"Über den algebraischen Inhalt topologischer Dualitätssätze," Math. Ann. **105** (1931), 165-205.

On the algebraic content of topological duality theorems

By

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Summary.

Let U and V be Abelian groups with finitely many generators, and let M be a (finite or infinite) cyclic group. The groups U and V define a *group pair* relative to M when any two elements u and v of U and V, resp., are associated with an element m of M – viz., the *product* of the two elements u and v – and the distributivity law for addition (as a group coupling in U and V) is valid; i.e., the relations $(u_1 + u_2) \cdot v = u_1 \cdot v + u_2 \cdot v$ and $u \cdot (v_1 + v_2) = u \cdot v_1 + u \cdot v_2$. The group pair U, V is called *primitive* when for any non-zero element of the one group an element of the second one can be found such that the product of the two elements is non-zero.

One then has the following *algebraic duality theorem:* Two groups that define a primitive pair are isomorphic.

If U and V are the r and $(n-r)$-dimensional Betti groups of a closed n-dimensional manifold, and one understands $u \cdot v$ to mean the intersection number (the intersection number, up to some fixed modulus μ, resp.) of the associated cycles then U and V define a primitive group pair (where M is either the group of all whole numbers or the residue classes mod μ), so they are isomorphic (Poincaré duality theorem).

When one chooses U and V to be the r-dimensional [$(n - r - 1)$-dimensional, resp.] Betti groups of a complex $K \subset R^n$ and its complementary set $R^n - K$, resp., and fixes one's attention on the linking numbers, instead of the intersection numbers, then the groups U and V, in turn, define a group pair, so they are likewise isomorphic (Alexander duality theorem).

In an analogous way, one obtains all of the topological duality theorems that are known up to now (that are expressed as generalizations of the two above).

Preface

In 1895, in his celebrated paper "Analysis Situs" [1]), Poincaré discovered the duality theorem that bears his name today, namely, the fact that for any r the r^{th} and $(n - r)^{\text{th}}$ Betti numbers of an oriented n-dimensional manifold are equal. At roughly the same time, Jordan expressed his curve theorem for the first time. However, at that time, nobody had any idea that these two totally different theorems belonged to the same circle of ideas, and in particular, that the second one would lead to broad and extremely significant generalizations. The path to these generalizations is, in all brevity, the one that follows.

In 1912, Brouwer [2]) proved the theorem of the invariance of a closed curve, which included Jordan's theorem as a very special case and generally asserted that the number of regions that a closed set determines in the plane depends upon only the topological properties of the set itself. In this way, the possibility was first suggested of separating the concept of a closed curve from that of the plane and of defining it invariantly. Thus, the path to adopting the invariants of the so-called combinatorial topology to the most general closed sets was already suggested, a path that has led, in the last five years, to a volume of new knowledge for which one has mainly has Alexandroff, Lefschetz, and Vietoris to thank [3]). *All of these results can be associated with the general duality theorem for closed sets, which the third chapter of the present paper is dedicated to.*

However, in this examination, one deals with not only the adaptation of theorems that were proved in an elementary context to more general contexts, but also with a generalization in regard to the dimensional relationships: What Jordan's theorem states in relation to dimension 2 (the plane) and 1 (the curve) will be formulated and proved, *mutatis mutandis*, for n and r, resp. One has Lebesgue to thank for taking the first step in this direction, who, in 1911, was the first to recognize [4]) that the property of an n-dimensional manifold (thus, for $n = 1$, it is a Jordan curve) separating $n+1$ space is a special case of the property of an r-dimensional manifold in n-dimensional space admitting an $n - r - 1$-dimensional linking. In this way, Lebesgue proved a part of the n-dimensional Jordan theorem; at the same time, Brouwer arrived at the proof of the remaining parts, as well as a complete and invariantly-defined theory of linking [5]).

One can thank Alexander [6]) for advancing the field in an essentially new way that opened up the widest perspective, who proved in an extraordinarily simple and elegant way that the $(n - r - 1)^{\text{th}}$ Betti number of the complementary space to an arbitrary complex (in R^n) [7]) equaled the r^{th} Betti number of the complex itself (*Alexander duality theorem*). That was a tremendous generalization of all the known theorems at the time on the circle of ideas of Jordan's theorems, insofar as they related to the topological images

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[1]) Journ. Ec. Poly. 1895.

[2]) Math. Ann. **72** (1912), 422-425.

[3]) Alexandroff, "Gestalt und Lage abgeschlossener Mengen," Ann. of Math. (2) **30** (1928), 101-187. There, one will also find references to the incisive papers of Lefschetz, Vietoris, et. al.

[4]) Comptes rendus Acad. Sciences Paris **154**, session on 27 March 1911.

[5]) Brouwer, "Beweis des Jordanschen Satzes für n-Dimensionen," Math. Annalen **71** (1911), 314-319, and "On looping coefficients," Proc. Akad. Amsterdam **15** (1912), 113-122.

[6]) "A proof and Generalization of the Jordan-Brouwer Theorem," Trans. Amer. Math. Soc. **23** (1922), 333-349.

[7]) In this paper, the n-dimensional Euclidian space, when extended by an infinitely distant point, was denoted by R^n throughout.

of polyhedra (and not to more general closed sets). The adaptation of Alexander's duality theorems to arbitrary closed sets was then carried out by Alexandroff [8]), in 1927 and, at roughly the same time, by Lefschetz and Frankl [9]). In this way, Lefschetz obtained results that related to the case of closed subsets of arbitrary manifolds; the essential tool that he used is a further construction of linking theory – i.e., in the final analysis, the theory of the so-called Kronecker intersection numbers – which he developed in sufficient generality as one might possibly desire for the new problems of topology.

On the other hand, in 1923, Veblen [10]) had already applied the theory of intersection numbers to the proof and generalization of Poincaré's duality theorems: Namely, he showed that one could always choose the r^{th} and $(n - r)^{\text{th}}$ Betti bases for a closed n-dimensional manifold so that the matrix of intersection numbers of the elements of the two bases would be the identity matrix, a fact that includes the Poincaré theorem and generalizes it essentially. When one compares the Poincaré-Veblen duality theorem, thus formulated, with a generalization of the Alexander duality theorem, which states that one can always choose the r-dimensional Betti basis of a complex K in R^n and the $n - r - 1$-dimensional Betti basis of the complementary space $R^n - K$ so that the matrix of linking numbers of the elements of the two bases would be the identity matrix [11]), a certain analogy emerged between these two theories with no further assumptions.

In the present paper, this analogy will be explained completely, by which, the two duality theorems – viz., those of Alexander, as well as Poincaré-Veblen – will follow from the application of one and the same purely algebraic principle to the Betti groups of the corresponding dimensions. This algebraic principle consists of the idea that for two Abelian groups U and V (which one thinks of as additive groups; i.e., the group operation is interpreted as addition), one introduces a new operation: the multiplication of an arbitrary element u of U with an arbitrary element v of V, for which, the product $u \cdot v$ is always an element of a third group of modulus M; M is therefore a finite or infinite cyclic group [12]).

The introduction of the just-described multiplication converts the system of two groups U and V into a *group pair* for the modulus M. Thus, a group pair is called primitive when for any non-zero element u (v, resp.) of the one group there is an element w of the other group such that $u \cdot w$ ($w \cdot v$, resp.) is non-zero.

The main theorem of primitive group pairs consists in the idea that the two groups of such a pair are isomorphic to each other. Now, in recent years it is indeed generally recognized that it is not the Betti numbers, but rather the Betti groups, that define the main focus of algebraic-topological investigations [13]), and that one must therefore also consider the so-called Betti groups, modulo μ. In order to avoid confusion of terminology, I will briefly summarize these basic notions here. The r-dimensional,

[8]) Gött. Nachr., Math.-Phys. Kl. 25 Nov. 1927.

[9]) Lefschetz, Ann. of Math. (2) **29** (1928), 232; Frankl, Wien. Ber., Dec. 1927, pp. 689.

[10]) Trans. Amer. Math. Soc. **25** (1923), 540.

[11]) Pontrjagin, Gött. Nachr., Math.-Phys. Kl., 25 Nov. 1927 and Frankl, *loc. cit.* [9]).

[12]) The infinite cyclic group (thus, the group of all whole numbers) will occasionally be referred to as the cyclic group of order zero. This manner of speaking will repeatedly prove to be very convenient in the course of this paper.

[13]) On this, cf., e.g., H. Hopf, "Eine Verallgemeinerung der Euler-Poincaréschen Formel," Gött. Nachr., Math.-Phys. Kl., 1928.

oriented sub-complexes of a given complex are linear forms with whole number coefficients in the r-dimensional elements of the complex; they define an Abelian group with finitely many generators (relative to addition) that might be called L^r. When one reduces the coefficients in the aforementioned linear forms, modulo a whole number $\mu > 1$, the group L^r_μ arises, which is the group of all sub-complexes, modulo μ. For any sub-complex, its boundary is defined to be the algebraic sum of the boundaries of its elements [14]), and the boundary, modulo μ, of a sub-complex, modulo μ, is defined likewise. Sub-complexes with boundary zero are called *cycles*, and similarly for sub-complexes, modulo μ. The cycles (cycles, modulo μ, resp.) define a subgroup Z^r (Z^r_μ, resp.) of L^r (L^r_μ, resp.). The group Z^r (Z^r_μ, resp.) includes a subgroup $\hat{H}^r$ ($\hat{H}^r_\mu$, resp.) of those cycles that take the form of boundaries (boundaries, modulo μ, resp.) of (r + 1-dimensional) sub-complexes (sub-complexes, modulo μ, resp.). The group $\hat{H}^r$ ($\hat{H}^r_\mu$, resp.) shall be called simply the group of bounding r-dimensional cycles (bounding cycles, modulo μ, resp.). The factor group $Z^r \mid \hat{H}^r$ is called the *r-dimensional Betti group* of the given complex, while the group $Z^r_\mu \mid \hat{H}^r_\mu$ is called the *Betti group, modulo* μ. The complete Betti group is the direct sum of two subgroups: The *torsion group*, which is generated by all elements of finite order, and the *reduced* Betti group, which is generated by all elements of the complete Betti group of infinite order. For the sake of simplicity, we refer to the reduced Betti group as the *Betti group, modulo zero*, such that now the numbers 0, 2, 3, … can appear as values of μ [15]).

With these preliminaries, we can express the generalization of the Poincaré duality theorem that we achieved quite simply: *The* r^{th} *and* $(n - r)^{\text{th}}$ *Betti groups, mod* μ, *define a primitive group pair relative to the cyclic group of order* μ *as its modulus* [12]). *The intersection number of the cycles in question is to be regarded as the product of two elements, where in the case* $\mu \neq 0$, *this intersection number is to be reduced modulo* μ. In an entirely analogous way, one also obtains the Alexander duality theorem in the following form: When K is a complex that lies in R^n, the r^{th} Betti group of K and the (n –

[14]) See the literature below in [25]).

[15]) Obviously, the groups L^n, Z^n, $\hat{H}^n$ can then be considered to be the groups L^n_μ, Z^n_μ, $\hat{H}^n_\mu$ with $\mu = 0$ (i.e., sub-complexes, cycles, boundaries, modulo zero). In the case $\mu = 0$, it is recommended that one further introduce the group $H^r = H^r_0$ of all those cycles Γ^r for which there is a positive, whole number $k \neq 0$ such that $k\,\Gamma^r$ bounds (so it is included in $\hat{H}^r = \hat{H}^r_0$). When one, in full generality, calls a subgroup U of an Abelian group G a *subgroup with division*, in which case, the inclusion $kx \subset U$ (x is an element of G, k is a positive, whole number) implies the inclusion $x \subset U$, one can define H^r_0 to be the smallest subgroup with division in $\hat{H}^r_0$. One easily sees that the reduced Betti group (hence, the Betti group, modulo zero) is nothing but the factor group $Z^r_0 \mid H^r_0$. By definition, for $\mu \neq 0$, we now set $H^r_\mu = \hat{H}^r_\mu$, and in the case of an *arbitrary* $\mu = 0, 2, 3, \ldots$ we introduce the term *homologous to zero* (in symbols, ~ 0) for all elements of H^r_μ. In the case $\mu \neq 0$, a cycle is homologous to zero if and only if it bounds (modulo μ), while in the case $\mu = 0$, we say that Γ^r is homologous to zero when there is a non-zero whole number k such that $k\,\Gamma^r$ bounds. *Therefore, the* r^{th} *Betti group, modulo* μ *(the case of* $\mu = 0$ *is not excluded) can be defined for every* μ *as the factor group* $Z^r_\mu \mid H^r_\mu$.

$r - 1)^{\text{th}}$ Betti group of $R^n - K$ define a primitive group pair, if one regards the linking number of the cycles in question as the product. Thus, the same arrangement is true relative to the various moduli that is true in the case of the Poincaré duality theorem [16]).

I call the aforementioned theorem *Alexander's theorem in the restricted sense;* it relates to complexes that lie in R^n. However, we shall also examine the more general case of a complex that is embedded in an arbitrary M^n. One also obtains a complete solution to the problem here; I call the corresponding theorem *Alexander's theorem in the broader sense*. It can be regarded as a generalization of the formulas that I already gave before for the case of "modulo 2" [17]). By the way, let it be remarked that in all of the present paper the concept of manifold is understood in a much more general sense than the usual one up to now. Namely, the so-called *h*-manifolds will be considered throughout, whose definition was found at roughly the same time by various authors – among them, Alexander, van Kampen, Vietoris, and the author – to be generalization of the classical concept of a manifold that rested upon only homological notions, and for that reason, was recognized to be invariant, and which is restated in § 1 of the second chapter.

