The Hamilton-Jacobi Theory for Double Integrals

(with an overview of the theory for simple integrals)

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Dedicated to

Herrn Professor Dr. Conrad H. Müller

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Introduction

To the extent that one deals with problems in mechanics with finitely-many degrees of freedom, the **Hamilton-Jacobi** theory plays a fundamental role in the investigations. The concept of the differential equations of motion being the **Euler-Lagrange** equations of a variational problem that is made possible by **Lagrange**'s introduction of the kinetic potential (i.e., the **Lagrangian** function) and the principle of least action finds its analytical expression in a variety of formulations, along with the conversion of those equations into a canonical system, as well as ultimately the introduction of the principle of varied action by **Hamilton**, and with that, the exhibiting of the **Hamilton-Jacobi** partial differential equation for the "characteristic function," have provided the key with which to open the door to the difficult problems of the mechanics of point-systems.

In the mechanics of continua, as in mathematical physics in general, it was indeed also shown a long time ago that the differential equations that regulate the detailed processes can be regarded as **Euler-Lagrange** differential equations for a variational problem, although the principle of varied action has not come into its own up to now. In fact, an essential difficulty will appear in the introduction of that principle as soon as one goes over to double or multiple integrals. In the theory of the simple integral, **Hamilton**'s "characteristic function" seems to be determined by the two limiting points of the path of integration, so it takes the form of a function (in the usual sense of the word) of finitely-many variables. By contrast, the analogous integral value for a multiple integral depends upon the boundary of the domain of integration, i.e., a curve, surface, or higher manifold. **Hamilton**'s "characteristic function" then no longer takes the form of a function, but that of a "functional."

Volterra has tackled the investigation of functions (i.e., line functions) ([1]), and has thus created the tool with which one can adapt the **Hamilton-Jacobi** theory to multiple integrals. He himself had also proposed that adaptation to be one of the most important goals of the new theory and made some first attempts ([2]). Thereupon, **Hadamard** took up the continued development of the theory of functions, and then **Fréchet** ([3]), who was influenced by him, attempted to continue **Volterra**'s adaptation of the **Hamilton-Jacobi** theory, while on the other hand, **P. Lévy** ([4]) constructed a general theory of partial functional differential equations, to which the partial functional differential equation for the "characteristic functional" that enters in place of **Hamilton**'s "characteristic function" belongs. However, from his general viewpoint, the former seems to be only a very special class of those equations, such that the relationship to the **Hamilton-Jacobi** theory recedes into the background completely in his presentation.

For a systematic construction of the **Hamilton-Jacobi** theory for multiple integrals, as we will attempt to do in what follows, it would seem preferable to start from the fact that the "limit formula" of the calculus of variations is the source of that theory, and all of its results can then be

([1]) **V. Volterra**, "Sopra le funzioni che dipendono da altre funzioni," Roma, Acc. Lincei Rend. (4) **3** 2 (1887), in various places.

([2]) **V. Volterra**, "Sopra una estensione della teoria **Jacobi-Hamilton** del calcolo delle variazioni," Roma, Acc. Lincei Rend. (4) **6** 1 (1890), pp. 127.

([3]) **M. Fréchet**, "Sur une extension de la méthode de **Jacobi-Hamilton**," Annali di matematica (3) **11** (1905), pp. 187.

([4]) **P. Lévy**, "Sur l'intégration des équations aux dérivées fonctionnelles partielles," Palermo circ. mat. Rend. **37** (1914), pp. 113.

derived from that immediately. Meanwhile, since that viewpoint has not been established completely in the theory of simple integrals in its usual presentations but is obscured by appealing to many other realms of analysis, a presentation of the **Hamilton-Jacobi** theory of simple integral will be prepended as an introductory section. The Table of Contents that precedes this introduction will, at the same time, allow the guiding concepts for the adaptation of the theory to multiple integrals to emerge.

Part I is organized into three sections. In § **1**, the variational problem will be introduced, and at the same time, **Hamilton**'s "characteristic function," for which the term *extremal integral* will be chosen, and which can be considered to be a function of the two limits of the integral. The extremal integral, as a *point-pair function*, satisfies the *two* **Hamilton** partial differential equations, which can also be combined into *one* from the standpoint of point-pair functions. However, if one would like to determine the extremal integral from those equations then the difficulty will arise of separating the extremal integral from the set of solution. **Jacobi** overcame it by replacing the extremal integral in a field of extremals with the "extremal integral value" (i.e., field integral). That extremal integral value satisfies *one* first-order partial differential equation that will be referred to as the **Hamilton-Jacobi** differential equation. In conjunction with the concept of a field, the connection between the **Hamilton-Jacobi** equation and the "independence fields" that were studied by **A. Mayer** and **D. Hilbert** will be summarized, and from that, the equivalence of knowing a complete integral of the **Hamilton-Jacobi** equation and knowing a solution of the boundary-value problem for the **Euler-Lagrange** equations will be inferred.

§ **2** generally treats the properties of the **Euler-Lagrange** equations (the equivalent canonical system that they can be converted into, according to **Poisson** and **Hamilton**, resp.). The limit formula for the calculus of variations will once more be consulted since it yields a direct connection with **Poincaré**'s theory of integral invariants. On the one hand, those integral invariants are closely connected with the **Hilbert** independence theorem, and on the other, they yield the reciprocity property of the **Jacobi** equations for the second variation (*équations aux variations*). However, the limit formula also provides an immediate understanding of the concept of a canonical system as an infinitesimal contact transformation in the **Lie** sense, and together with a theorem on the solutions of the **Jacobi** equations, it will further provide an understanding of the connections between the infinitesimal transformations of the canonical system into itself and its integrals, at which point the reciprocity property of the **Jacobi** equations will then lead to **Poisson**'s theorem.

The brief § **3** inverts **Hamilton**'s line of reasoning by solving the integration problem for a given partial differential equation by an extremal integral that is known by means of integrating the **Euler-Lagrange** equations; in other words, it treats the **Jacobi** method for integrating first-order partial differential equations.

With that organization and according to the given principle, the theory will be adapted to multiple integrals in Part Two. The presentation will be restricted to double integrals, and in the first chapter, it will be restricted to double integrals of one unknown function, while in the second chapter, it will be restricted to double integrals of two of them, but a theory that relates to n-fold integrals with m unknown functions would create no essential difficulties. Consistent with the present state of functional calculus, the implementation of that adaptation is given here only to the extent that no especially irregular behavior comes into question, since the theory of functionals itself still does not possess the same sharpness in its structure that is found in the theory of functions of a finite number of variables. The application of the arguments to well-defined problems of mathematical physics will of itself lead one to study such degenerate cases, and in that way, retroactively require the construction of the functional calculus.

CHAPTER ONE

The variational problem for simple integrals

§ 1.

The variational problem and the partial differential equations of Hamilton and Jacobi.

1) The variational problem and the Euler-Lagrange equations.

The realization of the theory might be coupled with the following problem in the calculus of variations:

Determine two functions $y(x)$ and $z(x)$ of the variables x in a given interval (x_1, x_2) such that the integral of a given (analytic) function f that depends upon those functions:

(1) $$I = \int_{x_1}^{x_2} f(y', z', y, z, x)\, dx \qquad \left(y' = \frac{dy}{dx}, \quad z' = \frac{dz}{dx}\right)$$

will be an extremum, while one might possibly make special demands on the "boundary values" of the functions $y(x)$ and $z(x)$ at the boundary points $x = x_1$, $x = x_2$ of the interval, which might be established in advance.

That problem is generally sufficient to allow all of the traits of the theory for a larger number of dependent variables to already emerge clearly, such as the introduction of auxiliary conditions, etc. The problem with only *one* dependent function still does not possess that generality. Indeed, its treatment generally runs parallel to the following developments, except that an essential simplification will enter at one point that will be noted in particular at that point in the development (rem., pp. 10).

