, which will give:

$$\sum \left(x_p \frac{dP}{dx_p} + y_p \frac{dP}{dy_p} \right) = \sum m_i m_k .$$

From the equations (1) [no. **68**], that relation is equivalent to:

$$\sum m_p \left(x_p \frac{dy_p}{dt} - y_p \frac{dx_p}{dt} \right) = \sum m_i m_k . \tag{10}$$

The left-hand side is the sum of the moments of the quantities of motion, while the right-hand side is constant.

77. We have thus determined three integrals of our differential equations (1); these properties of the equations permit us to integrate them by quadratures when there are only three vortex tubes.

Indeed, our equations have the form of Hamilton's canonical equations, which are integrated by quadratures when they contain $2n$ variables, and one will know n particular integrals. Now, when there exist three vortex tubes, the equations will contain six variables x_1, y_1, x_2, y_2, x_3, y_3, and we have found three particular integrals.

CHAPTER V

THE CASE OF TWO VORTEX TUBES. METHOD OF IMAGES.

78. Let a_1, a_2 be two vortex tubes whose moments are $2\pi m_1$ and $2\pi m_2$, resp. Their center of gravity G will be situated on the line $a_1\,a_2$ and will be determined by the condition:

$$\frac{Ga_1}{Ga_2} = -\frac{m_1}{m_2}.$$

(The segments Ga_1, Ga_2 are taken with their signs.)

From what we know [no. **65**], the point G will remain fixed. The velocity of the point a_2 will be the same as if the vortex a_1 existed by itself, namely:

$$\frac{m_1}{a_1\,a_2},$$

and it will be directed perpendicular to $a_1\,a_2$.

Since the point G is fixed, the three points a_1, G, a_2 will always be in a straight line, and the velocity of the point a_2 will be constantly normal to the radius vector Ga_2, and the trajectory of the point a_2 will be a circumference that has its center at G and a radius of Ga_2. The trajectory of the point a_1 will likewise be a circumference that has its center at G and a radius of Ga_1. Since the distance $a_1\,a_2$ remains constant, the velocities of the two points, which are equal to $\frac{m_1}{a_1\,a_2}$ and $\frac{m_2}{a_1\,a_2}$, respectively, will also be constant.

79. Suppose that m_1 and m_2 are of contrary sign. The point G will then be outside of $a_1\,a_2$, and will be further determined by the condition that:

$$\frac{\overline{Ga_1}}{\overline{Ga_2}} = -\frac{m_2}{m_1}.$$

In particular, if $m_1 = -m_2$ then the point G will be pushed out to infinity, and the trajectories of the points a_1 and a_2 will reduce to lines that are perpendicular to $a_1\,a_2$.

The two tubes displace with the same velocity:

$$\frac{m_1}{a_1\,a_2} = V.$$

If we consider the point M_1 that is the middle of $a_1\,a_2$ then the velocity that is communicated to that point by a vortex a_1 will be:

$$\frac{m_1}{a_1 M} = 2\frac{m_1}{a_1 a_2} = 2V.$$

The vortex a_2 likewise communicates a velocity of:

$$\frac{m_2}{Ma_2} = \frac{m_1}{a_1 M} = 2V$$

to it.

The resultant velocity of the point M is thus equal to four times the velocity that is common to the centers of the vortex tubes.

80. Liquid contained in a cylindrical vessel. – Imagine that the liquid is contained in a vessel that has the form of a cylinder whose generators are parallel to the z-axis. In that vessel, one finds a vortex tube that is formed by an infinitely-thin cylinder that is also parallel to Oz.

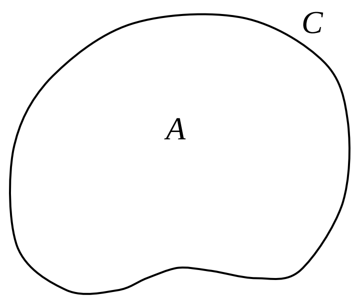

Figure 22.

Let C (Fig. 22) be the cross-section of the vessel in the xy-plane, and let A be the point to which the section of the vortex tube reduces.

The equation of continuity reduces to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Since the vorticity is zero everywhere, except at A:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0.$$

We have seen [no. **33**] that under these conditions there will exist a velocity function φ such that:

$$u = \frac{\partial \varphi}{\partial x}, \qquad v = \frac{\partial \varphi}{\partial y},$$

so the equation of continuity will become:

$$\Delta\varphi = 0.$$

In the present state, the boundary condition is that the contour C of the vessel must be a streamline; i.e., that at any point of that curve the velocity will be tangent to it. $v + \sqrt{-1}\, u$ must be a function of $x + \sqrt{-1}\, y$ in the interior of C. That function must behave regularly, except at the point A, where it becomes infinite.

The determination of u and v can be achieved by two methods:

1. The method of images, which applies to only a certain number of simple cases.

2. The method of conformal representation, which is much more general.

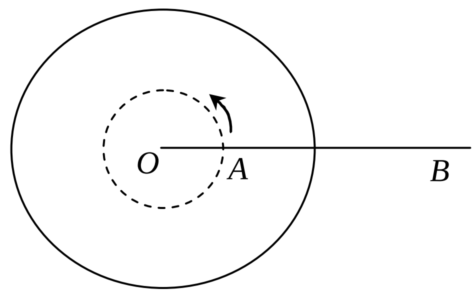

Figure 23.

81. Method of images. – Suppose that the vessel has the form of a circular cylinder that is parallel to Oz and a radius of R. Let C be the trace of that cylinder on the xy-plane. Let A be the trace of a vortex tube of moment 2π (Fig. 23).

Join the center O of the circle to the point A and take the point B along OA that is defined by the condition that:

$$OA \cdot OB = R^2.$$

The vortex tube that is parallel to Oz whose trace is B and whose moment is equal to 2π is called the *image* of A with respect to the circle C.

If the liquid is indefinite, and the tubes A and B exist simultaneously in reality then the streamlines will be circumferences, with respect to which, B and A will be conjugate [no. **64**].

In particular, the circumference C will be a streamline. The component of the velocity that is normal to that curve, and more generally, normal to the right cylinder that has C for its base, will be zero. The introduction of a solid wall that has the form of that cylindrical surface will not modify the motion inside of that surface.

The center of gravity A of the tube displaces with the same velocity as if the tube B existed by itself, and the liquid were indefinite. $AB / 2\pi$ will be the absolute magnitude of that velocity, which is constantly directed along the perpendicular to the radius vector OAB. The point A will thus describe a circumference that is concentric to C.

That trajectory is not the same as if the partition C did not exist, although the velocity will be the same. Indeed, if the liquid is indefinite then the point A will describe a line that is perpendicular to OA [no. **79**].

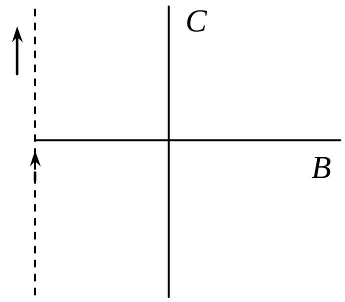

Figure 24.

82. If the radius of the circumference C increases indefinitely then that curve will conclude by reducing to a line, and B will be symmetric to A with respect to that line, or rather, to the plane C, to which the cylinder reduces. The trajectory of A, which is normal to AB, will be a line that is parallel to the trace of the plane C (Fig. 24).

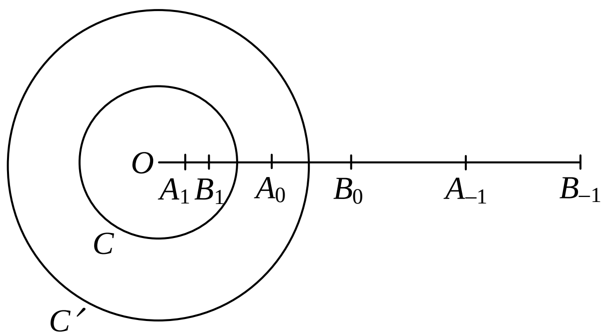

Figure 25.

83. Liquid contained between two concentric cylinders of revolution. – Let C and C' be the traces on the xy-plane of two cylinders of revolution whose axis is Oz; A_0 is the trace on the plane of an infinitely-thin vortex tube whose moment is 2π (Fig. 25).

Let B_0 be the image of A_0 with respect to C', B_1, the image of A_0 with respect to C, A_{-1}, the image of B_1 with respect to C', etc., in such a way that we have an infinitude of pairs of points that are conjugate with respect to C and C', which is indicated by the following table:

CONJUGATES

With respect to C'	With respect to C
A_0, B_0	B_1, A_0
A_{-1}, B_1	B_0, A_1
A_1, B_{-1}	B_2, A_{-1}
………	……..
A_i, B_{-i}	B_{-i}, A_i

in which the indices vary from $-\infty$ to $+\infty$. Consider each of the points A to be the trace of a vortex tube whose moment is 2π, and each of the points B to be the trace of a tube of moment -2π. These tubes will be conjugate.

Suppose that all of these tubes really exist, and that the liquid is indefinite. The velocity of the liquid will be the same as if the partitions existed in only the vortex tube A. We calculate that velocity by taking the sum of the velocities that are due to each tube separately. We then obtain a series, and we would like to know if that series is convergent.

From the relations:

$$OA_0 \cdot OB_0 = R'^2,$$
$$OA_1 \cdot OB_1 = R^2,$$

one will have:

$$\frac{OB_1}{OB_0} = \left(\frac{R}{R'}\right)^2.$$

Similarly:

$$\frac{OB_2}{OB_1} = \left(\frac{R}{R'}\right)^2, \text{ etc.}$$

In a general fashion:

$$OB_n = OB_0 \left(\frac{R}{R'}\right)^{2n}. \tag{1}$$

One proves, in a similar manner, that:

$$OA_n = OA_0 \left(\frac{R}{R'}\right)^{2n}. \tag{2}$$

Group the terms of the series as follows:

1. The terms that relate to the tubes A that are affected with negative indices; the sum of these terms will form a series. When the index n becomes very large, the point A_{-n} will become very distant, and the velocity that is communicated to a point M by that tube will become very small of order $1 / MA_{-n}$. If the point A_n is very distant then the difference $OA_n - MA_{-n}$ will be negligible, and the velocity will be of order of magnitude $1 / OA_{-n}$. The distance OA_{-n} increases according to a geometric progression with a ratio of $(R' / R)^2$. The series $1 / OA_{-n}$ is thus convergent.

2. The terms that relate to the tubes B that are affected negative indices. An argument that is identical to the preceding one will make it clear that the series $1 / OA_{-n}$ is convergent.

3. Group the tubes that are affected with positive indices into pairs:

$$A_0, B_0 - A_1, B_1 - \ldots A_n, B_n.$$

If one subtracts the corresponding sides of the equality (2) from those of the equality (1) then one will get:

$$A_n\,B_n = \mathrm{A}_0\,B_0\left(\frac{R}{R'}\right)^{2n}. \tag{3}$$

The tubes A_0, B_0, …, A_n , B_n have moments that are pair-wise equal and opposite in sign. The geometric sum of the velocities that are due to a group (A_n, B_n) has the same order of magnitude as $A_n\,B_n$. It thus decreases in a geometric progression with a ratio of $(R'/R)^2$, and will tend to O when n increases indefinitely; the series is then convergent.

Since the three partial series are convergent, the same thing will be true for the total series.

84. Furthermore, I say that the circumferences C and C' are streamlines. That is obvious. Indeed, all of the tubes are pair-wise conjugate with respect to the circle C and the circle C'. Assemble the tubes into groups of two that are conjugate with respect to C; the velocity that is due to each group will be tangent to C. Consequently, the total velocity will itself be tangent to C. The proof for C' is analogous.

85. The solution that we just found is not the only one, since the vessel is not simply-connected. In order to obtain the most general solution, it will suffice to add a tube whose trace has its center at O and an arbitrary moment. Indeed, the velocity that is due to that tube will be tangent to C and C', in particular, since all of the streamlines that are due to that tube will be circumferences that have O for their center.

The trajectory of A_0 is obviously a circumference whose center is at O, so the velocity that is due to all of the tubes will be constantly normal to the radius vector OA_0 .

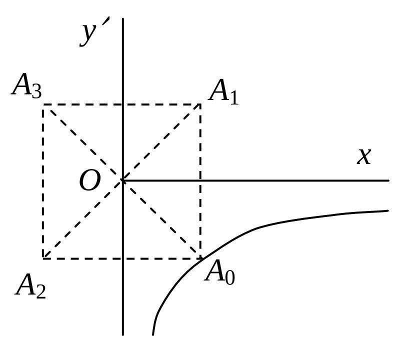

Figure 26.

86. Liquid contained between two rectangular planes. – Take the two planes that bound the liquid to be the *xz* and *yz*-planes. Let Ox and Oy be their traces, while A_0 is the trace of an infinitely-thin vortex tube whose moment is 2π (Fig. 26).

Take the symmetric points to A_0 with respect to Ox and Oy to be A_1 and A_2, resp., and then take the symmetric point to A_1 with respect to Oy and the symmetric point to A_2 with respect to Ox. Those two points will coincide with A_3, which is the symmetric point to A_0 with respect to the point O.

Attribute the moment:

$$\begin{aligned} &+ 2\pi \text{ to } A_0, \\ &- 2\pi \text{ to } A_1, \\ &- 2\pi \text{ to } A_2, \\ &+ 2\pi \text{ to } A_3 . \end{aligned}$$

Suppress the partition, and imagine that the four tubes A_0, A_1, A_2, A_3 exist simultaneously. The Ox and Oy axes will be streamlines, since the tubes are pair-wise conjugate with respect to those axes, and the introduction of the partitions into the xz and yz planes will not modify the motion.

What will the trajectory of the point A_0 be? We have defined the function P [no. **72**], and we have seen that in the case of an indefinite liquid that function will be proportional to the *vis viva*. Therefore, if the partition is suppressed, and the tubes four really exist then the total *vis viva* of the liquid will be equal to P, up to a constant factor. However, the *vis viva* of the liquid that is contained between the two planes is one-fourth the total *vis viva* that one would obtain by suppressing the partitions and given a real existence to the tubes A_1, A_2, A_3 .

Therefore: The *vis viva* of the *real* liquid that is contained between the two planes is again proportional to P, and the *vis viva* equation is written:

$$P = \text{const.}$$

Now:

$$P = \sum m_i\, m_k \log \rho_{ik} .$$

In the present case, $m_i = \pm 1$, and there are six terms, corresponding to six distances ρ that are pair-wise equal. The product $m_i\, m_k = + 1$ for the terms that correspond to the two opposite vertices of the rectangle; for the other four terms, $m_i\, m_k = -1$. Therefore:

$$\begin{gathered} P = 2 \log A_0 A_3 - 2 \log A_1 A_0 - 2 \log A_2 A_0 = \text{const.}, \\ A_0 A_3 = 2\sqrt{x^2 + y^2} , \\ A_1 A_0 = 2y, \\ A_2 A_0 = 2x. \end{gathered}$$

The equation for the trajectory of the point A_0 will then be:

$$2 \log 2\sqrt{x^2 + y^2} - 2 \log 4xy = \text{const.}$$

or:

$$\begin{gathered} \frac{x^2 + y^2}{x^2 y^2} = \text{const.}, \\ \frac{1}{x^2} + \frac{1}{y^2} = \text{const.} \end{gathered}$$

Therefore, the trajectory of point A_0 in the motion being studied will be represented by a curve that is asymptotic to the two axes.

CHAPTER VI

METHOD OF CONFORMAL REPRESENTATION

87. Definition of a conformal representation. – Let two planar areas be simply-connected, and let M (x, y) and $M'(x', y')$ be two points of those areas. Suppose that one has established a correspondence between M and M' such that x' and y' are functions of x and y; each point M corresponds to just one point M', and conversely. If x' and y' are continuous functions of x and y then M' will describe a curve when M describes a curve, and conversely. The various points of the contour of the first area correspond to the various points of the contour of the second one, and conversely. Upon conveniently choosing the functions x' and y', one can preserve angles; i.e., one can obtain representations of the curves that intersect at the same angle as the curves themselves. One then says that the representation is *conformal*. Two corresponding infinitely-small triangles will then be similar, and also two arbitrary corresponding infinitely-small figures, since one can decompose them into pair-wise similar triangles.

88. Consider the complex variable $x' + \sqrt{-1}\;y'$. It can happen that $x' + \sqrt{-1}\;y'$ is a function of $x + \sqrt{-1}\;y$. In that case, the angles will be preserved, and conversely. Indeed, the conditions that express that idea are:

$$(1)\qquad \frac{dx'}{dx} = \frac{dy'}{dy}, \qquad \frac{dx'}{dy} = -\frac{dy'}{dx}.$$

Can one define a conformal representation of a curve upon itself in this manner? For example, take a circumference.

1. One can rotate it around its center.

2. Consider a point M inside of the circle; let (x, y) be its coordinates. I can make it correspond to $M'(x', y')$, which is likewise interior to the circumference, by a conformal representation in such a manner that the center O will correspond to an arbitrary point O' that is interior to the circle.

Indeed, take the radius of the circumference to be unity. M is assigned to the imaginary quantity $x + \sqrt{-1}\;y$, and the equation of the circumference is:

$$|\, x + \sqrt{-1}\ y \,| = \text{const.} = 1.$$

Let $a + \sqrt{-1}\ b$ be assigned to O'. Consider the expression:

$$x' + \sqrt{-1}\ y' = \frac{\alpha(x+\sqrt{-1}y)+\beta}{\gamma(x+\sqrt{-1}y)+\delta}.$$

I can choose α, β, γ, δ in such a fashion that M' describes the circumference at the same time as M; i.e., that the modulus of $x' + \sqrt{-1}\ y'$ is equal to unity at the same as that of $x + \sqrt{-1}\ y$.

The point O' that is conjugate to O is assigned to $1 / (a - \sqrt{-1}\ b)$; set:

$$x' + \sqrt{-1}\ y' = \frac{x+\sqrt{-1}\,y-(a+\sqrt{-1}\,b)}{x+\sqrt{-1}\,y-\dfrac{1}{a+\sqrt{-1}\,b}}\,\frac{1}{a+\sqrt{-1}\,b}.$$

The condition:

$$|\, x + \sqrt{-1}\ y \,| = \text{const.} = 1$$

is equivalent to:

$$x^2 + y^2 = 1$$

or

$$\frac{1}{x-\sqrt{-1}\,y} = x + \sqrt{-1}\ y.$$

Therefore, for points (x, y) on the circumference, we can write:

$$x' + \sqrt{-1}\ y' = \frac{x+\sqrt{-1}\,y-(a+\sqrt{-1}\,b)}{\dfrac{1}{x-\sqrt{-1}\,y}-\dfrac{1}{a-\sqrt{-1}\,b}}\,\frac{1}{a-\sqrt{-1}\,b}$$

or

$$x' + \sqrt{-1}\ y' = \frac{x+\sqrt{-1}\,y-(a+\sqrt{-1}\,b)}{x-\sqrt{-1}\,y-(a-\sqrt{-1}\,b)}(x-\sqrt{-1}\,y)\frac{a+\sqrt{-1}\,b}{a-\sqrt{-1}\,b}.$$

The two fractions have unity modulus, since their two terms are conjugate. That of $x - \sqrt{-1}\ y$ is equal to unity. The modulus of $x' + \sqrt{-1}\ y'$ will then reduce to unity.

If one makes:

$$x + \sqrt{-1}\ y = a + \sqrt{-1}\ b$$

then one will have:

$$x' + \sqrt{-1}\ y' = 0$$

in such a way that the point O is the transform of the point O', which was chosen arbitrarily inside of the circumference.