After the so-to-speak classical case of the complex that is embedded in a manifold is dealt with, I turn to the case of an arbitrary closed set. Here, one can also immediately treat the general case of a closed set in an arbitrary manifold ("the case of F in M^n"). However, since the main difficulties of an algebraic nature already appear in the case of "K in M^n", and all of the set-theoretic difficulties appear in the case of "F in R^n", I have restricted myself to the latter case, in order to avoid all technical complications. The case of the closed sets will make the algebraic methods of this paper accessible, so one consequently appeals to the representation of a closed set by means of the Alexandroff projection spectra [18]). In this way, for any dimension r, one has, in place of a single Abelian group, a sequence of groups that each possess finitely many generators; these groups are the r^{th} Betti groups of the approximate complexes that enter into in the projection spectrum; the groups are linked to each other by homomorphic maps that correspond to the simplicial maps in the projection spectrum. In this way, the so-called "inverse sequences of homomorphisms" arise, which are definitive for the connectivity properties of closed sets. Indeed, these sequences of homomorphisms are defined with the aid of an arbitrarily chosen projection spectrum, but one finds that projection spectra that define homomorphic sets possess, in a certain sense, *equivalent* sequences of homomorphisms, such that one is justified in introducing the totality of all mutually equivalent sequences of homomorphisms as a new topological invariant, namely, the *r-dimensional cyclosis* of the set. One further finds that the *r*-dimensional cyclosis, which

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[16]) In the case $\mu = 0$, one can even prove a sharper result, namely, the so-called orthogonality of the two groups, which is a stronger property than primitivity, and which shall not, however, come under further consideration for us. Let it be remarked here that in the formulation of the two duality theorems one is necessarily compelled to regard the definitions of the Betti groups, modulo 0, as the reduced groups here. In fact, by the same method, one can show that the r^{th} torsion group of $R^n - K$ is not isomorphic to the $(n - r)^{\text{th}}$ torsion group, but to the $(n - r - 1)^{\text{th}}$ one; similarly, the r^{th} torsion group of K is isomorphic to the $(n - r - 1)^{\text{th}}$ torsion group of $R^n - K$ (when K is a complex that lies in R^n). As a consequence, in general, either the $(n - r)^{\text{th}}$ complete Betti group of an M^n is isomorphic to the r^{th} complete Betti group of the same manifold, or the $(n - r - 1)^{\text{th}}$ complete Betti group of $R^n - K$ is isomorphic to the r^{th} Betti group of K.

[17]) Gött. Nachr. 1927, pp. 323.

[18]) Alexandroff, *loc. cit.*, [3]), pp. 107.

is indeed not itself a group, defines a group in a unique way, namely, *the group that is dual to the cyclosis.* Furthermore, *this group is isomorphic to the* $(n - r - 1)$*-dimensional Betti group of the complementary space* $R^n - F$. The entire investigation may be again carried out for an arbitrary modulus μ, where, as always, in the case $\mu = 0$, one understands the Betti group, modulo zero, of $R^n - F$ to be the reduced Betti group. The most important consequence of this theory is undoubtedly the proof that it contains the fact that *the reduced Betti group of the complementary space to a closed set is a topological invariant of this set.* Moreover, with the help of the same methods, one can also prove the invariance of the torsion group of $R^n - F$; *by contrast, the question of the invariance of the complete Betti group of* $R^n - F$ under topological transformations of F remains undecided. Indeed, in general, a Betti group of $R^n - F$ does not have finitely many generators, and it also does not need to be representable as the direct sum of its reduced groups and the torsion groups.

The invariance of the Betti groups, modulo 2, of the $R^n - F$ was proved already by Alexandroff [19]). The proof was also true verbatim for an arbitrary prime number as modulus. For the case of modulus zero, a proof of invariance is included in the theorems of Lefschetz [20]), *which is, however, true only when the groups have finitely many generators;* in the cases mentioned, in fact, the invariance of the groups follows from the invariance of their ranks (Indeed, in the case of mod 0, one considers only the reduced, hence, only the free, groups). *By contrast, in the general case of infinitely many generators the isomorphism of the groups in no way follows from the equality of their ranks*, even in the case where there are no elements of finite order. The (additive) groups of all rational numbers, as well as the group of all dyadic fractions, already provide an example of two non-isomorphic groups whose rank is one, although they contain no element of finite order. Moreover, it will be proved in Appendix III that *any* Abelian group that consists of countably many elements with no elements of finite order can appear as the Betti group of some $R^n - F$ (even for $n = 3$). The invariance of this group cannot be proved mainly by methods that consider only the Betti numbers – hence, the ranks – such that our theorem is not in the slightest a self-explanatory extension of the known invariance of the Betti numbers, but lies fundamentally deeper. It is all the more interesting to also prove the invariance of the complete Betti groups of $R^n - F$.

In conclusion, I would like to mention that this paper was, to a large degree, inspired by a lecture of Alexandroff on combinatorial topology and a lecture of Emmy Noether on abstract algebra (both lectures were taught in the winter of 1928/29 at Moscow University). I also thank Alexandroff for much advice on the final editing of the present treatise.

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[19]) *loc. cit.* [8]).

[20]) Lefschetz, *loc. cit.*, [9])

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[21]) The things that are done in chap. II relate to polyhedral complexes; the case of continuous complexes (i.e., topological images of polyhedral complexes) will be treated in Appendix I.

Chapter I.

The algebraic fundamentals.

1. Let U and V be two Abelian groups with finitely many generators, and let M be a cyclic group of order μ; in the event that $\mu = 0$, this shall mean that M is the infinite cyclic group – i.e., the free (Abelian) group with one generator. We think of the groups U, V, M as being written additively. Since M can be replaced by an arbitrary isomorphic group in this investigation, we introduce, once and for all, the abbreviation that in the case $\mu = 0$, M will be represented by the (additive) group of all whole numbers, while for $\mu > 0$ the group M will be represented by the system of the smallest non-negative residues modulo μ. In this sense, an element of M is always a whole number.

2. Definition I. Two groups U and V define a *group pair* relative to M (the *modulus*) when any ordered pair of elements x, y – where x is an element of U and y is an element of V – is associated with an element k of M, namely, the *product* of the two elements x and y:

$$k = x \cdot y,$$

such that one always has:

$$(x + x') \cdot y = x \cdot y + x' \cdot y \qquad \text{(first distributivity law)}$$

and

$$x \cdot (y + y') = x \cdot y + x \cdot y' \qquad \text{(second distributivity law)}$$

(from this, it follows, in particular, that:

$$x \cdot 0 = 0 = 0 \cdot y \tag{1}$$

for any choice of x (y, resp.)).

3. Definition II. *Let A be an arbitrary subgroup of U; the totality of all elements y of V with the property that for any x in A one has:*

$$x \cdot y = 0$$

will be called the annihilator of A in V, and will be denoted by (V, A).

One defines the annihilator (U, B) for an arbitrary subgroup B of V in an analogous way.

It then follows from the first (second, resp.) distributivity law that for any choice of x and y one has:

$$x \cdot 0 = 0 = 0 \cdot y$$
$$(-x) \cdot y = -x \cdot y, \qquad x \cdot (-y) = -x \cdot y, \text{ resp.},$$

such that at the same time that one has $x \cdot y = 0$, one also has $(-x) \cdot y$ and $x \cdot (-y)$ equal to zero; in other words, an annihilator includes x, along with $-x$, y, along with $-y$, and the zero element also belongs to it. One then has the theorem:

I. *The annihilator is a subgroup of* U (V, *resp*.).

Definition III. *A group pair* U, V *is called primitive when the annihilator of each of the two groups in the other consists of only the zero element:*

$$(U, V) = 0, \qquad (V, U) = 0.$$

We also often say that U and V are *mutually primitive* (relative to M). One then has the following theorem:

II. *In the event that* U *and* V *define a primitive group pair, these groups may be decomposed into direct sums of cyclic subgroups:*

$$U = A_1 + A_2 + \ldots + A_n , \tag{2}$$

$$V = B_1 + B_2 + \ldots + B_n , \tag{3}$$

such that for the generators $a_1, a_2, \ldots, a_n$ *and* $b_1, b_1, \ldots, b_n$ *of the groups* A_i *and* B_i , *resp., one has the relations:*

$$\begin{aligned} a_i \cdot b_j &= 0 \qquad (\text{for } i \neq j) \\ a_i \cdot b_i &= k_i > 0, \end{aligned} \tag{4}$$

with $k_{i+1} = 0 \pmod{k_i}$; *thus,* k_i *are the divisors of* μ, *and* μ / k_i *is the order of* A_i *and* B_i .

Before we go into the proof of theorem II, we remark that as a result of it, one can represent the groups U and V as direct sums of one and the same number of cyclic groups (of equal order, resp.), such that one can formulate the following corollary:

Mutually primitive groups are isomorphic.

4. Proof of theorem II. One lets a_1 (b_1, resp.) denote those elements of U (V, resp.) (i.e., those "values" of x and y) such that the number $x \cdot y$ contains a smallest possible value of k_1 .

Then, for any choice of y (x, resp.), k_1 is a divisor of $a_1 \cdot y$ and $x \cdot b_1$; namely, if, e.g.:

$$a_1 \cdot y = qk_1 + r = q(a_1 \cdot b_1) + r,$$

with $r > 0$, then one would have:

$$a_1 \cdot (y - qb_1) = r, \qquad r < k_1 ,$$

so b_1 would have been chosen incorrectly.

One lets A_1 (B_1, resp.) denote the cyclic group that is generated by a_1 (b_1, resp.) and considers an arbitrary $x \in U$. From what we just proved, one infers the existence of a q such that $x \cdot b_1 = qk_1 = q(a \cdot b)$, so one has $(x - qa_1)\, b_1 = 0$; thus, $x - qa_1$ is an element x'' of (U, B_1), and one has:

(5) $$x = x' + x'',$$

with $x' \in A_1$, $x'' \in (U, B_1)$. Likewise, any element y of V can be represented in the form $y = y' + y''$, with $y' \in B_1$, $y'' \in (V, A_1)$. Let a be a common element of A_1 and $(U, B_1) = U_1$; we choose any $y \in V$ such that $y = y' + y''$. Therefore, one has $a \cdot y = a \cdot y' + a \cdot y''$. Now, one has, however, $a \cdot y' = 0$ when a is in (U, B_1) and y' is contained in B_1 . On the other hand, since $a \in A_1$, $y'' \in (V, A_1)$, one also has $a \cdot y''$. Therefore, for any $y \in V$, one has $a \cdot y = 0$, from which, the identity $a = 0$ follows, by means of the primitivity of the group pair. With that, we have proved that U is the direct sum of A_1 and U_1 . In the same way, one can prove that V is the direct sum of B_1 and $(V, A_1) = V_1$.

We now prove that the order of A_1 (B_1, resp.) equals μ / k_1 . Let s be the smallest positive number with the property that $sk_1 \equiv 0 \pmod{\mu}$; i.e., $sa_1 \cdot b_1 = 0$. The product of sa_1 with an arbitrary element of B_1 is then zero. If $sa_1 \in (U, B_1)$ then it follows that $sa_1 = 0$, since A_1 and (U, B_1) have only zero in their intersection. Therefore, the order of A is the smallest number s with the property that μ goes into sk_1 ; however, since $\mu \equiv 0$ (mod k_1) [22]), this number is equal to μ / k_1 .

As one easily recognizes, the groups U_1 and V again define a primitive group pair, and the process above yields the decompositions:

$$U_1 = A_2 + (U_1, B_2), \qquad V_1 = B_2 + (V_1, A_2).$$

Proceeding in this way, we obtain the direct sum decompositions:

$$U = A_1 + A_2 + \ldots + A_n + U_n ,$$
$$V = B_1 + B_2 + \ldots + B_n + V_n ,$$

with

$$U_{i+1} = (U_i , B_{i+1}),\ V_{i+1} = (V_i, A_{i+1}); \qquad U_i = A_{i+1} + U_{i+1} ,\ V_i = B_{i+1} + V_{i+1} ,$$

where for any i, U_i and V_i are mutually primitive. The process terminates after finitely many steps (since U and V indeed have finitely many generators); i.e., for a certain n, perhaps V_n is the zero group. However, since U_n and V_n define a primitive group pair, U_n must also be the zero group. Thus, the process terminates with U_i and V_i *simultaneously*, and one must ultimately get $U = A_1 + A_2 + \ldots + A_n$, $V = B_1 + B_2 + \ldots + B_n$.

It only remains for us to show that $k_{i+1} \equiv 0 \pmod{k_1}$. Now, however, it would follow from $k_{i+1} = a_{i+1} \cdot b_{i+1} = d \cdot k_1 + r$, with $0 < r < k_i$ that $(- da_i + a_{i+1}) \cdot (b_i + b_{i+1}) = - d(a_i \cdot b_i) + a_{i+1} \cdot b_{i+1} = - dk_i + k_{i+1} = r$, which contradicts the definition of a_i and b_i .

All of the parts of Theorem II are then proved.

5. Definition IV. The sum decompositions (2), (3) – in the event that they satisfy the conditions of Theorem II – define a *characteristic representation of the group pair U, V*. The constants k_i that thus appear are called the *invariant factors of the group pair.*

The term "invariant factors" will be justified by the following remark: Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be two linearly independent systems of generators of the groups U

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[22]) When $\mu \not\equiv 0 \bmod k_1$, so perhaps $\mu = qk_1 - r$ (with a positive $r < k_1$), one has $qk_1 = q(a_1 \cdot b_1) = \mu + r$ – i.e., $qa_1 \cdot b_1 = r < k_1$ – and the element $a_1 \cdot b_1$ would have been chosen incorrectly.

and V. Theorem II states that one can go from the x_i and y_i to new generators a_i and b_i such that one thus has $a_i \cdot b_j = 0$ (for $i \neq j$), $a_i \cdot b_i = k_i$, and $k_{i+1} \equiv 0 \pmod{k_i}$. However, since the transition from one system of generators to another one results from a unimodular substitution, this means nothing but the following:

Corollary II to Theorem II. *The numbers k_i are the elementary divisors of the matrix $(x_i \cdot y_j)$, where the x_i (y_i, resp.) define an arbitrary linearly independent system of generators of U (V, resp.).*

The invariant factors of the group pair are then determined uniquely by the group pair.

6. One now considers a group pair U, V, a subgroup A of U, and a subgroup B of V. When $A \subset (U, V)$ and $B \subset (V, U)$, and the elements x and x' of U, as well as the elements y, y' of V belong to the same residue class of A (B, resp.), one has (when one sets $\alpha = x' - x \in A$ and $\beta = y' - y \in B$):

$$x \cdot \beta = \alpha \cdot y = \alpha \cdot \beta = 0,$$

and therefore:

$$x' \cdot y' = (x + \alpha) \cdot (y + \beta) = x\, y.$$

In the case of $A \subset (U, V)$, $B \subset (V, U)$, the multiplication law for the group pair U, V induces a multiplication law (of the same modulus) for the factor groups $U \mid A$ and $V \mid B$. In particular, when $A = (U, V)$ and $B = (V, U)$, the group pair $U \mid A$, $V \mid B$ is primitive.

7. We now consider the case $\mu = 0$ in detail and introduce the following definition:

Definition V. *In the case $\mu = 0$, a primitive group pair is called orthogonal when its invariant factors are all equal to* 1.