Thus, now let the boundary values of the unknown functions $y(x)$, $z(x)$ in the Problem (1) be given as fixed:

(2) $$y_1 = y(x_1), \quad z_1 = z(x_1); \quad y_2 = y(x_2), \quad z_2 = z(x_2).$$

In a known way, the calculus of variations will then give a first criterion for the occurrence of an extremum, which is the existence of the **Euler-Lagrange** equations for the unknown functions $y(x)$, $z(x)$:

(3) $$\begin{cases} \dfrac{d}{dx}(f_{y'}) - f_y = 0, & \left[f_{y'} = \dfrac{\partial f}{\partial y'}, \quad f_y = \dfrac{\partial f}{\partial y}, \right. \\[2ex] \dfrac{d}{dx}(f_{z'}) - f_z = 0, & \left. f_{z'} = \dfrac{\partial f}{\partial z'}, \quad f_z = \dfrac{\partial f}{\partial z} \right]. \end{cases}$$

From known existence theorems in the theory of ordinary differential equations, there is, in general, one and only one integral curve of those equations (3) that goes through two points P_1 and P_2 that are given in equations (2). The integral curves of (3) are called the *extremals* in the calculus of variations, so an extremal is generally determined uniquely by a *point-pair* P_1 (x_1, y_1, z_1) and P_2(x_2, y_2, z_2).

2) The "extremal integral" (i.e., principal function) as a "point-pair function" and its derivatives.

We now imagine that two points P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) are given arbitrarily in a suitable region of space and connected by an extremal. We would like to calculate the value of the integral (1) when it is taken along the extremal from P_1 to P_2 that was constructed. That integral will have a well-defined value for any positions of the points P_1 and P_2, so it will take the form of a function of the point-pair P_1, P_2:

$$I(P_1, P_2) = I(x_1, y_1, z_1; x_2, y_2, z_2) = \int_{x_1, y_1, z_1}^{x_2, y_2, z_2} E\; f(y', z', y, z, x)\, dx\,, \tag{4}$$

in which the symbol E shall suggest that the values of the functions y and z along the (associated) extremal are substituted in f. The significance of that point-pair function was first emphasized by **Hamilton** ([1]), who called it the "principal function." We would like to refer to its by the more intuitive name of "extremal integral."

The following remark about point-pair functions might be made here: By analogy with **Volterra** ([2]), we would like to call a point-pair function S (P_1, P_2) *simple* ("semplice," "di primo grado") when it satisfies the condition:

$$S(P_1, P_3) = S(P_1, P_2) + S(P_2, P_3)\,. \tag{5}$$

The extremal integral I (P_1, P_2) that was just introduced is generally *not simple* as a point-pair function because since the integral depends upon the extremals and they are completely independent of each other, the value of the extremal integral when taken along the extremal $P_1 P_3$ will be completely different from the sum of the integrals that are taken along the extremals $P_1 P_2$ ($P_2 P_3$, resp.). However, one can easily construct a simple point-pair function from an ordinary "function of position" V (x, y, z) in space when one sets:

$$S(P_1, P_2) = V(x_2, y_2, z_2) - V(x_1, y_1, z_1)\,. \tag{5*}$$

That would be essentially identical to the Ansatz of a simple point-pair function that is defined by a curve integral:

$$S(P_1, P_2) = \int_{x_1, y_1, z_1}^{x_2, y_2, z_2} L(x, y, z)\, dx + M(x, y, z)\, dy + N(x, y, z)\, dz \tag{5**}$$

([1]) **W. R. Hamilton**, "On a general method in dynamics," London Phil. Trans. (1834), 247-308. "Second essay on a general method in dynamics," *ibid.* (1835), 95-144.

([2]) **V. Volterra**, cf., the survey treatise: "Sur une généralisation de la théorie des fonctions d'une variable imaginaire," Acta math. **12** (1889), 233-286.

that is independent of the path of integration. It can be proved with no effort that every simple point-pair function can be represented by a path-independent curve integral (with the help of *one* function of position, resp.) in that way.

From the so-called "limit formula" of the calculus of variations, the differential of our extremal integral (4), which takes the form of a function of six variables x_1, y_1, z_1 ; x_2, y_2, z_2 , is:

$$\delta I = \left[f_{y'}\,\delta y + f_{z'}\,\delta z + (f - y' f_{y'} - z' f_{z'})\,\delta x \right]_2 - \left[f_{y'}\,\delta y + f_{z'}\,\delta z + (f - y' f_{y'} - z' f_{z'})\,\delta x \right]_1 , \tag{6}$$

which will then imply the following six variables for its derivatives:

$$\begin{cases} \dfrac{\partial I}{\partial x_2} = \quad f - y' f_{y'} - z' f_{z'}\big|_2 , & \dfrac{\partial I}{\partial y_2} = \quad f_{y'}\big|_2 , \quad \dfrac{\partial I}{\partial z_2} = \quad f_{z'}\big|_2 , \\ \dfrac{\partial I}{\partial x_1} = -(f - y' f_{y'} - z' f_{z'})\big|_1 , & \dfrac{\partial I}{\partial y_1} = - f_{y'}\big|_1 , \quad \dfrac{\partial I}{\partial z_1} = - f_{z'}\big|_1 , \end{cases} \tag{6*}$$

in which the indices mean that the functions $y(x)$ and $z(x)$ that determine the extremal in question are evaluated at the locations $x = x_2$ ($x = x_1$, resp.).

Since the direction coefficients $y'(x)$ and $z'(x)$ of the extremal at one of the two points P_1 and P_2 depend upon the position of the other one, the derivatives (6^*) are once more point-pair functions.

For simple point-pair functions, the derivative (6^*) are just functions of position, as one will verify immediately from (5^*) [(5^{**}), resp.].

An asymmetry enters into the formulas (6) [(6^*), resp.] in such a way that one point of the point-pair appears as the lower limit of a certain integral in the point-pair function, while the other appears as the upper limit. Therefore, the increment dx might be, say, positive on both sides of an increasing interval at the point P_2 , but negative at the point P_1 . The symmetry in the concept will then be regained when we replace dx_1 with $(-\,dx_1)$ at P_1 when constructing all derivatives with respect to x. The sign in the second group of formulas in (6^*), which deviates from that of the first group in (6^*), will then, in fact, invert, so we can say that the derivatives of the point-pair function $I(P_1, P_2)$ are:

$$\frac{\partial I}{\partial x} = f - y' f_{y'} - z' f_{z'} , \qquad \frac{\partial I}{\partial x} = f_{y'} , \qquad \frac{\partial I}{\partial y} = f_{z'} \tag{7}$$

at *both* of those points.

3) The two Hamiltonian equations.

Hamilton ([1]) concluded from formulas (6^*) that the integration of the **Euler-Lagrange** equations (3) for given boundary values (2) could be achieved when the extremal integral was

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([1]) **Hamilton**, *loc. cit.*, pp. 10.

known in some way as a function of its two boundary points. That is because we can calculate $y'(x_1)$, $z'(x_1)$, $y'(x_2)$, $z'(x_2)$ from the four following equations in (6*):

$$(6^{**})\qquad \begin{aligned} \frac{\partial I}{\partial y_2} &= \quad f_{y'}\big|_2\,, & \frac{\partial I}{\partial z_2} &= \quad f_{z'}\big|_2\,, \\ \frac{\partial I}{\partial y_1} &= -f_{y'}\big|_1\,, & \frac{\partial I}{\partial z_1} &= -f_{z'}\big|_1\,, \end{aligned}$$

and thus construct an extremal from each of the two boundary points in a known way. We know, *a priori*, that each of the two extremals thus-constructed must also go through the other point of the point-pair, and therefore both of them must be identical.