89. Schwartz gave the means to construct the conformal representation of an arbitrary planar area onto a circle. However, the procedure is generally very complicated, except in some relatively simple cases.

Suppose that we know how to make a conformal representation of an area A onto a circle in such a manner that the point M of the area A corresponds to a point M' of the circle, and the point P_0 of the area A will correspond to the center of the circle. I say that one can also find another representation of the area A on the same circle that will be such that another point P of the area A will go to the center of the circle. Indeed, let P' be the point of the circle that corresponds to P under the first representation. Make a conformal representation of the circle onto itself in such a manner that the point M' will correspond to the point M'', and the point P' will correspond to the center of the circle. That will always be possible [no. **88**].

We will again have a conformal representation of the original area A. Indeed, let (x, y), (x', y'), and (x'', y'') be the coordinates of the point M, M', and M'', resp. x'' and y'' are functions of (x', y'), and in turn, functions of (x, y), like x' and y' themselves. On the other hand, the angles have not been altered, since the two representations that were made in succession are both conformal representations. Finally, if the point M goes to P then the point M' will go to P', and the point M'' will go to the center of the circle.

90. (Application to) the Helmholtz problem. – Suppose that the section of the vessel is a curve C. Let A_0 be the trace of a vortex tube that has a moment that is equal to 2π. As we have established [no. **60**]:

$$v + \sqrt{-1}\, u$$

is a function of $x + \sqrt{-1}\, y$.

I can then set:

$$v + \sqrt{-1}\, u = \frac{d(\psi + \sqrt{-1}\varphi)}{d(x + \sqrt{-1}\, y)}, \tag{2}$$

so:

$$v = -\frac{d\psi}{dy} = \frac{d\varphi}{dx},$$

$$u = \frac{d\psi}{dx} = \frac{d\varphi}{dy}. \tag{3}$$

Set:

$$x' + \sqrt{-1}\, y' = e^{\psi + \sqrt{-1}\varphi}.$$

That expression will be a function of $x + \sqrt{-1}\ y$. The functions u and v behave regularly inside of the curve C, except at a point A_0, where u and v become infinitely large of first order.

I add that the difference:

$$v + \sqrt{-1}\ u - \frac{1}{x+\sqrt{-1}\,y}$$

remains finite. The function:

$$\psi + \sqrt{-1}\ \varphi - \log\,(x + \sqrt{-1}\ y) = f_1\,(x + \sqrt{-1}\ y)$$

also remains finite, even at the point A_0. Consequently:

$$e^{\psi+\sqrt{-1}\varphi} = (x + \sqrt{-1}\ y)\ e^{f_1}$$

will admit no singular point, since the two factors behave regularly at the point A_0. ψ = const. along the curve C, which is a streamline. Now, e^{ψ} is the modulus of $e^{\psi+\sqrt{-1}\varphi}$, or $x' + \sqrt{-1}\ y'$. Therefore, the modulus of $x' + \sqrt{-1}\ y'$ is constant along C:

(1) $$x'^2 + y'^2 = \text{const.}$$

Consider a point M (x, y) inside of the curve C. When that point traverses all of the area that is bounded by C, the point (x, y) will traverse the area that is bounded by the curve that corresponds to C. Now, from equation (1), that curve will be a circumference whose center corresponds to A_0. The representation will be conformal, since $x' + \sqrt{-1}\ y'$ is a function of $x + \sqrt{-1}\ y$.

91. *Conversely*, if one makes a conformal representation of the area C then one can solve the Helmholtz problem. One will then know $x' + \sqrt{-1}\ y'$, so one can set:

$$e^{\psi+\sqrt{-1}\varphi} = x' + \sqrt{-1}\ y',$$

$$u = \frac{d\psi}{dx},\ \ldots,\ v = -\frac{d\psi}{dy}.$$

φ will be the velocity function (outside of the tube A_0).

92. In order to determine the trajectory of the center of gravity A_0 of the tube, it will be most convenient to appeal to the electrostatic comparison.

Consider an electric field that is determined by a certain number of lines that are perpendicular to the xy-plane, whose lengths $2l$ are very large with respect to their separation distances, and which are uniformly electrified. We suppose that the extremities of all of these lines are in the two planes $z = l$ and $z = -l$.

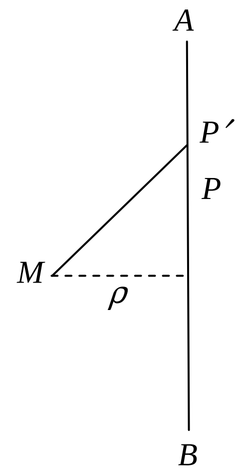

Figure 27.

Let AB be one of these lines (Fig. 27), let P be an arbitrary point on that line that has the coordinates x', y', z', and let P' be an infinitely-close point whose coordinates are x', y', $z' + dz'$.

If δ is the charge per unit length then the charge on PP' will be $\delta\, dz'$, and the potential of the line AB at a point such as M (x, y, z) will be:

$$V = \int_{-l}^{+l} \frac{\delta\, dz'}{MP'}.$$

If ρ is the distance from the point M to the line:

$$\overline{MP}^2 = \rho^2 + (z - z')^2,$$

$$V = \int_{-l}^{+l} \frac{\delta\, dz'}{\sqrt{\rho^2 + (z - z')^2}}.$$

Set:

$$z' = \zeta + z, \qquad \alpha = l - z, \qquad \beta = l + z,$$

$$V = \int_{-\beta}^{-\alpha} \frac{\delta\, d\zeta}{\sqrt{\rho^2 + \zeta^2}},$$

$$V = \delta \log \frac{\alpha + \sqrt{\rho^2 + \alpha^2}}{-\beta + \sqrt{\rho^2 + \beta^2}} = \delta \log \frac{(\alpha + \sqrt{\rho^2 + \alpha^2})(-\beta + \sqrt{\rho^2 + \alpha^2})}{\rho^2},$$

or, since α and β are very large in comparison to ρ, one will have approximately:

$$V = \delta \log \frac{4\alpha\beta}{\rho^2} = 2\delta \log \frac{2\sqrt{l^2 - z^2}}{\rho}.$$

93. If the electrified lines are arbitrary in number then:

$$V = \sum 2\delta_i \log \frac{2\sqrt{l^2 - z^2}}{\rho_i}$$

or

(4) $$V = -2 \sum \delta_i \log \rho_i + 2 \log 2\sqrt{l^2 - z^2} \sum \delta_i .$$

If the sum of the charges is zero:

$$\sum \delta_i = 0$$

then the potential:

(5) $$V = -2 \sum \delta_i \log \rho_i$$

will no longer depend upon either l or z.

The component dV / dz of the electromotive force that is parallel to Oz is zero.

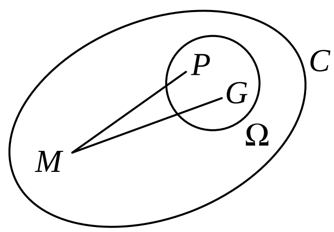

Figure 28.

94. Application to hydrodynamics. – Let C be the section of the vessel, and let Ω be that of the vortex tube (Fig. 28). The vorticity ζ varies in an arbitrary manner inside of Ω, and is zero outside of it. Let 2π be the total moment of the tube:

(6) $$\int 2\zeta \, d\omega = 2\pi.$$

The equations that must be satisfied are: the continuity equation:

(7) $$\frac{du}{dx} + \frac{dv}{dy} = 0,$$

and the equation:

(8) $$\frac{dv}{dx} - \frac{du}{dy} = 2\zeta .$$

The relation (1) expresses the idea that u and v are the derivatives of the same function $\psi(x, y)$:

$$u = -\frac{d\psi}{dy}, \qquad v = \frac{d\psi}{dx},$$

$$v\,dx - u\,dy = d\psi.$$

If one substitutes these values of u and v into (2) then one will get:

$$\Delta\psi = 2\zeta,$$

and there is a velocity function. The curve C must be a streamline; i.e., along that line:

$$\frac{dx}{u} = \frac{dy}{v}$$

or

$$\frac{d\psi}{dx}dx + \frac{d\psi}{dy}dy = d\psi = 0.$$

Therefore, one will have:

$$\psi = \text{const.}$$

along C.

Since ψ is defined by only its derivatives, one can always arrange them in such a way that the constant is zero.

95. Now, replace each infinitely-thin tube in the tube Ω with an electrified line of length $2l$, and a charge density along the line that is proportional to ζ. Suppose that the space that is enclosed between Ω and C is filled with a dielectric, and exterior to C, a conductor is bounded on the inside by the cylinder that admits that curve for its cross-section. If that conductor is connected to the ground then its potential will be zero, and one will have $\psi = 0$ all along C.

However, a layer of electricity will spread along C that forms a charge that is equal and opposite in sign to that on Ω (Faraday's theorem). Upon conveniently choosing the proportionality factor that couples the density to ζ, the potential will be represented by the function ψ that we have defined previously [no. **94**].

Inside of C, inside of the dielectric:

$$\Delta\psi = 0.$$

Inside of the cylinder:

$$\Delta\psi = -4\pi\mu''.$$

It will then suffice to take:

(9) $$\mu'' = -\frac{\zeta}{2\pi}$$

or

$$\int \mu''\,d\omega = -\tfrac{1}{2}.$$

The function ψ will then satisfy the same conditions as the function ψ that is defined by the Helmholtz problem. ψ depends upon only x and y, and the sum of the charges is zero, since the charges on C and Ω are equal and of opposite sign.

At a point of the surface of the cylinder C, the surface density μ' will be such that:

$$\int \mu' \, d\omega = \tfrac{1}{2},$$

upon calling the arc of the curve C.

The potential then has the expression:

$$\begin{aligned} \psi &= -2 \int \mu' ds \log \rho - \int \mu'' d\omega \log \rho \\ &= \psi' + \psi'' \end{aligned} \tag{10}$$

(upon denoting the first integral by ψ' and the second one by ψ'').

It must indeed be remarked that ψ' is not the potential of the cylinder C by itself, and ψ'' is not the potential of the cylinder Ω by itself, since the charges on these cylinders, when they are considered to be isolated, are no longer zero.

The expression for the potential C will be [no. **93**]:

$$-2 \int \mu' ds \log \rho + 2 \log 2\sqrt{l^2 - z^2} \int \mu' ds$$

or

$$-\psi' + \log 2\sqrt{l^2 - z^2},$$

and for the cylinder Ω:

$$\psi'' - \log 2\sqrt{l^2 - z^2},$$

upon remarking that $\int \mu' ds = +1/2$ and $\int \mu'' ds = -1/2$.

The components of the electromotive force are:

$$v = \frac{d\psi}{dx}, \qquad u = -\frac{d\psi}{dy},$$

so one must make a 90 degree rotation in order to obtain the component of the velocity, whose values will consequently be:

$$\begin{aligned} u &= -\frac{d\psi'}{dy} - \frac{d\psi''}{dy}, \\ v &= \frac{d\psi'}{dx} + \frac{d\psi''}{dx}. \end{aligned} \tag{11}$$

96. Suppose that the tube becomes infinitely-thin, so its trace reduces to the point G, and its moment remains equal to 2π.

Any electricity will be concentrated on the line G that is perpendicular to the xy-plane. The electric force – and consequently, the velocity – will become large in the neighborhood of that line. The expression for ψ'' will become:

$$\psi'' = -2 \log \rho_0 \int \mu'' \, ds = \log \rho_0 ,$$

in which ρ_0 is the distance from the point M to the line that is drawn through the center of gravity of the tube and parallel to the z-axis.

Indeed, ρ is equal to ρ_0, up to higher-order infinitesimals. [MP becomes equal to MG (Fig. 28)]. If the point considered M approaches G indefinitely then ρ_0 will tend to 0, and ψ'' will become infinitely large. On the contrary, ψ' will remain finite, since ρ remains finite even when the point M approaches G indefinitely.

Consequently, u and v increase indefinitely, even though their first terms $d\psi' / dx$ and $d\psi'' / dx$ remain finite; in other words, the functions:

$$u + \frac{d \log \rho_0}{dy},$$

$$v - \frac{d \log \rho_0}{dx}$$

will remain finite, even at the point G.

The function ψ must then satisfy the following conditions:

$$u + \frac{d \log \rho_0}{dy}, \quad v - \frac{d \log \rho_0}{dx}.$$

The problem thus-posed involves only one solution; that solution will be given to us by conformal representation.

97. Indeed, assume that we have obtained the conformal representation of the area C on the surface of a circle of radius unity and its center at the origin. Suppose that the point G corresponds to the center of the circle. The point M (x, y) of the area C will correspond to $M'(x', y')$ on the circle: $x' + \sqrt{-1}\, y'$ is a function of $x + \sqrt{-1}\, y$. Set:

$$\log (x' + \sqrt{-1}\, y') = \psi + \sqrt{-1}\, \varphi.$$

I say that the function:

$$\psi = \log \sqrt{x'^2 - y'^2}$$

satisfies the required conditions.

Indeed:

$$\Delta \psi = 0,$$

since ψ is the real part of an analytic function.

$\psi = 0$ along C, since C is represented by the circle:

$$x'^2 + y'^2 = 1.$$

$\psi = \log \rho_0$ remains finite. It can be infinite only at the point G that corresponds to the point O, for which:

$$x'^2 + y'^2 = 0.$$

Let x_0, y_0 be the coordinates of the point G, so:

$$\rho_0 = | x + \sqrt{-1}\, y | - | x_0 + \sqrt{-1}\, y_0 |,$$

$$\psi - \log \rho_0 = \text{real part of } \log \frac{x' + \sqrt{-1}\, y}{x + \sqrt{-1}\, y - (x_0 + \sqrt{-1}\, y_0)}.$$

$x' + \sqrt{-1}\, y'$ is annulled for $x + \sqrt{-1}\, y = x_0 + \sqrt{-1}\, y_0$, and it will be a simple zero; the quantity under the log sign will thus no longer be annulled at the point G.

98. Velocity of the point *G*. – The velocity of the point G is determined by the equation (1) [no. **65**]:

$$\frac{dx_0}{dt} \int \zeta\, d\omega = \int u\zeta\, d\omega$$

or

$$\frac{dx_0}{dt} \int \zeta\, d\omega = - \int \frac{d\psi'}{dy} \zeta\, d\omega - \int \frac{d\psi''}{dy} \zeta\, d\omega .$$

Now, $\int \frac{d\psi''}{dy} \zeta\, d\omega = 0$. Indeed, if the partition C does not exist then the components of the velocity will be:

$$u = - \frac{d\psi''}{dy}, \qquad v = \frac{d\psi''}{dz}.$$

However, in this case, the center of gravity G of the vortex tube will be fixed:

$$\frac{dx_0}{dt} = - \frac{\int \frac{d\psi''}{dy} \zeta\, d\omega}{\int \zeta\, d\omega} = 0,$$

so:

$$\int \frac{d\psi''}{dy} \zeta\, d\omega = 0.$$

What will then remain is:

$$\frac{dx_0}{dt}\int \zeta\, d\omega = -\int \frac{d\psi'}{dy}\zeta\, d\omega$$

dx_0 / dt will be one of the values that $- d\psi' / dy$ takes inside of the section Ω of the vortex tube. If the tube is infinitely-thin then that value will differ slightly from the value that the function takes at the point G.

In order to calculate dx_0 / dt, we then take the derivative $- d\psi' / dy$ and substitute the coordinates x_0, y_0 of the point G for x and y. We calculate $\frac{dy_0}{dt} = \left(\frac{d\psi'}{dx}\right)_0$.

99. Electrostatic comparison. – Consider a point M in the xy-plane. The electric force that acts at that point is situated in that plane, by reason of symmetry (since the extremities of the electric lines are supposed to lie in the planes $z = \pm l$, which are equidistant from the xy-plane), so its components will have the expressions $d\psi / dx$, $d\psi' / dy$. It is due to the charge on the cylinder Ω and the surface C. That force can thus be regarded as the resultant of two other ones, one of which is due to the charge on C and whose components will be:

$$\frac{d\psi'}{dx},\quad \frac{d\psi'}{dy},\quad \frac{d\log 2\sqrt{l^2 - z^2}}{dz}.$$

If the point M is in the xy-plane then the third component will be annulled. The second force that is due to Ω will have the components:

$$\frac{d\psi''}{dx},\quad \frac{d\psi''}{dy},\quad -\frac{d\log 2\sqrt{l^2 - z^2}}{dz}.$$

The second term in the potential of Ω will no longer depend upon x and y.

The third components are further annulled in the xy-plane, and we will have two forces whose three components are:

$$\frac{d\psi'}{dx},\ \frac{d\psi'}{dy},\ 0,$$

and

$$\frac{d\psi''}{dx},\ \frac{d\psi''}{dy},\ 0;$$

the latter will become very large near the point G. The former will remain finite, and from the preceding section, one will obtain the velocity of the point G upon making a 90 degree rotation.

100. Trajectory of the point G. – In order to find that trajectory (or rather one of its principal properties, since it is not always possible to obtain its equation in a finite form), it is convenient to once more appeal to the electrostatic comparison.

101. I first recall some theorems of electrostatics that I will make use of.

THEOREM I. – Let an electric field contain conductors that possess charges M_1, M_2, …, M_n at potentials V_1, V_2, …, V_n , respectively. If that field experiences a change then the work that is done by the electric forces will be equal to the increase in the sum:

$$\tfrac{1}{2}(M_1V_1 + M_2V_2 + \ldots + M_nV_n) = \tfrac{1}{2}\sum MV .$$

THEOREM II. – Let S and S' be two systems of conductors that exert mutual actions upon each other (abstracting from the forces that the conductors of one system exert upon the conductors of the same system).

Suppose, to simplify the statement, that the conductors are very small and reduce to points. If that is not the case then one must decompose each conductor into infinitely-small elements.

Let M_1, M_2, …, M_n be the electric charges on the conductors that compose the system S, and let V_1', V_2', …, V_n', respectively, be the potentials that the system S' produces at the points where these charges are found. Let M_1', M_2', …, M_n' be the charges on the conductors in the system S', and let V_1, V_2, …, V_n , resp., be the potentials that the system S produces at these points. The force that is done by S on S', when added to the work that is done by the forces that S' exerts upon S, is equal to the increase in the function:

$$M_1V_1' + M_2V_2' + \ldots + M_nV_n' = M_1'V_1 + M_2'V_2 + \ldots + M_n'V_n$$

or

$$\sum MV' = \sum M'V .$$

If the electric masses are infinite in number then the theorem will still be true; however, one will replace the sums $\sum$ with integrals $\int$.

102. Apply these theorems to the study of the electric field that we have defined [no. **95**].

The forces that are exerted in this field define four groups:

The forces F_1 that the charges μ'' exert on each other.
The forces F_2 that are the actions of the charges μ' on the charges μ''.
The forces F_2' that are the actions of the charges μ'' on the charges μ'.
The forces F_3 that are the actions of the charges μ'' on each other.

Decompose the cylinder Ω into infinitesimal elements in the following manner: We divide the cylinder Ω into infinitely-thin cylinders of section $d\omega$ that are parallel to Oz, and we then cut up these cylinders with planes that are parallel to the xy-plane; the volume of each slice will be $d\omega\,dz$, and its charge will be $\mu''\,d\omega\,dz$.