A simple calculation shows that when U, V are orthogonal to each other, in the sense that we just formulated, and one has a linearly independent system of generators – say, x_1, x_2, …, x_n – of the one group – say, U – then one can find a system of generators y_1, y_2, …, y_n of the other group such that $x_i \cdot y_j = \delta_{ij}$, where – as usual – $\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ii} = 1$.

(If a_i and b_j are the generators of the characteristic representation and $x_i = \lambda_i^j a_j$ expresses the x_i in terms of the a_i then one has the following defining equation for the $y_i = \mu_i^h b_h$:

$$\sum_{(j)} \lambda_i^j \mu_k^j = \delta_{ik},$$

which (since $\left\| \lambda_i^j \right\| = 1$) is single-valued and soluble by whole numbers).

One can also naturally introduce the concept of orthogonality in precisely the same way for the case $\mu > 0$; however, it will be show that this is undesirable, because in this case, the usual primitive group pairs already serve the same purpose as the orthogonal group pairs in $\mu = 0$. In the majority of the following theorems, primitive group pairs with $\mu > 0$ will thus appear in parallel with the orthogonal group pairs with $\mu = 0$, which will be expressed by saying that we write "primitive" and "orthogonal" in brackets, where the convention will be introduced for all cases that the former adjective refers to the case $\mu > 0$ and the latter, to the case $\mu = 0$.

Definition VI. *When the factor groups $U \mid (U, V)$ and $V \mid (U, V)$ for a group pair U, V with $\mu = 0$ are not only mutually primitive, but also orthogonal, one calls U, V a conjugate group pair.*

In the case $\mu > 0$, all group pairs already serve the same purpose as the conjugate group pairs do in the case $\mu = 0$. Correspondingly, in the sequel, we will speak of "properties of (conjugate) group pairs" in the sense that the property in question is present for all group pairs in the case $\mu > 0$, but generally only for conjugate group pairs in the case $\mu = 0$.

We finally remark that in the case $\mu = 0$, we always understand the term "subgroup A of U" to mean a subgroup *with division* [23]).

8. *Let U, V be a (conjugate) group pair, and let $z(v)$ be a homomorphic map of V into the group M, under which all elements of (V, U) are mapped onto the zero element of M. This homomorphism can then be generated by an element x_0 of U, in the sense that for all $v \in V$:*

$$z(v) = x_0 \cdot v.$$

Thus, in the case where the group pair (U, V) is primitive (orthogonal, resp.), the element x_0 can be determined in only one way.

One first assumes that U and V are mutually primitive (orthogonal, resp.).

Let $a_1, a_2, \ldots, a_n$ ($b_1, b_2, \ldots, b_n$, resp.) be the generators of the characteristic representation of U and V, and let $k_1, k_2, \ldots, k_n$ be its invariant factors. One sets $h_i = z(b_i)$ and then proves that k_i goes to h_i. This is clear for $\mu = 0$, and indeed the $k_i = 1$ in this case. When $\mu > 0$, one has – since μ / k_i is the order of b_i:

$$0 = z\left(\frac{\mu}{k_i} b_i\right) = \frac{\mu}{k_i} z(b_i) = \frac{\mu h_i}{k_i} \qquad (\text{mod } \mu),$$

from which it follows that μ goes to $\mu\, h_i / k_i$, and therefore that h_i / k_i is a whole number.

One now sets:

$$x_0 = \sum_{i=1}^{n} \frac{h_i}{k_i} a_i .$$

If $v = \sum_{j=1}^{n} \mu_j b_j$ is an arbitrary element of V then one has:

[23]) Subgroups with division are defined in footnote [15]).

$$x_0 \cdot v = \sum_{i,j} \frac{h_i}{k_i} a_i \cdot \mu_j b_j = \sum_{i,j} \frac{h_i}{k_i} \mu_j (a_i \cdot b_j) = \sum_{i,j} \frac{h_i}{k_i} \mu_i (a_i \cdot b_i)$$
$$= \sum_i h_i \mu_i = z\left(\sum_i \mu_i b_i\right) = z(v),$$

with which, our assertion is proved.

If there are two elements x and x' that satisfy the condition above then one would have $(x - x')\ v = 0$ for any v, which is only consistent with the primitivity of the group pair when $x = x'$.

Now, let U, V be a (conjugate) – but not necessarily primitive – group pair. From the conditions of our theorem, it follows immediately that all v that belong to the same residue class of (V, U) have the same value, such that the map z defines a homomorphic map of the factor group $V_1 = V \mid (V, U)$ that satisfies our conditions. However, V_1 is primitive (orthogonal, resp.) to $U_1 = U \mid (U, V)$, so it follows that there exists an element x_0 of U_1 such that for any η in V_1 one has $x_0 \cdot \eta = z(\eta)$. From the definition of the multiplication of residue classes, (§ 6) it then follows that for all elements ξ_0 of U that belong to the residue class x_0 and for any element y of V one has:

$$x_0 \cdot y = \xi_0 \cdot \eta = z(\eta) = z(y),$$

where η means the residue class that belongs to y. Our theorem is thus proved:

9. Lemma. *When U, V are a primitive (orthogonal, resp.) group pair and A is a subgroup of U then A, V is a (conjugate) group pair.*

The lemma is trivial for $\mu \neq 0$. In the case $\mu = 0$, let $u_1, u_2, \ldots, u_n$ be a linearly independent system of generators of U that are so arranged that, perhaps, $u_1, u_2, \ldots, u_r$ generates the subgroup A. From § 7, the system of generators $v_1, v_2, \ldots, v_n$ can be determined in such a way that one has $u_i \cdot v_j = \delta_{ij}$. Since $u_i \cdot v_h = 0$ for arbitrary $i \leq r$ and $h > r$, the $v_{r+1}, \ldots, v_n$ all belong to (V, A); on the other hand, when $v = c^i\, v_i$ is any element of V such that c^h is non-zero for some $h \leq r$, $u_h \cdot v$ is non-zero. Therefore, all of the elements that are generated by just the $v_{r+1}, \ldots, v_n$ belong to (V, A); in other words, $v_{r+1}, \ldots, v_n$ define an independent system of generators for (V, A).

Since (A, V) obviously consists of only the zero element, we have to show that A, $V \mid (V, A)$ is an orthogonal group pair. From what we just proved, under the homomorphic map of V onto $V \mid (V, A)$, the elements that are generated by the $v_{r+1}, \ldots, v_n$ go to zero, while the $v_1, \ldots, v_r$, by contrast, go to elements $\beta_1, \ldots, \beta_r$ that are all different from each other and from zero and define a system of generators for $V \mid (V, U)$. Furthermore, since one shall set $u_i \cdot \beta_j = u_i \cdot v_j = \delta_{ij}$ (for $i, j \leq r$), the groups A and $V \mid (V, A)$ are mutually orthogonal, and the lemma is proved.

10. Theorem III. *If U, V is a primitive (orthogonal, resp.) group pair, A is a subgroup of U, and $B = (V, A)$ then $A = (U, B)$.*

One denotes the group (U, B) by A'; it then follows from the definition of B that for any choice of elements $x \in A$, $y \in B$, one has $x \cdot y = 0$, such that in any case the relation $A \subset A'$ is valid. In order to prove the converse inclusion, one considers – under the assumption that it is not applicable – any element z of $A - A'$. The product $z \cdot v$ is determined uniquely for this element z and an arbitrary v in V, so for all y in B one must have $z \cdot y = 0$. Therefore, a homomorphic map $z(v) = z \cdot v$ is defined, to which the theorem of § 6 can be applied, where A now takes on the role of U (which is permissible if the groups A and V are indeed conjugate to each other, as in the lemma). Therefore, there exists an element x_0 of A such that for all $y \in V$ one has $x_0 \cdot y = z \cdot y$, so $(x_0 - x) \cdot y = 0$; since the groups U and V are mutually primitive, it follows from the latter equation that $x_0 - x \in A$, contrary to the definition of z. Theorem III is then proved by contradiction.

11. As a generalization of the lemma of § 9, we then prove the following:

Theorem IV. *U and V might define a (conjugate) group pair; if A and B are the subgroups of U and V that contain the annihilators* $A' = (U, B)$ *and* $B' = (V, A)$, *resp., then (on the basis of the multiplication law that is defined for U and V) A and B also define (conjugate) group pair.*

Proof. In the case $\mu \neq 0$, the assertion is trivial. Therefore, let $\mu = 0$. We next prove our theorem under the assumption that U and V are not only conjugate, but orthogonal.

Let:

(1) $$a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_{k+r}, a_{k+r+1}, \ldots a_n,$$

(2) $$b_1, b_2, \ldots, b_k, b_{k+1}, \ldots, b_{k+r}, b_{k+r+1}, \ldots b_n$$

be two systems of generators of U (V, resp.) with $a_i \cdot b_j = \delta_{ij}$. We enumerate the elements of the systems (1) and (2) such that $a_1, \ldots, a_k$ is a system of generators of A' and $a_1, \ldots, a_{k+1}$ is a system of generators of A.

Since, by definition, $A' = (U, B)$, one then has, by means of Theorem III that $B = (V, A')$. An element b of V then belongs to B if and only if for all a_i with $i \leq k$ one has $a_i \cdot b = 0$; however, this condition is satisfied only for linear combinations of the b_j with $j > k$. As a result, the aforementioned b_j define a system of generators for B. In an analogous way, b belongs to $B' = (V, A)$ when for all a_i with $i \leq k + r$ one has $a_i \cdot b = 0$, from which it follows, in turn, that the $b_{k+r+1}, \ldots b_n$ define a system of generators of B'.

We now consider the factor groups $A \mid A'$ and $B \mid B'$ and the associated homomorphisms. Let α_i be the image of a_i and let β_j be the image of b_j under these homomorphisms. Therefore, $\alpha_1 = \ldots = \alpha_k = 0$, while the $\alpha_{k+1}, \ldots, \alpha_{k+r}$ are different from each other and non-zero and define an independent system of generators for $A \mid A'$. Likewise, $\beta_{k+r+1} = \ldots = \beta_n = 0$ and the $\beta_{k+1}, \ldots, \beta_{k+r}$ define an independent system of generators for $B \mid B'$. Furthermore, $\alpha_i \cdot \beta_j = a_i \cdot b_j = \delta_{ij}$, such that $A \mid A'$ is orthogonal to $B \mid B'$. The theorem is thus proved in the special case of orthogonal group pairs U, V.

Now, let U, V be a conjugate group pair for which orthogonality will not be assumed. We let A'' (B'', resp.) denote the annihilators (U, V) [(V, U), resp.]. One then has the inclusions:

$$A'' \subset A' \subset A, \quad B'' \subset B' \subset B,$$

from which, all of these groups are subgroups with division.

We now define the factor groups $\bar{U} = U \mid A''$ and $\bar{V} = V \mid B''$, and consider the associated homomorphisms f and g:

$$\bar{U} = f(U), \qquad \bar{V} = g(V).$$

Therefore, let:

$$\bar{A} = f(A), \qquad \bar{A}' = f(A') \qquad \text{and} \qquad \bar{B} = g(B), \qquad \bar{B}' = g(B');$$

these groups are once more subgroups with division.

Due to the isomorphism theorem [24]), $\bar{A} \mid \bar{A}'$ is isomorphic to $A \mid A'$ and $\bar{B} \mid \bar{B}'$ is isomorphic to $B \mid B'$. Furthermore, $\bar{U}$ and $\bar{V}$ are mutually orthogonal, and one has the relations:

$$\bar{A}' = (\bar{U}, \bar{B}) \subset \bar{A}, \qquad \bar{B}' = (\bar{V}, \bar{A}) \subset \bar{B}.$$

From the orthogonality of $\bar{U}$, $\bar{V}$, it then follows, on the basis of the just-proved orthogonality of $\bar{A} \mid \bar{A}'$ and $\bar{B} \mid \bar{B}'$ – by means of the isomorphisms between $\bar{A} \mid \bar{A}'$ and $A \mid A'$ and between $\bar{B} \mid \bar{B}'$ and $B \mid B'$ – that $A \mid A'$ and $\bar{B} \mid \bar{B}'$ are orthogonal. With that, Theorem IV is proved completely.

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[24]) Cf., say, E. Noether, Math. Zeit. **30**, pp. 648.

Chapter II

The generalized duality theorems of Poincaré-Veblen and Alexander.

1. Geometric preliminaries [25]).

1. Let K^n be a simplicial complex, let a^r be a simplex of K, and let a_1^n, a_2^n, …, a_r^n be the simplexes of K^n that the a^r carry on their boundaries; these simplexes define the *star* of a^r. The totality of the opposite sides of the a_i^n for a simplex a^r defines the *neighboring complex* (Umgebungskomplex) $Z^{n-r-1}(a^r)$ of a^r in K^n.

A connected complex M^n is called a *manifold* when all of the neighboring complexes in it are homeomorphic to spheres of the corresponding dimension. Since the topological invariance of the definition that was just formulated (i.e., its independence of the particular simplicial decomposition that is present in M^n) is unproved up to now, in the sequel, we will appeal to the more general concept of the so-called h-manifolds [26]).

A connected complex M^n will be called an *h-manifold* when any $n - 1$-dimensional element of M^n is linked with precisely two n-dimensional simplexes, while the neighboring complex of any a^k ($k < n - 1$) has the following property: $Z^{n-r-1}(a^r)$ is a connected complex in which the r-dimensional cycle ($0 < r < n - k - 1$) bounds and for the most part a single $n - k -$ 1-dimensional cycle exists that is not homologous to zero at that place. The invariance of this concept may be proved easily [26]).

An h-manifold (which is usually always a pseudo-manifold, in the Brouwer sense) is called *orientable* when its n-dimensional elements can be oriented such that the algebraic sum of its oriented boundaries is equal to zero. In what follows, we will allow only such orientations.