However, there is initially no advantage to be derived from the approach that was taken up to now, since one must first integrate the **Euler-Lagrange** equations if one is to define the extremal integral. Meanwhile, **Hamilton** arrived at an Ansatz for that integral *directly* as a solution to two partial differential equations. In order to do that, he started from equations (6*). Once he had solved the last two equations in each row for y' and z', he substituted each of the expressions that were obtained in the first equation of that group of formulas. He then set:

$$(8)\qquad f - y'\,f_{y'} - z'\,f_{z'} = H\left(x, y, z, \frac{\partial I}{\partial y}, \frac{\partial I}{\partial z}\right),$$

to abbreviate, and obtained the *two* partial differential equations for the point-pair function I (P_1, P_2):

$$(9)\qquad \begin{aligned} \frac{\partial I}{\partial x_2} &= H\left(x_2, y_2, z_2\,;\ \frac{\partial I}{\partial y_2}, \frac{\partial I}{\partial z_2}\right), \\ -\frac{\partial I}{\partial x_1} &= H\left(x_1, y_1, z_1\,; -\frac{\partial I}{\partial y_1}, -\frac{\partial I}{\partial z_1}\right). \end{aligned}$$

If we replace dx_1 with $(-\,dx_1)$, as we have shown to be natural above, then both of equations (9) will coincide in *one* equation:

$$(9^*)\qquad \frac{\partial I}{\partial x} = H\left(x, y, z\,;\ \frac{\partial I}{\partial y}, \frac{\partial I}{\partial z}\right)$$

that must be fulfilled for *each* of the two boundary points P_1 and P_2 of the point-pair function.

Hamilton then raised the question of whether any point-pair function that is a solution to each of the **Hamilton** equations (9) can play the role of an extremal integral in the arguments of this Chapter and lead to an integration of the **Euler-Lagrange** equations for prescribed boundary values. One can immediately convince oneself that this is not the case. To that end, we construct a point-pair function that satisfies equations (9) in the following way: We imagine fixing one point of the extremal integral (say, the upper limit) at a well-defined point A_1 of the region considered and then connect any other point P in the region to A_1 by an extremal. If the points P, as a type of covering of space, were then assigned the *values* of the extremal integral I (A_1, P) then a *function of position* V_1 (x, y, z) would now be defined in the region of space (in contrast to a point-pair function), and that function of position would be a solution to the partial differential equation:

$$\frac{\partial V_1}{\partial x} = H\left(x, y, z; \frac{\partial V_1}{\partial y}, \frac{\partial V_1}{\partial z}\right), \tag{10}$$

which coincides formally with (9*).

In the same way, we can connect a second fixed point A_2 in the region with all points P by an extremal and define a second function of position V_2 (x, y, z) by the associated extremal integral value that likewise satisfies the partial differential equation (10).

If we then define the following point-pair function for the two arbitrary points P_1 and P_2 with the help of those two functions of position V_1 (x, y, z) and V_2 (x, y, z) :

$$T(P_1, P_2) = V_2(P_2) - V_1(P_1) = V_2(x_2, y_2, z_2) - V_1(x_1, y_1, z_1), \tag{11}$$

then it (which is naturally not a simple point-pair function) will satisfy the two **Hamilton** differential equations (9). However, from the way that the point-pair function T (P_1, P_2) was created, it is clear that the partial derivatives $\frac{\partial T}{\partial y_2}, \frac{\partial T}{\partial z_2}$ ($\frac{\partial T}{\partial y_1}, \frac{\partial T}{\partial z_1}$, resp.) have nothing to do with the direction coefficients of the extremals that connect the points P_1 and P_2, so they cannot be used for the desired integration of the **Euler-Lagrange** equations. We then see that the difficulty lies precisely in separating the extremal integral itself from the set of all possible solutions of the partial differential equations (9).

4) The Hamilton-Jacobi differential equation.

Perhaps in order to get around that difficulty, **Jacobi** ([1]), who pursued **Hamilton**'s investigations further, broke with **Hamilton**'s conception of the extremal integral as a point-pair function. He started from precisely the idea that we have just used by fixing the one endpoint of the extremal integral and then arrived at a covering of space with the values of the extremal integral V (x, y, z). As we saw, that covering yields a solution to *one* of the partial differential equations (10) that replaced the two **Hamilton** differential equations in **Jacobi**'s representation and which we would then like to call the ***Hamilton-Jacobi*** *equation*.

The function of position considered – viz., the *extremal integral value* V (x, y, z) – is a solution to that equation that depends upon two essential constants (actually three, but the third one is only additive) as a result of the arbitrary position of the point A from which the extremal integral is

([1]) **C. G. J. Jacobi**, Cf., esp., the survey paper: "Probleme der Mechanik bei Existenz einer Kräftefunktion und bei die Theorie der Störungen," *Werke*, Bd. V, pp. 217-395.

extended and is then a complete integral of (10). The extremal integral value itself can, with no further analysis, take the place of **Hamilton**'s extremal integral in the determination (6^{**}) of the solution to the **Euler-Lagrange** equations above. Namely, if we assume that the point A has the coordinates $x = a$, $y = b$, $z = c$, so the complete integral will be $V = V(x, y, z, a, b, c)$, then from the developments on pp. 6, we will need to form only the derivatives of that function with respect to b and c in order to find the initial direction of the extremal that radiates from A and goes through P. When we set the derivatives equal to new arbitrary constants:

$$\frac{\partial V}{\partial b} = \beta, \qquad \frac{\partial V}{\partial c} = \gamma, \tag{12}$$

we will obtain the entire family of extremals that radiate from A. Equations (12) will then yield the general (i.e., depending upon four constants) solution to the **Euler-Lagrange** equations. At the same time, the boundary-value problem that was posed above will also be resolved insofar as the constants β and γ can be determined in such a way that the extremals (12) go through a second arbitrarily-given point in addition to A.

Here, we shall also ask the converse question of whether an *arbitrary* complete integral of (10) can always be represented by an extremal integral value (whether that can happen, resp.). **Jacobi** ([1]) expressed that by saying that the general solution to equations (3) can be obtained from an arbitrary complete integral of the partial differential equation (10) by means of equations (12).

If we would like to arrive at that theorem from the variational problem directly (which **Jacobi** derived by calculation) then some further analysis will be necessary. In order to accomplish that, we shall once more clearly emphasize the difference between **Hamilton**'s original conception and the modification that **Jacobi** carried out.

Hamilton simultaneously considered the whole four-parameter family of integral curves to the **Euler-Lagrange** equations, each of which could serve as a certain choice of the extremal integral for him. **Jacobi** then selected a two-parameter family from that four-parameter family, namely, all of the ones that radiated from a fixed point A. He then first obtained the four-parameter set of extremals in such a way that he again let the point A vary afterwards, which would introduce two new parameters.

5) *The Hamilton-Jacobi differential equation and the extremal field.*

In order to get **Jacobi**'s theorem later, we appeal to the picture of a two-parameter family of extremals that fill up space simply and with no gaps, so they define a *field* in the **Weierstrass**. **Jacobi**'s family of extremals is one such special field that leads to a solution of the **Hamilton-Jacobi** differential equation. In general, we would like to characterize the extremal fields from which a solution to the **Hamilton-Jacobi** differential equation can be obtained.

If we imagine that we are given an arbitrary field of extremals:

$$y = Y(x, b, c), \qquad z = Z(x, b, c) \tag{13}$$

([1]) **Jacobi**, *loc. cit.*, pp. 240-241.

then we can also define a function of position W (x, y, z) in it in manner that is similar to what **Jacobi** did. We need only to lay any surface through the field that cuts each extremal at one and only one point. Any point P in space can then be associated with a functional value that is the value that the extremal integral takes when it is extended along the extremal of the field that goes through P from the intersection point of that extremal with the surface to the point P.

However, the function W (x, y, z) thus-defined does not generally satisfy the **Hamilton-Jacobi** differential equation.