The components of the force F_2 that relates to that element will be:

$$\mu''\,d\omega\,dz\,\frac{d\psi'}{dx},$$
$$\mu''\,d\omega\,dz\,\frac{d\psi'}{dy},$$
$$\mu''\,d\omega\,dz\,\frac{d\log 2\sqrt{l^2-z^2}}{dz}.$$

We remark that the section of the cylinder Ω was assumed to be very small. $d\psi'/dx$ and $d\psi'/dy$ have values that are reasonably constant inside of that section. The point of application of the resultant of the forces F_2 is situated inside of Ω; it is thus very close to the line that is drawn through the point G and parallel to Oz. Upon letting l denote the total charge on Ω, which is supposed to be concentrated on that line, the resultant of the forces F_2 will have the projections:

$$-l\,\frac{d\psi'}{dx}, \qquad -l\,\frac{d\psi'}{dy}, \qquad 0.$$

103. The evaluation of the work that is done by these forces begins with the total work that is done by the four kinds of electrostatic forces F_1, F_2, F_2', F_3.

The work that is done by the electrostatic forces is represented, as we just recalled, by the increase in:

$$\tfrac{1}{2}\int V\,dm,$$

in which V is the potential at which one finds the infinitely-small charge dm, and the integral is taken over all the masses. In the case that we are occupied with, the potential will be ψ. The desired work will then be equal to the increase in:

$$\tfrac{1}{2}\int \psi\,\mu''\,d\omega\,dz + \tfrac{1}{2}\int \psi\,\mu'\,ds\,dz,$$

in which $d\omega$ is the section of one of the elementary cylinders into which we have decomposed Ω, and ds is an element of the contour C, in such a way that the surface C is decomposed into rectangles whose areas are $ds\,dz$.

Upon remarking that ψ, μ', and μ'' do not depend upon z, and that one must integrate between $z = -l$ and $z = +l$, that expression can be written:

$$l\int \psi\,\mu'' d\omega + l\int \psi\,\mu' ds\,.$$

Set:

$$P = \int \psi\,\mu'' d\omega + \int \psi\,\mu' ds\,.$$

The work that is done by the electrostatic forces will be represented by:

$$d\mathcal{T} = l\,dP.$$

104. Make the same calculation for the forces F_1 by themselves. The charge on a small cylinder is $\mu'' d\omega\,dz$, and its potential is:

$$\psi'' - \log 2\sqrt{l^2 - z^2}\,.$$

One must then have:

$$d\mathcal{T} = \tfrac{1}{2} d\int [\psi'' - \log 2\sqrt{l^2 - z^2}\,]\,\mu'' d\omega dz$$

$$= \tfrac{1}{2} d\int \psi''\mu'' d\omega dz - \tfrac{1}{2} d\int \mu'' d\omega \log 2\sqrt{l^2 - z^2}\,]\,dz$$

$$= ld\int \psi''\mu'' d\omega - \tfrac{1}{2}\int \mu'' d\omega \int_{-l}^{+l} \log 2\sqrt{l^2 - z^2}\,dz\,,$$

since μ'' does not depend upon z. The last two integrals are constants. Consequently, upon setting:

$$P' = \int \psi''\mu'' d\omega\,,$$

the force that is done by the forces F_1 will have the expression:

$$l\,dP'.$$

The work that is done by the other forces F_2, F_2', and F_3 will be represented by the difference:

$$l\,d\,(P - P').$$

Now:

$$P - P' = \int \psi'\mu'' d\omega + \int \psi\,\mu' ds\,.$$

ψ represents the potential at a point C; that potential is zero, since C is connected to ground. The second term is therefore zero. The first integral must be taken over the entire area Ω. Since that area is infinitely small, ψ' will have a value that is constant in all of its extent, up to a constant, which will be the value ψ_0' that it has at the point G. We can then write:

$$P - P' = \psi_0' \int \mu'' d\omega = -\,\psi_0'\,.$$

105. Suppose that the tube displaces, and the velocity of its center of gravity has the components:

$$\frac{dx_0}{dt} = -\left(\frac{d\psi'}{dy}\right)_0, \qquad \frac{dy_0}{dt} = \left(\frac{d\psi'}{dx}\right)_0 .$$

$\left(\frac{d\psi'}{dy}\right)_0$ denotes the value that $\frac{d\psi'}{dy}$ takes when one substitutes the coordinates x_0, y_0 of the point G for x and y.

The masses m will then displace, and that displacement will be modified at the same time as the electric distribution on the surface C.

The forces F_2 have:

$$\frac{d\psi'}{dx}, \quad \frac{d\psi'}{dy}$$

for their components in the xy-plane. The resultant of the forces F_2 is then normal to the trajectory of the point G, and therefore does no work; the same thing will be true for the forces F_2' and F_3 . Indeed, the resultant of these forces is normal to the conductor C (because the lines of force always contact the conductors normally for an electrostatic distribution). The masses μ' will displace, *but while remaining on the surface of the conductor*, and consequently, in a direction that is perpendicular to that of the force. That force will do no work.

The total work that is done by the forces F_2, F_2', and F_3 will then be zero.

It then results that:

$$d(P - P') = 0$$

or

$$P - P' = \text{const.},$$

and

$$\psi_0' = \text{const.}$$

106. The trajectory is a closed curve. – Suppose that one makes a conformal representation of the area C onto the circle K that has its center at O (Fig. 29). The point G (x_0, y_0) corresponds to G' (x_0', y_0'), and the point M (x, y) corresponds to the point M' (x', y'). Set:

$$\begin{aligned} x' + \sqrt{-1}\,y' &= Z', \\ x + \sqrt{-1}\,y &= Z, \\ x_0' + \sqrt{-1}\,y_0' &= Z_0', \\ x_0 + \sqrt{-1}\,y_0 &= Z_0 . \end{aligned}$$

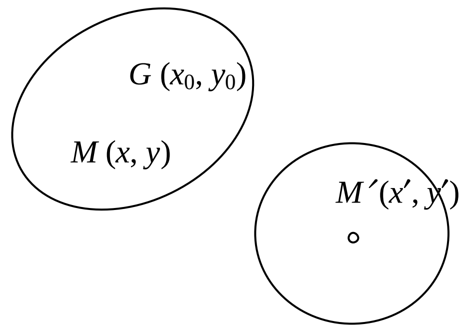

Figure 29.

Z is a certain function of Z, namely, $f(Z)$, and Z_0' is the same function of Z_0, namely, $f(Z_0)$.

Further set:

$$x_0' - \sqrt{-1}\, y_0' = U_0'.$$

In order to be able to apply the formulas, we must find another conformal representation such that the point G corresponds to the point O, and M, to a point $M''(x'', y'')$, such that:

$$x'' + \sqrt{-1}\; y'' = Z'' = \varphi(Z).$$

During the displacement of the point G, the form of the function φ will vary, but that of f will remain the same, because the point G plays no role in the definition of f.

We know [no. **88**] that it suffices to takes:

$$Z'' = \frac{Z' - Z_0'}{Z' U_0' - 1}.$$

Indeed: If mod $Z' = 1$ then one will have mod $Z'' = 1$.

If $Z = Z_0$ then $Z' = Z_0'$, and $Z_0' = 0$; i.e., the point G will indeed correspond to the point O.

Z'' will be a function of x and y:

$$Z'' = e^{\psi + \sqrt{-1}\varphi},$$

$$\psi \;= p \text{ real part of } \log Z'' = \log |Z''|,$$

$$\psi'' = \log \rho_0,$$

if one takes:

$$\rho_0 = MG = |Z - Z_0|,$$

$$\psi' = \log\left|\frac{Z''}{Z - Z_0}\right| = \log\left|\frac{Z' - Z_0'}{Z - Z_0}\right| - \log|Z' U_0' - 1|.$$

ψ_0' is the value of that expression for $Z = Z_0$, so upon applying l'Hôpital's rule:

$$\psi_0' = \log\left|\frac{dZ_0'}{dZ_0}\right| - \log|Z_0' U_0' - 1| = \log\left|\frac{dZ_0'}{dZ_0}\right| - \log[1 - |Z_0'|^2]\,.$$

The equation ψ_0' = const. can then be written:

$$\frac{\left|\dfrac{dZ_0'}{dZ_0}\right|}{1 - |Z_0'|^2} = \text{const.}$$

when one passes from logarithms to numbers.

The trajectory is therefore always a closed curve.

CHAPTER VII

MOTION OF VORTEX TUBES. GENERAL THEOREMS. TUBES OF REVOLUTION

107. Vortex tubes of revolution. – Suppose that vortex tubes exist in an indefinite liquid that are tubes of revolution around the *z*-axis. If that condition is satisfied at the time origin then it will persist constantly; any plane that passes through the *z*-axis will be a symmetry plane and remain one.

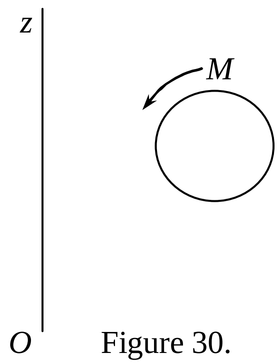

Figure 30.

Let *M* be an arbitrary point of the liquid; draw the meridian plane that passes through that point (Fig. 30), and take the image of the system with respect to that plane. The velocity of the point *M* should not change (by reason of symmetry), so it is necessary that it should continue in that meridian plane.

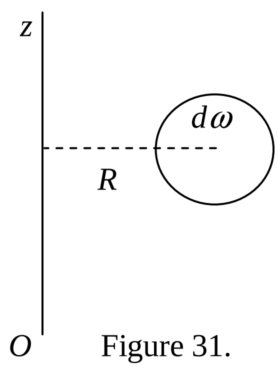

Figure 31.

108. Consider an infinitely-thin tube that forms a kind of torus (Fig. 31), let $d\omega$ be its cross-section, and let *R* be the distance from the center of gravity of that section to the *Oz* axis. The volume of the tube will then be:

$$2\pi R\, d\omega,$$

and it must remain constant. Since the vorticity σ is perpendicular to the meridian plane, the moment of the tube will have a value:

$$2\sigma\, d\omega,$$

which is constant all along the tube. Since $d\sigma$ is constant, it is necessary that σ should also be so; σ depends upon only z and R. Upon setting:

$$x = R\cos\varphi,$$
$$y = R\sin\varphi,$$
$$z = \sigma,$$

it will become:

$$\sigma = f(R, z).$$

On the other hand, the moment of the tube must remain constant in time; consequently, the same thing will be true for σ / R.

109. We have to find functions u, v, w that satisfy the equations:

$$(1)\qquad \begin{aligned} 2\xi &= \frac{dw}{dy} - \frac{dv}{dz}, \\ 2\eta &= \frac{du}{dz} - \frac{dw}{dx}, \\ 2\zeta &= \frac{dv}{dx} - \frac{du}{dy}, \\ \frac{du}{dx} + \frac{dv}{dy} &+ \frac{dw}{dz} = 0. \end{aligned}$$

These equations have the same form as those of Maxwell. They will coincide with Maxwell's equations by assuming that one has replaced the vortex tubes with the currents whose components are $\xi / 2\pi$, $\eta / 2\pi$, $\zeta / 2\pi$; u, v, w will be the components of the magnetic field that is determined by these currents.

Maxwell introduced what one calls the potential vector whose components F, G, H are defined by the conditions:

$$(2)\qquad u = \frac{dH}{dy} - \frac{dG}{dz}, \text{ etc.},$$

and which verify the condition:

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0.$$

Upon eliminating u, v, w, one will find that:

$$\Delta F + H\pi u = 0$$

or

$$\Delta F + 4\pi \cdot \frac{\xi}{2\pi} = 0.$$

F is then the potential of an attracting material whose density is $\xi / 2\pi$. Let x, y, z be the coordinates of a point of the field, let x', y', z' be the coordinates of the center of gravity of a volume element $d\tau'$, let ξ', η', ζ' be the values of ξ, η, ζ at that point, and let r be the distance between the points x, y¸ z and x', y', z'. From that, we will have:

$$(3) \qquad \left\{ \begin{aligned} F &= \int \frac{\xi' \, d\tau}{2\pi r}, \\ G &= \int \frac{\eta' \, d\tau}{2\pi r}, \\ H &= \int \frac{\zeta' \, d\tau}{2\pi r}. \end{aligned} \right.$$

These formulas give F, G, H, and in turn u, v, w, when ξ, η, ζ are known. (Cf., *Électricité et optique,* I, page 144, *et seq*.).

110. Expressing the *vis viva* of the liquid. – The total *vis viva* of the liquid will have the expression:

$$(4) \qquad T = \tfrac{1}{2} \int (u^2 + v^2 + w^2) \, d\tau,$$

upon supposing that the density of the liquid is taken to be unity. In the electrodynamical comparison, if the medium in which one finds the currents is non-magnetic (μ = 1) then the electrokinetic energy will be:

$$(5) \qquad \frac{1}{8\pi} \int (u^2 + v^2 + w^2) \, d\tau = \frac{T}{4\pi}.$$

That electrokinetic energy is susceptible to another expression. Indeed, consider a current element on one of the circuits that generates the field. Let ds be that element, i, the current intensity, and let P be the projection of the potential vector onto the direction of the element ds, so the sum:

$$\tfrac{1}{2} \int i \, ds \, P,$$

when extended over all current elements, will represent the kinetic energy. If, for the moment, we (with Maxwell) denote the components of the current by u, v, w then Maxwell showed that:

$$(6) \qquad \tfrac{1}{2} \int i \, ds \, P = \tfrac{1}{2} \int (Fu + Gv + Hw) \, d\tau.$$

(Cf., *Électricité et optique*, I, page 153.)

With our present notations, the components of the current are:

$$\frac{\xi}{2\pi},\ \frac{\eta}{2\pi},\ \frac{\zeta}{2\pi},$$

so

$$\tfrac{1}{2}\int i\,ds\,P = \tfrac{1}{2}\int\left(F\frac{\xi}{2\pi}+G\frac{\eta}{2\pi}+H\frac{\zeta}{2\pi}\right)d\tau = \frac{1}{4\pi}\int (F\xi+G\eta+H\zeta)\,d\tau,$$

and

(7) $$T = \int (F\xi+G\eta+H\zeta)\,d\tau.$$

111. The identity of these two expressions is proved directly, moreover. Indeed.

$$\int (F\xi+G\eta+H\zeta)\,d\tau = \int \sum F\xi\,d\tau = \int \sum F\frac{dw}{dy}d\tau - \int \sum F\frac{dv}{dz}d\tau.$$

We remark that $d\tau = dx\,dy\,dz$, and integrate by parts:

$$\int F\frac{dw}{dy}dx\,dy\,dz = \int Fw\,dx\,dy - \int w\frac{dF}{dy}d\tau.$$

We must integrate from $-\infty$ to $+\infty$. Now, F and w are zero at infinity. All of the known terms are then zero at the limits, and what will remain is:

$$\int F\frac{dw}{dy}d\tau = -\int w\frac{dF}{dy}d\tau.$$

Similarly:

$$\int F\frac{dv}{dz}d\tau = -\int v\frac{dF}{dz}d\tau.$$

Therefore:

$$T = -\tfrac{1}{2}\int \sum w\frac{dF}{dy}d\tau + \tfrac{1}{2}\int \sum v\frac{dF}{dz}d\tau,$$

namely, upon developing:

$$T = \tfrac{1}{2}\int d\tau\left(v\frac{dF}{dz}+w\frac{dG}{dx}+u\frac{dH}{dz}-w\frac{dF}{dy}-u\frac{dG}{dz}-v\frac{dH}{dx}\right)$$

$$= \tfrac{1}{2}\int d\tau\left[u\left(\frac{dH}{dz}-\frac{dG}{dz}\right)+v\left(\frac{dF}{dz}-\frac{dH}{dx}\right)+w\left(\frac{dG}{dx}-\frac{dF}{dy}\right)\right].$$

or, upon referring to equations (2):

$$T = \tfrac{1}{2}\int (u^2+v^2+w^2)\,d\tau.$$

112. Mutual actions of the current elements that replace the vortex tubes. – Let a current element *MM′* of length *ds* and intensity *i* be placed in a magnetic field. Let *MP* be the vector that represents the magnetic force at *M*, and let *MC* be a vector that is tangent to *MM′* and proportional to *i ds*. As one knows, the element *MM′* is subject to a force that is perpendicular to the plane *MTC* and equal to the area of the parallelogram that is constructed on *MT* and *MC*. Let *dx*, *dy*, *dz* be the projections of *ds* onto the three axes, let α, β, γ be those of the magnetic force *MT*, and let *i dx*, *i dy*, *i dz* be the projections of the vector *MC*. The projections of the electrodynamical force onto the axis will be:

$$(Ox) \qquad i\,dz \cdot \beta - i\,dy \cdot \gamma,$$
$$(Oy) \qquad i\,dx \cdot \gamma - i\,dz \cdot \alpha,$$
$$(Oz) \qquad i\,dy \cdot \alpha - i\,dx \cdot \beta.$$

In the case that we are occupied with, the quantities that correspond to *i dx*, *i dy*, *i dz* are:

$$\frac{\xi\,d\tau}{2\pi},\ \frac{\eta\,d\tau}{2\pi},\ \frac{\zeta\,d\tau}{2\pi},$$

and those that correspond to α, β, γ are u, v, w. The components of the electrodynamical force will then be:

$$\frac{d\tau}{2\pi}(v\,\zeta - w\,\eta) = X\,d\tau,$$
$$\frac{d\tau}{2\pi}(w\,\xi - u\,\zeta) = Y\,d\tau,$$
$$\frac{d\tau}{2\pi}(u\,\eta - v\,\xi) = Z\,d\tau,$$

upon setting:

$$X = \frac{1}{2\pi}(v\,\zeta - w\,\eta),\ \ldots,\ \text{etc.}$$

113. THEOREM. – The forces (X, Y, Z), which represent the mutual actions of the fictitious current elements, and with which we have replaced our vortex tubes, must be pair-wise equal and opposite. Therefore:

The sum of their projections onto an arbitrary axis will be zero:

$$(8) \qquad \sum X\,d\tau = 0, \qquad \text{or:} \qquad \sum (v\zeta - w\eta)\,d\tau = 0.$$

Similarly, the sum of their moments with respect to an arbitrary axis will be zero:

$$(9) \qquad \sum (Xy - Yx)\,d\tau = 0.$$

114. Direct proof of the equation $\sum X\,dt = 0$. – Let S be a surface; for example, a sphere whose center is at the origin and whose radius R is very large. Let $d\omega$ be an element on that surface, and let l, m, n be the direction cosines of the normal to that element. I say that the integral:

$$\int d\omega\left[\frac{l}{2}(u^2+v^2+w^2)-u\,(lu+mv+nw)\right]$$

is zero.

Indeed, we have supposed that our vortex tubes are all at finite distances. A point of the surface that is situated at a very large distance R from the origin will also be at a very large distance from the vortex tubes that is of order R. The vector (u, v, w) represents the velocity or the magnetic force. One knows that this magnetic force varies like $1 / R^3$. If we then regard R as infinite of first order then u, v, w will infinitesimals of third-order, like $1 / R^3$, and u^2, v^2, w^2 will be of sixth order. Granted, the surface over which one extends the integral is infinitely large, but only of second order; the integral is therefore zero.

