“Sur les lignes élastiques à double coubure,” Correspondance sur l’École Poly. **3** (1816), 355-360.

On elastic lines of double curvature

By Siméon Denis Poisson

Translated by D. H. Delphenich

In this article, we will need to know the angles that the perpendicular to the osculating plane of a curve makes with the three aces of coordinates. In order to determine them, let:

$$Ax' + By' + Cz' = 0$$

be the equation of that plane. Call the required angles α, β, γ, in such a way that we will have, from known formulas:

$$\cos\alpha = \frac{A}{\sqrt{A^2+B^2+C^2}}, \qquad \cos\beta = \frac{B}{\sqrt{A^2+B^2+C^2}}, \qquad \cos\gamma = \frac{C}{\sqrt{A^2+B^2+C^2}}.$$

Furthermore, let x, y, z be the coordinates of the point on the curve to which the osculating plane is referred. Since it must pass through that point and two consecutive points of the same curve, we will have these three condition equations:

$$\begin{aligned} A\,x + B\,y + C\,z &= D, \\ A\,dx + B\,dy + C\,dz &= 0, \\ A\,d^2x + B\,d^2y + C\,d^2z &= 0\,; \end{aligned}$$

hence, one can infer the values of A, B, C. If one performs the elimination one takes the quantity D, which is indeterminate, to be equal to the common denominator of A, B, C, and in order to simplify those values, then one will have simply:

$$A = dz\,d^2y - dy\,d^2z, \qquad B = dy\,d^2x - dx\,d^2y, \qquad C = dx\,d^2z - dz\,d^2x,$$

and consequently:

$$\cos\alpha = \frac{dz\,d^2y - dy\,d^2z}{K}, \qquad \cos\beta = \frac{dx\,d^2z - dz\,d^2x}{K}, \qquad \cos\gamma = \frac{dy\,d^2x - dx\,d^2y}{K},$$

in which we have set:

$$(dz\,d^2y - dy\,d^2z)^2 + (dx\,d^2z - dz\,d^2x)^2 + (dy\,d^2x - dx\,d^2y)^2 = K^2,$$

to abbreviate.

Now consider an elastic line whose points are subjected to given forces and which is in equilibrium. Denote that curve (without it being necessary to illustrate this) by *AmB*, in such a manner that *A* and *B* are its extremities and *m* is an arbitrary point that answers to the coordinates x, y, z. Suppose that the part *mB* of the curve is made inflexible and fixed, and that the other part *mA* becomes only inflexible, while preserving the freedom to turn around the point *m*. The equilibrium of the entire line must still persist. Consequently, the given forces that act on the part *mA* and the elastic forces that take place at the point *m* must be in equilibrium around the fixed point, which will demand that the sums of the moments of those forces, when taken with respect to the three axes that are drawn through the point *m*, must be zero.

Now, the elasticity at the point *m* will tend to produce two distinct effects. First of all, it will tend to put the two elements of the curves that are bounded by the that point back into a straight line, or more generally, if the natural form of that curve is not a straight line then the elasticity will tend to make the angle of contingency at *m* take on a greater or lesser value than the one that it had in the natural state of the curve. Therefore, let *E* refer to the moment of that force when it is taken with respect to the point *m*. The perpendicular axis to its plane will make angles with the coordinate planes that we just denoted by α, β, γ. Hence, the moments of the forces will decompose according to the same laws as those of the forces themselves (*), so it will follow that the moments of the force that we consider, when referred to the lines that are drawn through the point *m* and parallel to the x, y, and z axes, will be:

$$E \cos \alpha, \qquad E \cos \beta, \qquad E \cos \gamma,$$

respectively, or rather, upon setting $E / K = u$ and replacing the cosines with their preceding values:

$$u\,(dz\,d^2y - dy\,d^2z), \quad u\,(dx\,d^2z - dz\,d^2x), \quad u\,(dy\,d^2x - dx\,d^2y)\,.$$

When the elastic line is twisted on itself, the elasticity at the point *m* will tend to produce a second effect that consists of making the moving part *mA* of curve turn around the indefinite prolongation of the element that is bounded by the point *m* and belongs to the fixed part *Bm*. We attribute this second effect to a force that one can call the *torsion*, and which is exerted in a plane that perpendicular to the tangent at the point *m*. Let θ be its moment, when it is taken with respect to that tangent. The cosines of the angle that the line will make with the x, y, z axes are dx / ds, dy / ds, dz / ds, where the element of the curve is represented by ds. Consequently, the moments of the torsion with respect to the same axes will be:

$$\frac{\theta\,dx}{ds}, \quad \frac{\theta\,dy}{ds}, \quad \frac{\theta\,dz}{ds}.$$