Namely, let x_0, y_0, z_0 be the coordinates of a point of the initial surface and let the equation of that surface read:

$$x_0 = x\,(y_0, z_0)\,. \tag{14}$$

y_0 and z_0, and also x_0, by the intermediary of that equation, will then take the form of functions of the coordinates x, y, z of an arbitrary point because every point of the field is associated with a point on the initial surface by the extremal that goes through it. One will then have:

$$y_0 = y_0\,(x, y, z)\,, \qquad z_0 = z_0\,(x, y, z)\,, \qquad x_0 = \xi\,(y_0\,(x, y, z), z_0\,(x, y, z))\,. \tag{15}$$

If we apply the limit formula (6) of the calculus of variations to the function:

$$W\,(x, y, z) = \int\limits_{\xi(y_0, z_0), y_0, z_0}^{x, y, z} E\ f\,(y', z', y, z, x)\,dx \tag{16}$$

then that will give:

$$\delta W = (f - y'\,f_{y'} - z'\,f_{z'})\,\delta x + f_{y'}\,\delta y + f_{z'}\,\delta z - \left\{(f - y'\,f_{y'} - z'\,f_{z'})\Big|^{\xi}\delta x_0 + f_{y'}\Big|^{\xi}\delta y_0 + f_{z'}\Big|^{\xi}\delta z_0\right\}. \tag{17}$$

The y' and z' in this mean the derivatives of the extremal of the field that goes through the point P. They can also be regarded as functions of position in the field, so they will be called the *slope functions* of the field; they might then be denoted by p (x, y, z) and q (x, y, z). If we introduce those notations and recall (15) then we will get the following values for the partial derivatives of the function W (x, y, z) :

$$\tag{17*} \left\{\begin{aligned}
\frac{\partial W}{\partial x} &= f - p\,f_p - q\,f_q - \left\{\left[(f - p\,f_p - q\,f_q)\frac{\partial \xi}{\partial y_0} + f_p\right]^{\xi}\frac{\partial y_0}{\partial x} + \left[(f - p\,f_p - q\,f_q)\frac{\partial \xi}{\partial z_0} + f_q\right]^{\xi}\frac{\partial z_0}{\partial x}\right\}, \\
\frac{\partial W}{\partial y} &= f_p - \left\{\left[(f - p\,f_p - q\,f_q)\frac{\partial \xi}{\partial y_0} + f_p\right]^{\xi}\frac{\partial y_0}{\partial y} + \left[(f - p\,f_p - q\,f_q)\frac{\partial \xi}{\partial z_0} + f_q\right]^{\xi}\frac{\partial z_0}{\partial y}\right\}, \\
\frac{\partial W}{\partial z} &= f_q - \left\{\left[(f - p\,f_p - q\,f_q)\frac{\partial \xi}{\partial y_0} + f_p\right]^{\xi}\frac{\partial y_0}{\partial z} + \left[(f - p\,f_p - q\,f_q)\frac{\partial \xi}{\partial z_0} + f_q\right]^{\xi}\frac{\partial z_0}{\partial z}\right\}.
\end{aligned}\right.$$

It is immediately clear from the form of those derivatives that $W(x, y, z)$ can satisfy the **Hamilton-Jacobi** differential equation only when the expressions that are included in brackets vanish. However, that means that the initial surface must be determined in such a way that it is an integral of the total linear differential equation:

$$(f - p\, f_p - q\, f_q)\, d\xi + f_p\, dy_0 + f_q\, dz_0 = 0\,, \tag{18}$$

so it must be a surface that cuts all extremals of the field *transversally*.

6) The extremal field as an "independence field."

In order for one to be able to determine such a transverse surface, the differential equation (18) must be completely integrable.

For the variational problem with only one unknown function, the differential equation:

$$(f - p\, f_p)\, d\xi + f_p\, dy_0 = 0 \tag{18.a}$$

will appear in place of (18). Since it is an ordinary total differential equation in ξ and y_0 it will *always* possess a solution. Therefore, *any* field is of the desired type here. The specialization that was spoken of on pp. 3 is based upon that fact.

The integrability condition for equation (18) reads:

$$(f - p\, f_p - q\, f_q)\left[\frac{\partial f_p}{\partial z} - \frac{\partial f_q}{\partial y}\right] + f_p\left[\frac{\partial f_q}{\partial x} - \frac{\partial}{\partial z}(f - p\, f_p - q\, f_q)\right] + f_q\left[\frac{\partial}{\partial y}(f - p\, f_p - q\, f_q) - \frac{\partial f_p}{\partial x}\right] = 0\,. \tag{19}$$

Furthermore, the following partial differential equations for the slope functions p and q are true in *any* case ([1]):

$$\left\{\begin{aligned} &\frac{\partial}{\partial x}(f_q) - \frac{\partial}{\partial z}(f - p\, f_p - q\, f_q) + q\left[\frac{\partial}{\partial z}(f_p) - \frac{\partial}{\partial y}(f_q)\right] = 0, \\ &\frac{\partial}{\partial x}(f_q) - \frac{\partial}{\partial z}(f - p\, f_p - q\, f_q) + p\left[\frac{\partial}{\partial y}(f_p) - \frac{\partial}{\partial z}(f_q)\right] = 0. \end{aligned}\right. \tag{20}$$

Equations (19) and (20) represent three homogeneous linear equations for the expressions:

$$\frac{\partial f_p}{\partial z} - \frac{\partial f_q}{\partial y}\,, \qquad \frac{\partial f_p}{\partial x} - \frac{\partial}{\partial y}(f - p\, f_p - q\, f_q)\,, \qquad \text{and} \qquad \frac{\partial f_q}{\partial x} - \frac{\partial}{\partial z}(f - p\, f_p - q\, f_q)\,.$$

([1]) Cf., **A. Mayer**, "Über den Hilbertschen Unabhängigkeitssatz," Math. Ann. **62** (1906), pp. 339 or **O. Bolza**, "Über den Hilbertschen Unabhängigkeitssatz," Palermo circ. mat. Rend. **31** (1911), 258-261.

Since their determinant:

$$\begin{vmatrix} f-p\,f_p-q\,f_q & -f_q & f_p \\ q & 1 & 0 \\ -p & 0 & 1 \end{vmatrix} = \begin{vmatrix} f & -f_q & f_p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = f$$

does not vanish identically, it will then follow that the slope functions p and q of the fields that we consider here must satisfy the conditions:

$$(21)\qquad \begin{cases} \dfrac{\partial f_p}{\partial z}-\dfrac{\partial f_q}{\partial y}=0, \\ \dfrac{\partial f_q}{\partial x}-\dfrac{\partial (f-p\,f_p-q\,f_q)}{\partial z}=0, \\ \dfrac{\partial f_p}{\partial x}-\dfrac{\partial (f-p\,f_p-q\,f_q)}{\partial y}=0. \end{cases}$$

However, those are the conditions for the *Hilbert integral*:

$$(22)\qquad \begin{aligned} I^* &= \int (f-p\,f_p-q\,f_q)\,dx+f_p\,dy+f_q\,dz \\ &= \int [f+(y'-p)\,f_p+(z'-q)\,f_q]\,dx \end{aligned}$$

to be independent of the path of integration in the field. The **Hilbert** independence theorem is true for our field then, so we shall call it an *independence field*. We can then formulate the result as follows: If we would like to obtain a solution to the **Hamilton-Jacobi** equation from the value of the extremal integral in a field then the field must be an independence field and the value of the extremal integral must be calculated from a transversal of the independence field.

However, with that, we have then found the connection to the aforementioned theorem of **Jacobi**, as we will discuss in more detail in the third section because if we fix the constants in an arbitrary complete integral of the **Hamilton-Jacobi** equation then we can give an independence field, i.e., a field that possesses a transversal surface for which the value of the extremal field that is calculated from a transversal surface will coincide with the given integral ([1]).