Transform that integral with the known formula:

$$\int l\,F\,d\omega = \int \frac{dF}{dx}\,d\tau\,,$$

and it will become:

$$\int d\tau\left[\begin{array}{c} u\dfrac{du}{dx}+v\dfrac{dv}{dx}+w\dfrac{dw}{dx} \\ -u\dfrac{du}{dx}-v\dfrac{dv}{dy}-w\dfrac{dw}{dz} \\ -u\dfrac{du}{dx}-v\dfrac{du}{dy}-w\dfrac{du}{dz} \end{array}\right] = 0.$$

The second line is zero, by virtue of the equation of continuity. Upon taking into account equations (1), what will remain is:

$$2\int d\tau\,(v\,\zeta - w\,\eta) = 0$$

or

$$\sum X\,d\tau = 0.$$

115. The theorem of moments gives an analogous equation:

$$\int (Xy - Yx)\,d\tau = 0$$

or

$$\int \begin{vmatrix} y & x & 0 \\ u & v & w \\ \xi & \eta & \zeta \end{vmatrix} d\tau = 0. \tag{9}$$

116. Another expression for the *vis viva*. – In order to obtain that expression, first recall the following theorem of electrodynamics: If one displaces the currents without changing their intensity then the work that is done by the electrodynamic forces will be equal to the increase in the electrokinetic energy. Let a current element *ds* have an intensity *i*; suppose that the coordinates *x*, *y*, *z* of that element experience the variations δx, δy, δz:

$$\int (X\,\delta x + Y\,\delta y + Z\,\delta z)\,d\tau = \frac{\delta T}{4\pi}.$$

In particular, assume that δx, δy, δz are proportional to *x, y¸ z*:

$$\delta x = \varepsilon\, x, \qquad \delta y = \varepsilon\, y, \qquad \delta z = \varepsilon\, z.$$

Since ε is an infinitely-small constant, the transformation will amount to multiplying all of the distances by $1 + \varepsilon$. Suppose, to fix ideas, that there are only two currents; in that case:

$$T = \tfrac{1}{2}(L\, i^2 + 2M\, i\, i' + N\, i'^2),$$
$$\delta T = \tfrac{1}{2}(\delta L\, i^2 + 2\delta M\, i\, i' + \delta N\, i'^2).$$

The currents will remain homothetic with respect to the origin under the particular change that we have performed; *L*, *M*, *N* are their lengths, in the electromagnetic system. By reason of homogeneity, these lengths must be multiplied by $1 + \varepsilon$. Therefore:

$$\delta L = L\varepsilon, \qquad \delta M = M\varepsilon, \qquad \delta N = N\varepsilon,$$

and finally:

$$\delta T = T\varepsilon.$$

That formula will obviously remain true for an arbitrary number of currents, and even an infinitude of them. Therefore:

$$\int (X\,x + Y\,y + Z\,z)\,\varepsilon = \frac{T\varepsilon}{4\pi},$$

or upon suppressing ε and replacing *X*, *Y*, *Z* with their values:

$$= \int \begin{vmatrix} x & y & z \\ u & v & w \\ \xi & \eta & \zeta \end{vmatrix} d\tau = \frac{T}{2}, \tag{10}$$

or, upon representing the determinant by D:

$$\int D\,d\tau = \frac{T}{2}.$$

117. Liquid enclosed in a vessel. – If one is dealing with a liquid that is enclosed in a vessel and fills it completely then one can again appeal to the electrodynamical comparison on the condition that one must replace the vessel with a perfect conductor.

Maxwell has shown that the currents in such a conductor are localized to the surface, and that surface will form an electrodynamical screen (i.e., a current sheet). The theorems that were stated previously will remain true if one takes that current sheet into account.

118. Consider a point on the surface of the vessel: The molecules of liquid that are situated inside of the vessel have a velocity that is situated in the tangent plane, while the liquid is at rest at a point that is infinitely close to the surface, but situated on the other side. The velocity will then be discontinuous. That discontinuity can be replaced with the introduction of a vortex tube. Indeed, take the particular case of a planar surface – for example, the *xy*-plane – such that the liquid is beneath that plane. The velocity will be zero beneath the plane, so the vortex tube will be constant and parallel to *Ox*.

Suppose that the variation of that velocity does not happen impulsively, but in a continuous fashion, although very rapidly. u will be a certain function of z in the transition layer:

$$u = f(z),$$

and

$$\frac{du}{dz} = f(z)$$

will be non-zero. From the hypotheses that were made, u will be a function of only z: $v = w = 0$. Therefore, ξ and η will be zero, but:

$$\eta = \frac{1}{2}\frac{du}{dz}$$

will be non-zero and even very large.

The vortex that replaces the discontinuity will then be parallel to the separation plane and perpendicular to the velocity.

119. The theorem will still be true when the separation surface is curved, and the velocity is variable. In order to prove this, it will suffice to decompose the surface into elements that are small enough that one can consider them to be planar and the velocity as being constant over all of their extent. It is always possible to choose the thickness of the transition layer to be itself very small with respect to these elements; we can then replace the surface with a sheet of the vortex tube. From the preceding proof, each of the tubes will be directed in the plane of the element – i.e., in the plane that is tangent to the surface – and its boundary will be on that surface itself and perpendicular to the velocity at the point considered.

120. The force (X, Y, Z) that represents the electrodynamical action on an element of our fictitious current must be perpendicular to both the current and the magnetic force. The current is in the plane that is tangent to the surface of vessel. The magnetic force, which is directed like the velocity, is also situated in the tangent plane. The force (X, Y, Z) is then normal to the surface of the vessel.

121. In order to apply the theorems that we have proved in the case of an indefinite liquid [no. **113-116**] to the present case, we must take into account two groups of electrodynamical forces, namely, the ones that act upon the currents that we have substituted for the vortex tubes and the ones that act upon the current sheet that replaces the surface.

However, in a certain number of particular cases, these complementary terms that are provided by these latter forces, which we must add to our equations, will have a zero sum.

For example, if the vessel has the form of a cylinder whose generators are parallel to *Ox* then the forces that act upon the current sheet (being normal to the surface) will be normal to *Ox*, and the sum of the projections onto *Ox* will be zero. The first theorem will remain true without modification [no. **113**].

If the vessel is a figure of revolution around *Oz* then the sum of the moments of the complementary forces with respect to *Oz* will be zero, since all of these forces will meet *Oz*. The second theorem [no. **113**] will still be true.

If the vessel is a sphere or the space between two concentric spheres then the second theorem will be true with respect to an arbitrary axis that passes through the center when the sphere is a figure of revolution around an arbitrary axis.

If the vessel is bounded by two planes that are parallel to the *xy*-plane, for example, then one can consider it to be a figure of revolution around the *z*-axis or a cylinder that is parallel to *Ox* and *Oy*, and apply the remarks that relate to these various cases.

122. The vortex tubes are cylinders that are parallel to *Oz*. – Under these conditions:

$$\xi = \eta = 0, \qquad w = 0.$$

ζ, u, v depend upon only z.

The motion will not be modified [no. **56**] if we bound the liquid with two planes that are parallel to the *xy*-plane; for example: $z = 0$, $z = 1$. The two theorems will again be applicable. Here is how we define the element $d\tau$: Decompose the *xy*-plane into surface elements $d\omega$, and take each of these elements to be the base of a cylinder that is parallel to *Oz* and bounded by the planes $z = 0$ and $z = 1$. The space that is found between the two planes will be divided into an infinitude of these cylinders in that fashion. We can then draw planes that are parallel to the *xy*-plane with a distance of dz between them. It is the slice of one of these cylinders that is bounded by two of those planes that we take to be the element $d\tau$. We will then have:

$$d\tau = d\omega\, dz,$$

and the statement of the first theorem will become:

$$\int v\,\zeta\, dz\, d\omega = 0.$$

Since v, ζ do not depend upon z, we can integral over z between the limits $z = 0$ and $z = + 1$, and get:

(11) $$\int v\,\zeta\, d\omega = 0;$$

in the same manner, we will find that:

(12) $$\int u\,\zeta\, d\omega = 0.$$

Upon developing the determinant [no. **116**], the second theorem will become:

$$\int (xu + yv)\,\zeta\, dz\, d\omega = 0,$$

or upon integrating over z, as above:

(13) $$\int (xu + yv)\,\zeta\, d\omega = 0.$$

123. We thus find the theorems that we have proved in the preceding chapters.

The center of gravity of all of the vortex tubes remains fixed, which is expressed by the equations:

$$\int x\,\zeta\, d\omega = \text{const.},$$
$$\int y\,\zeta\, d\omega = \text{const.},$$

and the moment of inertia of the tubes with respect to an arbitrary axis that is parallel to Oz will remain constant. For example, with respect to Oz itself:

(14) $$\int (x^2 + y^2)\,(x^2 + y^2)\, dz\, d\omega = \text{const.}$$

In fact, upon differentiating these equations with respect to t, one will get:

$$\int \frac{dx}{dt}\,\zeta\, d\omega = \int v\,\zeta\, d\omega = 0,$$

$$\int \frac{dy}{dt}\zeta\,d\omega = \int v\,\zeta\,d\omega = 0,$$

$$\int \left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)\zeta\,d\omega = \int (xu + yv)\,\zeta\,d\omega = 0.$$

We recover the equations (11), (12), and (13) that were written above.

Finally, upon writing that the sum of the moments of the quantities of motion of the tubes with respect to Oz is zero, we will get:

$$\int (uy - vx)\;\zeta\,d\omega = \text{const.}$$

or

$$\int \begin{vmatrix} x & y & 0 \\ u & v & 0 \\ 0 & 0 & \zeta \end{vmatrix} d\omega = \text{const.},$$

which is nothing but equation (10) of no. **116** in the case that we have placed ourselves in:

$$\int D\;d\tau = \text{const.}$$

124. Direct proof of the relation $\int D\;d\tau = T / 2$. – Now, return to the general case. We have proved the relation (10):

$$\int D\,d\tau = \frac{T}{2}$$

by the electrodynamical comparison.

That relation can also be established directly, as we shall now see.

Set:

$$h = \frac{u^2 + v^2 + w^2}{2},$$

to abbreviate, or:

$$T = \int h\,d\tau\;.$$

Develop the determinant D in elements of the first row:

$$D = Ax + By + Cz,$$

upon setting:

$$A = \eta\,w - \zeta\,v,$$
$$B = \zeta\,u - \xi w,$$
$$C = \xi\,v - \eta\,u.$$

For example, calculate A:

$$\begin{aligned}
2A &= 2\eta\, w - 2\zeta\, v = w\left(\frac{du}{dz}-\frac{dw}{dx}\right) - v\left(\frac{dv}{dx}-\frac{du}{dy}\right),\\
&= u\frac{du}{dx}-u\frac{du}{dx}+v\frac{du}{dy}-v\frac{du}{dy}+w\frac{du}{dz}-w\frac{du}{dz},\\
&= u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}-\frac{dh}{dx},
\end{aligned}$$

so the expression for B and C can be deduced from this by symmetry.

Imagine the integral:

$$\int_z \left[(lx+my+nz)\frac{u^2+v^2+w^2}{2}-(lu+mv+nw)(xu+yv+zw)\right] d\omega.$$

That integral will be zero when it is taken over the surface of a sphere of very large radius, from an argument that we have made several times already (see, in part, **114**). We transform it with the formula that we have also already made use of:

$$\int l\, X\, d\omega = \int \sum \frac{dX}{dx} d\tau$$

or

$$\int \sum l\, X\, d\omega = \int \sum \frac{dX}{dx} d\tau,$$

and get:

$$X = x\,\frac{u^2+v^2+w^2}{2} - u\,(x\,u+y\,v+z\,w) = xh - uK,$$

upon setting:

$$K = xu + yv + zw,$$

and consequently:

$$\frac{dX}{dx} = x\frac{dh}{dx}+h-\frac{du}{dx}K-u^2-u\left(x\frac{du}{dx}+y\frac{dv}{dx}+z\frac{dw}{dx}\right),$$

$$\frac{dY}{dy} = y\frac{dh}{dy}+h-\frac{dv}{dy}K-v^2-v\left(x\frac{du}{dy}+y\frac{dv}{dy}+z\frac{dw}{dy}\right),$$

$$\frac{dZ}{dz} = z\frac{dh}{dz}+h-\frac{du}{dz}K-w^2-w\left(x\frac{du}{dz}+y\frac{dv}{dz}+z\frac{dw}{dz}\right),$$

where the last two equations are obtained by symmetry. We must then write that:

$$\int \sum \frac{dX}{dx} d\tau = 0.$$

Upon performing the sum $\sum$, we will find, after the obvious reductions, and upon taking the equation of continuity into account, that:

$$\sum \frac{dX}{dx} = h - 2\,(Ax + By + Cz) = h - 2D,$$

so

$$\int h\,d\tau - \int 2D\,d\tau = 0,$$

$$\int D\,d\tau = \tfrac{1}{2}\int h\,d\tau = \frac{T}{2}.$$

125. The vortex tubes are figures of revolution around *Oz*. – Suppose that the vortex tubes are figures of revolution around *Oz*. In that case, we adopt semi-polar coordinates by setting:

$$x = \rho \cos \varphi,$$
$$y = \rho \sin \varphi,$$
$$z = z.$$

By hypothesis, the vorticity is perpendicular to the meridian at each point.

Therefore, if σ is the magnitude of the vorticity then:

$$\xi = -\ \sigma \sin \varphi,$$
$$\eta = \ \ \sigma \cos \varphi,$$
$$\zeta = 0.$$

By reason of symmetry [no. **107**], the velocity (u, v, w) will be situated in the meridian plane:

$$u = \frac{d\rho}{dt} \cos \varphi,$$
$$v = \frac{d\rho}{dt} \sin \varphi,$$
$$w = \frac{dz}{dt}.$$

Moreover, σ / ρ is constant, as we have seen [no. **108**].

Substitute these values in our equations; a certain number of them will reduce to identities. However, the following one will remain:

$$\int (u\,\eta - v\,\xi)\,d\tau = 0, \tag{15}$$

in particular.

In order to define the element $d\tau$, consider a meridian plane; for example, the *zy*-plane. Decompose it into surface elements $d\omega$. Each of these elements will generate a volume when it is revolved around *Oz*. If we draw meridian planes that have an angular separation of $d\varphi$ then those planes will cut out sections of the volumes that amount to cylinders of section $d\omega$ and height $\rho\, d\varphi$. The volume of these sections will be:

$$d\tau = \rho\, d\omega\, d\varphi\,.$$

Equation (15) will become:

$$\int \sigma \frac{d\rho}{dt} (\cos^2 \varphi + \sin^2 \varphi) = \rho\, d\omega\, d\varphi = 0,$$

or

(16) $$\int \sigma \frac{d\rho}{dt} \rho\, d\omega\, d\varphi = 0.$$

The integral must be extended over all elements $d\omega$ on one-half of the *zy*-plane and between the limits $\varphi = 0$ and $\varphi = 2\pi$.

Since the coefficient of $d\varphi$ does not depend upon φ, one can integrate over φ and write:

(17) $$\int \sigma\rho \frac{d\rho}{dt} d\omega = 0.$$

Now, transform the determinant D by multiplying it with another one that is equal to 1:

$$D = \begin{vmatrix} x & y & z \\ \xi & \eta & \zeta \\ u & v & w \end{vmatrix} \times \begin{vmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x\cos\varphi + y\sin\varphi & -x\sin\varphi + y\cos\varphi & z \\ \xi\cos\varphi + \eta\sin\varphi & -\xi\sin\varphi + \eta\cos\varphi & \zeta \\ u\cos\varphi + v\sin\varphi & -u\sin\varphi + v\cos\varphi & w \end{vmatrix}$$

or

$$D = \begin{vmatrix} \rho & 0 & z \\ 0 & \sigma & 0 \\ \dfrac{d\rho}{dt} & 0 & \dfrac{dy}{dy} \end{vmatrix} = \sigma\left(\rho\frac{dz}{dt} - z\frac{d\rho}{dt}\right).$$

If we substitute that value of D in the relation (10) then it will become:

$$\int D\, d\tau = \int D\rho\ d\omega d\varphi = \frac{T}{2},$$

$$2\pi \int D\rho\ d\omega = \frac{T}{2}.$$

(18)([1]) $$4\pi \int \sigma\rho\, d\omega \left(\frac{dz}{dt} - z\frac{d\rho}{dt} \right) = T\,.$$

126. Magnitude of the velocity. – Let an infinitely-thin tube of circular cross-section exist by itself; its radius will remain constant. The relation (17):

$$\int \sigma\rho \frac{d\rho}{dt} d\omega = 0$$

expresses the idea that:

$$\int \sigma\rho^2\, d\omega = \text{const.}$$

Indeed, if we differentiate the latter with respect to t, upon remarking that $\sigma\, d\omega$ represents the moment of the tube and is constant, then we will recover equation (17):

Set:

(1) $$\int \sigma\, d\omega = M,$$

(2) $$\int \sigma\rho^2\, d\omega = MR^2.$$

M will be a constant, as well as MR^2.

If σ has the same sign everywhere then R will be found between the extreme values of ρ, namely, ρ_0 and ρ_1 .

Indeed, let ρ_0 be the largest value of ρ:

$$MR^2 < \int \sigma\rho_0^2\, d\omega$$

or

$$MR^2 < \rho_0^2 \int \sigma\, d\omega = M\rho_0^2$$

$$R^2 < \rho_0^2\,;$$

one likewise proves that:

$$R^2 > \rho_1^2\,.$$

Consequently:

$$\rho_0 > R > \rho_1\,.$$

If the tube is infinitely thin then ρ_0 and ρ_1 will differ very little from each other and the mean radius of the tube. That mean radius will also differ very slightly from R, and

---

([1]) Helmholtz found:

$$4\pi \int \sigma\rho\, d\omega \left(\rho\frac{dz}{dt} - 2z\frac{d\rho}{dt} \right) = T,$$

instead of this equation, as a result of an error in calculation. However, the presence of the factor 2 will not change the results that we shall present, since they are based upon only the consideration of the order of magnitude of the different factors.

can be regarded as constant, just like R. The only motion that the tube can undergo will thus reduce to a displacement parallel to Oz. What will the velocity of that displacement be? It is not obvious *a priori* that it will be constant, because the position of the tube relative to Oz will not change under the displacement; the velocity can depend upon the form of the section.

That is not true: The velocity will be constant and very large, as Helmholtz showed by appealing to equation (3) and taking the order of magnitude of the quantities:

$$T, u, v, \text{ and } \frac{d\rho}{dt}$$

into account.

If the section of the tube is infinitely thin then a point that is situated at a finite distance from the tube will have a finite velocity. However, a point that is situated in a neighborhood of that tube will have a very large velocity. The radius of the section of the tube and the distance from that point to the boundary of that section will always be very small relative to R. One will obtain a sufficient approximation by making the neighboring part of the tube into a cylinder and applying the formulas for the cylindrical tubes to it.

127. Consider an infinitely-thin rectilinear tube. In order to determine the velocity, we replace the tube with an indefinite rectilinear current. The velocity is represented by the same vector as the magnetic force; it will then vary inversely with the distance form the point considered to the axis of the current. The velocity will be infinitely large at a point that is infinitely close to that line. Consequently, the *vis viva* T will be infinite.

The potential vector (F, G, H) is defined by the relations:

$$F = \int \frac{\xi}{2\pi}\, dl, \text{ etc.}$$

Suppose that our rectilinear tube is parallel to the x-axis.

In the present case, $\xi / 2\pi$ will be a constant. F will be the potential of a uniformly-electrified, indefinite line. That potential will be of order $\log \rho$ at a point that is infinitely close to the line, if ρ is the distance from the point to the line.

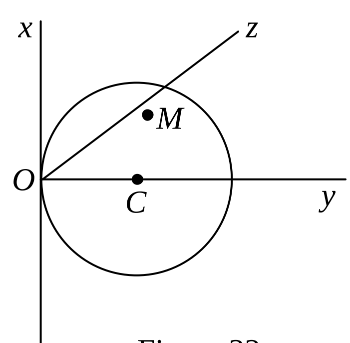

Figure 32.

128. Now, let there be a circular tube or a circular current that replaces it. In the plane that is normal to the circle at the point *O*, I take a point *M* that is very close to *O*. I would like to look for the magnetic force and the vector potential that is generated by the current at the point *M* and compare them to the ones that are generated by a rectilinear current that is directed along the tangent to *O* and has the same intensity as the circular current.