2. Let M^n be an n-dimensional h-manifold. One considers a barycentric subdivision of M^n and orients the elements of it as follows: Let $a^n = \varepsilon(a_0, a_1, \ldots, a_n)$ be a positively-oriented n-dimensional simplex of M^n and let $a^r = \eta(a_0, a_1, \ldots, a_r)$ be a likewise positively-oriented side of a^r. Finally, let β_i be the center of mass of $(a_0, a_1, \ldots, a_i)$ (thus, i is arbitrary, hence, independent of r). The positive orientation of the barycentric simplex that is determined by the vertices of $\beta_r, \beta_{r+1}, \ldots, \beta_n$ is then, by definition, $\varepsilon \cdot \eta(\beta_r, \beta_{r+1}, \ldots, \beta_n)$. When one permutes the sequence of vertices $a_r, a_{r+1}, \ldots, a_n$ in all possible ways, one obtains $(n - r)!$ different barycentric simplexes; they lie in a^n and are called *dual* to a^r. When one carries out this construction for all simplexes a^n that are connected to a^r, one obtains the totality of all barycentric simplexes that are dual to a^r. The

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[25]) For basic concepts of the topology of complexes and manifolds, cf., say, E. R. van Kampen, "Die kombinatorische Topologie und die Dualitätssätze," Dissertation, Leiden (1929), as well as Alexander, "Combinatorial Analysis Situs," Trans. Amer. Math. Soc. **28** (1926), 301-329 and Ann. Math. (2) **21** (1930), 292-320 and Lefschetz, "Intersections and transformations of complexes and manifolds," Trans. Amer. Math. Soc. **28** (1926), 1-49. Further literature is in van der Waerden, "Kombinatorische Topologie," Jahresber. d. Deutsche Math. Ver. **39** (1930), pp. 121. In the sequel, one will observe the terminological conventions of the preface, in particular, footnote [15]).

[26]) Cf., Ann. Math. (2) **31**, pp. 307 and Vietoris, Monat. f. Math. u. Phys. **35** (1927), pp. 165, as well as van Kampen, *loc. cit.*, pp. 13.

algebraic sum of these simplexes (when oriented with the prescription above) defines the *barycentric star that is dual to* a^r; we denote it by $b^{n-r}(a^r)$.

One easily sees that the boundary of an r-dimensional barycentric star is composed of barycentric stars of dimension $r - 1$. Sign convention: If:

$$a^k \to \varepsilon\, a^{k-1} + \dots$$

and b^{n-k} (b^{n-k+1}, resp.) are the barycentric stars dual to a^k (a^{k-1}, resp.) then:

$$b^{n-k+1} \to (-1)^k \cdot b^{n-k} + \dots$$

3. We now consider two kinds of "building blocks," from which we will construct sub-complexes of M^n: The building blocks of the first kind (which lead to sub-complexes of the first kind) are the elements of the given simplicial decomposition of M^n; i.e., the simplexes of various dimensions; for dimension r, let them be, say, a_1^r, a_2^r, …, $a_{\alpha_r}^r$. A sub-complex of the first kind is, correspondingly, regarded as a linear form of the form $\lambda^i a_i^r$. The building blocks of the second kind are the barycentric stars. Since the r-dimensional barycentric stars correspond to the $n - r$-dimensional simplexes of M^n in a one-to-one way, they may be enumerated thus:

$$b_1^r,\ b_2^r,\ \dots,\ b_{\alpha_{n-r}}^r;$$

therefore, $b_i^r = b^r(a_i^{n-r})$. An r-dimensional sub-complex of the second kind is then, by definition, a linear form of the form $\lambda^i b_i^r$.

Now, let two sub-complexes $A = A^r = \lambda^i a_i^r$ and $B = B^{n-r} = \mu^i b_i^{n-r}$ of the first (second, resp.) kind be given. The number:

$$\chi(A, B) = \sum_{i=1}^{\alpha_r} \lambda^i \mu^i$$

is called the *Kronecker characteristic* or the *intersection number* of the two complexes A and B.

4. One now considers two *continuous complexes* A^r and B^{n-r} that are embedded in M^n (with possible singularities) [27]). It will be assumed that none of the two complexes A^r and B^{n-r} meet the boundary of the other one, so the minimal distance from one complex to the boundary of the other one is a positive number σ. One can then assume that the simplexes of M^n are all smaller than $\frac{1}{100}\sigma$. The complexes A^r and B^{n-r} can be approximated arbitrarily well by sub-complexes of the first (second, resp.) kind in M^n.

One proves the following facts with no effort:

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[27]) A single-valued (but not necessarily one-to-one), continuous image of a polyhedral complex in M^n is called an embedded complex in M^n (with possible singularities). In the event that the map is one-to-one, one speaks of a singularity-free embedding.

1. When A' is a sub-complex of the first kind, B' is a sub-complex of the second kind, and A' (B', resp.) approximates the complexes A^r and B^{n-r} sufficiently well, the number $\chi(A', B')$ has a value that is independent of the particular choice of the approximating complexes A' and B'; it is called the *Kronecker characteristic (the intersection number)* $\chi(A^r, B^{n-r})$ of the complexes A^r and B^{n-r}.

2. The Kronecker characteristic of two complexes A^r and B^{n-r} does not depend upon the choice of simplicial approximation of M^n; it thus represents a relative invariant of A and B relative to M^n.

3. $$\chi(A^r, B^{n-r}) = (-1)^{r(n-r)}\,\chi(B^{n-r}, A^r).$$

Remark. We denote the (oriented) boundary of the (oriented) complex K^r by $\dot{K}^{r-1}$. One then has:

Theorem I. *If $\dot{A}^{r-1}$ is disjoint from $\dot{B}^{n-r}$ then one has:*

$$\chi(A^r, \dot{B}^{n-r}) = (-1)^r\,\chi(\dot{A}^{r-1}, B^{n-r+1}). \tag{1}$$

It suffices to prove this assertion for the case where A^r is a simplex and B^{n-r+1} is a barycentric star. The assertion is then trivial in the event that A^r and B^{n-r+1} are disjoint. One assumes that A^r and B^{n-r+1} have a non-vacuous intersection. In this case, the barycentric star B^{n-r+1} is dual to a side A^{r-1} of A^r; therefore, if, perhaps:

$$A^r \to \varepsilon\cdot A^{r-1} + \ldots, \qquad B^{n-r} = b^{n-r}(A^r)$$

then one has:

$$\chi(\dot{A}^{r-1}, B^{n-r+1}) = \varepsilon, \qquad B^{n-r+1} \to (-1)^r\cdot\varepsilon\cdot B^{n-r} + \ldots, \qquad \chi(A^r, \dot{B}^{n-r}) = (-1)^r\,\varepsilon,$$

from which, the assertion follows.

Theorem II. *If A^r and B^{n-r} are two cycles, at least one of which bounds in M^n then one has:*

$$\chi(A^r, B^{n-r}) = 0.$$

Proof. In fact, let, e.g., $B' \to B^{n-r}$. One then has:

$$\chi(A^r, B^{n-r}) = \chi(A^r, \dot{B}') = \pm\,\chi(\dot{A}^r, B').$$

Q. E. D.

4. Let A^r and B^s be two disjoint cycles in M with $r + s = n - 1$. These cycles might, moreover, bound in M^n, and indeed let:

$$A' \to A^r, \qquad B' \to B^r.$$

The number $\chi(A^r, B')$ is called the *linking number* of A^r with B^s and will be denoted by $\mathfrak{v}(A^r, B^s)$. *It does not depend upon the choice of* B', because if B'' is a second complex that is bounded by B^s then $B' - B''$ is a cycle and A^r is a bounding cycle. It then follows from Theorem II that:

$$\chi(A^r, B' - B'') = 0, \qquad \text{i.e.,} \qquad \chi(A^r, B') = \chi(A^r, B'').$$

Q. E. D.

Theorem III. *Again, let A^r and B^s be two disjoint bounding cycles in M^n with $r + s = n - 1$. One then has:*

$$\mathfrak{v}(A^r, B^s) = (-1)^{rs+n}\,\mathfrak{v}(B^s, A^r).$$

Proof. As always, let $A' \to A^r$, $B' \to B^s$. By using what was already proved, one has:

$$\mathfrak{v}(A^r, B^s) = \chi(\dot{A}', B') = (-1)^{r+1}\chi(A', \dot{B}') = (-1)^{r+1}(-1)^{(r+1)s}\chi(\dot{B}', A') = (-1)^{rs+n}\chi(B^s, A')$$
$$= (-1)^{rs+n}\mathfrak{v}(B^s, A^r).$$

Q. E. D.

If the cycles A^r and B^s, $r + s = n - 1$ are disjoint and are homologous to zero in M^n such that, e.g., cA^r and dB^s bound in M^n (where c and d are suitably chosen whole number coefficients) then one can define the linking number $\mathfrak{v}(A^r, B^s)$ to be $\dfrac{\mathfrak{v}(cA^r, dB^s)}{cd}$; in general, one obtains rational linking numbers in this way. In our later presentation, the ratios will be so arranged that we can always arrive at whole number linking numbers.

5. We conclude these preliminaries with the following closely-related theorem:

Theorem IV. *Let M^n be an h-manifold, K, a complex composed of simplexes in M^n, L, the complex that composed of all of the barycentric stars* (= *building blocks of the second kind*) *that are disjoint to K, and let Γ be an arbitrary cycle that lies in $M^n - K$. Under these condition, Γ is a sub-cycle Δ that is homologous L* (*which is therefore a sub-complex of the second kind in M^n*) *in $M^n - K$. In the event that Γ is a sub-cycle of L that bounds in $M^n - K$, Γ is the boundary of a sub-complex of L (that is composed of building blocks of the second kind).*

Proof. We first prove the following lemma:

Any closed set $F \subset M^n - K$ may be converted into a set F' that lies in L by means of a continuous deformation inside of $M^n - K$, and indeed, in such a way that throughout the entire deformation process all of the points that belong to L remain fixed.

Let $\mathfrak{R}$ be the system of all barycentric stars in M^n that have a non-vacuous intersection with K, so they are dual to the simplexes of K. We let s denote the highest dimension of the stars in the system $\mathfrak{R}$ that contain points of F in their interiors; let S be one of these stars. Since S is dual to a simplex that contains no point of F, one transports all of the points of F that belong to S onto the boundary of the star by central projection from the center of mass of the simplex ("the center of the star"). This "cleaning out" is a deformation of F that fixes all of the points of this set that are exterior to it or on the boundary of S, and during the entire time F and K stay disjoint. A repeated application of this cleaning process converts F into a closed set F' that possesses no point in the interior of a star that belongs to $\mathfrak{R}$. Let x be an arbitrary point of F'; it is an interior point of a barycentric star that – since it is disjoint to K, from what was just proved – must be contained in L, with which the lemma is proved.

Now, let $\bar{L}$ be the complex of the first kind (which coincides with L geometrically) that one obtains when one again breaks up the building blocks of the second kind that L is composed of into simplexes. From the lemma, it then follows that each cycle Γ^r that belongs to $M^n - K$ lies in $\bar{L}$, so it can be further converted into a cycle γ^r that is built up from simplexes of $\bar{L}$ by a homotopy. One now considers the simplexes of γ^r to be sub-simplexes of the building blocks of the second kind that define the complex L. If such a building block S^r were only partially contained in γ^r then one would be able to remove it in such a way that one would carry the part of γ^r that lies in it to the boundary of S^r (which always happens effortlessly). After one has repeated this finitely many times, γ^r is converted into a cycle that is constructed from building blocks of the second kind, and thus, into a sub-cycle of $\bar{L}$. Precisely the same process can be applied to the homology carrier that was mentioned in the conclusion of Theorem IV, which then yields the proof of the two assertions of this theorem.

II. Formulation and proof of the two duality theorems.

1. Let M^n be an orientable and oriented h-manifold. Let $a_1^r, a_2^r, \ldots, a_{\alpha_r}^r$ be the r-dimensional elements of the given simplicial decomposition of M^n, and let $\mathfrak{b}_1^{n-r}, \mathfrak{b}_2^{n-r}, \ldots, \mathfrak{b}_{\alpha_r}^{n-r}$ be the barycentric stars that are dual to them; we set $\chi(a_i^r, \mathfrak{b}_j^{n-r}) = \delta_{ij}$. Let K be a complex that is constructed out of simplexes of M^n, where the simplexes that appear in K might be $a_1^r, a_2^r, \ldots, a_{h^r}^r$. Then, among the $\mathfrak{b}_k^{n-r}$, the ones with $k \leq hr$, and only these ones, have a non-vacuous intersection with K.

The entire investigation that follows is based upon a fixed number μ as modulus that is equal to zero or greater than 1. Let the groups $L_\mu^r, Z_\mu^r, \hat{H}_\mu^r, H_\mu^r$ that pertain to K be denoted simply by $L^r, Z^r, \hat{H}^r, H^r$. Let the Betti groups mod μ of K be denoted simply by B^r, where the conventions that were made in the preface [in particular, the ones in footnote [15])] will preserve their validity throughout.

We further let $\mathfrak{L}^r$ denote the Abelian group that is generated by the elements $\mathfrak{b}_k^r$, $k = 1, 2, \ldots, h^{n-r}$ with the single relation $\mu\mathfrak{b}_k^r = 0$. The boundary mod μ of an element of $\mathfrak{L}^r$ (which is indeed a complex of the second kind) can be written as a linear form in the $\mathfrak{b}_k^{r-1}$. When we keep only the terms with $k \leq h^{n-r+1}$ in the linear form, we get a complex of the second kind: the *reduced boundary* of the chosen elements of $\mathfrak{L}^r$. The complexes that appear as the reduced boundaries of the complexes that exist in $\mathfrak{L}^r$ define a subgroup of $\mathfrak{L}^{r-1}$ that we denote by $\hat{\mathfrak{H}}^{r-1}$; analogous to the previous notations, we set $\mathfrak{H}^r = \hat{\mathfrak{H}}^r$ when $\mu \neq 0$, while in the case $\mu = 0$, $\mathfrak{H}^r$ will be defined as the smallest subgroup with division of $\mathfrak{L}^r$ over $\hat{\mathfrak{H}}^r$. Furthermore, there exists a homomorphic map of $\mathfrak{L}^r$ onto $\mathfrak{H}^{r-1}$. The *kernel* [28]) of this homomorphism shall be denoted by $\mathfrak{Z}^r$; one easily sees that $\mathfrak{H}^r$ is a subgroup of $\mathfrak{Z}^r$.

2. A multiplication law shall now be established for the two groups L^r and $\mathfrak{L}^{n-r}$. This happens simply by setting the product of the element $a \in L^r$ and $\mathfrak{b} \in \mathfrak{L}^{n-r}$ equal to the intersection number $\chi(a, \mathfrak{b})$ mod μ. By means of this multiplication, the groups L^r and $\mathfrak{L}^{n-r}$ define a primitive (orthogonal, resp.) group pair if the two groups indeed possess some system of generators a_i^r ($\mathfrak{b}_j^{n-k}$, resp.) with $a_i^r \cdot \mathfrak{b}_j^{n-k} = \delta_{ij}$.