7) *The independence field and the integration of the Euler-Lagrange equations.*

The independence field unites into just a two-fold infinitude of extremals. The boundary-value problem for the **Euler-Lagrange** equations cannot be solved with the help of the value of the extremal integral that such a thing defines, since four available parameters are, in fact, necessary.

([1]) Cf., also **J. Hadamard**, *Calcul des variations*, Paris, 1910, t. 1, pp. 161-162.

Naturally, the same thing is true for the **Hilbert** independence integral that belongs to the field, which indeed represents a point-pair function, but also a *simple* one.

Meanwhile, we can provide the necessary number of parameters in such a way that we do not consider *one* independence field by itself, but a family of fields that depends upon two parameters ([1]).

In order to exhibit such a thing, we shall give a two-parameter family of surfaces and construct an independence field for each of them by drawing all of the extremals that are transverse to them. The value of the extremal integral that belongs to each individual independence field then takes the form of a function of the two parameters of the family of surfaces $I = I(x, y, z, a, b)$. Since it includes two essential parameters, it will then be a complete integral of the **Hamilton-Jacobi** differential equation (10). In the third section, we will likewise show that any complete integral of that equation can be represented in the given way.

If the two-parameter family of surfaces is given by the equations:

$$x_0 = x_0(x, y, z, a, b), \qquad y_0 = y_0(x, y, z, a, b), \qquad z_0 = z_0(x, y, z, a, b)$$

then the limit formula of the calculus of variations will yield the derivatives of the value of the extremal integral $I(x, y, z, a, b)$ with respect to the parameters (for fixed x, y, z) in the form of:

$$\text{(23)} \qquad \begin{cases} \dfrac{\partial I}{\partial a} = -(f_{y'})_0 \dfrac{\partial y_0}{\partial a} - (f_{z'})_0 \dfrac{\partial z_0}{\partial a} - (f - y' f_{y'} - z' f_{z'})_0 \dfrac{\partial x_0}{\partial a}, \\ \dfrac{\partial I}{\partial b} = -(f_{y'})_0 \dfrac{\partial y_0}{\partial b} - (f_{z'})_0 \dfrac{\partial z_0}{\partial b} - (f - y' f_{y'} - z' f_{z'})_0 \dfrac{\partial x_0}{\partial b}. \end{cases}$$

Those expressions will not change as long as the point x, y, z remains on *the same* extremal of the field, so the equations:

$$\text{(24)} \qquad \frac{\partial I}{\partial a} = c, \qquad \frac{\partial I}{\partial b} = d,$$

in which one understands c and d constants, will then represent the individual extremals of the variational problem.

Now in order to solve the boundary-value problem for the **Euler-Lagrange** equations for two arbitrarily-given points P_1 and P_2 , we shall determine the values of the extremal integrals $I_1 = I(P_1, a, b)$ and $I_2 = I(P_2, a, b)$ that belong to P_1 and P_2, and initially for a fixed choice of the parameters a and b. We will then get:

$$\text{(25)} \qquad \begin{cases} \dfrac{\partial I_1}{\partial a} = c_1, \quad \dfrac{\partial I_1}{\partial b} = d_1, \\ \dfrac{\partial I_2}{\partial a} = c_2, \quad \dfrac{\partial I_2}{\partial b} = d_2 \end{cases}$$

([1]) **D. Hilbert**, "Zur Variationsrechnung," Math. Ann. **62** (1906), pp. 361.

for the extremal of the field that goes through P_1 (P_2, resp.).

Should the field be chosen in such a way that P_1 and P_2 would lie along the same extremals then one would need to have $c_1 = c_2$, $d_1 = d_2$, i.e., the parameters a and b would have to be calculated from the equations:

$$\frac{\partial(I_1 - I_2)}{\partial a} = 0\,, \qquad \frac{\partial(I_1 - I_2)}{\partial b} = 0\,. \tag{26}$$

The directions of the desired extremals at the point P_1 or P_2 are then determined from the relations

§ 2.

Integration theory of the Euler-Lagrange equations (the associated canonical system, resp.)

The integration of the two **Euler-Lagrange** equations:

$$\begin{cases} \dfrac{d}{dx}(f_{y'}) - f_y = 0, \\ \dfrac{d}{dx}(f_{z'}) - f_z = 0 \end{cases} \tag{1}$$

was accomplished in the first section with the help of the **Hamilton-Jacobi** partial differential equation where it was used, in particular, to treat the boundary-value problem for those equations. However, the fact that they arise from the variational problem:

$$I = \int_{x_1}^{x_2} f(y', z', y, z, x)\, dx = \text{extremum} \tag{2}$$

also brings with it certain advantages for the *general* integration theory of those differential equations. We would like to summarize them in an overview here.

1) The canonical system.

In the previous section, we first started by solving a boundary-value problem for the equations (1), namely, we determined an integral curve in such a way that it would go through two given points of the region considered. In the sense of the general integration theory of differential equations, it would seem simpler to give only the values of y and z and their derivatives y' and z' at a point in the region. That convention will seem quite natural when one decomposes the two equations (1) into a system of two first-order differential equations (as one likes to do in the integration theory of ordinary differential equations). Formally, one can achieve that conversion simply by introducing:

$$y' = p\,, \qquad z' = q$$

as new variables, by which the system (1) will go to a system of four first-order ordinary differential equations.

Meanwhile, for such a conversion, it is expedient to apply a **Legendre** transformation and introduce the quantities:

(3) $$\pi = f_p\,, \qquad \kappa = f_q$$

in place of p and q as new variables. If we calculate p and q as functions of π and κ using those two equations and introduce the new function:

(4) $$H(\pi, \kappa, y, z, x) = f - p \cdot f_p - q \cdot f_q\,,$$

while once more substituting the calculated values as we did in the previous section [cf., equation (8)], then if we consider the relations:

(4*) $$\frac{\partial f}{\partial y} = \frac{\partial H}{\partial y}\,, \qquad \frac{\partial f}{\partial z} = \frac{\partial H}{\partial z}\,,$$
$$-p = \frac{\partial H}{\partial \pi}\,, \qquad -q = \frac{\partial H}{\partial \kappa}\,,$$

which follow immediately from (4), the **Euler-Lagrange** equations (1) will go to the *canonical system:*

(5) $$\left\{\begin{aligned} \frac{d\pi}{dx} &= \frac{\partial H}{\partial y}, & \frac{dy}{dx} &= -\frac{\partial H}{\partial \pi}, \\ \frac{d\kappa}{dx} &= \frac{\partial H}{\partial z}, & \frac{dz}{dx} &= -\frac{\partial H}{\partial \kappa}, \end{aligned}\right.$$

which belongs to the system of **Euler-Lagrange** equations of the variational problem:

(6) $$I = \int_{x_1}^{x_2} \left\{ \pi \frac{dy}{dx} + \kappa \cdot \frac{dz}{dx} + H(\pi, \kappa, y, z, a) \right\} dx = \text{extremum}$$

intrinsically. While varying that integral, we can prescribe the values of y, z ; π, κ at only one of the endpoints, say, for $x = x_0$.

We can associate any group of values of the canonical variables π, κ ; y, z, x geometrically with a point in a five-dimensional space. One and only one integral curve of the canonical system will then run through each point in that space, which is determined completely when we are given that point, such that we will immediately have in mind a simple covering of that five-dimensional space by the set of all solutions to the system (5) [and therefore equations (1), as well].

By contrast, with the **Euler-Lagrange** equations (1), we have only three variables y, z, x, which we must interpret in a three-dimensional space. An integral curve of equations (1) is first determined then when we prescribe, not only a point x, y, z, but also its direction y', z' at that point. Therefore, a two-fold infinitude of extremals will run through each point in space. Here, we would like to have a simple covering of space, so we must select one such two-parameter family that defines a field from the four-parameter set of extremals. When we speak of the integral curve of equations (1) in what follows, we would always like to imagine that such a field has been given.

2) The relative integral invariant and the independent integral.