Take the point *O* to be the origin (Fig. 32), the tangent to be the *x*-axis, and the diameter *OC* to be the *y*-axis. The point *M* will be in the *yz*-plane.

Suppose that the intensity of the current is equal to unity. Let *ds* be a current element, *dx*, its projection onto *Ox*, and let *r* be the distance from *ds* to the point *M*:

$$F = \int \frac{dx}{r}.$$

The integral must be taken from – *R* to + *R*, where *R* is the radius of the circle.

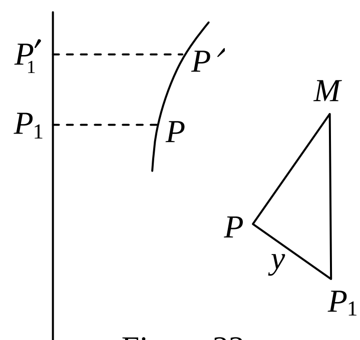

Figure 33.

Let $PP' = ds$, let $P_1P_1' = dx$ be the projection of PP' onto the *x*-axis (Fig. 33), and let *r* be the distance from P_1P_1' to *M*. The vector potential that is due to the rectilinear current will have the expression:

$$F_1 = \int \frac{dx}{r_1}.$$

The integral must be taken from – ∞ to + ∞, but, since we are proposing to study only the order of magnitude of F_1, we can take it between the limits – *R* and + *R*, like the first one.

Indeed, the elements that are situated at a finite distance from *M* will contribute finite terms to the expression for F_1 that are negligible in comparison to the very large terms that are given by the elements that are close to *M*.

It then results that we have the right to write:

$$F - F_1 = \int dx\left(\frac{1}{r} - \frac{1}{r_1}\right).$$

We can then find an upper limit for the element of that integral. Indeed, let:

$0, y_0, z_0$ be the coordinates of M,
$x, y, 0$ " " P,
$x, 0, 0$ " " P_1 .

In the triangle MPP_1:

$$MP = r, \qquad MP_1 = r_1 , \qquad PP_1 = y.$$

Now:

$$\left|\frac{1}{r}-\frac{1}{r_1}\right| = \frac{r_1 - r}{r r_1}$$

is smaller than:

$$(r_1 - r)\left(\frac{1}{r^2}+\frac{1}{r_1^2}\right).$$

Since $r_1 - r < y$, one will finally have:

$$\left|\frac{1}{r}-\frac{1}{r_1}\right| < y\left(\frac{1}{r^2}+\frac{1}{r_1^2}\right)$$

and

$$F - F_1 < \int \frac{y\,dx}{r^2}+\int \frac{y\,dx}{r_1^2} .$$

Moreover:

$$r^2 = \overline{MP}^2 = x^2 + (y - y_0)^2 + z_0^2 ,$$

$$\int \frac{y\,dx}{r^2} = \int \frac{y\,dx}{x^2+(y-y_0)^2+z_0^2},$$

so

$$\int \frac{y\,dx}{r^2} < \int \frac{y\,dx}{x^2} .$$

When the point P approaches 0 indefinitely, y / x^2 will tend to a finite limit $2R$, where R is the radius of the circle. The integral $\int \frac{y\,dx}{r^2}$, and likewise, the integral $\int \frac{y\,dx}{r_1^2}$, will thus remain finite, and as a result, the difference $F - F_1$ will be finite.

129. Order of magnitude of the potential vector. – Since the difference $F - F_1$ is finite, we can replace the circular tube with a rectilinear tube in order to find the order of magnitude of the potential vector.

First, suppose that one is dealing with a unique tube and that the vorticity is constant. The rectilinear tube will be a cylinder with a circular section. The potential vector will be equal to the potential of an attracting mass that is distributed on the cylinder, and whose density will be equal to $\xi / 2\pi$.

At a point exterior to the cylinder, the potential will be the same as if all of the attracting mass were concentrated on the axis. If ρ_0 is the radius of the cylinder then everything will take place at an exterior point – i.e., at a point whose distance from the ρ-axis is larger than ρ_0 – as if there existed an attracting mass along the axis with a density of:

$$\pi\rho_0^2 \frac{\xi}{2\pi} = \frac{\xi\rho_0^2}{2},$$

and the potential at that point will be:

$$\int \zeta\rho_0^2 \log\rho \quad (\rho > \rho_0). \tag{3}$$

For a point that is interior to the cylinder – i.e., for $\rho < \rho_0$ – the potential will be obtained by decomposing the cylinder into two parts by a cylindrical surface that has the same axis as the cylinder and passes through the point considered. The annular layer has no effect upon the point. The other part has the same effect as if all of the mass were concentrated along its axis.

Consequently, the attraction will be equal to:

$$\begin{aligned} &-\frac{\zeta\rho_0^2}{\rho} && \text{for } \rho > \rho_0, \\ &-\zeta\rho_0 && \text{“ } \rho = \rho_0, \\ &-\zeta\rho && \text{“ } \rho < \rho_0. \end{aligned}$$

The potential will then have the expression:

$$-\frac{\zeta\rho^2}{2} + C, \tag{4}$$

so the two formulas (3) and (4) will agree for $\rho = \rho_0$. That condition will determine the constant C:

$$-\frac{\zeta\rho^2}{2} + C = -\zeta\rho_0^2 \log\rho_0,$$

$$C = \zeta\rho_0^2\left(\tfrac{1}{2} - \log\rho_0\right).$$

Suppose that the moment $\zeta\rho_0$ of the tube is finite. If ρ_0 is very small then C, and consequently, the potential vector, will have the same order of magnitude as $\log\rho_0$.

130. Order of magnitude of the *vis viva*. – Let P be the potential vector, and let σ be the vorticity, which are both perpendicular to the meridian. The *vis viva* will have the expression:

$$I = \int \sigma P\, d\tau = \int \sigma P\, d\omega\, d\varphi,$$

or, upon integrating over φ:

$$I = 2\pi \int \sigma P \rho \, d\omega .$$

Now, $\int \sigma \, d\omega$ is finite, since it is the moment of the tube, which we regard as finite, by hypothesis. P has the order of $\log \rho_0$. The *vis viva* will also have that order, and consequently, it will be very large. As for the velocity, it will have the same order of magnitude as the attraction of the cylinder that we considered above, and consequently, it will be $1 / \rho_0$.

131. Velocity of motion. – Set:

(5) $$A = \int \sigma\rho^2 z \, d\omega = MR^2 z_0 ,$$

upon taking:

(6) $$MR^2 = \int \sigma\rho^2 \, d\omega = \text{const.}$$

z_0 will be assigned to a point that is situated inside of the meridian section of the tube. Indeed, let z_1 and z_2 be the extreme ordinates of that section.

I say that:

$$z_1 > z_0 > z_2 .$$

Indeed:

$$\int \sigma\rho^2 z_1 \, d\omega = z_1 \int \rho^2 \sigma \, d\omega = z_1 MR^2.$$

On the other hand:

$$\int \sigma\rho^2 z \, d\omega < \int \sigma\rho^2 z_1 \, d\omega ,$$

or:

$$z_0 MR^2 < z_1 MR^2,$$

and:

$$z_0 < z_1 .$$

If one differentiates A with respect to t then one will get:

$$\frac{dA}{dt} = MR^2 \frac{dz_0}{dt} = \int \sigma \, d\omega \left(2\rho z \frac{d\rho}{dt} + \rho^2 \frac{dz}{dt} \right),$$

because $\sigma \, d\omega$, which represents the moment of the tube, is a constant. On the other hand, we infer from equation (3) [no. **125**] that:

$$\int \sigma\rho^2 \frac{dz}{dt} d\omega = \int \sigma\rho z \frac{d\rho}{dt} d\omega + \frac{T}{4\pi} .$$

Consequently:

(7) $$\frac{dA}{dt} = MR^2 \frac{dz_0}{dt} = 3 \int \sigma\rho z \frac{d\rho}{dt} d\omega + \frac{T}{4\pi} .$$

One deduces from equation (1), upon remarking that z_0 is a constant:

(8) $$0 = \int \sigma\rho \frac{d\rho}{dt} z_0 \, d\omega,$$

and, upon adding (7) and (8):

(9) $$MR^2 \frac{dz_0}{dt} = 3\int \sigma\rho \frac{d\rho}{dt}(z - z_0)\, d\omega + \frac{T}{4\pi}.$$

The first term in the right-hand side is finite. Indeed, $\int \sigma\, d\omega$ is finite, as is ρ. $d\rho / dt$ is the component of the velocity along the radius vector; it is a very large quantity that has the same order as $1 / \varepsilon$. $z - z_0$ is smaller than the diameter ε of the section of the tube. Therefore, $d\rho / dt\,(z - z_0)$ is finite.

One can thus neglect the first term in comparison to the second one, and one will be limited to writing:

$$\frac{dz_0}{dt} = \frac{T}{4\pi} \times \frac{1}{MR_0^2}.$$

It results from this equation that:

1. The velocity dz_0 / dt is very large and has the same order of magnitude as the *vis viva* T.

2. It is reasonably constant, since T is constant. Granted, the first term is variable, but we have shown that it is negligible in comparison to T.

The vortex tube will thus displace with a very large velocity parallel to Oz.

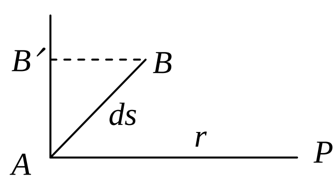

Figure 34.

132. Order of magnitude of the velocity. Direct proof. – As we know, the velocity (u, v, w) is represented by the same vector as the magnetic force. Let $AB = ds$ be a current element (Fig. 34), and let P be a magnetic pole that is equal to unity. The force that the element AB exerts upon that pole will be perpendicular to the plane PAB, and will have the expression:

$$\frac{AB' \times i}{r^2}.$$

AB' is the projection of AB onto a perpendicular PA, i is the current intensity, and r is the distance AP.

In absolute value:

$$\frac{AB' \times i}{r^2} < \frac{i\,ds}{r^2}.$$

Decompose our vortex tube into volume elements and replace each of them with a current element. If σ is the intensity of the vorticity then we must give an intensity $\sigma\,d\omega$ / 2π [no. **43**], and consequently:

$$i\,ds = \frac{\sigma\,ds\,d\omega}{2\pi} = \frac{\sigma\,d\tau}{2\pi},$$

in which $d\omega$ is the element of the section, and $d\tau$ is the volume element of the tube.

The velocity will then have the upper limit:

$$\int \frac{\sigma\,d\tau}{2\pi r^2}.$$

Decompose the tube into elements in the following manner:

Draw some concentric spheres with their centers at P. These spheres cut out slices from the tube that have the form of spherical caps. In particular, consider one of these slices that is bounded by spheres of radius r and $r + dr$, and the integral:

$$\int \frac{\sigma\,d\tau}{r^2},$$

which is extended over all of the volume of that slice. Let σ_1 be the largest value that σ takes in the slice. Since all of the elements in the integral have the same sign:

$$\int \frac{\sigma\,d\tau}{r^2} < \int \frac{\sigma_1\,d\tau}{r^2}.$$

r can be regarded as constant over the entire thickness of the slice. That permits us to write:

$$\int \frac{\sigma\,d\tau}{r^2} < \frac{\sigma_1}{r^2} \int d\tau.$$

Now, $\int d\tau$ is the volume of the slice, which is equal to $\lambda\,dr$, if one calls the section of that slice λ. As a result:

$$\int \frac{\sigma\,d\tau}{r^2} < \frac{\sigma_1 \lambda\,dr}{r^2}.$$

Now, let $d\omega$ be an element of the section of the tube, and consider the elementary tube that is generated by revolution around the axis of that element $d\omega$, so $d\lambda$ will be the section of that tube that is made by the sphere of radius r.

I shall divide these spheres into two groups:

1. The ones whose radius is smaller than a certain upper limit that is, at most, the diameter ε of the section of the total tube. For example, I will take that limit to be 2ε.

2. The ones whose radius is greater than 2ε.

The upper limit on the velocity will then be:

$$\int_0^{2\varepsilon} \frac{\sigma_1 \lambda\, dr}{r^2} + \int_{2\varepsilon}^{\alpha} \frac{\sigma_1 \lambda\, dr}{r^2}. \tag{10}$$

In the first integral, λ is obviously smaller than $4\pi r^2$, which is the area of the entire sphere. Therefore:

$$\int_0^{2\varepsilon} \frac{\sigma_1 \lambda\, dr}{r^2} < 4\pi \int_0^{2\varepsilon} \sigma_1\, dr$$

or

$$\int_0^{2\varepsilon} \frac{\sigma_1 \lambda\, dr}{r^2} < 8\pi\, \sigma_1\, \varepsilon.$$

As always, we suppose that the moment of the tube has a finite magnitude, so $\sigma_1 \Omega$ will be finite. σ_1 will then have the same order of magnitude as $1 / \Omega$ – i.e., $1 / \varepsilon^2$ – and $\sigma_1 \varepsilon$ will have order $1 / \varepsilon$.

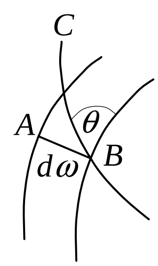

Figure 35.

In the second integral:

$$d\lambda = \frac{d\omega}{\sin\theta},$$

in which θ is the angle by which the sphere cuts the tube (Fig. 35); θ is always greater than a certain limit θ_0, which is different from 0. Indeed, no sphere of the second group can be tangent to the tube. The contact will take place in the meridian section, and the radius of the sphere will be less than ε.

As a result:

$$d\lambda < \frac{d\omega}{\sin\theta_0}$$

and

$$\lambda < \frac{\Omega}{\sin\theta_0},$$

$$\int_{2\varepsilon}^{\alpha} \frac{\sigma_1 \lambda \, dr}{r^2} < \int_{2\varepsilon}^{\alpha} \frac{\sigma_1 \Omega}{\sin \theta_0} \frac{dr}{r^2} \,;$$

the first factor $\dfrac{\sigma_1 \Omega}{\sin \theta_0}$ is finite, while the second one is:

$$\int_{2\varepsilon}^{\alpha} \frac{\sigma_1 \Omega}{\sin \theta_0} \frac{dr}{r^2} = \frac{1}{2\varepsilon} - \frac{1}{a} \,.$$

The integral again has the order of magnitude 1 / ε.

The two terms in the expression (10) then have the same order of magnitude as 1 / ε, and the velocity itself has an order of magnitude that is at most equal to that of 1 / ε.

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CHAPTER VIII

STABILITY CONDITIONS FOR PERMANENT MOTION

133. Permanent motion. – Suppose that the liquid is indefinite, and the vortex tubes are parallel to *Oz*. The motion will obviously be permanent if the ζ are functions of only the distance $\rho = \sqrt{x^2 + y^2}$.

There will be a series of concentric layers around *Oz*, inside of which the vorticity will have a constant value.

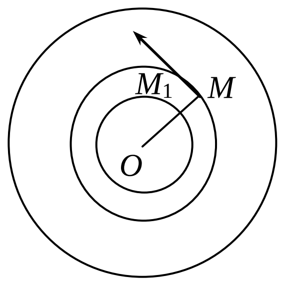

Figure 36.

Let M be an arbitrary point (Fig. 36). The velocity of that point will be directed perpendicular to the radius vector OM, which is drawn from the origin to that point. The point M will go to M_1, which is situated at the same distance from O, during the time dt. M will therefore describe a circumference with its center at O. Let ζ be the value of the vorticity at the point M and the instant t, while ζ' is its value at M_1 at time $t + dt$, and ζ_1 is its value at M at time dt.

I say that:

$$\zeta = \zeta'.$$

Indeed, the molecule that is found at M at time t will go to M_1, which is on the same tube, at time $t + dt$. Now, ζ = const. for the same tube.

Indeed, the two points M and M_1 will be on the same circumference whose center is at O at time t, and ζ will depend upon only the distance to the point O, by hypothesis.

The intensity of the vorticity at M_1 is therefore constant. Since M_1 is an arbitrary point, the same thing will be true for every other point, and the motion will be permanent.

134. Stability of motion. – Is that permanent motion stable?

In other words, if an arbitrary cause tends to deform these concentric layers infinitely little then will that deformation be exaggerated, or will the liquid tend to revert to its original state of motion?

In order to resolve that question, we must study the variation of the velocity (u, v) under these transformations.

We have found [no. **94**], in a general manner, that:

$$u = -\frac{d\psi}{dy}, \qquad\qquad v = \frac{d\psi}{dx},$$

upon setting:

$$\psi = -\int \frac{\zeta' \, d\omega'}{\pi} \log \rho.$$

If we pass to polar coordinates by setting:

$$x = r \cos \varphi,$$
$$y = r \sin \varphi$$

then the components of the velocity will become:

$$\frac{dr}{dt} = -\frac{d\psi}{r\,d\varphi},$$
$$r\frac{d\varphi}{dt} = \frac{d\psi}{dr}.$$

135. Special case. – Suppose that there exists just one cylindrical tube whose cross-section is a circle, and that ζ is constant inside of that circle. The velocity at an arbitrary point will be ζr inside of the cylinder and $\zeta r_0^2 / r$ outside of it, if we call the radius of the cross-section r_0 .

Furthermore:

$$\psi = \zeta r_0^2 \log r .$$

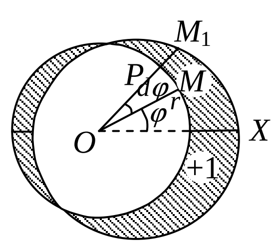

Figure 37.

Suppose that the cylinder experiences a small deformation – for example, the point M goes to M_1 – and the radius vector $OM = s$ becomes a function of time t and the angle φ that it makes with a certain diameter through the origin OX (Fig. 37).

During the time dt, the point M_1 will go to M_1', and its polar coordinates φ and s will experience increments of:

$$d\varphi = \frac{d\varphi}{dt} dt \, \frac{d\psi}{r\,dr} = dt,$$

and

$$ds = -\frac{d\psi}{r\,d\varphi}\;dt.$$

On the other hand, since s is a function of φ and t:

$$ds = \frac{ds}{d\varphi}d\varphi + \frac{ds}{dt}dt\,,$$

so, upon replacing ds and $d\varphi$ with their values and solving for ds / dt, one will get:

(11) $$\frac{ds}{dt} = -\frac{d\psi}{r\,d\varphi} - \frac{d\psi}{r\,dr}\frac{ds}{d\varphi}\,.$$

Develop s in multiples of φ using Fourier's formula:

(12) $$s = r_0 + \sum a_n \cos n\varphi + \sum b_n \sin n\varphi\,.$$

The total surface of the tube must not vary. Therefore, the constant must be equal to r_0, up to second-order infinitesimals. a_n and b_n are functions of t that are independent of φ and very small, since we have assumed that the transformation is very small. The function ψ is, up to a constant, the potential of an attracting mass that is distributed with a density $\zeta / 2\pi$ throughout the entire cylinder that forms the tube. In order to simplify the notation, we shall suppose that $\zeta = 1$ in what follows.

The potential ψ can be considered to be composed of two parts: The one ψ_0 is due to the undeformed cylinder, while the other $\delta\psi$ is due to the deformed part (which is shaded in Fig. 38). Since the thickness of these deformed portions is very small, the potential will have the same value as if, instead of attributing it to a constant cubic density, one gave it a surface density that was proportional to the thickness $s - r_0$.