3. We now prove the relations:

$$(1) \qquad (L^r, \mathfrak{H}^{n-r}) = Z^r, \qquad (\mathfrak{L}^{n-r}, H^r) = \mathfrak{Z}^{n-r},$$

from which, due to chap. I, § 10, Theorem III, the relations then follow:

$$(2) \qquad (\mathfrak{L}^{n-r}, Z^r) = \mathfrak{H}^{n-r}, \qquad (L^r, \mathfrak{Z}^{n-r}) = H^r.$$

A. Let $a \in Z^r$ and $\mathfrak{b} \in \mathfrak{H}^{n-r}$; we prove that $\chi(a, \mathfrak{b}) = 0$ (mod μ). When $\mu \neq 0$, there exists a $\mathfrak{c} \in \mathfrak{L}^{n-r+1}$ whose reduced boundary is $\mathfrak{b}$. It follows that:

$$\mathfrak{c} \rightarrow \mathfrak{b} + \mathfrak{b}', \qquad \mathfrak{b}' \text{ is disjoint to } K;$$

$kb + b'$ is again homologous to zero, so $\chi(a, k\mathfrak{b} + \mathfrak{b}') = 0$, which, since $\chi(a, \mathfrak{b}) = 0$, yields:

$$0 = \chi(a, k\mathfrak{b}) = k\chi(a, \mathfrak{b}),$$

and ultimately yields $\chi(a, \mathfrak{b}) = 0$.

B. Let a be an element of L^r that does not belong to Z^r; we propose to find an element of $\mathfrak{H}^{n-r}$ that possesses a non-zero intersection number with a. Since a does not belong to Z^r – hence, it is not a cycle – one has:

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[28]) When a group A is mapped homomorphically onto a group B, the subgroup of A that consists of all elements that are mapped to the identity (or zero)-element of B is called the *kernel* of the homomorphic map.

$$a \to b \in H^{r-1}, \qquad b \neq 0;$$

L^{r-1} and $\mathfrak{L}^{n-r+1}$, however, define a primitive (orthogonal, resp.) group pair; it follows that there exists a $\mathfrak{c} \in \mathfrak{L}^{n-r+1}$ such that $\chi(b, \mathfrak{c}) \neq 0$. If we let $\dot{\mathfrak{c}}$ denote the boundary and let $\hat{\mathfrak{c}}$ denote the reduced boundary of $\mathfrak{c}$, and remark that due to Theorem I of this chapter $\chi(b, \mathfrak{c}) = \pm\, \chi(a, \dot{\mathfrak{c}})$, while, on the other hand, one obviously has $\chi(a, \dot{\mathfrak{c}}) = \chi(a, \hat{\mathfrak{c}})$, then we see that $\chi(a, \hat{\mathfrak{c}}) \neq 0$. The element $\hat{\mathfrak{c}}$ thus satisfies our requirements, and the first of the two formulas (1) is therefore proved; the second one is proved in precisely the same way.

We connect these procedures with the following remark: Since L^r and $\mathfrak{L}^{n-r}$ define a primitive (orthogonal, resp.) group pair and:

$$(L^r, \mathfrak{Z}^{n-r}) \in Z^r, \qquad (\mathfrak{L}^{n-r}, Z^r) \in \mathfrak{Z}^{n-r},$$

it follows from chap. I, § 6 that the factor groups $Z^r \mid H^r$ and $\mathfrak{Z}^{n-r} \mid \mathfrak{H}^{n-r}$ are likewise mutually primitive (orthogonal, resp.).

4. We now obtain the theorem of Poincaré-Veblen when we set $K = M^n$. In this case, the group $Z^r \mid H^r$ is the r-dimensional Betti group and $\mathfrak{Z}^{n-r} \mid \mathfrak{H}^{n-r}$ and is the $n - r$-dimensional Betti group of M^n. We thus obtain:

The generalized Poincaré-Veblen duality theorem: *The reduced Betti groups of dimension r and n – r of an n-dimensional h-manifold are mutually primitive (orthogonal, resp.) as long as one considers the image of the intersection number to be the multiplication law; in particular, the two groups are isomorphic to each other.*

5. We now go on to the Alexander duality theorem. We first prove the Alexander theorem in the restricted sense (which actually defines a generalization of the result that Alexander himself proved), when we assume that in all cycles bound in the h-manifold M^n. We call such h-manifolds (generalized) *Poincaré-Veblen spaces.* Then, let K be a complex (that is constructed from the elements of the given simplicial decomposition), for which, we assume only that it does not coincide with M^n.

In the following investigation of the bounding relations in $M^n - K$, we can restrict ourselves to the consideration of the complexes that are built up from barycentric stars $\mathfrak{b}$. We shall do this without mentioning it.

Let $\mathfrak{a}$ be an s-dimensional cycle in $M^n - K$. Since $\mathfrak{a}$ bounds in M^n, there exists a $\mathfrak{c} \to \mathfrak{a}$; $\mathfrak{c}$ is a linear form in the $\mathfrak{b}_i^{s+1}$ and can be represented in the form $\mathfrak{c} = \mathfrak{b} + \mathfrak{d}$, where $\mathfrak{b}$ is an element of $\mathfrak{Z}^{s+1}$ and $\mathfrak{d}$ is disjoint to K. Furthermore, one has:

$$\mathfrak{d} = \mathfrak{b} - \mathfrak{c} \to \dot{\mathfrak{c}} - \dot{\mathfrak{b}} = \mathfrak{a} - \dot{\mathfrak{b}},$$

so $\mathfrak{a} \sim \dot{\mathfrak{b}}$ in $M^n - K$.

Let $\mathfrak{a}'$ be any other cycle in $M^n - K$ that is homologous to $\mathfrak{a}$ in that space. If $m \neq 0$ then there is an $\mathfrak{e} \in M^n - K$ such that:

$$\mathfrak{e}' \to \mathfrak{a}' - \mathfrak{a}.$$

On the other hand, a' is contained in M^n, which leads to:

$$\mathfrak{c}' \to \mathfrak{a}', \qquad \mathfrak{c}' = \mathfrak{b}' + \mathfrak{d}'.$$

Furthermore, one has:

$$\mathfrak{c} - \mathfrak{c}' + \mathfrak{e} \to \mathfrak{a} - \mathfrak{a}' + \mathfrak{a}' - \mathfrak{a} = 0,$$

such that $\mathfrak{c} - \mathfrak{c}' + \mathfrak{e}$ is a cycle (that bounds in M^n). As a result, there is an $\mathfrak{f}$ with:

$$\mathfrak{f} \to \mathfrak{c} - \mathfrak{c}' + \mathfrak{e}, \qquad \mathfrak{f} = \mathfrak{f}' + \mathfrak{f}'',$$

where $\mathfrak{f}' \in \mathfrak{L}^{r+s}$, $\mathfrak{f}'' \in M^n - K$. As one easily sees, the reduced boundary of $\mathfrak{f}'$ is $\mathfrak{d} - \mathfrak{d}'$. In other words: *From the homology* $\mathfrak{a} \sim \mathfrak{a}'$ *in* $M^n - K$*, it follows that the corresponding complexes* $\mathfrak{b}$ *and* $\mathfrak{b}'$ (*whose boundaries* $\mathfrak{a}$ *and* $\mathfrak{a}'$*, resp., are homologous in* $M^n - K$) *are elements of* $\mathfrak{Z}^{s+1}$ *that belong to the same residue class relative to* $\mathfrak{H}^{s+1}$. One also finds the same result in the case $\mu = 0$: In fact, one then has (for a certain $k \neq 0$):

$$\mathfrak{e} \to k(\mathfrak{a}' - \mathfrak{a}) \qquad (\text{in } M^n - K),$$
$$\mathfrak{f} \to k(\mathfrak{c} - \mathfrak{c}') + \mathfrak{e}, \qquad \mathfrak{f} = \mathfrak{f}' + \mathfrak{f}'', \qquad \mathfrak{f}' \in \mathfrak{L}^{r+s}, \qquad \mathfrak{f}'' \in M^n - K;$$

this time, the reduced boundary of $\mathfrak{f}'$ is $k(\mathfrak{b} - \mathfrak{b}')$, and the assertion above keeps its validity.

We once more summarize: Any s-dimensional cycle $\mathfrak{a} \in M^n - K$ is homologous to the boundary of an element $\mathfrak{b}$ of $\mathfrak{Z}^{s+1}$ in that space, where, if $\mathfrak{a}$ and $\mathfrak{a}'$ are homologous in $M^n - K$ then $\mathfrak{b}$ and $\mathfrak{b}'$ belong to the same residue class relative to $\mathfrak{H}^{s+1}$. However, an isomorphism follows from this between the s-dimensional Betti group of $M^n - K$, namely, $\mathfrak{B}^s(M^n - K)$, and the group $\mathfrak{Z}^{s+1} \mid \mathfrak{H}^{s+1}$. Nevertheless, the r-dimensional Betti group of K – i.e., the group $Z^r \mid H^r$ – is primitive (orthogonal, resp.) to $\mathfrak{Z}^{n-r} \mid \mathfrak{H}^{n-r}$, and the multiplication is then given by the intersection map. It follows that $Z^r \mid H^r$ and $\mathfrak{B}^{n-r-1}(M^n - K)$ also define a primitive (orthogonal, resp.) group pair, where the intersection number of the corresponding $\mathfrak{b}$ with the r-dimensional cycle in K defines nothing but the linking number of this cycle with the corresponding cycle $\mathfrak{a}$ in $M^n - K$. We thus obtain the following generalized:

Alexander duality theorem in the restricted sense. *Let* M^n *be a generalized Poincaré space of dimension n. If K means a complex in* M^n *then the r-dimensional Betti* *group of K is primitive (orthogonal, resp.) to the* $n - r - 1$*-dimensional Betti group of* $M^n - K$*, as long as one regards the map of the linking numbers of corresponding cycles as the multiplication; in particular, the two groups are isomorphic to each other.*

6. Now, let M^n be an arbitrary *h*-manifold. We consider any complex K that is constructed from simplexes of M^n, for which we shall assume only that it does not coincide with M^n. Let B^r, $\mathfrak{B}^r$, W^r be the *r*-dimensional Betti groups of K, $M^n - K$, and M^n, resp. Since $K \subset M^n$, B^r will be mapped homomorphically onto a subgroup $\hat{V}^r$ of W^r and $\mathfrak{B}^r$ will be mapped to a subgroup $\hat{\mathfrak{B}}^r$; let the kernels of these homomorphisms be A^r and $\mathfrak{A}^r$. For $\mu \neq 0$, we set $V^r = \hat{V}^r$ and $\mathfrak{B}^r = \hat{\mathfrak{B}}^r$, while for $\mu = 0$ V^r ($\mathfrak{B}^r$, resp.) will denote the smallest subgroup with division of W^r over $\hat{V}^r$ ($\hat{\mathfrak{B}}^r$, resp.). Obviously, A^r consists of all those elements – i.e., residue classes of Z^r mod H^r – that only contain cycles that are homologous to zero in M^n. Correspondingly, we let $\overset{*}{A}{}^r$ denote the group of all those elements of B^r that are residue classes that contain cycles that bound in M^n. Obviously, $\overset{*}{A}{}^r$ is a subgroup of A^r that coincides with A^r in the case $\mu \neq 0$; moreover, in the case $\mu = 0$, A^r is the smallest supergroup *) with division for the group $\overset{*}{A}{}^r$ that is contained in B^r. The group $\overset{*}{\mathfrak{A}}{}^r$ is also defined analogously. The generalized Alexander duality theorem can then be expressed in the form of the following assertion:

Alexander duality theorem in the broader sense.

Preliminary remark. Due to the Poincaré-Veblen theorem, W^r, W^{n-r} is a group pair.

First statement:

(3) $$(W^r, \mathfrak{B}^{n-r}) = V^r; \qquad (W^{n-r}, V^r) = \mathfrak{B}^{n-r};$$

Second statement:

A^r *and* $\overset{*}{\mathfrak{A}}{}^{n-r-1}$ ($\overset{*}{A}{}^r$ *and* $\mathfrak{A}^{n-r-1}$, *resp.*) *define a primitive (orthogonal, resp.) group pair (where the linking number of the corresponding cycles appears as the multiplication).*

Proof of the first statement: By means of Theorem III of chap. I (§ 10), it suffices to prove one of the two formulas (3); e.g., the second one. To that end, we next remark that any $n - r$-dimensional cycle that lies outside of K (as well as any cycle that is homologous to it in M^n) obviously has an intersection number of zero with any *r*-dimensional cycle in K. From this, it follows immediately that $\mathfrak{B}^{n-r} \subset (W^{n-r}, V^r)$. In order to verify the converse inclusion, we consider any $n - r$-dimensional cycle $\mathfrak{a}$ of M^n and show that in the event that $\mathfrak{a}$ has an intersection number of zero with all *r*-

*) [Trans. note: as opposed to a subgroup; from the German *Obergruppe*.]

dimensional cycles of K, there exists a cycle $\mathfrak{a}'$ in $M^n - K$ that is homologous to $\mathfrak{a}$ in the case $\mu \neq 0$ and to $k\mathfrak{a}$, $k \neq 0$ in the case $\mu = 0$. One can then restrict oneself to the case in which $\mathfrak{a}$ is a linear form in the barycentric stars that are dual to the r-dimensional simplexes of M^n. One then has:

$$\mathfrak{a} = \mathfrak{b} + \mathfrak{b}', \qquad \mathfrak{b} \in \mathrm{Z}^{n-r}, \qquad \mathfrak{b}' \in M^n - K.$$