In order to arrive at the principle from which we can derive the advantage in integrating equations (1) that arises from their relationship to the variation problem, we shall now focus on the limit formula for the calculus of variations and consider the differential expression:

$$f_{y'}\,\delta y + f_{z'}\,\delta z + (f - y' f_{y'} - z' f_{z'})\,\delta x, \tag{7}$$

in which the differentials δx, δy, δz are arbitrary increments, while y' and z' are the well-defined direction coefficients of the extremal of the field that goes through the point in question. In terms of the canonical variables, that expression would take the form:

$$\pi\,\delta y + \kappa\,\delta z + H\,\delta x\,. \tag{7*}$$

We take the line integral of the expression (7), as well as (7*), along an arbitrary closed curve in the associated three-dimensional (five-dimensional, resp.) space. An extremal will emanate from each point of one such closed curve, and we imagine that the curve has been chosen especially to make no two curve points lie along one and the same extremal. The set of all extremals that are lined up along the curve will then generate a type of tubular surface. If we lay a second curve of that kind around that tube of extremals then each point of the first curve will be associated with a well-defined point of the second curve, namely, the one that lies along the same extremal as it. We can establish the points of the first curve x_1, y_1, z_1 [when we couple them with the expression (7)] as functions of one parameter α, so the points x_2, y_2, z_2 on the second curve that they are associated with will also be given as functions of α. If we now take the extremal integral that we introduced in the previous section [equation (4)] along the piece of the extremal field that connected the two associated points then it will likewise take the form of a function of the parameter α:

$$I(x_1, y_1, z_1\,;\, x_2, y_2, z_2) = I(x_1(\alpha), y_1(\alpha), z_1(\alpha)\,;\, x_2(\alpha), y_2(\alpha), z_2(\alpha)) = I(a)\,. \tag{8}$$

From the limit formula of the calculus of variations, we will then have:

$$\text{(9)}\qquad \begin{aligned}\frac{\delta I}{\delta\alpha} &= \left\{(f_{y'})_2\frac{\delta y_2}{\delta\alpha}+(f_{z'})_2\frac{\delta z_2}{\delta\alpha}+(f-y'\,f_{y'}-z'\,f_{z'})_2\frac{\delta x_2}{\delta\alpha}\right\}\\ &\quad -\left\{(f_{y'})_1\frac{\delta y_1}{\delta\alpha}+(f_{z'})_1\frac{\delta z_1}{\delta\alpha}+(f-y'\,f_{y'}-z'\,f_{z'})_1\frac{\delta x_1}{\delta\alpha}\right\}.\end{aligned}$$

If we then take the integral of that expression along the given closed curve then we will have $\oint\frac{\delta I}{\delta\alpha}$ = 0 on the left-hand side, and we will then get:

$$\text{(10)}\qquad \left[\oint f_{y'}\,\delta y+f_{z'}\,\delta z+(f-y'\,f_{y'}-z'\,f_{z'})\,\delta x\right]_2=\left[\oint f_{y'}\,\delta y+f_{z'}\,\delta z+(f-y'\,f_{y'}-z'\,f_{z'})\,\delta x\right]_1.$$

Poincaré ([1]) expressed the fact that our integral of the differential expression (7), which is taken over a closed curve, will keep the same value when we take it over a second arbitrary curve that encircles the tube of extremals that emanate from the first curve by calling that integral an *integral invariant* of the **Euler-Lagrange** equations, and indeed it is, in particular, a *relative integral invariant*, with his terminology, and it is relative in the sense that the domain of integration must be a closed curve.

In the same way, one will find that the integral of the expression (7*) for the canonical system (5):

$$\text{(10*)}\qquad \oint\pi\,\delta y+\kappa\,\delta z+H\,\delta x$$

is a relative integral invariant.

Knowing that relative integral invariant is the foundation for the theory of integration for the **Euler-Lagrange** equations (the canonical system, resp.) that **Poincaré** discussed in detail. Here, we shall investigate how that relative integral invariant is connected with the **Hilbert** independence theorem.

If we consider those curves that can contract to a point on a tube of extremals, as we have characterized it up to now, and assume, for the sake of simplicity, that they are intersected by each extremal at two points then we can also associate the points on the curves that lie along the same extremal here, as well. We will then get the points of the two segments of the curves that arise in that way as functions of one parameter α. In that way, α will run from one value α' to another value α'', which correspond to the points at which the curve is tangent to the extremals of the field, in order to describe that curve. We will then have $I(\alpha')$ = 0, as well as $I(\alpha'')$ = 0, and it will then follow from (9) that:

$$0=\int_{\alpha'}^{\alpha''}(f_{y'})_2\,\delta y_2+(f_{z'})_2\,\delta z_2+(f-y'\,f_{y'}-z'\,f_{z'})_2\,\delta x_2-\int_{\alpha'}^{\alpha''}(f_{y'})_1\,\delta y_1+(f_{z'})_1\,\delta z_1+(f-y'\,f_{y'}-z'\,f_{z'})_1\,\delta x_1$$

or

([1]) **H. Poincaré**, *Les methodes Nouvelles de la mécanique celeste*, Paris (1892-99), t. 3, Chap. 22.

(11) $$\oint f_{y'}\,\delta y + f_{z'}\,\delta z + (f - y' f_{y'} - z' f_{z'})\,\delta x = 0 ,$$

in which the integral (11) is once more extended over a closed curve ([1]). Naturally, a corresponding statement would be true for the canonical system.

The latter case always presents itself when the variational problem (2) includes only one unknown function. That is because we will then have a field in the plane, and in that case, any closed curve will be of the kind that can be contracted to a point. That says nothing more than the fact that the **Hilbert** integral:

(12*) $$\int \left\{ f(y', y, x) + \left(\frac{\delta y}{\delta x} - y' \right) f_{y'} \right\} \delta x$$

is independent of the path of integration for any field in the plane. By contrast, a field in space is an independence field only when the integral in the formula (10) has the value zero for any tube of extremals along which the curve of integration *cannot* be contracted to a point. The **Hilbert** integral:

(12) $$\int \left\{ f(y', z', y, z, x) + \left(\frac{\delta y}{\delta x} - y' \right) f_{y'} + \left(\frac{\delta z}{\delta x} - z' \right) f_{z'} \right\} \delta x$$

will, in fact, be independent of the path, here as well. The independence integral can then be regarded as a special type of integral invariant ([2]).

3) *The associated "absolute integral invariants" and the "Jacobi equations."*

We can immediately construct absolute integral invariants (of order two) from the relative integral invariants (of order one) that we have considered up to now by means of **Stokes**'s theorem. From the previous section, we can distort a given closed curve that serves to define the relative integral invariant arbitrarily along the associated tube of extremals without changing the value of the invariant. With no loss of generality in the argument, we can then assume from the outset that the curve lies in a plane x = const. The relative integral invariant will then take on the form:

(13) $$\oint f_{y'}\,\delta y + f_{z'}\,\delta z = \text{const.},$$

or

(13*) $$\oint \pi\,\delta y + \kappa\,\delta z = \text{const.},$$

resp.

([1]) A theorem of **H. Hahn**: "Über den Zusammenhang zwischen den Theorien den zweiten Variation und der **Weierstraß**'schen Theorie der Variationsrechnung," Palermo circ. mat. Rend. **29** (1910), 49-78.

([2]) That fact is essential if one is to gain the advantage in integration that comes from knowing the independence integral. In fact, all of the results of **D. C. Gillespie** (Dissertation, Göttingen, 1906), who pursued that topic further, are immediately contained in **Poincaré**'s theorems on integral invariants.

If we then lay an arbitrary surface through the closed curve along which the integral is taken, which we can naturally choose to be the plane x = const. that goes through curve, for the sake of simplicity, then from **Stokes**'s theorem, the curve integral will go to the double integral:

(14) $$\iint \left\{ \frac{\partial}{\partial z}(f_{y'}) - \frac{\partial}{\partial y}(f_{z'}) \right\} \delta y \, \delta z = \text{const.},$$

or

(14*) $$\iint \delta\pi \, \delta y + \delta\kappa \, \delta z = \text{const.},$$

resp., in which the integrals are extended over the surface patch in the plane x = const. that is enclosed by the curve ([1]).