That surface density will be $\dfrac{s-r_0}{2\pi}$:

(13) $$\frac{s-r_0}{2\pi} = \frac{\sum a_n \cos n\varphi + \sum b_n \sin n\varphi}{2\pi}\,.$$

The potential $\delta\psi$ will then be a function of r and φ, and we can write:

$$\delta\psi = \sum c_n \cos n\varphi + \sum d_n \sin n\varphi\,.$$

For a point that is exterior to the active layers, $\delta\psi$ will verify the Laplace equation:

$$\Delta\,(\delta\psi) = 0.$$

Consequently:

$$\Delta\,(c_n \cos n\varphi) = 0,$$
$$\Delta\,(d_n \sin n\varphi) = 0.$$

Since c_n and d_n are functions of only r:

$$c_n = c'_n\, r^n + c''_n\, r^{-n},$$
$$d_n = d'_n\, r^n + d''_n\, r^{-n},$$

in which c'_n, c''_n, d'_n, d''_n are constants.

Since $\delta\psi$ must not become infinite at the same time as r, it is necessary that:

$$c'_n = d'_n = 0.$$

The function is different for an interior point. It is necessary that it must remain finite for $r = r_0$, so in the interior:

$$c''_n = d''_n = 0.$$

It then results that for an exterior point, we must set:

$$\delta\psi = \sum g_n \left(\frac{r_0}{r}\right)^n \cos n\varphi + \sum h_n \left(\frac{r_0}{r}\right)^n \sin n\varphi \tag{14}$$

and for an interior point:

$$\delta\psi = \sum g'_n \left(\frac{r_0}{r}\right)^n \cos n\varphi + \sum h'_n \left(\frac{r_0}{r}\right)^n \sin n\varphi . \tag{15}$$

The potential remains continuous when one crosses the attracting surface. The two formulas (14) and (15) must then give the same value for $r = r_0$, which demands that:

$$g'_n = g_n, \qquad h'_n = h_n .$$

However, when one crosses the surface, the force will undergo a brief variation that is equal to the product of the density by 4π. That force will have the expression:

1. At an exterior point:

$$\frac{d\,\delta\psi}{dr} = \sum -n\left(\frac{r_0}{r}\right)^n \frac{1}{r}(g_n \cos n\varphi + h_n \sin n\varphi) . \tag{16}$$

2. At an interior point:

(17) $$\frac{d\delta\psi}{dr} = \sum n\left(\frac{r}{r_0}\right)^n \frac{1}{r}(g_n \cos n\varphi + h_n \sin n\varphi)\,.$$

For $r = r_0$, the difference between these two expressions must be equal to 4π, multiplied by the expression (13) for the density.

One then has the relation:

$$-2\sum \frac{n}{r_0}(g_n \cos n\varphi + h_n \sin n\varphi) = 2\sum (a_n \cos n\varphi + b_n \sin n\varphi)\,.$$

Upon identifying the coefficients of cos $n\varphi$ and sin $n\varphi$, one will get:

$$g_n = -\frac{a_n\, r_0}{n}\,,$$
$$h_n = -\frac{b_n\, r_0}{n}\,.$$

If we substitute these values into the expression for $\delta\psi$ then we will get the following expressions for the potential:

1. At an exterior point:

(18) $$\psi = \psi_0 + \delta\psi = \psi_0 - \sum \frac{r_0\,\zeta}{n}(a_n \cos n\varphi + b_n \sin n\varphi)\left(\frac{r_0}{r}\right)^n.$$

2. At an interior point:

(19) $$\psi = \psi_0 - \sum \frac{r_0\,\zeta}{n}(a_n \cos n\varphi + b_n \sin n\varphi)\left(\frac{r}{r_0}\right)^n.$$

(when we re-establish the factor ζ that we have suppressed).

The two formulas will agree for $r = r_0$, and will both give (upon once more supposing that $\zeta = 1$):

$$\psi = \psi_0 - \sum \frac{r_0}{n}(a_n \cos n\varphi + b_n \sin n\varphi)\,,$$

while equation (11) must be verified, moreover.

For $r = r_0$, one will have:

$$\frac{d\psi}{d\varphi} = \sum r_0(a_n \sin n\varphi - b_n \cos n\varphi)\,,$$
$$\frac{d\psi}{r\,d\varphi} = \sum (a_n \sin n\varphi - b_n \cos n\varphi)\,,$$

$$\frac{ds}{d\varphi} = r_0 \sum (-na_n \sin n\varphi + nb_n \cos n\varphi) .$$

$ds / d\varphi$ is a first-order infinitesimal. Since we are neglecting second-order infinitesimals, it will suffice for us to take finite quantities in the coefficients of $ds / d\varphi$.

To that degree of approximation, that coefficient will reduce to:

$$\frac{d\psi_0}{r_0\, d\varphi_0} = 1.$$

Make the replacement in equation (11):

$$\frac{ds}{dt} = - \sum (a_n \sin n\varphi - b_n \cos n\varphi) - \sum (-na_n \sin n\varphi + nb_n \cos n\varphi)$$
$$= \sum \frac{da_n}{dt} \cos n\varphi + \frac{db_n}{dt} \sin n\varphi .$$

Upon identifying them, we will get the conditions:

$$\text{(20)} \qquad \frac{da_n}{dt} = (1 - n)\, b_n , \qquad \frac{db_n}{dt} = -(1 - n)\, a_n .$$

These equations admit the integrals:

$$\text{(21)} \qquad a_n = A \sin (1 - n)\, t + B, \qquad b_n = A \cos (1 - n)\, t + B.$$

These equations show that if a_n and b_n are small at time $t = 0$ then they will always remain very small. The motion will then be stable.

136. Special deformations. – Let there be a deformation such that all of the coefficients are zero to begin with, except for a_n and b_n . All of the other ones will remain constantly zero. The deformed curve will have the equation:

$$\text{(22)} \qquad s = r_0 + a_n \cos n\varphi + b_n \sin n\varphi .$$

The radius vector will then present n maxima and n minima, and the curve will present a series of festoons (Fig. 38).

The curve will keep the same form when t varies, except that it will appear to rotate around Oz with a velocity that is equal to $(1 - n)\, \zeta$.

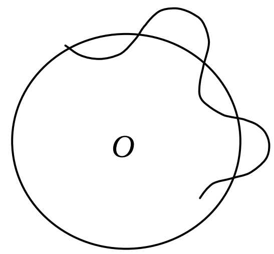

Figure 38.

If the curve is more complex – i.e., if it has more than two non-zero coefficients – then one must decompose it into simple curves that each correspond to a value of n and each rotate around Oz with a particular velocity.

137. Suppose that $n = 1$, so that:

$$s = r_0 + a_1 \cos \varphi + b_1 \sin \varphi \,.$$

That equation, up to second-order infinitesimals, represents a circle whose center has the coordinates a_1 and b_1 . In that case, $1 - n$ will be equal to O, so:

$$\frac{da_1}{dt} = 0, \qquad \frac{db_1}{dt} = 0.$$

The center of the circle (a_1, b_1) is therefore fixed.

Let $n = 2$.

$$s = r_0 + a_2 \cos 2\varphi + b_2 \sin 2\varphi \,.$$

Up to second-order infinitesimals, that equation is the equation of an ellipse whose center is at the origin and whose eccentricity is very slight. On the other hand, $1 - n = 1$, so:

$$a_2 = A \sin t + B,$$
$$b_2 = A \cos t + B,$$

and the ellipse will appear to rotate with a uniform motion.

138. That theorem is still true in the case of an arbitrary ellipse. Indeed, let ζ be the value of the vorticity inside the ellipse, which is assumed to be constant.

Take rectangular coordinates again by choosing the axes to be the axes of the ellipse at a well-defined instant.

The components of the velocity will be:

$$u = -\frac{d\psi}{dy}, \qquad\qquad v = \frac{d\psi}{dx},$$

which will be the components of the attraction that is exerted by an attracting mass that fills the cylinder and possesses a constant density of $\zeta / 2\pi$. Now, we can consider an elliptic cylinder to be an ellipsoid whose one axis is infinite.

Let a homogeneous ellipsoid be:

$$ax^2 + by^2 + cz^2 = 1.$$

Its attraction at an interior point will have the components:

$$Ax, \qquad By, \qquad Cz,$$

in which A, B, C are constants.

The equation of the cylinder will reduce to:

$$ax^2 + by^2 = 1.$$

Therefore:

$$\frac{d\psi}{dx} = Ax,$$
$$\frac{d\psi}{dy} = By.$$

At the beginning of time, the interior point considered will experience a displacement whose components are:

$$u\,dt = -\frac{d\psi}{dy}\,dt = -\,By\,dt,$$
$$v\,dt = -\frac{d\psi}{dx}\,dt = \quad Ax\,dt.$$

I can always determine two numbers α and β such that:

$$A = \alpha + \beta a,$$
$$B = \alpha + \beta b,$$

as long as $a \neq b$ and $A \neq B$.

The displacement will then decompose into two other ones that will have the components:

$$(1) \quad \begin{cases} -\beta by\,dt = dx, \\ \beta ax\,dt = dy, \end{cases} \qquad\qquad (2) \quad \begin{cases} -\alpha y\,dt = dx, \\ \alpha x\,dt = dy, \end{cases}$$

respectively.

The displacement (1) will not alter the form of the ellipse. Indeed, differentiating the equation of the ellipse will give:

$$2ax\,dx + 2by\,dy = -\,2ax\beta by\,dt + 2by\beta ax\,dt = 0.$$

The displacement (2) represents a rotation around *Oz*. The ellipse rotates with deformation after the time *dt*, and consequently, after an arbitrary time.

139. Concentric vortex tubes. – Consider a vortex tube that is bounded by two cylindrical surfaces of revolution *C* and *C′* around *Oz* (Fig. 39). We assume that the vorticity has a constant value $\zeta + \zeta'$ inside of the cylinder *C* whose radius is r_0, another constant value ζ' between the two cylinders, and finally, that the vorticity is zero outside the cylinder *C′* whose radius is r_0'.

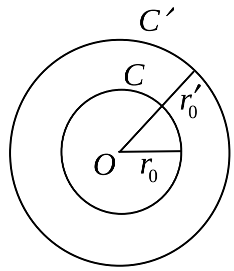

Figure 39.

The effect of the two concentric tubes will be equal to the sum of the effects of the two tubes, one of which has a radius r_0 with a value of the vorticity that is equal to ζ, and the other of which has a radius of r_0' and a vorticity ζ'.

Each of these tubes will give rise to a permanent motion [no. **133**]; if one superimposes their effects then the motion will again be permanent.

140. Stability conditions. – Is that motion stable?

In order for us to answer that question, we shall proceed by a method that is analogous to the one that we employed previously.

Let ψ_0 be the value of ψ. ψ_0 depends upon only *r*, so the velocity will perpendicular to the radius vector and equal to $d\psi_0 / dr$.

If $r < r_0$ then the point will be interior to the two cylinders, so:

$$\frac{d\psi_0}{dr} = \zeta r + \zeta' r.$$

If $r_0 < r < r_0'$ then the point will be exterior to the first cylinder *C* and interior to the second one *C′*, and:

$$\frac{d\psi_0}{dr} = \frac{\zeta r_0^2}{r} + \zeta' r.$$

Finally, if $r > r_0'$ then the point will be exterior to both cylinders, and:

$$\frac{d\psi_0}{dr} = \frac{\zeta r_0^2}{r} + \frac{\zeta' r_0'^2}{r}.$$

For $r = r_0$ and $r = r_0'$, these formulas will become:

$$\frac{d\psi_0}{dr} = (\zeta + \zeta')\, r_0 ,$$
$$\frac{d\psi_0}{dr_0'} = \frac{\zeta r_0^2}{r_0'^2} + \zeta' r_0'.$$

Upon setting $r_0' / r = \varepsilon$, they will become:

$$\frac{d\psi_0}{r_0\, dr_0} = \zeta + \zeta',$$
$$\frac{d\psi_0}{r_0'\, dr_0'} = \zeta \varepsilon^2 + \zeta'.$$

Endow C and C' with a small deformation, in such a way that their radius vectors become:

$$s = r_0 + \sum a_n \cos n\varphi + b_n \sin n\varphi,$$
$$s = r_0' + \sum a_n' \cos n\varphi + b_n' \sin n\varphi,$$

respectively.

We have found that for just one tube the value of ψ after deformation will be:

$$\psi = \psi_0 - \sum \frac{r_0 \zeta}{n} (a_n \cos n\varphi + b_n \sin n\varphi) \left(\frac{r}{r_0}\right)^{\pm n},$$

upon agreeing to give n a sign that makes the factor $\left(\frac{r}{r_0}\right)^{\pm n}$ always be < 1.

For tubes, we will have:

$$\psi = \psi_0 - \sum \frac{r_0 \zeta}{n} (a_n \cos n\varphi + b_n \sin n\varphi) \left(\frac{r}{r_0}\right)^{\pm n} - \sum \frac{r_0' \zeta'}{n} (a_n' \cos n\varphi + b_n' \sin n\varphi) \left(\frac{r}{r_0'}\right)^{\pm n}.$$

If we write the differential equation (11) for each of the tubes then it will be:

$$\frac{ds}{dt} = -\frac{d\psi}{r_0\, d\varphi} - (\zeta + \zeta') \frac{ds}{d\varphi},$$

$$\frac{ds'}{dt} = -\frac{d\psi}{r'_0\,d\varphi} - (\zeta\varepsilon^2 + \zeta')\frac{ds'}{d\varphi}.$$

Develop the terms and identify the coefficients of cos $n\varphi$ and sin $n\varphi$ for $r = r_0$ and $r = r'_0$. We will then find the equation (upon suppressing the indices n as being superfluous):

$$(13)\qquad \begin{aligned}
\frac{da}{dt} &= \quad b\zeta + b'\zeta'\varepsilon^{n-1} - nb\,(\zeta + \zeta'),\\
\frac{da'}{dt} &= \quad b\zeta\varepsilon^{n+1} + b'\zeta' - nb\,(\zeta\varepsilon^2 + \zeta'),\\
\frac{db}{dt} &= -a\zeta - a'\zeta'\varepsilon^{n-1} + na\,(\zeta + \zeta'),\\
\frac{db'}{dt} &= -a\zeta\varepsilon^{n+1} - a'\zeta' + na'\,(\zeta\varepsilon^2 + \zeta').
\end{aligned}$$

We thus obtain four linear differential equations with constant coefficients in order to determine the unknowns a, b, a', b'. The integrals of these equations come down to sums of exponentials of the form $e^{\alpha t}$.

If α is real and positive then that exponential will increase indefinitely with time, and the motion will be unstable, which will accentuate the deformation.

If the exponentials are of the form $e^{-\alpha t}$, where α is real and positive, then the deformation will tend to 0, and one can believe that the motion will then be stable. That is not the case, however, because since the characteristic equation has roots that are equal with opposite signs, we cannot have exponentials like $e^{-\alpha t}$ without having exponentials like $e^{\alpha t}$ at the same time.

Consequently, one will have instability whenever the characteristic equation has a real root.

If the roots are complex, and of the form $\alpha + \sqrt{-1}\,\beta$, then the exponentials will have the form:

$$e^{(\alpha+\sqrt{-1}\beta)\,t} = e^{\alpha t}(\cos\beta t + \sqrt{-1}\,\sin\beta t),$$

and there will again be at least one of them whose modulus increases indefinitely. The motion will again be unstable.

The necessary and sufficient condition for there to be stability is then that all of the roots of the characteristic equations must have the form:

$$\alpha\sqrt{-1},$$

where α is real. The integrals are then a sum of terms such as:

$$e^{\sqrt{-1}\,\alpha t} = \cos \alpha t + \sqrt{-1}\,\sin \alpha t,$$

which will remain finite.

We thus have to look for conditions under which that will be the case.

To abbreviate, set:

$$\begin{aligned}
\alpha &= \zeta - n\,(\zeta + \zeta'),\\
\beta &= \zeta'\,\varepsilon^{n-1},\\
\gamma &= \zeta'\,\varepsilon^{n+1},\\
\delta &= \zeta' - n\,(\zeta\,\varepsilon^2 + \zeta').
\end{aligned}$$

Equations (23) become:

$$(24)\qquad \begin{aligned}
\frac{da}{dt} &= \alpha\,b + \beta\,b',\\
\frac{da'}{dt} &= \gamma b + \delta\,b',\\
\frac{db}{dt} &= -\,\alpha\,a - \beta\,a',\\
\frac{db'}{dt} &= -\,\gamma a - \delta\,a'.
\end{aligned}$$

Furthermore, set:

$$\begin{aligned}
\lambda\,a + \lambda'\,a' &= x,\\
\lambda\,b + \lambda'\,b' &= y,
\end{aligned}$$

in which λ and λ' are two numbers that we shall determine conveniently, at some point.

Multiply the first two equations (24) by λ and λ', and add them. That will give:

$$\frac{dx}{dt} = b\,(\lambda\alpha + \lambda'\gamma) + b'\,(\lambda\beta + \lambda'\delta).$$

I now choose λ and λ' in such a fashion that the right-hand side reduces to Sy.

λ and λ' will be given by the equations:

$$\begin{aligned}
\lambda\alpha + \lambda'\gamma &= S\lambda,\\
\lambda\beta + \lambda'\delta &= S\lambda'.
\end{aligned}$$

If one finds λ and λ' that satisfy these conditions then:

$$\frac{dx}{dt} = Sy.$$

Upon treating the last two equations in the system (24) similarly, one will find:

$$\frac{dy}{dt} = -\,Sx.$$

We obtain the value of S by writing that the determinant of the homogeneous equations in λ and λ' is zero.

S will then be a root of the equation:

$$\begin{vmatrix} \alpha - S & \gamma \\ \beta & \delta - S \end{vmatrix} = 0. \tag{25}$$

That equation has degree two; let S and S_1 be its roots, and let:

λ, λ', x, y be the values that correspond to S,
$\lambda_1, \lambda_1', x_1, y_1$ " " S_1.

We will have:

$$\lambda_1\, a + \lambda_1'\, a' = x_1,$$
$$\lambda_1\, b + \lambda_1'\, b' = y_1,$$
$$\frac{dx_1}{dt} = \quad S_1\, y_1,$$
$$\frac{dy_1}{dt} = -\,S_1\, x_1.$$

The general equation of our equations will then be:

$$\begin{aligned} x &= A \sin(S\,t + B), \\ y &= A \cos(S\,t + B), \\ x_1 &= A_1 \sin(S_1\,t + B), \\ y_1 &= A_1 \cos(S_1\,t + B)\,. \end{aligned}$$

If S is real then the sines and cosines will remain finite, and there will be stability.

If S is imaginary then $S = s + \sqrt{-1}\;u$:

$$\sin S\,t = e^{ut}\,(\cos st + \sqrt{-1}\;\sin st).$$

The modulus increases indefinitely with t; the motion is then unstable.

The necessary and sufficient condition for the motion to be stable is then that the roots S are real.

Develop the equation in S:

$$S^2 - S\,(\alpha + \delta) + \alpha\delta - \beta\gamma = 0.$$

The roots will be real if:

$$(\alpha + \delta)^2 - 4\,(\alpha\delta - \beta\gamma) > 0$$

or

$$(\alpha + \delta)^2 - 4\beta\gamma > 0.$$

Replace α, β, γ, δ by their values:

$$[\zeta - n\,(\zeta + \zeta') - \zeta' + n\,(\zeta\,\varepsilon^2 + \zeta')^2] + 4\zeta\zeta'\,\varepsilon^{2n} > 0.$$

That inequality must be verified for all integer values of n.

We first remark that if the vorticities ζ and ζ' have the same sign then. The motion will be stable in that case.