Since the intersection number of $\mathfrak{a}$ with all of the cycles in K is zero, one has $\mathfrak{b} \in \mathfrak{H}^{n-r}$, and as a result, there exists an element $\mathfrak{c}$ of $\mathfrak{L}^{n-r+1}$ and a positive whole number k that is different from 1, at most in the case $\mu = 0$, such that $k\mathfrak{b}$ defines the reduced boundary of $\mathfrak{c}$. However, $\mathfrak{a}' = k\mathfrak{a} - \dot{\mathfrak{c}}$ is then a cycle that lies in $M^n - K$ (since obviously $\dot{\mathfrak{c}} \sim 0$ in M^n):

$$\mathfrak{a}' \sim k\mathfrak{a} \qquad (\text{in } M^n).$$

Q. E. D.

Proof of the second statement: First and foremost, we prove that A^r and $\overset{*}{\mathfrak{A}}{}^{n-r-1}$ are mutually primitive (orthogonal, resp.). It is then clear that any s-dimensional cycle on $M^n - K$ that bounds in M^n is homologous to the boundary of an element of $\mathfrak{Z}^{s+1}$ in $M^n - K$. For that reason, it suffices to consider those cycles that are boundaries of elements in $\mathfrak{Z}^{n-r}$. We further let $\bar{H}^r$ ($\dot{\mathfrak{H}}^s$, resp.) denote the subgroup of Z^r ($\mathfrak{Z}^s$, resp.) that consists of all elements that are homologous to zero in M^n (whose boundaries are homologous to zero in $M^n - K$, resp.). We shall now prove the formula:

$$(4) \qquad (\mathfrak{Z}^{n-r}, \bar{H}^r) = \dot{\mathfrak{H}}^{n-r}.$$

The inclusion $\dot{\mathfrak{H}}^{n-r} \subset (\mathfrak{Z}^{n-r}, \bar{H}^r)$ is again trivial. In order to prove the converse, one must show that whenever there is some $\mathfrak{a} \in \mathfrak{Z}^{n-r}$ such that for all γ in $\bar{H}^r$ one has $\chi(\mathfrak{a}, \gamma) = 0 \pmod{\mu}$, $\dot{\mathfrak{a}}$ is necessarily homologous to zero in $M^n - K$. To that end, for any element $\mathfrak{a} \in \mathfrak{Z}^{n-r}$, we define the product $y \cdot \mathfrak{a}$ to be the intersection number (mod μ) of an arbitrary cycle of the residue class y with $\mathfrak{a}$. From this definition, it then follows that for the group (residue class) A^r, when regarded as an element y_0 of B^r, one has $y_0 \cdot \mathfrak{a} = 0$, which yields a natural definition of the product $x \cdot \mathfrak{a}$ when x means an arbitrary element of the factor group $B^r \mid A^r = \hat{V}^r$. In the case $\mu \neq 0$, there is thus a multiplication law $x \cdot \mathfrak{a}$ that is automatically defined for the element x of V^n (in this case, V^r is then indeed identical to $\hat{V}^r$). By contrast, when $\mu = 0$, one defines the analogous multiplication law as follows: Let x be an arbitrary element of V^n. There then exists a $z \in \hat{V}^r$ with $h \cdot x = z$, and we set $x \cdot \mathfrak{a} = \dfrac{z \cdot \mathfrak{a}}{h}$. Thus, fractions can appear, but since V possesses finitely many generators, all of the denominators are restricted such that for a suitable choice k, any choice of $x \in$

V^r, $x \cdot k\,\mathfrak{a}$ is a whole number. For the sake of unity in the formulas, we also introduce the coefficient k in the case $\mu \neq 0$, except that, by definition, it shall then be set to 1. Since W^r and W^{n-r} are mutually primitive (orthogonal, resp.), V^r and W^{n-r} (by the force of the lemma in § 9, of chap. I) are a conjugate group pair. Since one has, moreover, $(W^r, W^{n-r}) = 0$, the assumptions of § 8 of chap. I are abundantly fulfilled, and there is an element $b \in W^{n-r}$ such that for any $x \in V^r$ one has $x \cdot k\,\mathfrak{a} = x \cdot b$, but this means that when β is an arbitrary cycle of K, $\chi(x, k\mathfrak{a} - \beta) = 0$ is no longer true. We now decompose the cycle β, as usual, into the sum of two complexes $\mathfrak{a}'$ and $\mathfrak{a}''$, where $\mathfrak{a}'$ is an element of $\mathfrak{Z}^{n-r}$ and $\mathfrak{a}''$ is disjoint to K. From what was just proved, it then follows that $k\mathfrak{a} - \mathfrak{a}'$ is an element of $(\mathfrak{Z}^{n-r}, Z^r)$. Thus (for a k' that is possibly different from 1 only in the case $\mu = 0$) $k'(k\mathfrak{a} - \mathfrak{a}')$ is the reduced boundary of a certain $\mathfrak{c}$ in $\mathcal{L}^{n-r+1}$.We further have the bounding relation:

$$k'\,k\mathfrak{a} - \dot{\mathfrak{c}} - k'\beta \to k'\,k\,\dot{\mathfrak{a}}\,, \tag{5}$$

but, on the other hand, from the definition of $\mathfrak{a}'$ and $\mathfrak{c}$, the left-hand side of (5) is a complex that lies in $M^n - K$, such that the homology:

$$k'\,k\,\dot{\mathfrak{a}} \sim 0 \qquad \text{in } M^n - K$$

follows from (5), as we would like to prove.

After formula (4) is proved, it only remains for us to take a simple step, and the second statement of Alexander duality theorem will be disposed of. Next, from § 2 of this section, L^r is primitive (orthogonal, resp.) to $\mathcal{L}^{n-r}$. Since [due to § 3, formula (22)], one has:

$$(L^r, \mathfrak{Z}^{n-r}) = H^r \subset Z^r, \qquad\qquad (\mathfrak{L}^{n-r}, Z^r) = \mathfrak{H}^{n-r} \subset \mathfrak{Z}^{n-r},$$

from Theorem IV of chap. I, Z^r and $\mathfrak{Z}^{n-r}$ is a conjugate group pair. Furthermore, since $(Z^r, \mathfrak{Z}^{n-r}) = H^r \subset \bar{H}^r$, due to Theorem IV, chap. I, the groups $\bar{H}^r$ and $\mathfrak{Z}^{n-r}$ are conjugate, such that the factor groups of the annihilators $(\bar{H}^r, \mathfrak{Z}^{n-r}) = H^r$ and $(\mathfrak{Z}^{n-r}, \bar{H}^r) = \dot{\mathfrak{H}}^{n-r}$ are primitive (orthogonal, resp.) to each other. In other words: The groups $\bar{H}^r$, H^r and $\mathfrak{Z}^{n-r} \mid \mathfrak{H}^{n-r}$ are mutually primitive (orthogonal, resp.). However, the first of these groups is, by definition, the group A^r; an element ζ of the second group is a residue class of $\mathfrak{Z}^{n-r}$, mod $\dot{\mathfrak{H}}^{n-r}$. Any element $\mathfrak{z}$ of this residue class will be associated with its boundary $\mathfrak{a}^{n-r+1}$, and all of the boundaries are homologous to each other in $M^n - K$ (the residues are indeed taken modulo $\dot{\mathfrak{H}}^{n-r}$!). Thus, the entire residue class ζ will be associated with an element of $\mathfrak{B}^{n-r-1}$, and in turn, an element of $\overset{*}{\mathfrak{A}}{}^{n-r-1}$ (which certainly bounds $\mathfrak{a}^{n-r-1}$ in M^n, by its construction, because it was defined to be a boundary). Conversely, if an element of $\overset{*}{\mathfrak{A}}{}^{n-r-1}$ is given then it is a residue class modulo $\mathfrak{H}^{n-r-1}$ in the group of all cycles in M^n –

K, and this residue class contains a cycle that bounds in M^n; it is homologous to the boundary of an element of $\mathfrak{Z}^{n-r}$. In this, cycles that originate from one and the same element of $\overset{*}{\mathfrak{A}}{}^{n-r-1}$ are associated with elements of $\mathfrak{Z}^{n-r}$ that belong to the same residue class modulo $\dot{\mathfrak{H}}^{n-r}$, and, as a result, one and the same element of $\mathfrak{Z}^{n-r} \mid \dot{\mathfrak{H}}^{n-r}$. Thus, one can exhibit an isomorphism between $\overset{*}{\mathfrak{A}}{}^{n-r-1}$ and $\mathfrak{Z}^{n-r} \mid \dot{\mathfrak{H}}^{n-r}$, which allows one to carry over immediately the multiplication that was defined for $A^r = \bar{H}^r \mid H^r$ and $\mathfrak{Z}^{n-r} \mid \dot{\mathfrak{H}}^{n-r}$ to A^r and $\overset{*}{\mathfrak{A}}{}^{n-r-1}$. This multiplication law yields the linking number of a cycle in the residue class a with one in the residue class $\mathfrak{a}$ as the product of the elements $a \in A^r$ and $\mathfrak{a} \in \overset{*}{\mathfrak{A}}{}^{n-r-1}$.

7. It is thus proved that A^r and $\overset{*}{A}{}^{n-r-1}$ define a primitive (orthogonal, resp.) group pair. One can regard this as a statement that relates to both the complexes K and L if any boundary that is found in $M^n - K$ can be taken away from L. Now, however, the elements (of the first kind) of K and the elements (of the second kind) of L appear in our proof in a symmetric fashion, such that one can exchange their roles. Then, however, the arguments of the last paragraphs would lead to the proof of the primitivity (orthogonality, resp.) of the group pairs $\overset{*}{A}{}^r$ and $\mathfrak{A}^{n-r-1}$. Q.E.D.

8. In conclusion, we would like to show how one derives duality formulas from the previous proofs that were previously presented in the case of mod 2 [29]). Therefore, we understand that we are considering modulo μ, which is zero or an arbitrary prime number.

If one lets $\mathfrak{r}(G)$ generally denote the rank of the Abelian group G then one has:

(6) $$\mathfrak{r}(\overset{*}{\mathfrak{A}}{}^{n-r-1}) = \mathfrak{r}(\mathfrak{A}^{n-r-1}) = \mathfrak{r}(\overset{*}{A}{}^r) = \mathfrak{r}(A^r)$$

and

(6′) $$\mathfrak{r}(\dot{\mathfrak{B}}^{n-r}) = \mathfrak{r}(\mathfrak{B}^{n-r}), \qquad \mathfrak{r}(\hat{V}^r) = \mathfrak{r}(V^r).$$

Furthermore, since $(W^r, \mathfrak{B}^{n-r}) = V^r$, while $(\mathfrak{B}^{n-r}, W^r)$ consists of the zero element, $W^r \mid V^r$ is isomorphic to $\mathfrak{B}^{n-r}$, which yields, if one observes the general relation $\mathfrak{r}(W^r) = \mathfrak{r}(W^r \mid V^r) + \mathfrak{r}(V^r)$:

(7) $$\mathfrak{r}(W^r) = \mathfrak{r}(V^r) + \mathfrak{r}(\mathfrak{B}^{n-r}).$$

We thus have:

(8) $$\mathfrak{r}(\mathfrak{B}^{n-r-1}) = \mathfrak{r}(\mathfrak{A}^{n-r-1}) + \mathfrak{r}(\mathfrak{B}^{n-r-1} \mid \mathfrak{A}^{n-r-1}) = \mathfrak{r}(\mathfrak{A}^{n-r-1}) + \mathfrak{r}(\hat{\mathfrak{B}}^{n-r-1}).$$

[29]) Pontrjagin, "Zum Alexanderschen Dualitätssatz, zweite Mitteilung," Gött. Nachr., 1927, pp. 446.

From (6′) and (7), it further follows that:

$$\mathfrak{r}(\hat{\mathfrak{B}}^{n-r-1}) = \mathfrak{r}(W^{r+1}) - \mathfrak{r}(V^{r+1}). \tag{9}$$

If one substitutes this in (6) and (8) and observes that:

$$\mathfrak{r}(\mathfrak{B}^r) = \mathfrak{r}(A^r) + \mathfrak{r}(B^r \mid A^r) = \mathfrak{r}(A^r) + \mathfrak{r}(V^r)$$

then one finally obtains:

$$\mathfrak{r}(\mathfrak{B}^{n-r-1}) = \mathfrak{r}(A^r) + \mathfrak{r}(W^{r+1}) - \mathfrak{r}(V^r) - \mathfrak{r}(V^{r+1}).$$

Q. E. D.

Chapter III

The general duality theorem for closed sets.

I. Direct and inverse sequences of homomorphisms.

1. Let:

(1) $$U_1, U_2, \ldots, U_m, \ldots$$

be an infinite sequence of groups, each of which U_m is mapped to it successor U_{m+1} by means of a homomorphism φ_m [30]); the sequence:

(2) $$\varphi_1, \varphi_2, \ldots, \varphi_m, \ldots$$

is then called a *direct sequence of homomorphisms.* It determines a new group U – viz., the *limit of the sequence* (1) *relative to the sequence* (2), or, more briefly, the *limit group* – as follows: First, one calls any sequence of the form:

(3) $$x_1, x_2, \ldots, x_m, \ldots$$

a *fundamental sequence* when x_m is an element of U_m, and therefore one always has $x_{m+1} = \varphi_m(x_m)$. Two fundamental sequences (3) and:

(4) $$y_k, y_{k+1}, \ldots, y_m, \ldots$$

are called *cofinal* when there is a κ such that for $m > \kappa$; one has $x_m = y_m$. Obviously, the totality of all fundamental sequences decomposes into classes of mutually cofinal fundamental sequences. These classes will be composed of elements of the group U. The group operations in U will then be defined as follows: Let α and β be two classes. One chooses a fundamental sequence in each of them, say:

$$a = (x_k, x_{k+1}, \ldots, x_m, \ldots)$$

and

$$b = (y_h, y_{h+1}, \ldots, y_m, \ldots),$$

with perhaps $h \geq k$. The class γ that is determined by the fundamental sequence:

$$c = (x_h \cdot y_h, x_{h+1} \cdot y_{h+1}, \ldots, x_m \cdot y_m, \ldots)$$

is then called the *product* (in the sense of the group operations in U) of the elements α and β. Obviously, γ is uniquely determined by α and β (i.e., γ does not depend upon the choice of the sequences a and b in the classes α and β). If e_m is the identity element of

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[30]) If any element a of a set A is associated with an element b of a set B, and therefore any b corresponds to at least one a, then one speaks of a map of A *onto* B. A map of A onto a proper or improper subset of M is called (following van der Waerden) a map of A *into* B.

U_m then the element in U that is defined by $(e_1, e_2, \ldots, e_m, \ldots)$ is the identity element for the group operations thus defined. Moreover, one gets an inverse a^{-1} to any element a of U when one replaces all of the elements in the sequences of the class α with their inverses. Since our operation satisfies the associative law, in addition, all group postulates are fulfilled, and U is a group. Before we go further, we introduce the following notation: φ_s^r shall mean the map $\varphi_{s-1}(\ldots(\varphi_{r-1}(\varphi_r))$ of U_r into U_s (thus, one naturally has $s > r$).

If:

(5) $$U_{m_1}, U_{m_2}, \ldots, U_{m_q}, \ldots$$

is a subsequence of the sequence (1) then it corresponds to the sequence of homomorphisms:

(6) $$\varphi_1', \varphi_2', \ldots, \varphi_q', \ldots \qquad \text{with } \varphi_q' = \varphi_{m_{q+1}}^{m_q},$$

and the group U', which appears as the limit of (5) relative to (6), is easily recognized to be isomorphic to the limit of (1) relative to (2). In this case, we say of the direct sequence of homomorphisms (2) that it *incorporates* (umfasse) the sequence (6). Two direct sequences of homomorphisms are called *equivalent*, moreover, when one can find two subsequences in each of them that are incorporated in a third sequence. This concept of equivalence satisfies the equality axioms (viz., reflexivity, symmetry, and transitivity) [31]), so one can speak of classes of mutually equivalent homomorphisms. Furthermore, since two sequences, one of which incorporates the other one, determine isomorphic groups in the limit, this yields the following theorem:

I. *Equivalent sequences of homomorphisms have isomorphic limit groups.*

From now on, a direct sequence of homomorphisms (1), (2) will always be denoted by $\mathcal{F}(U_m, \varphi)$.