If we next recall (14) then we can bring that formula into either of the forms:

$$\iint \left\{ f_{y'y'} \frac{\partial y'}{\partial z} + f_{y'z'} \frac{\partial z'}{\partial z} + f_{y'z'} - f_{y'z'} \frac{\partial y'}{\partial z} - f_{z'z'} \frac{\partial y'}{\partial z} - f_{z'y'} \right\} \delta y \, \delta z = \text{const.}$$

or

(15) $$\iint f_{y'y'} \, \delta y \, \delta y' + f_{y'z'} \, \delta y \, \delta z' + f_{y'z'} \, \delta z \, \delta y' + f_{z'z'} \, \delta z \, \delta z' + (f_{y'z'} - f_{z'y'}) \, \delta y \, \delta z = \text{const.}$$

by differentiating it.

In the consideration of the basic field of extremals, let the individual extremals be characterized by the two parameters a and b. The individual points in each plane x = const. can also be established by those parameters then. If we introduce the parameters into the integral (15) then we will get:

$$\iint \left\{ f_{y'y'} \left(\frac{\partial y}{\partial a} \frac{\partial y'}{\partial b} - \frac{\partial y}{\partial b} \frac{\partial y'}{\partial a} \right) + \cdots + (f_{y'z'} - f_{z'y'}) \left(\frac{\partial y}{\partial a} \frac{\partial z}{\partial b} - \frac{\partial y}{\partial b} \frac{\partial z}{\partial a} \right) \right\} da \, db = \text{const.}$$

along each extremal, so we will also have that the integrand satisfies:

(16) $$f_{y'y'} \left(\frac{\partial y}{\partial a} \frac{\partial y'}{\partial b} - \frac{\partial y}{\partial b} \frac{\partial y'}{\partial a} \right) + \cdots + (f_{y'z'} - f_{z'y'}) \left(\frac{\partial y}{\partial a} \frac{\partial z}{\partial b} - \frac{\partial y}{\partial b} \frac{\partial z}{\partial a} \right) = \text{const.}$$

(because the domain of integration is completely arbitrary). For the canonical system, one has the corresponding result that one has:

([1]) Should the field in question be an independence field, then it would follow from (14) that the expression $\frac{\partial}{\partial z}(f_{y'}) - \frac{\partial}{\partial y}(f_{z'})$ must vanish identically. The quantities $f_{y'}$ and $f_{z'}$ must then be the partial derivatives of a function $\Omega(y, z, x)$. That is the condition that **A. Mayer** (*loc. cit.* on pp. 10) gave for the validity of the **Hilbert** independence theorem.

(16*) $$\left(\frac{\partial \pi}{\partial a}\frac{\partial y}{\partial b}-\frac{\partial y}{\partial a}\frac{\partial \pi}{\partial b}\right)+\left(\frac{\partial \kappa}{\partial a}\frac{\partial z}{\partial b}-\frac{\partial z}{\partial a}\frac{\partial \kappa}{\partial b}\right) = \text{const.}$$

along the extremals.

Those results can be given yet another setting: The **Euler-Lagrange** equations (1) are associated with certain linear differential equations that we would like to call the *Jacobi equations*. (In **Poincaré**'s terminology, they are called the "équations aux variations.") They belong to the second variation of the integral (2), and they are the **Euler-Lagrange** equations of that integral. If:

(17) $$I = \int_{x_1}^{x_2} \Phi(\mathfrak{y}', \mathfrak{z}', \mathfrak{y}, \mathfrak{z}, x)\, dx$$

is that second variation, in which one has:

(18) $$\begin{cases} 2\Phi(\mathfrak{y}', \mathfrak{z}', \mathfrak{y}, \mathfrak{z}, x) & = f_{y'y'}\,\mathfrak{y}'^2 + 2 f_{y'z'}\,\mathfrak{y}'\mathfrak{z}' + f_{z'z'}\,\mathfrak{z}'^2 \\ & + 2 f_{y'y}\,\mathfrak{y}'\mathfrak{y} + 2 f_{y'z}\,\mathfrak{y}'\mathfrak{z} + 2 f_{z'y}\,\mathfrak{z}'\mathfrak{y} + 2 f_{z'z}\,\mathfrak{z}'\mathfrak{z} \\ & + f_{yy}\,\mathfrak{y}^2 + 2 f_{yz}\,\mathfrak{y}\mathfrak{z} + f_{zz}\,\mathfrak{z}^2, \end{cases}$$

then the **Jacobi** equations will read:

(19) $$\begin{cases} \dfrac{d}{dx}(\Phi_{\mathfrak{y}'}) - \Phi_{\mathfrak{y}} = 0, \\ \dfrac{d}{dx}(\Phi_{\mathfrak{z}'}) - \Phi_{\mathfrak{z}} = 0. \end{cases}$$

As **Jacobi** showed, any solution $y = y\,(x, a)$, $z = (z, a)$ of the **Euler-Lagrange** equations (1) that includes one parameter will immediately imply a solution to the equations (2) when one differentiates them with respect to that parameter: $\mathfrak{y} = \dfrac{\partial y}{\partial a}$, : $\mathfrak{z} = \dfrac{\partial z}{\partial a}$. We can the derive the following two systems of solutions to the **Jacobi** equations from the two-parameter family of field extremals directly:

(20) $$\begin{cases} \mathfrak{y}_1 = \dfrac{\partial y}{\partial a}, \quad \mathfrak{z}_1 = \dfrac{\partial z}{\partial a}, \\ \mathfrak{y}_2 = \dfrac{\partial y}{\partial b}, \quad \mathfrak{z}_2 = \dfrac{\partial z}{\partial b}. \end{cases}$$

Those solutions are coupled by the relation (16), which says that the **Jacobi** system (19) is self-adjoint. It would likewise follow from (16) that the **Jacobi** system that belongs to the canonical system (5):

$$(21)\qquad \begin{cases} \dfrac{d\mathfrak{p}}{dx} = \dfrac{\partial^2 H}{\partial y^2}\mathfrak{y} + \dfrac{\partial^2 H}{\partial y\,\partial \pi}\mathfrak{p} + \dfrac{\partial^2 H}{\partial y\,\partial z}\mathfrak{z} + \dfrac{\partial^2 H}{\partial y\,\partial \kappa}\mathfrak{k}, \\ \dfrac{d\mathfrak{y}}{dx} = -\dfrac{\partial^2 H}{\partial \pi\,\partial y}\mathfrak{y} - \dfrac{\partial^2 H}{\partial \pi^2}\mathfrak{p} - \dfrac{\partial^2 H}{\partial \pi\,\partial z}\mathfrak{z} - \dfrac{\partial^2 H}{\partial \pi\,\partial \kappa}\mathfrak{k}, \\ \dfrac{d\mathfrak{k}}{dx} = \dfrac{\partial^2 H}{\partial z\,\partial y}\mathfrak{y} + \dfrac{\partial^2 H}{\partial z\,\partial \pi}\mathfrak{p} + \dfrac{\partial^2 H}{\partial z^2}\mathfrak{z} + \dfrac{\partial^2 H}{\partial z\,\partial \kappa}\mathfrak{k}, \\ \dfrac{d\mathfrak{z}}{dx} = -\dfrac{\partial^2 H}{\partial \kappa\,\partial y}\mathfrak{y} - \dfrac{\partial^2 H}{\partial \kappa\,\partial \pi}\mathfrak{p} - \dfrac{\partial^2 H}{\partial \kappa\,\partial z}\mathfrak{z} - \dfrac{\partial^2 H}{\partial \pi^2}\mathfrak{k} \end{cases}$$

is also self-adjoint. As will be explained in subsection 5), that reciprocity property leads to **Poisson**'s theorem.