141. We shall not make a complete discussion of the inequality. We shall consider only the particular case for which:

$$\zeta\,\varepsilon^2 + \zeta' = 0.$$

That condition expresses the idea that the velocity $(\zeta\,\varepsilon^2 + \zeta')\ r_0'^2 / r$ is zero at an exterior point before the deformation. Choose the units in such a manner that $\zeta = 1$, so $\zeta' = -\,\varepsilon^2$. The motion will be stable if the aforementioned inequality is verified for all values of n. Upon writing that it is true for $n = 2$, we will have the necessary condition for stability:

$$[1 - 2\,(1 - \varepsilon^2) + \varepsilon^2] - 4\,\varepsilon^6 > 0$$

or

$$(1 - \varepsilon^2)(1 - 4\varepsilon^2) > 0.$$

Since the first factor is essentially positive, one must have that:

$$1 - 4\varepsilon^2 > 0,$$

or, since ε is positive:

$$\varepsilon < \frac{1}{2}.$$

As a result, if the radius of the interior cylinder C is larger than the mean radius of the exterior cylinder C' then there will be no stability.

142. Explanation of an experimental fact. – Imagine that two currents exist in a liquid that either have opposite senses or just different velocities. The two liquid masses, when animated with different velocities, will rub again each other, and that will give rise to a separation surface of small vortices. Ordinarily, if one wishes to explain the formation of these vortices then one will be content to say that they are due to the friction between the two veins of liquid. That explanation is not sufficient. Indeed, to begin with, we will have two liquid masses that are animated with different velocities, which, for more simplicity, we consider to be constant in magnitude and direction. That state cannot persist, due to the friction that comes from viscosity in the liquid. However, it first seems that it must produce a transition layer in which the velocity varies gradually and the vortices are distributed uniformly. Now, that is not what one actually observes, as opposed to the formation of small vortices that seem to have a tendency to collect into separate tubes. That amounts to saying that under the conditions in which we find ourselves, the state in which the vortices are distributed uniformly is unstable, as we can show by appealing to the preceding discussion.

Indeed, let C be a cylinder. Endow the liquid inside of the cylinder with a uniform rotational velocity ($\zeta + \zeta'$), while the liquid will remain at rest outside. The velocity will be discontinuous at the surface of the cylinder, and in turn, friction must produce a transition zone that is limited by two cylinders that are concentric to C.

As in the case that we just studied, there will be two concentric cylinders: The value of the vorticity inside the first one will be $\zeta + \zeta'$, while it is zero outside of the second one, and finally, it will vary gradually in the annular zone. Nonetheless, for more simplicity, I will suppose that the vorticity has a constant value between 0 and $\zeta + \zeta'$ in that annular zone; let that value be ζ'.

Set:

$$\zeta + \zeta' = a,$$

and attribute a mean constant value to ζ', as in the preceding example, such that:

$$\zeta \varepsilon^2 + \zeta' = 0,$$

which expresses the idea that the liquid is at rest outside.

We infer from this that:

$$\zeta' = \frac{a\varepsilon^2}{\varepsilon^2 - 1}.$$

Since the transition zone is very thin to begin with, ε will be very close to 1, and ζ' will be very large. However, since ε is larger than 1 / 2, these conditions will be unstable, from what we saw in [no. **141**].

Suppose that the liquid on the outside possesses a certain velocity b, instead of being at rest, so:

$$\zeta \varepsilon^2 + \zeta' = b,$$

and there is friction again, since the velocity is discontinuous. Upon substituting the values of ζ and ζ' into the inequalities of the condition, we will again find that there is instability.

CHAPTER IX

FLUIDS THAT PRESENT A FREE SURFACE

143. Up to now, we have studied the motion of liquids that are either indefinite or fill a vessel that encloses them completely. We shall now occupy ourselves with the case in which the liquids no longer fill the vessel completely and possess a free surface that is in contact with the fluid.

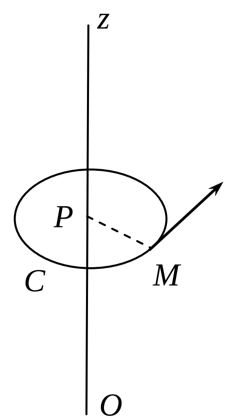

Figure 40.

Imagine that the molecules describe circumferences whose centers are along the *z*-axis and lie in planes that are perpendicular to that axis. If such a motion is possible then it will be necessarily permanent.

If the pressure and the density are the same all along one of the circumferences then the equation of continuity will be verified. The entire system will be a figure of revolution around *Oz*, so the velocity of the molecule *M* (Fig. 40) at an arbitrary instant will be directed along the tangent to the circumference *C*, and it will have the same magnitude for all points on that circumference.

Take a small surface element in the plane *ZOM*; that element will generate a small surface under revolution around *Oz*. The fluid that occupies the volume that is bounded by that surface will undergo a rotation around *Oz*, and its volume will not be altered.

Set:

$$MP = r = \sqrt{x^2 + y^2}\ .$$

The pressure *p* and the density *r* are functions of only *r* and *z*, from the hypotheses that we have made. Consequently:

$$w = 0,$$
$$ux + vy = 0.$$

One-half the square of the velocity, namely:

$$T = \frac{u^2 + v^2}{2},$$

will also be a function of only r and z.

If the weight is the only external force that acts upon the fluid (the z-axis being vertical) then we can set [no. **4**]:

$$V = gz,$$

$$\psi = -\int \frac{dp}{\rho} + V.$$

The function ψ exists on the condition that r is a function of only p. That is what happens when the fluid is a homogeneous liquid or a gas that experiences isothermal or adiabatic transformations.

In the Lagrange system of notations, the components of the acceleration have the expression:

$$(1)\qquad \frac{\partial\psi}{\partial x} = \frac{du}{dt}, \quad \frac{\partial\psi}{\partial y} = \frac{dv}{dt}, \quad \frac{\partial\psi}{\partial z} = \frac{dw}{dt}.$$

(See § **4**.)

On the other hand, if a molecule describes a circumference in a uniform motion then the acceleration will reduce to the normal acceleration:

$$\frac{u^2 + v^2}{2} = \frac{2T}{r},$$

which is directed along MP. Its components are:

$$-\frac{2T}{r}\frac{x}{r}, \qquad -\frac{2T}{r}\frac{y}{r}, \qquad 0.$$

Moreover:

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial r}\frac{dr}{dx},$$

and as a result:

$$(2)\qquad \frac{\partial\psi}{\partial r} = -\frac{2T}{r}, \qquad \frac{\partial\psi}{\partial z} = 0.$$

It must then be the case that ψ depends upon only r; the same thing will be true for $\partial\psi / \partial r$ and T.

144. Simple special cases. – First, let us study some simple special cases:

1. The velocity is inversely proportional to the distance r. T will then be inversely proportional to r^2, or:

$$T = \frac{\alpha}{r^2},$$

in which α is a constant.

$$\frac{d\psi}{dr} = -\frac{2\alpha}{r^2},$$

and upon integrating, one will have:

$$\psi = \frac{\alpha}{r^2} + C = T + C.$$

Therefore:

(3) $$\psi - T = \text{const.}$$

We recover Bernoulli's equation, which is easy to predict. Indeed, that equation was proved [no. **24-25**] in the case for which there exists a velocity function – i.e., the vorticity is zero – and consequently, for which, the velocity varies like $1 / r$. Presently, we suppose that there is only one vortex tube that has Oz for its axis, and outside of which the vorticity is zero; we have then satisfied the aforementioned conditions.

145.

2. The velocity is proportional to the distance r; in other words, the liquid possesses a rotational motion around the z-axis with a constant angular velocity: It moves in the manner of a solid body. One obviously has:

$$T = \alpha r^2,$$

$$\frac{d\psi}{dr} = -2\alpha r,$$

$$\psi = -\alpha r^2 + \text{const.},$$

or

(4) $$\psi + T = \text{const.}$$

That result could also be predicted.

Indeed, recall that we have let J denote the integral [no. **5**]:

$$J = \int u\,dx + v\,dy + w\,dz,$$

which is taken along an arc of the curve. We have shown that:

$$\frac{dJ}{dt} = \int (d\psi + dT)\,.$$

Since $d\psi + dT$ is an exact differential, dJ / dt will be zero when the integration curve is closed. Here, from equation (4):

$$\psi + T = \text{const.}$$

or

$$\frac{dJ}{dt} = 0,$$

even when the integration curve is not closed.

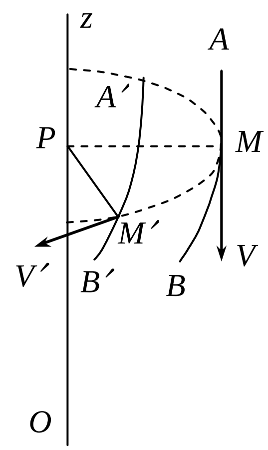

Figure 41.

Indeed, we have assumed that the liquid rotates around Oz in a uniform motion; the integration curve will also rotate around Oz with deforming. Draw a vector MV from the point M that will represent the velocity (Fig. 41). Upon considering that vector to be a force, J will be the work that is done by that force while the point traverses the integration curve AB. When that curve takes the position $A'B'$ in its rotation around Oz, the vector $M'V'$ will keep the same magnitude and position relative to $A'B'$. Indeed, $M'V'$ will be obtained by rotating MV around Oz until M agrees with M'. The work J that the force $M'V'$ does when the point M' traverses $A'B'$ will then be the same as the work that the force MV does when the point M describes AB.

146. Form of the free surface. – Suppose that the only external force that acts upon the fluid is its weight.

Take the z-axis to be vertical, and measure z as positive upwards. Under these conditions:

$$V = -\,gz,$$

$$\psi = -\,gz - \int \frac{dp}{\rho}\,.$$

If the fluid is a homogeneous liquid then:

$$p = \text{const.}$$

and

(5) $$\psi = -gz - \frac{p}{\rho}.$$

If one is dealing with a gas whose temperature is constant then the density ρ will be proportional to the pressure, and upon denoting a constant by β:

(6) $$\rho = \beta p, \qquad \psi = -gz - \frac{1}{\beta}\ln p.$$

Finally, for a gas that is subject to adiabatic transformations:

$$\rho = \beta p^{\gamma},$$

upon letting γ denote the ratio c / C of the specific heat c at constant volume to the specific heat C at constant pressure. Then:

(7) $$\psi = -gz - \frac{1}{\beta(1-\gamma)} p^{1-\gamma}.$$

146 (cont.). – Let a homogeneous liquid have a free surface upon which atmospheric pressure is exerted. If we call the excess of real pressure over atmospheric pressure p then one must make:

$$p = 0$$

in equation (5), and write:

(8) $$\psi = -gz.$$

ψ is a function of r, and that equation will be the equation of the free surface of the liquid.

Suppose that there exists just one vortex tube that has the form of a cylinder of revolution around Oz; the vorticity is constant inside of the cylinder and zero outside of it. The cylinder will then possess a uniform rotational motion.

Let r_0 be its radius. Inside of its surface (i.e., for $r < r_0$), the velocity will be proportional to r, and:

$$T = \alpha r^2.$$

Outside of it (i.e., for $r > r_0$), there will be a velocity function, and:

$$T = \frac{\alpha'}{r^2}.$$

These two expressions for T must take the same value at a point on the surface of the cylinder (i.e., for $r = r_0$). Therefore:

$$\alpha r_0^2 = \frac{\alpha'}{r_0^2},$$

so:

$$\alpha' = \alpha r_0^4 .$$

We calculate the values of ψ by means of these expressions.

In the interior:

$$\psi + T = \text{const.},$$

so:

$$\psi = -\,\alpha\, r^2 + C,$$

and exterior:

$$\psi - T = \text{const.},$$

so:

$$\psi = \frac{\alpha r_0^4}{r^2} + C'.$$

These formulas must give the same value for ψ when one makes $r = r_0$, which will give a relation between the constants C and C':

$$-\,\alpha r_0^2 + C = \frac{\alpha r_0^4}{r^2} + C'.$$

I may assign the constant C' arbitrarily: Changing its value amounts to displacing the xy-plane parallel to itself, which simply has the effect of adding a constant to z, and in turn, to ψ. I shall take $C' = 0$. With that choice, we will have:

$$\psi = 0$$

for $r = \infty$, and consequently:

$$z = 0.$$

The free surface of the liquid thus admits an asymptotic plane; it will be the plane that we have chosen to be the xy-plane. It will be the level of the liquid at a very large distance from the axis. The equation for the free surface, when referred to the rotational axis and to the asymptotic plane, will then be:

1. Inside of the vortex tube:

(9) $$gz = -\,\alpha\,(r^2 - r_0^2).$$

2. Outside of the tube:

(10) $$gz = -\,\frac{\alpha r_0^4}{r^2}.$$

The first one is represented by a paraboloid. Inside of the tube, the meridian of the free surface will be a small arc of a parabola.

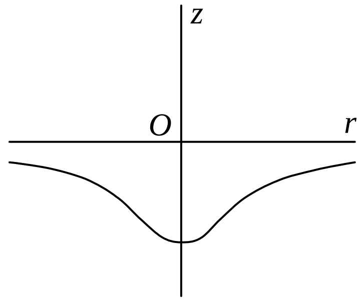

Figure 42.

The second equation represents a surface whose meridian is composed of two branches that are asymptotic to the z-axis (Fig. 42). The two curves will agree on the section of the vortex tube.

We remark that z is always negative. Consequently, the free surface will be situated completely below the xy-plane.

That circumstance does not take into account the special hypotheses that we have made; it is a general fact, as I shall show.

Indeed, ψ is zero for $r = \infty$. Hence:

$$\psi = \int_{+\infty}^{r} -2T\,\frac{dr}{r} = \int_{r}^{+\infty} 2T\,\frac{dr}{r}.$$

Since T and r are essentially positive, the same thing will be true for ψ, and consequently, z will be negative, from equation (8).

That result does not seem to conform to observation. Indeed, everyone that has had occasion to observe a whirlwind will agree that the liquid is, on the contrary, raised towards the center of the vortex in such a manner that it forms a sort of bead beneath the free surface. This disagreement between calculation and observation is probably due, in part, to the fact that in the calculations we have assumed that the pressure is uniform on the surface of the liquid, while that condition is probably not satisfied in the case of a whirlwind. Meanwhile, there is no doubt that this hypothesis has a very great influence on the results of the calculations if it is to make the difficulty that we just pointed out disappear.

147. Distribution of pressure in a gas. – If the fluid in motion is a gas then we can determine its state by studying the manner by which the pressure p varies in a plane that is parallel to the xy-plane. If the gas keeps a constant temperature then one must appeal to formula (6), and one will find that:

$$\ln p = -\beta g\, z_0 - \beta\,\psi$$

in the plane $z = z_0$.

ψ becomes zero at infinity; let p_0 be the corresponding value of p:

$$\ln p_0 = -\beta g z_0 ,$$

so:

$$\ln \frac{p}{p_0} = -\beta \psi.$$

The right-hand side is negative, since ψ is always positive, as we just saw. p / p_0 is then smaller than 1, and as a consequence, there will be a depression inside of the vortex.

If the gas is subject to adiabatic transformation then it will be formula (7) that is useful. The pressure p will be given by:

$$p^{1-\gamma} = -\beta(1-\gamma) z_0 - \beta(1-\gamma)\psi$$

in the plane $z = z_0$.

ψ is zero at infinity, and p is equal to p_0; consequently:

$$p_0^{1-\gamma} = -\beta(1-\gamma) z_0 ,$$

so:

$$p^{1-\gamma} - p_0^{1-\gamma} = -\beta(1-\gamma)\psi.$$

The quantities β, $1-\gamma$, ψ are positive, and consequently:

$$p^{1-\gamma} - p_0^{1-\gamma} < 0$$

or

$$p < p_0 .$$

There is a depression inside of the vortex again.

148. Case of several superimposed liquids. Form of the separation surface. – Suppose that there are only two liquids. Let ρ_1 be the density of the first one, and let p_1 be its pressure, while ρ_2 and p_2 are the density and pressure, resp., of the second one. If one lets ψ_1 and ψ_2 be the functions ψ that relate to the two liquids then:

$$\psi_1 = -g z_1 - \frac{p_1}{\rho_1} ,$$

$$\psi_2 = -g z_2 - \frac{p_2}{\rho_2} .$$

$z_1 = z_2$ at the separation surface, and the pressures must be equal, so $p_1 = p_2$. However, we know only one thing about ψ_1 and ψ_2, namely, that their derivatives $d\psi_1 / dr$ and $d\psi_2 / dr$ must have the same value. From that condition:

$$\frac{d\psi_1}{dr} = \frac{d\psi_2}{dr} ,$$

one deduces that:

$$\psi_1 - \psi_2 = \text{const.}$$

or

$$\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) p = \text{const.}$$

The pressure is then constant on the separation surface, and that surface will have the same form as the free surface.

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CHAPTER X

INFLUENCE OF FLUID VISCOSITY

149. Hypotheses. Notations. – When one is dealing with a viscous fluid – i.e., one whose molecules can move only with a certain amount of rubbing against each other – there will no longer exist a force function; in fact, the forces of friction will depend upon the velocity. Helmholtz's theory, as we have proved it, will therefore no longer be applicable.

Up to today, one can subject that case to calculation only by appealing to certain hypotheses that are more or less likely, but which are generally adopted.

One first assumes that the force that is due to the viscosity has the components:

$$K\,\Delta u, \quad K\,\Delta v, \quad K\,\Delta w$$

in the system of Lagrange variables, where K is a constant.

One then assumes that the surface of the vessel of the liquid is at rest, so in other words:

$$u = v = w = 0.$$

Finally, one assumes that an element $d\omega$ of that surface is acted upon by surface forces whose components are:

$$K\frac{du}{dn}d\omega, \qquad K\frac{dv}{dn}d\omega, \qquad K\frac{dw}{dn}d\omega.$$

Suppose that the external forces that act upon the liquid admit a force function V. Upon introducing the force of viscosity, the Lagrange equations (1) will become:

$$(1) \qquad \begin{aligned} \frac{\partial p}{\rho\,\partial x} &= -\frac{du}{dt} + \frac{\partial V}{\partial x} + K\,\Delta u, \\ \frac{\partial p}{\rho\,\partial y} &= -\frac{dv}{dt} + \frac{\partial V}{\partial y} + K\,\Delta v, \\ \frac{\partial p}{\rho\,\partial z} &= -\frac{dw}{dt} + \frac{\partial V}{\partial z} + K\,\Delta w. \end{aligned}$$

When a volume element $d\tau$ of the liquid is subjected to a displacement whose projections are dx, dy, dz, the work that is done by the forces that admit the function V will be:

$$\rho\,d\tau\,dV,$$

and upon adding this to the work that is done by the force of viscosity, one will obtain the real work:

$$d\mathcal{T} = \rho\,d\tau\,[dV + K\,(\Delta u\,dx + \Delta v\,dy + \Delta w\,dz)].$$

We again set:

$$dT = \rho\, d\tau \cdot dV',$$

where:

(2) $$dV' = dV + K\,(\Delta u\, dx + \Delta v\, dy + \Delta w\, dz).$$

However, it is essential to remark that the notation dV' has only a purely symbolic significance, since dV' is no longer an exact differential.

We then write:

(3) $$d\psi = dV' - \frac{dp}{\rho},$$

and recover equations (1) in a simplified form that is analogous to the one that we obtained in § **4**:

(4) $$\frac{\partial \psi}{\partial x} = \frac{du}{dt}, \quad \frac{\partial \psi}{\partial y} = \frac{dv}{dt}, \quad \frac{\partial \psi}{\partial z} = \frac{dw}{dt},$$

upon remarking, as above, that one is dealing with only symbols. $d\psi$ is not a total differential, but it is defined by the relation (3); $\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial z}$ are not derivatives of the same function $\psi(x, y, z)$, but only the coefficients of dx, dy, dz in the expression for $d\psi$.