Finally, let X, Y, Z denote the components of the given forces along the x, y, z axes that act upon the point of the curve that corresponds to those coordinates. The sum of the

(*) *See* my *Traité de Mécanique*, Tome Premier, page 3. Here, we intend the moment relative to an axis to mean the moment of the forces when it is projected onto a plane perpendicular to that axis.

moments with respect to the x-axis of the similar forces that act upon the part mA of the those curves will be given by the integral $\int (zY - yZ)\, dm$, in which dm represents the material element of the curve, and if one would like to refer the moments of those forces to the line that is drawn through the point m parallel to the x-axis then it will be easy to see that one must add the quantity $y \int Z\, dm - z \int Y\, dm$ to that integral. One will get some similar results relative to the y and z axes. Hence, the sums of the moments of the given forces, when taken with respect to three lines that are drawn through the point m and parallel to the x, y, z axes will be expressed by these formulas:

$$\begin{aligned}
&\int (zY - yZ)\, dm + y \int Z\, dm - z \int Y\, dm, \\
&\int (xZ - zX)\, dm + z \int X\, dm - x \int Z\, dm, \\
&\int (yX - xY)\, dm + x \int Y\, dm - y \int X\, dm\,.
\end{aligned}$$

The six integrals that are contained in them are supposed to each refer to an arbitrary constant that is provided by the particular forces that can be applied to the point A.

Upon now adding the moments of the given forces and the elastic forces that all refer to the same axis and equating the sums to zero, we will have the three equations of equilibrium of the elastic line with double curvature that is twisted on itself, namely:

$$\left.\begin{aligned}
u\,(dz\, d^2y - dy\, d^2z) + \frac{\theta\, dx}{ds} + \int (zY - yZ)dm + y\int Z\, dm - z\int Y\, dm = 0, \\
u\,(dx\, d^2z - dz\, d^2x) + \frac{\theta\, dy}{ds} + \int (xZ - zX)\, dm + z\int X\, dm - x\int Z\, dm = 0, \\
u\,(dy\, d^2x - dx\, d^2y) + \frac{\theta\, dz}{ds} + \int (yX - xY)\, dm + x\int Y\, dm - y\int X\, dm = 0.
\end{aligned}\right\} \tag{1}$$

If one differentiates these equations then one will get:

$$dz\, d\,(u\, d^2y) - dy\, d\,(u\, d^2z) + d \cdot \frac{\theta\, dx}{ds} + dy \int Z\, dm - dz \int Y\, dm = 0,$$

$$dx\, d\,(u\, d^2z) - dz\, d\,(u\, d^2x) + d \cdot \frac{\theta\, dy}{ds} + dz \int X\, dm - dx \int Z\, dm = 0,$$

$$dy\, d\,(u\, d^2x) - dx\, d\,(u\, d^2y) + d \cdot \frac{\theta\, dz}{ds} + dx \int Y\, dm - dy \int X\, dm = 0,$$

and if one adds these, after multiplying them by dx / ds, dy / ds, dz / ds, respectively, then one will have:

$$\frac{dx^2 + dy^2 + dz^2}{ds^2} \cdot d\theta + \left(\frac{dx}{ds} d \cdot \frac{dx}{ds} + \frac{dy}{ds} d \cdot \frac{dy}{ds} + \frac{dz}{ds} d \cdot \frac{dz}{ds}\right)\theta = 0.$$

However, one has:

$$\frac{dx^2 + dy^2 + dz^2}{ds^2} = 1 \quad \text{and} \quad \frac{dx}{ds} d\cdot\frac{dx}{ds} + \frac{dy}{ds} d\cdot\frac{dy}{ds} + \frac{dz}{ds} d\cdot\frac{dz}{ds} = 0$$

identically, so it will result that $d\theta = 0$, which is an equation that shows that the moment of the force of torsion is a constant quantity under all extensions of the elastic curve in equilibrium.