2. We again consider a sequence:

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[31]) The fact that our notion of equivalence is transitive comes from the following two remarks, of which the first one is self-explanatory, and the second can be verified effortlessly:

1. When I ≡ II and one has I′ ⊃ I, II′ ⊃ II, one then has I′ ≡ II′.
2. If I = II and II′ ⊂ II then one has I′ ≡ II′.

It then follows from I ≡ II and II ≡ III that there exists subsequences I_1 and II_1, II′ and III′, as well as sequences IV and V, with:

$$\mathrm{I}_1 + \mathrm{II}_1 \subset \mathrm{IV}, \qquad \mathrm{II}' + \mathrm{III}' \subset \mathrm{V}.$$

By means of remarks 1 and 2, this then yields, in turn:

$$\mathrm{I} \equiv \mathrm{II}', \mathrm{I} \equiv \mathrm{V}, \mathrm{I} \equiv \mathrm{III}', \mathrm{I} \equiv \mathrm{III}.$$

Q. E. D.

(1) $$U_1, U_2, \ldots, U_m, \ldots,$$

but we now assume that U_{m+1} is mapped into U_m by means of the homomorphism π_m. The sequence:

(7) $$\pi_1, \pi_2, \ldots, \pi_m, \ldots$$

[in this, the map $\pi_r(\ldots,\pi_{s-2}(\pi_{s-1}))$ of U_s into U_r ($s > r$) will be demoted by π_s^r] is then called an *inverse sequence of homomorphisms.* Precisely as before, one can also introduce the concepts of incorporated (equivalent, resp.) sequences of inverse sequences of homomorphisms. There is then no analogue to Theorem I, since an inverse sequence of homomorphisms certainly possesses no limit group.

An inverse sequence of homomorphisms (1), (7) will always be denoted by $\bar{\mathcal{F}}(U_m, \pi)$. A sequence of homomorphisms in (1), of which, one does not know whether it is direct or inverse, shall be loosely denoted by $\mathcal{F}(U_m)$.

3. From now on, we restrict ourselves to commutative groups and thus avail ourselves of the additive notation, as before.

Lemma. *Let U, A* (*V, B, resp.*) *be two primitive* (*orthogonal, resp.*) *group pairs relative to the modulus M. Furthermore, let a homomorphism* φ *of U into V be given. There is then one and only one homomorphic map* ψ *of B into A that satisfies the following condition: If u* (*b, resp.*) *is any element of U* (*B, resp.*) *then one has:*

(8) $$u \cdot \psi(b) = \varphi(u) \cdot b.$$

Proof. If u runs through the entire group U then $\varphi(u) \cdot b$ assumes certain values for which one always has $\varphi(u) \cdot b + \varphi(u') \cdot b = \varphi(u + u') \cdot b$.

One now considers the homomorphic map:

$$z(u) = \varphi(u) \cdot b$$

of U into M. From chap. I, § 9, it then follows that there is a single element a of A such that for all u one has

$$u \cdot a = z(u) = \varphi(u) \cdot b.$$

We denote this a by $\psi(b)$. From:

$$u \cdot (\psi(b) + \psi(b')) = u \cdot \psi(b) + u \cdot \psi(b') = \varphi(u) \cdot b + \varphi(u) \cdot b' = \varphi(u) \cdot (b + b')$$
$$= u \cdot \psi(b + b'),$$

it follows that:

$$u \cdot ((\psi(b) + \psi(b')) - \psi(b + b')) = 0,$$

so (from the primitivity of the group pair U, A):

$$\psi(b) + \psi(b') = \psi(b + b');$$

ψ is then a homomorphism. Q. E. D.

4. Definition. Let $\mathcal{F}(U_m\ ,\ \varphi)$ and $\bar{\mathcal{F}}(V_m\ ,\ \pi)$ be given. These two sequences of homomorphisms will be called *mutually orthogonal* (relative to the modulus M) when the following conditions are fulfilled:

1. U_m and V_m define a primitive (orthogonal, resp.) group pair (relative to M).

2. When u and v are arbitrary elements of U_m (V_m, resp.) one has:

$$\varphi_m(u) \cdot v = u \cdot \pi_m(v).$$

From the lemma of § 3, the theorem follows immediately:

Theorem II. *Let $\mathcal{F}(U_m)$ be given, where the U_m are so arranged that for any group U_m there exists a group V_m that is primitive* (*orthogonal, resp.*) *to it* (*which is then determined uniquely up to isomorphism*). *A sequence of homomorphisms $\bar{\mathcal{F}}(V_m)$ in the V_m can be defined uniquely* (*in any event, up to isomorphism*) *such that $\mathcal{F}$ and $\bar{\mathcal{F}}$ are mutually orthogonal* (*relative to the modulus M*).

Remark. When two sequences of homomorphisms are mutually orthogonal the same thing is true of two subsequences that consist of mutually corresponding (i.e., provided with the same index m) terms of the two sequences.

Theorem III. *Let $\mathcal{F}(U_m)$ and $\mathcal{F}(V_m)$ be mutually orthogonal, just like $\mathcal{F}'(U'_m)$ and $\mathcal{F}'(V'_m)$; if $\mathcal{F}(U_m)$ is equivalent to $\mathcal{F}'(U'_m)$, moreover, then $\mathcal{F}(V_m)$ is equivalent to $\mathcal{F}'(V'_m)$.*

Proof. There exist subsequences $\mathcal{F}_0$ and $\mathcal{F}'_0$ in $\mathcal{F}(U_m)$ and $\mathcal{F}'(U'_m)$ that are subsequence of one and the same $\mathcal{F}_1$. We may then assume of the sequence of homomorphisms $\mathcal{F}_1$ that it consists of only elements of $\mathcal{F}_0$ and $\mathcal{F}'_0$ (when we simply delete all of the possible remaining elements). We further choose the subsequences $\bar{\mathcal{F}}_0$ and $\bar{\mathcal{F}}'_0$ in $\mathcal{F}(V_m)$ and $\mathcal{F}'(V'_m)$ that correspond to the subsequences $\mathcal{F}_0$ and $\mathcal{F}'_0$. Since one can construct an element primitive (orthogonal, resp.) to any element of $\mathcal{F}_1$, by the force of Theorem II, there exists a sequence of homomorphisms $\bar{\mathcal{F}}_1$ that is orthogonal to $\mathcal{F}_1$. $\mathcal{F}_0$ and $\mathcal{F}'_0$ can be regarded as subsequences of $\bar{\mathcal{F}}_1$, so $\mathcal{F}(V_m)$ and $\mathcal{F}'(V'_m)$ are equivalent.

5. We especially emphasize: An inverse sequence of homomorphisms $\bar{\mathcal{F}}(U_m\ ,\ \pi)$ uniquely defines – under the assumption that there is a group V_m (therefore, essentially only one) that is primitive (orthogonal, resp.) to any U_m – a direct sequence of homomorphisms $\mathcal{F}(V_m, \varphi)$ that is orthogonal to $\bar{\mathcal{F}}(U_m, \pi)$, and it defines the limit group V uniquely. *This group that is determined uniquely by the sequence of homomorphisms* $\bar{\mathcal{F}}(U_m, \pi)$ *is called the group that is dual to* $\bar{\mathcal{F}}(U_m, \pi)$. *Equivalent sequences of homomorphisms possess isomorphic dual groups.*

II. Formulation and proof of the general duality theorem.

6. Let F be a closed, compact set that lies in R^n. One considers any projection spectrum [32]):

(1) $$A = (A_1, A_2, \ldots, A_m, \ldots)$$

of F and the associated simplicial maps π_m of A_{m+1} onto A_m.

Let $B_m = B^r(A_m)$ be the r^{th} Betti group of A_m. A homomorphism of B_{m+1} into B_m (which we likewise denote by π_m) arises by means of the simplicial map π_m, and it follows that there is an inverse sequence of homomorphisms $\bar{\mathcal{F}}(B_m, \pi)$. One then has:

Lemma. *Let two different projection spectra* (1) *and:*

(2) $$A' = (A'_1, A'_2, \ldots, A'_m, \ldots)$$

be given on a set F. The sequences of homomorphisms $\bar{\mathcal{F}}(B_m, \pi)$ *and* $\bar{\mathcal{F}}(B'_m, \pi)$ *that correspond to these spectra are equivalent.*

For the moment, we assume that the lemma has been proved already.

The totality of all mutually equivalent sequences of homomorphisms $\bar{\mathcal{F}}(B_m, \pi)$ that are determined by the projection spectra of F is obviously a topological invariant of the set F. We call it the *r-dimensional cyclosis of the set F*. The cyclosis determines a group uniquely, namely, the unique (up to isomorphism) group that is dual to all sequences of homomorphisms of the cyclosis, which we briefly call the *group dual to the r-dimensional cyclosis.* This group obviously has a likewise topologically invariant meaning for the set F. Now, the main point of this chapter consists in the proof of the following theorem:

General duality theorem. *If F is a closed, compact set in* R^n *then the group that is dual to the r-dimensional cyclosis of F is isomorphic to* $(n - r - 1)^{\text{th}}$ *Betti group of the complementary space* $R^n - F$.

From this, one finds, with no further assumptions, the:

Invariance theorem. *The Betti groups* [33]) *of the complementary space to a closed set F in* R^n *are topological invariants of the set F.*

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[32]) Alexandroff, *loc. cit.* [2]), pp. 107.

[33]) Cf., footnote [15]) and Appendix III.

Proof. We carry out the proof of the lemma and the duality theorem simultaneously. Let:

(3) $$Q_1, Q_2, \ldots, Q_i, \ldots$$

be a decreasing sequence of polyhedral neighborhoods of the set F that converge to that set.

Let G_i be the open set that is complementary to Q_i. This set increases with i, and the union of these sets is identical with $G = R^n - F$. We let β_i denote the r^{th} Betti group of Q_i, and let $\bar{\beta}_i$ denote the $(n - r - 1)^{\text{th}}$ Betti group of G_i. A homomorphic map $\tilde{\omega}_i$ from β_{i+1} to β_i follows from the fact that $Q_i \supset Q_{i+1}$, and a homomorphic map φ_i of $\bar{\beta}_i$ into $\bar{\beta}_{i+1}$ follows from the fact that $G_{i+1} \supset G_i$. The sequences of homomorphisms $\bar{\mathcal{F}}(\beta_i, \tilde{\omega})$ and $\mathcal{F}(\bar{\beta}_i, \varphi)$ that arise in this way are mutually orthogonal if β_i is indeed primitive (orthogonal, resp.) to $\bar{\beta}_i$. The limit group $\bar{\beta}$ that is determined by the sequence of homomorphisms $\mathcal{F}(\bar{\beta}_i, \varphi)$, as one easily recognizes, is isomorphic to $(n - r - 1)^{\text{th}}$ Betti group of G. We thus need only to show the following if we are to prove everything:

For any choice of projection spectrum (1) *for the set* F, the sequence of homomorphisms $\bar{\mathcal{F}}(\beta_i, \pi)$ is equivalent to $\mathcal{F}(\bar{\beta}_i, \varphi)$.

8. We now turn to the proof of the latter assertion.

First, it is advantageous to arrive, by a slight gimmick, at the idea that the complex A_i is geometric, and indeed realized without singularities. To this end, we consider the topological product $Z = R^n \times E$ of R^n with a sufficiently high-dimensional simplex E. Let the center of mass of E be ξ. We assume that R^n is identical with $R^n \times \xi$, and consider a sequence of concentric simplexes $E_1, E_2, \ldots, E_h, \ldots$ about ξ that converge to that point. The:

$$Q_1 \times E_1, Q_2 \times E_2, \ldots, Q_h \times E_h, \ldots$$

then define a sequence of polyhedral neighborhoods of F in Z that converge to F. Since $Q_i \times E_i$ can be continuously taken to Q_i inside of itself (such that Q_i remains point-wise fixed in the process), the Betti groups of $Q_i \times E_i$ are isomorphic to those of Q_i, the corresponding homomorphisms are then the same, and one can quietly replace Q_i with $Q_i \times E_i$, and thus replace a neighborhood of F in R^n with a neighborhood of F in Z. In the $Q_i \times E_i$, however, one can – as long as the dimension of E is sufficiently large – realize all of the A_m (with the possible exception of finitely many of them) without singularities. From now, we simply denote $Q_i \times E_i$ the by Q_i.

Let i be arbitrary and let q be large enough that A_i lies in Q_i. The existence of a homomorphic map f_1 of B_q into β_i follows from this. We write it thus:

$$f_1(B_q) \subset \beta_i.$$

Now, it is known that when a complex K lies in a sufficiently close neighborhood of F and is constructed from sufficiently small simplexes, one can map this complex simplicially by means of a small displacement of its vertices into A_q [34]). When one takes j to be sufficiently large and the simplexes of Q_j to be sufficiently small, Q_j can be chosen for such a K. One denotes the aforementioned simplicial map of Q_j into A_q by g; we would like to construct it as follows: First, let s be sufficiently large that A_s lies in Q_j . One now chooses a simplicial decomposition $\mathfrak{z}$ of Q_j such that a certain subdivision A_s^* of A_s can be regarded as a sub-complex of the decomposition $\mathfrak{z}$. The simplicial map g of Q_j into A_q shall be based upon the simplicial decomposition $\mathfrak{z}$. Thus, let a be any vertex of $\mathfrak{z}$. First and foremost, we examine the case where a is simultaneously a vertex of A_s^*. In the case, a is an interior point of a certain (possible also zero-dimensional) simplex T of A_s . A vertex a' shall be chosen to be the image of a, such that under the projection of A_s onto A_q any vertex of T will be mapped to it. The remaining (hence, not belonging to A_s^*) vertices of $\mathfrak{z}$ might be mapped into any one of the next-lying vertices to them in A_q . As one easily recognizes, the thus-defined map g onto the complex A_s agrees with the projection of A_s onto A_q algebraically [35]). Finally, it can still be assumed that g is realizable by means of a continuous deformation inside of Q_i .