150. Helmholtz's theorem [no. **5-6**] is expressed by the relation:

(5) $$\frac{dJ}{dt} = \int_C (d\psi + dT) = 0.$$

The integral will be zero along a closed curve when $d\psi + dT$ is a total differential; i.e., when one is dealing with an inviscid fluid.

However, if one neglects the viscosity then $d\psi + dT$ will no longer be a total differential. From relations (2) and (3), one will then have:

(6) $$\frac{dJ}{dt} = \int \left(dV - \frac{dp}{\rho} + dT \right) + K \int (\Delta u\, dx + \Delta v\, dy + \Delta w\, dz).$$

From Helmholtz's theorem [no. **6**], the first integral will be zero when it is extended over a closed contour. What will then remain is:

(7) $$\frac{dJ}{dt} = K\int (\Delta u\, dx + \Delta v\, dy + \Delta w\, dz)\,.$$

Make an arbitrary closed surface pass through the contour of integration *C*. The curve *C* will bound a certain area *A*. Let *l*, *m*, *n* be the direction cosines of the normal to the elements $d\omega$ of the area *A*. Upon applying Stokes's theorem [no. **8**], we found that:

$$J = 2\int (l\xi + m\eta + n\zeta)\, d\omega,$$

in which the integral was taken over all elements $d\omega$ of the area *A*, and ξ, η, ζ were defined by relations (1) of § 9.

Transform the integral (7) by the same theorem, and get:

$$\int (\Delta u\, dx + \Delta v\, dy + \Delta w\, dz)$$
$$= \int d\omega \left[l\left(\frac{d\Delta w}{dy} - \frac{d\Delta v}{dz}\right) + m\left(\frac{d\Delta u}{dz} - \frac{d\Delta w}{dx}\right) + n\left(\frac{d\Delta v}{dx} - \frac{d\Delta u}{dy}\right)\right].$$

We remark that one can invert the order of differentiations and write:

$$\frac{d\Delta w}{dy} = \Delta\frac{dw}{dy},$$
$$\frac{d\Delta v}{dz} = \Delta\frac{dv}{dz},$$

or, upon subtracting corresponding sides:

$$\frac{d\Delta w}{dy} - \frac{d\Delta v}{dz} = \Delta\left(\frac{dw}{dy} - \frac{dv}{dz}\right) = 2\,\Delta\xi.$$

Upon transforming the other terms similarly, one will finally arrive at the formula:

(8) $$\frac{dJ}{dt} = 2K\int (l\,\Delta\xi + m\,\Delta\eta + n\,\Delta\zeta)\, d\omega.$$

151. Necessary conditions for Helmholtz's theorem to still be applicable. – We have proved [no. **14**] that, from Helmholtz's theorem, the vortex surfaces (and consequently, the vortex lines) will be conserved in the motion of a liquid if one neglects the forces of viscosity or friction. If one takes these latter forces into account then that statement can no longer be true, in general. The theorem can still be true only under some special conditions that we propose to determine.

Helmholtz's theorem [no. **6**] is expressed by the condition:

$$\frac{dJ}{dt} = 0.$$

Along a curve that is traced on a vortex surface, one will have:

$$J = 0,$$

since all along that curve the vortex will be represented by a vector that is tangent to the surface.

In order for dJ / dt to remain zero when one introduces viscous forces, it will be necessary that:

$$l\,\Delta\xi + m\,\Delta\eta + n\,\Delta\zeta = 0,$$

from equation (7).

That relation expresses the idea that the vector ($\Delta\xi$, $\Delta\eta$, $\Delta\zeta$) is found in the plane of the element $d\omega$. Trace out that vector: If it is tangent to the surface $J = 0$ after time dt then one will again have $J = 0$. Since dJ / $dt = 0$, the vortex surface will be conserved. If we would desire that the vortex lines should be conserved then it will be necessary that an arbitrary element of these lines should remain constantly tangent to the vortex vector. It would then be necessary that the plane of the element $d\omega$ should contain both of the two vectors, and since that must be true for an *arbitrary* element $d\omega$ that passes through the vortex, it would be necessary that these two vectors should agree in direction; in other words, that:

$$\frac{\Delta\xi}{\xi} = \frac{\Delta\eta}{\eta} = \frac{\Delta\zeta}{\zeta}. \tag{9}$$

In general, that condition will not be fulfilled, and the vortex lines will not be conserved.

152. In the particular case where there exists a velocity function, the vorticity will be zero, so one will have:

$$\xi = \eta = \zeta = 0,$$

and as a result $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ will be zero identically. For an arbitrary curve, we can then write:

$$J = 0, \quad \frac{dJ}{dt} = 0,$$

and the velocity function will then persist at an arbitrary instant.

That consequence of our argument seems to constitute an objection to the hypotheses that served as its point of departure.

153. Special case in which the vortex lines are conserved. – Suppose that the vortex lines in an indefinite liquid are lines that are parallel to the z-axis, and that ζ and η

are zero, as well as $\Delta\xi$ and $\Delta\eta$, but that ζ and $\Delta\zeta$ are non-zero. The conditions (9) will then be fulfilled, and the vortex lines will be conserved. Moreover, in order to prove that, it will suffice to appeal to symmetry considerations.

Indeed, consider an arbitrary plane that is parallel to the *xy*-plane. That plane is a symmetry plane, for which, one does or does not take friction into account. If the streamlines are planar at the onset of the motion and situated in planes that are parallel to the *xy*-plane then they will always remain in those planes, by reason of symmetry, and independently of friction.

However, under the present conditions, ζ will no longer preserve its value, and dJ / dt will no longer be zero.

Indeed, take the contour of integration to be the cross-section of a vortex tube. Because of that choice, one must make:

$$l = m = 0, \qquad n = 1,$$

and one will get:

$$J = 2\int \zeta\, d\omega, \tag{10}$$

$$\frac{dJ}{dt} = -\,2K\int \Delta\zeta\, d\omega. \tag{11}$$

The section of the tube $d\omega$ is constant; indeed, the volume that is bounded by that tube and two planes $z = z_1$ and $z = z_2$ will be constant, from the equation of continuity. That volume will be equal to:

$$(z_1 - z_2)\, d\omega.$$

z_1 and z_2 remain constant, since the velocity is always parallel to the *xy*-plane, and $d\omega$ will therefore be constant.

Differentiate equation (9) with respect to t and get:

$$\frac{dJ}{dt} = 2\int \frac{d\zeta}{dt}\, d\omega. \tag{12}$$

Compare these two expressions for dJ / dt; since the integrals are extended over the same area, it will be necessary that:

$$\frac{d\zeta}{dt} = K\,\Delta\zeta. \tag{13}$$

The derivative $d\zeta / dt$ is calculated with the Lagrange variables; i.e., by following a molecule in its motion.

Equation (13) is analogous to the one that represents the propagation of heat by conduction, except that in the latter problem one ordinarily regards the molecules as immobile. Here, on the contrary, ζ will vary as the temperature of the liquid varies, if it possesses the same motion, and K is its coefficient of conductivity. However, under those conditions, a transport of heat by convection will also be produced.

154. (Extension of some) general theorems. – We have proved [nos. **65**, *et seq.*, **113**, *et seq.*] some theorems that are applicable to liquids in which no friction is produced; some of them will still be true.

Indeed, let an indefinite liquid contain vortex tubes that are cylinders that are parallel to Oz.

We have seen [no. **126**] that upon considering ζ to be the density of an attracting mass that is spread over the xy-plane, the total mass $M = \int \zeta \, d\omega$ of that fictitious matter will be constant (when the integral is taken over all elements $d\omega$ in the xy-plane).

The mass thus-defined will still be constant when the liquid is devoid of friction.

Indeed, differentiate:

$$M = \int \zeta \, d\omega$$

with respect to time, upon remarking that $d\omega$ is constant; one will get:

$$\frac{dM}{dt} = \int \frac{d\zeta}{dt} d\omega = K \int \Delta\zeta \, d\omega . \tag{14}$$

I say that this integral is zero. In order to prove that, apply Green's formula:

$$\int \left(u \frac{dv}{dn} - v \frac{du}{dn} \right) ds = \int (u\,\Delta v - v\Delta u)\, d\omega$$

to a circle of very large radius upon supposing that the functions u and v, or just one of them, are annulled at infinity. The integral on the left-hand side will be zero, and what will remain is:

$$\int u\,\Delta v = \int v\Delta u .$$

If we now make:

$$u = 1, \quad v = \zeta$$

then we will find that:

$$\int \Delta\zeta \, d\omega = 0. \tag{15}$$

Therefore:

$$M = \text{const.}$$

155. The center of gravity of these fictitious masses is fixed, even when there is friction.

Indeed, the coordinates of that center of gravity are defined by the equations:

$$M\,x_0 = \int \zeta\,x\,d\omega ,$$
$$M\,y_0 = \int \zeta\,y\,d\omega .$$

Differentiate the first one with respect to t and get:

$$M\,\frac{dx_0}{dt} = \int \zeta u\,d\omega + \int \frac{d\zeta}{dt}\,x\,d\omega,$$

since $d\omega$ is constant. The first integral is zero, because the existence of friction will influence only the value of the derivatives of u and v, but not the values themselves of those functions. What will then remain is:

$$M\,\frac{dx_0}{dt} = \int \frac{d\zeta}{dt}\,x\,d\omega = K\int \Delta\zeta\,x\,d\omega\,.$$

Once more, apply Green's theorem, while setting:

$$u = x, \quad v = \zeta;$$

x has degree one, so Δx will be zero.

Consequently:

(16) $$\int x\cdot\Delta\zeta\,d\omega = \int \Delta\zeta\,x\,d\omega = 0.$$

Hence, $dx_0 / dt = 0$, and x_0 is a constant.

The same argument will lead to the same conclusion for y_0 .

156. The moment of inertia I of the fictitious mass with respect to an axis that is parallel to Oz will be constant when there exists no friction. However, when friction does exist that same moment will vary in proportion to time.

Indeed:

$$I = \int \zeta(x^2 + y^2)\,d\omega$$

and

$$\frac{dI}{dt} = \int 2\zeta(xu + yv)\,d\omega + \int \frac{d\zeta}{dt}(x^2 + y^2)\,d\omega\,.$$

The first integral is zero, as if there were no friction, since the presence of friction does not affect the values of u and v. Upon taking equation (12) into account, one has:

$$\frac{dI}{dt} = \int \frac{d\zeta}{dt}(x^2 + y^2)\,d\omega = K \int \Delta\zeta\,(x^2 + y^2)\,d\omega\,.$$

For $u = x^2 + y^2$, $v = z$, Green's theorem will give:

$$\int (x^2 + y^2)\,\Delta\zeta\,d\omega = \int \zeta\,\Delta(x^2 + y^2)\,d\omega,$$

when one remarks that:

$$\Delta\,(x^2 + y^2) = 4,$$

$$\int (x^2 + y^2)\,\Delta\zeta\,d\omega = \int \zeta\,d\omega = 4M,$$

and finally:

(17) $$\frac{dI}{dt} = 4KM.$$

The derivative dI / dt is constant, and in turn, I will vary in proportion to time; it will be easy, moreover, to calculate the rapidity of that variation.

157. Application to a simple case. – Suppose that, to begin with, ζ depends upon only the distance to the z-axis, $r = \sqrt{x^2+y^2}$. By reason of symmetry, that condition will always persist. The velocity at a point will be perpendicular to the radius vector that is based at a point on Oz and perpendicular to it. The point will describe a circumference that has its center on the Oz axis and is situated in a plane that is perpendicular to that axis. Due to symmetry, the point will remain on that circumference, even when there is friction; however, in the latter case, the velocity will cease to be uniform. Indeed, ζ is a function of r and t. Therefore:

$$\frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} + \frac{\partial\zeta}{\partial r}\frac{dr}{dt}.$$

However, since r = const., dr / dt will be zero, and one will have simply:

$$\frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} = K\,\Delta\zeta,$$

or, from a well-known formula, since ζ does not depend upon r:

(18) $$\frac{\partial\zeta}{\partial t} = K\left(\frac{d^2\zeta}{dr^2} + \frac{1}{r}\frac{d\zeta}{dr}\right).$$

We must now integrate that differential equation.

Consider the integral:

$$Z = -\int_{-\infty}^{+\infty} e^{-\alpha^2/h} F(x+\alpha\sqrt{t})\,d\alpha\,,$$

in which α is a constant. We will have:

$$\frac{dZ}{dt} = \int_{-\infty}^{+\infty} e^{-\alpha^2/h} F' \frac{\alpha}{2\sqrt{t}}\,d\alpha\,.$$

Integrate this by parts:

$$\frac{dZ}{dt} = \left[e^{-\alpha^2/h}\left(-\frac{h}{2}\right)\frac{F'}{\sqrt{t}}\right]_{-\infty}^{+\infty} + \frac{h}{4}\int_{-\infty}^{+\infty} e^{-\alpha^2/h} F''\,d\alpha\,.$$

The integrated term is zero at the limits, so this expression will reduce to:

$$\frac{dZ}{dt} = \frac{h}{4}\int_{-\infty}^{+\infty} e^{-\alpha^2/h} F'' \, d\alpha \, .$$

On the other hand:

$$\frac{d^2 Z}{dt^2} = \int_{-\infty}^{+\infty} e^{-\alpha^2/h} F'' \, d\alpha \, ,$$

so:

$$\frac{dZ}{dt} = \frac{h}{4}\frac{d^2 Z}{dx^2} \, .$$

In a more general manner, consider the integral:

$$Z = \iint e^{-\frac{\alpha^2+\beta^2}{h}} F(x+\alpha\sqrt{\tau}, y+\alpha\sqrt{\tau'}) \; d\alpha \, d\beta.$$

If, for the moment, we regard y and τ' as constants then Z will be a function of only x and τ, and from the preceding discussion:

$$\frac{dZ}{d\tau} = \frac{h}{4}\frac{d^2 Z}{dx^2} \, .$$

If we similarly regard x and τ as constants then:

$$\frac{dZ}{d\tau'} = \frac{h}{4}\frac{d^2 Z}{dy^2} \, .$$

On the other hand, we have:

$$\frac{dZ}{dt} = \frac{dZ}{d\tau}\frac{d\tau}{dt} + \frac{dZ}{d\tau'}\frac{d\tau'}{dt} = \frac{dZ}{d\tau} + \frac{dZ}{d\tau'}$$

or:

$$\frac{dZ}{dt} = \frac{h}{4}\Delta Z.$$

In order to identify that equation with equation (12), it will suffice to set:

$$h = 4K,$$

in which K is essentially positive.

Upon taking, in turn:

$$\zeta = Z,$$

we can calculate ζ at an arbitrary instant.

In order to choose the function F, one must know the value of ζ at the initial time, since for $t = 0$:

$$\zeta_0 = \iint e^{-\frac{\alpha^2+\beta^2}{h}} F(x, y)\, d\alpha\, d\beta = F(x, y) \iint e^{-\frac{\alpha^2+\beta^2}{h}} d\alpha\, d\beta .$$

Now:

$$\iint e^{-\frac{\alpha^2+\beta^2}{h}} d\alpha\, d\beta = \frac{\pi}{4},$$

and consequently:

$$\zeta_0 = \frac{\pi}{4} F(x, y),$$

which is an equation that determines F.

If the initial value of ζ depends upon only r then F will no longer depend upon r, and it will be possible to determine ζ at an arbitrary epoch, if nothing perturbs the symmetry or if the stability conditions are fulfilled.

158. Helmholtz's theorem for relative motion. – Helmholtz's theorem expresses the idea that the integral:

$$J = \int u\, dx + v\, dy + w\, dz$$

is constant when there is a force function. In the case of relative motion, there is no longer a force function, so the theorem will no longer be true. One has:

$$\frac{dJ}{dt} = \int d\psi + dT$$

with

$$d\psi = dV - \frac{dp}{\rho},$$

when there exists a potential V.

If there no longer exists a potential then:

$$d\psi = X\, dx + Y\, dy + Z\, dz - \frac{dp}{\rho}.$$

If the dragging motion is, for example, a rotation around the terrestrial axis with an angular velocity of ω_0 then one will have:

$$X\, dx + Y\, dy + Z\, dz = dV + 2\omega_0\, (v\, dx - u\, dy),$$

from Coriolis's theorem, if the potential V includes the ordinary centrifugal force, where the z-axis is the axis rotation. One will then get:

$$\frac{dJ}{dt} = \int (dV + dT) + \int 2\omega_0 (v\, dx - u\, dy) .$$

The first integral is zero, and what will remain is:

$$\frac{dJ}{dt} = 2\omega_0 \int (v\,dx - u\,dy)\,.$$

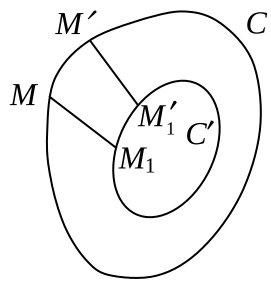

Figure 43.

Let C be the integration curve. Project it onto the xy-plane. Let A be the area that is bounded by the projection (Fig. 43), and let M and M' be two infinitely close points; the projections of MM' onto the three axes are dx, dy, dz. After the time dt, the various molecules that cross C will go to C', and in particular, M will go to M_1, and M' will go to M_1'. The projections of MM_1 are $u\,dt$, $v\,dt$, $w\,dt$. The quadrilateral $MM'M_1M_1'$ can be regarded as a parallelogram whose projection onto the xy-plane bounds an area that is equal to:

$$dt\,(v\,dx - u\,dy).$$

The integral:

$$dt \int (v\,dx - u\,dy)$$

thus represents the variation dA / dt of the area A during the time dt. Consequently:

$$\frac{dJ}{dt} = 2\omega_0 \frac{dA}{dt}$$

and

$$J = 2\omega_0\, A + \text{const.},$$

and if J_0 and A_0 are the initial values of J and A then one will have:

$$J_0 = 2\omega_0\, A_0 + \text{const.},$$
$$J - J_0 = 2\omega_0\, (A - A_0).$$

Let (Fig. 44) be a circle of radius r_0. The molecules that are situated on that circumference are originally in relative equilibrium with respect to the surface of the Earth. Therefore:

$$A_0 = \pi r_0^2 \sin \lambda,$$

in which λ is the latitude, and:

$$J = 0.$$

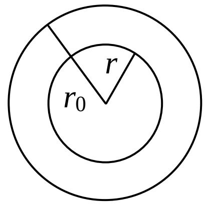

Figure 44.

If a perturbation is produced that pushes the air towards the center of the circle then the molecules will occupy a closed contour after a certain time that amounts to a circumference of radius r. In that new position:

$$A = \pi r^2 \sin \lambda,$$
$$J = 2\omega_0 \pi (r^2 - r_0^2) \sin \lambda.$$

A rotation will then be produced: Let ω be the angular velocity or the vertical component of the vortex, so one will have:

$$J = 2\int \omega \, d\sigma ,$$

in which $d\sigma$ is a surface element, or upon supposing that ω is constant:

$$J = 2\omega \int d\sigma = 2\omega \pi r^2.$$

Upon equating the two expressions for J, one will find that:

$$\omega = \omega_0 \sin \lambda \left(1 - \frac{r_0^2}{r^2}\right).$$

If r / r_0 is very small then ω will become very large and will always have the same sign – i.e., the rotation will always be directed in the same sense.

That is one of the explanations that have been proposed for the formation of atmospheric cyclones.

FIN