Hence, the torsion is not a force whose law one can determine by a hypothesis, as one ordinarily does for elasticity, properly speaking. The torsion depends upon neither the form of the curve nor forces such as weight or other ones that act at all of its points. It is produced by a force that is applied to one or the other extremity, and whose moment with respect to the extreme tangent will determine the value of θ. Once that quantity is given, it will remain the same for all of the other points of the curve, in such a manner that if one would like to cut the curve at an arbitrary point then if one wishes to prevent it from untwisting, one must employ a force whose moment with respect to the tangent at that point will be equal to the moment of the extreme force that produces the torsion. Binet was the first to look into the torsion to which elastic curves are susceptible (*), but he did not explain the nature of that force, and showed that its moment would be constant in the equilibrium state. In his *Mécanique analytique* (**), Lagrange gave some equations for the elastic line with double curvature that he found by an analysis that was very different from ours, and which nonetheless come down to our equations (1) when one supposes that $\theta = 0$.

Upon adding those three equations, after multiplying them by dx / ds, dy / ds, and dz / ds, the u will disappear, and one will have:

$$\frac{d^2x}{ds}\cdot\int (zY - yZ)\,dm + \frac{d^2y}{ds}\cdot\int (xZ - zX)\,dm + \frac{d^2z}{ds}\cdot\int (yX - xY)\,dm$$

$$+ \frac{y\,d^2x - x\,d^2y}{ds}\cdot\int Z\,dm + \frac{x\,d^2z - z\,d^2x}{ds}\cdot\int Y\,dm + \frac{z\,d^2y - y\,d^2z}{ds}\cdot\int X\,dm = 0.$$

However, that equation is a result of the preceding one, as is easy to verify when one differentiates it under the hypothesis that ds is constant and observes that $d\theta = 0$.

It results from this that in order to determine the elastic curve, one can take equation (2), combined with one of the equations (1) or some combination of those three equations that one prefers, provided that it again refers to the variable u. As for that quantity, one has $u = E / K$, and one commonly supposes that the moment E of the elasticity at the point m is proportional to the square of the thickness of the curve multiplied by the excess of the angle of contingency that exists at that point in the equilibrium state over the one that exists at the same point in the natural state of the curvature. Since those angles are

---

(*) Journal de l'École Polytechnique, 17th Cahier, pp. 418, *et seq.*.

(**) Second edition, Tome premier, page 154.

inversely proportional to the radii of curvature that they correspond to, that hypothesis amounts to setting:

$$E = a\,\varepsilon^2 \left(\frac{1}{\rho} - \frac{1}{r}\right),$$

in which a is a coefficient that depends upon the matter that the curve is made of, ε is its thickness at the point m, ρ is its radius of curvature at the same point, and r is the radius of curvature that exists at that point in the natural state of the curve. Since it is supposed to be inextensible, it will then follow that the arc length s, when counted from the extremity A and stopping at the point m, must not change when the curve is flexed by the forces that are applied to it. Hence, the radius r can be regarded to be a function of s in each particular case. The expression for the radius ρ for an arbitrary curve is $\rho = ds^3 / K$, where K has the same significance as before. One will then have:

$$u = \frac{E}{K} = \frac{a\,\varepsilon^2}{ds^3} \cdot \left(1 - \frac{ds^2}{r\,K}\right)$$

for the value of u that one must substitute in the second equation of the elastic curve. The integration of those two simultaneous equations is impossible in general, and one will succeed in separating the variables only in some very special cases that are the simplest ones that one can treat.

We conclude this article with a remark that can often be useful: When everything is similar with respect to the three coordinate axes x, y, z in a question of geometry or mechanics, and if one has an equation that relates to one of those axes then there will exist analogous equations that refer to the other two that can be deduced from the given equation by simple permutations of the variables x, y, z, and all of the other quantities that refer to them. However, in order to not risk being wrong, and in order for the analogous quantities to keep the same significance and not change sign, it will be necessary to perform that permutation in a certain way that we shall indicate, and the reason for which one will easily recognize. One arranges the letters x, y, z, and everything that corresponds to them in this manner:

$$\begin{array}{lll} x, & y, & z, \ldots, \\ z, & x, & y, \ldots \end{array}$$

One then replaces each letter of the top line with the one that is found below it in the bottom line in such a way that x takes the place of y, y takes the place of z, and z takes the place of x. The given equation will be changed into another one by that permutation, and upon performing the same permutation on it, one will have the analogous equation with respect to the third axis. That is how, for example, we deduced the third equation (1), which refers to the z-axis, from the first one, which relates to the x-axis, and then deduced the second equation by a second permutation, and the third equation by a third permutation.

____