One thus obtains the following homomorphisms:

$$f_2(B_s) \subset \beta_j$$

(A_s is then indeed contained in Q_j):

$$f_3(\beta_j) \subset B_q ,$$

(by means of the map g of Q_j in A_q);

$$\tilde{\omega}(\beta_j) \subset \beta_i ,$$

(Q_j is then contained in Q_i):

$$\pi(B_s) \subset B_q$$

(projection). One thus has:

(4) $$f_1 f_2(\beta_j) = \tilde{\omega}(\beta_j),$$

(5) $$f_3 f_2(B_s) = \pi(B_q),$$

such that the groups are mapped to each other as follows:

$$\beta_i \leftarrow B_q \leftarrow \beta_j \leftarrow B_s .$$

When one begins this process with $i = 1$ and advances ever further, one effortlessly obtains a sequence of homomorphisms that, due to (4) and (5), incorporate certain subsequences of the two sequences $\bar{\mathcal{F}}(B_m, \pi)$ and $\bar{\mathcal{F}}(B_m, \tilde{\omega})$. Therefore, the latter sequences of homomorphisms are equivalent. Q. E. D.

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[34]) Alexandroff, *loc. cit.* [8]), pp. 117 (cor. I).

[35]) The proof is by induction on the dimension of the elements of the complex; cf., e.g., Alexandroff, Trans. Amer. Soc. **28**.

Appendix I

The duality theorem for continuous complexes.

Let K be a (singularity-free) continuous complex in R^n. One considers a polyhedral complex Q that is homeomorphic to K in a sufficiently high-dimensional R^n. Since K and Q are homeomorphic closed sets, the groups that are dual to the r-dimensional cycloses of them, and therefore also the $(n - r - 1)^{\text{th}}$ Betti groups of $R^n - K$ and the $(m - r - 1)^{\text{th}}$ Betti group of $R^n - Q$, are isomorphic to each other. However, the latter group is isomorphic to the r-dimensional Betti group of Q (hence, also to that of K).

In other words:

If $K \subset R^n$ is a continuous complex then the $(n - r - 1)^{\text{th}}$ Betti group of $R^n - K$ is isomorphic to the r-dimensional Betti group of K.

In order to derive this isomorphism from the basic (i.e., derived from linking) primitivity (orthogonality, resp.) [36]) of the two groups, one considers a polyhedral neighborhood V of K that satisfies the following two conditions:

a) Any cycle of K that is homologous to zero in V also has that property in K itself.

b) For any cycle $z \subset R^n - K$ there is a cycle $z' \subset R^n - F$ with the property that $z \sim z'$ in $R^n - K$.

Furthermore, let $U \subset V$ be a polyhedral neighborhood of K such that for any cycle $z \subset U$ there is a cycle $z' \subset K$ such that $z \sim z'$ in V.

We let B (B', resp.) denote the r-dimensional Betti group of K (U, resp.) and let β (β', resp.) denote the $n - r - 1$-dimensional Betti group of K (U, resp.). Due to condition a), it follows from $K \subset U$ that there is an isomorphic map of B onto a subgroup of B' that we will likewise denote by B. We next prove that in the case of $\mu = 0$ *this subgroup is a subgroup with division of B'*. In fact, let $z \subset B$, $y \subset B'$, $x = ky$, $k \neq 0$; we will show that one also has $y \subset B$. To that end, we let $\bar{x}$ ($\bar{y}$, resp.) denote cycles in K (U, resp.) that belong to the homology class of x (y, resp.). One then has:

$$\bar{x} \sim k\bar{y} \qquad \text{(in } U\text{)}.$$

From the definition of U, it then follows that there is a cycle $\bar{y}'$ in K that is $\sim \bar{y}$ in V, such that:

$$\bar{x} \sim k\bar{y}' \qquad \text{(in } V\text{)},$$

so (due to a)):

$$\bar{x} \sim k\bar{y}' \qquad \text{(in } K\text{)}.$$

[36]) Cf., the formulation of the Alexander duality theorem in the restricted sense (chap. II, II, § 5).

If we let y' denote the element of B that is generated by the cycle $\overline{y}'$ then $x = ky'$; on the other hand, since $x = ky$ and the group B' is free, one has $y = y'$, and therefore $y \subset B$.

The groups B' and β' are mutually primitive (orthogonal, resp.), while B and β are isomorphic. From the inclusion $R^n - U \subset R^n - K$ and condition b), this yields a homomorphic map of β' onto the entire group β. We denote the kernel of this homomorphism by, perhaps, A and prove that $A = (\beta', B)$. Now, it is clear that $A \subset (\beta', B)$ since a cycle of $R^n - K$ that links with a cycle of K cannot be homologous to zero in $R^n - K$. However, A can also not be a proper subgroup of (β', B), since otherwise the group $\beta' | A \approx \beta$ (according to whether $\mu \neq 0$ or $= 0$) would have a higher order (rank, resp.) than the group $\beta' \mid (\beta', B)$, which would contradict the isomorphism between β and B. Therefore, $A = (\beta', B)$, and since $\beta \approx \beta' | (\beta', B)$, B and β define a primitive (orthogonal, resp.) group pair. Q. E. D.

Appendix II

Relationship between the Lefschetz duality theorem for closed sets and the theory of Chapter III. [37])

Definition I. Let $\bar{\mathcal{F}}(V_n, \pi)$ be an inverse sequence of homomorphisms. A sequence:

(1) $$y_k, y_{k+1}, \ldots, y_n, \ldots$$

where y_n is an element of V_n is called a *chain* (Kette) when for any $n \geq k$ one has:

$$\pi_n(y_{n+1}) = t_n\, y_n\,,$$

and therefore t_n is a positive whole number.

Definition II. A system of chains is called *linearly independent* when corresponding (i.e., having the same index) elements of this chain, when considered as elements of the groups to which they belong, are linearly independent in these groups. When one can find arbitrarily many mutually independent chains in a given sequence of homomorphisms, we say that this sequence of homomorphisms has an *infinite rank*, while otherwise the highest number of independent chains that appear in the sequence of homomorphisms is called its *rank*. The rank of a direct sequence of homomorphisms shall be defined to be the rank of its limit group.

The relationship between Lefschetz's theorem and our theory is thus given by the following theorem:

Two (*modulo zero*) *orthogonal sequences of homomorphisms have the same rank.*

We next remark that *equivalent sequences of homomorphisms have the same rank*. For direct sequences of homomorphisms, this follows from the fact that equivalent sequences have the same limit group. As far as inverse sequences of homomorphisms are concerned, our assertion follows from the analogous assertion for two sequences, the first of which is a subsequence of the second one. However, for this special case, the assertion can be established immediately.

We now call an inverse sequence of homomorphisms *complete* when for any n the only subgroup with division of V_n that contains $\pi_n(V_{n+1})$ is the group V_n itself.

When $\bar{\mathcal{F}}(V_n, \pi)$ is a complete sequence of homomorphisms and a_n is any free element of V_n, there is an element a_{n+1} of V_{n+1} such that $\pi_n(a_{n+1}) = t\, a_n$ for a suitably chosen whole number $t \neq 0$. However, that means that for any free element a_n of an arbitrary group V_n in $\bar{\mathcal{F}}(V_n, \pi)$ one can find at least one chain that begins with a_n. If one has a system of linearly-independent elements $a_n^1, a_n^2, \ldots, a_n^k$ of V_n, moreover, then the chains that begin with it are also linearly independent. From this, it follows that the rank

[37]) We consider only the case in which M^n is a generalized Poincaré space. The case of a more general M^n can be disposed of in an analogous manner.

of a complete sequence of homomorphisms is equal to the finite or infinite upper limit of the ranks of the individual groups V_n .

We now prove the following two lemmas.

Lemma I. *To any inverse sequence of homomorphisms* $\bar{\mathcal{F}}(V_n, \pi)$ *there is a complete sequence of homomorphisms that is equivalent to it.*

In fact, let V_{nk} be smallest subgroup with division in V_n that contains $\pi_n^{n+k} V_{n+k}$. When k increases, V_{nk} can only decrease. As a result (V_n indeed has finitely many generators), amongst the V_{nk} (n fixed!) there exists a smallest subgroup, and it shall be called V_n'. One now sets $V_{s_1} = V_1$, so one also has $V_{s_1}' = V_n'$, and one assumes that V_{s_i} has already been found. By definition, V_{s_i}' is the smallest of the groups $V_{s_i k}$, so it is a well-defined group $V_{s_i s}$. We set $V_{s_{i+1}} = V_{s_i + g}$. For any $h > s_{i+1}$, V_{s_i}' is the smallest subgroup with division of V_{s_i} that contains $\pi_{s_i}^h V_h$ or also only $\pi_{s_i}^h V_h'$. From this, it follows that:

$$
\begin{cases} V_{s_1}', V_{s_2}', \ldots, V_{s_n}', \ldots \\ \pi_{s_1}^{s_2}, \pi_{s_2}^{s_3}, \ldots, \pi_{s_n}^{s_{n+1}}, \ldots \end{cases} \tag{2}
$$

is a complete sequence of homomorphisms. However, one can also describe:

$$
V_{s_1},\ V_{s_1}',\ V_{s_2},\ V_{s_2}',\ \ldots,\ V_{s_n},\ V_{s_n}',\ \ldots \tag{3}
$$

as a sequence of homomorphisms when one maps V_{s_n} into $V_{s_{n-1}}'$ by means of the maps $\pi_{s_{n-1}}^{s_n}$ and V_{s_n}' into V_{s_n} by means of the identity map. The sequence (3) incorporates (2) and the subsequence V_{s_1}, V_{s_2}, …, V_{s_n}, … of the originally-given sequence $\bar{\mathcal{F}}(V_n, \pi)$, so (2) is equivalent to $\bar{\mathcal{F}}(V_n, \pi)$, with which Lemma I is proved.

Lemma II. *If* $\bar{\mathcal{F}}(V_n, \pi)$ *is a complete sequence of homomorphisms and* $\mathcal{F}(U_n, \varphi)$ *is a sequence of homomorphisms that is orthogonal to* $\bar{\mathcal{F}}(V_n, \pi)$ *then the homomorphisms* φ *are all isomorphisms.*

In fact, let a be a non-zero element of U_n that gets mapped to the zero element of U_{n+1} by means of φ_n . As a result of the orthogonality of the group pairs U_n, V_n, there is an element b of V_n such that $ab \neq 0$. Since $\bar{\mathcal{F}}(V_n, \pi)$ is complete, there exists a $b' \in V_{n+1}$ such that $\pi_n(b') = tb$ (with $t > 0$). Now, one has, however, $\varphi_n(a)\, b' = a\, \pi_n(b') = tab \neq 0$, so it also follows that $\varphi_n(a) \neq 0$. Lemma II is proved by this contradiction.

If one now has two orthogonal sequences of homomorphisms $\mathcal{F}$ and $\bar{\mathcal{F}}$ then one next replaces them with two likewise orthogonal sequences $\mathcal{F}_1$ and $\bar{\mathcal{F}}_1$ that are equivalent to $\mathcal{F}$ ($\bar{\mathcal{F}}$, resp.), of which, the first one is a complete sequence of homomorphisms and the second one is then a direct sequence of isomorphisms. The rank of a sequence of isomorphisms is obviously equal to the upper bound of the ranks of the groups in question. Since the analogous statement is also true for complete sequences of homomorphisms and the mutually corresponding groups of the two sequences are orthogonal, hence, isomorphic, the two sequences have the same rank. Q. E. D.

Appendix III

Example of a curve in R^3 whose complementary space has an arbitrary countable Abelian group with no elements of finite order for its first Betti group.

Let U be an arbitrary Abelian group that consists of countably many free elements a_1, a_2, …, a_n, … Let U_n be the subgroup of U that is generated by a_1, a_2, …, a_n. Since U_n is a subgroup of U_{n+1}, one has a homomorphism φ_n of U_n into U_{n+1}, and therefore a direct sequence of homomorphisms $\mathcal{F}(U_n, \varphi)$. The corresponding limit group is, as one easily recognizes, isomorphic to U.

Since U_n is a free group with finitely many generators, there exists a group V_n that is orthogonal to U_n, and as a consequence, an (inverse) sequence of homomorphisms $\bar{\mathcal{F}}(V_n, \pi)$ that is orthogonal to $\mathcal{F}(U_n, \varphi)$. If $x_n^1, x_n^2, \ldots, x_n^{\alpha_n}$ is a system of free generators for the group V_n then (for a suitably-chosen whole number ${}_n c_j^i$) one has:

$$\pi_n(x_{n+1}^i) = {}_n c_j^i x_n^j. \tag{1}$$

Now, let K_n be the line segment complex that one obtains when one identifies all of the vertices of an α_n-vertex figure with each other. We let $x_n^1, x_n^2, \ldots, x_n^{\alpha_n}$ denote the system of one-dimensional cycles of K_n that are chosen in the simplest manner and define a one-dimensional basis for this complex. Now, a continuous map f_n of K_{n+1} into K_n can be easily be given for which:

$$f_n(x_{n+1}^i) \sim {}_n c_j^i x_n^j.$$

One now embeds K_1 as a singularity-free polygonal line segment complex in R^3 and chooses a polyhedral neighborhood Q_1 of K_1 such that any cycle of Q_1 is homologous to a cycle of K_1 there and thus arrives at an isomorphism between the Betti groups of K_1 and Q_1. The image $f_1(K_2)$ of K_3 lies in K_1, and one can, by an arbitrarily small change in f_1, take that map to one that maps K_2 onto a polygonal complex $\bar{K}_2$ that lies in Q_1 singularity-free and is homeomorphic to K_2. One now chooses a polyhedral neighborhood Q_2 of $\bar{K}_2$ such that any cycle that lies in Q_2 is homologous to a cycle of $\bar{K}_2$ there and the Betti groups of Q_2 and K_2 are isomorphic, and then begins this process. A sequence of connected polyhedral regions $Q_1, Q_2, \ldots, Q_n, \ldots$ that are nested inside of each other thus arises that can be chosen such that their intersection is a curve F. Therefore, the Betti groups of Q_{n+1} will be homomorphically mapped into those of Q_n according to formulas (1). The group that is dual to the thus-arising inverse sequence of homomorphisms (which coincides with the group that is dual to the cyclosis of F) is obviously the group U, which is therefore isomorphic to the first Betti group of $R^3 - F$.

In conclusion, it might be remarked that already the set of Abelian groups of rank 1 that consist of countably many free elements and are pair-wise different from each other, in the sense of isomorphism, has the cardinality of the continuum. One sees from this the high degree to which the theorem of the invariance of the Betti groups is much richer in content than the invariance theorem for the Betti numbers.

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