4) The canonical system as a contact transformation.

Lie ([1]) addressed the integration theory of the **Euler-Lagrange** equations (the canonical system, resp.) from an ostensibly-different viewpoint, namely, the theory of transformation groups. The limit formula of the calculus of variations will also imply those arguments directly, and in particular, mediate its connection with the previous results.

If we keep the abscissas x_2 (x_1, resp.) of both points of the point-pair constant in the extremal integral I [§ **1**, eq. (4)] then I will take the form of a function of four variables $I\,(y_2, z_2\,;\, y_1, z_1)$, or as we would like to write it:

$$(22)\qquad I\,(Y, Z\,;\, y, z)\,.$$

When we consider (3) and further set $f_{y'} = \Pi$, $f_{z'} = \mathrm{K}$, the limit formula from the calculus of variations will imply that:

$$(23)\qquad \delta I = \Pi\,\delta Y + \mathrm{K}\,\delta Z - \pi\,\delta y - \kappa\,\delta z\,.$$

From **Lie**'s theory, that fundamental relation says that the extremals of our variational problem can be regarded as the trajectories of a contact transformation. The function (22) characterizes that contact transformation, which we initially considered to be a function of two points on the trajectory. If we give only a point and the associated tangent in order to fix the extremal instead of two points then the characteristic function I will take the form of a function of y, z, π, and κ:

$$(22^*)\qquad I\,(Y, Z, y, z) = \overline{I}\,(\pi, \kappa, y, z)\,.$$

([1]) **S. Lie**, "Die Störungstheorie und die Berührungstransformationen," Christiania, Arch. for Math. og Naturw. **2** (1877), pp. 129.

With that argument, the defining equations of the associated infinitesimal contact transformations will be identical to the canonical system (5). In order to verify that directly, we let the two points (Y, Z) and (y, z) on the extremal move infinitely-close to each other. From (23), we will then have:

$$\delta I = d\,(\pi\,\delta y + \kappa\,\delta z)\,, \tag{24}$$

and δI will become:

$$\delta\left[\left(\pi\frac{dy}{dx}+\kappa\frac{dz}{dx}+H\,(\pi,\kappa,y,z,x)\right)dx\right]$$

in this case. We will then get:

$$\delta\left(\pi\frac{dy}{dx}+\kappa\frac{dz}{dx}+H\,(\pi,\kappa,y,z,x)\right)=\frac{d}{dx}(\pi\,\delta y+\kappa\,\delta z)$$

or

$$\frac{d\pi}{dx}\delta y+\frac{d\kappa}{dx}\delta z-\frac{dy}{dx}\delta\pi-\frac{dz}{dx}\delta\kappa=\delta H\,(\pi\,\delta y+\kappa\,\delta z)\,, \tag{25}$$

from which the canonical system (5) will emerge immediately.

The two functions that always appear next to each other in the theory of contact transformations, one of which characterizes the finite transformations, while the other characterizes the infinitesimal transformations, are I and H here. The former can then be defined by the latter when one puts the integral into the form (6) and determines it as the extremal integral.

5) *The transformation of the canonical system into itself. Poisson's theorem.*

Any point of the five-dimensional space (x, y, z, π, κ) is associated with a certain point y_0, z_0, π_0, κ_0 in an arbitrary hyperplane $x = x_0$ by means of the extremal of the system (5) that goes through it. One must now ask, in the spirit of **Lie**'s theory, whether there are one-parameter groups that leave the canonical system (5) invariant, i.e., permute the associated extremals amongst each other. (In order to do that, it would obviously suffice that they only permute the points at which the extremals meet the hyperplane $x = x_0$, because a transformation group that permutes those initial values amongst each other will, at the same time, permute the extremals amongst each other, since they are indeed determined by the initial values.) Let:

$$\begin{cases} \delta y=\mathfrak{a}\,(x,y,z,\pi,\kappa)\,\delta\alpha\,, & \delta z=\mathfrak{b}\,(x,y,z,\pi,\kappa)\,\delta\alpha\,, \\ \delta\pi=\mathfrak{m}\,(x,y,z,\pi,\kappa)\,\delta\alpha\,, & \delta\kappa=\mathfrak{n}\,(x,y,z,\pi,\kappa)\,\delta\alpha\,, \end{cases} \tag{26}$$

be the infinitesimal transformations of the group that act upon a point in space. From pp. 19, $\mathfrak{y} = \mathfrak{a}$, $\mathfrak{z} = \mathfrak{b}$, $\mathfrak{p} = \mathfrak{m}$, $\mathfrak{k} = \mathfrak{n}$ must then be a system of solutions of the **Jacobi** equations (21). Furthermore, the relative integral invariant:

$$\oint \pi\, dy + \kappa\, dz$$

must also remain a relative invariant under the transformation, i.e., by means of (26), one must have:

$$\delta(\pi\, dy + \kappa\, dz) = dg\ \delta\alpha\,, \tag{27}$$

in which g is an arbitrary function of x, y, z, π, κ. We will then have:

$$\mathfrak{m}\ dy - \mathfrak{a}\ d\pi + \mathfrak{n}\ dz - \mathfrak{b}\ d\kappa = d\,(g - \pi\,\mathfrak{a} - \kappa\,\mathfrak{b})\,. \tag{27*}$$

Hence, if we set:

$$g - \pi\,\mathfrak{a} - \kappa\,\mathfrak{b} = Q\,(x, y, z, \pi, \kappa) \tag{28}$$

then we will have:

$$\mathfrak{m} = \frac{\partial Q}{\partial y}, \qquad \mathfrak{n} = \frac{\partial Q}{\partial z}\ ; \qquad \mathfrak{a} = -\frac{\partial Q}{\partial \pi}, \qquad \mathfrak{b} = -\frac{\partial Q}{\partial \kappa}\,. \tag{29}$$

In order to accomplish the desired permutation of the extremals amongst each other, the transformation (26) must then have the form:

$$\left\{\begin{aligned} \frac{\delta\pi}{\delta\alpha} &= \frac{\partial Q}{\partial y}, \quad & \frac{\delta\kappa}{\delta\alpha} &= \frac{\partial Q}{\partial z}, \\ \frac{\delta y}{\delta\alpha} &= -\frac{\partial Q}{\partial \pi}, \quad & \frac{\delta z}{\delta\alpha} &= -\frac{\partial Q}{\partial \kappa}, \end{aligned}\right. \tag{26*}$$

in which the function Q is to be determined such that:

$$\mathfrak{p} = \frac{\partial Q}{\partial y}, \qquad \mathfrak{y} = -\frac{\partial Q}{\partial \pi}\ ; \qquad \mathfrak{k} = \frac{\partial Q}{\partial z}, \qquad \mathfrak{z} = -\frac{\partial Q}{\partial \kappa} \tag{30}$$

is a solution of the **Jacobi** equations (21). However, if:

$$\pi = \pi\,(x, a)\,, \quad y = y\,(x, a)\,, \quad \kappa = \kappa\,(x, a)\,, \quad z = z\,(x, a)$$

are any solutions of the canonical system that include the parameter a then:

$$\mathfrak{p} = \frac{\partial \pi}{\partial a}, \qquad \mathfrak{y} = \frac{\partial y}{\partial a} \; ; \qquad \mathfrak{k} = \frac{\partial \kappa}{\partial a}, \qquad \mathfrak{z} = \frac{\partial z}{\partial a}$$

will also be solutions of the **Jacobi** equations (21). From (16*), the relation must then exist for any system of solutions of (5) that includes one parameter *a* that:

$$\frac{\partial Q}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial Q}{\partial \pi}\frac{\partial \pi}{\partial a} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial a} + \frac{\partial Q}{\partial \kappa}\frac{\partial \kappa}{\partial a} = \text{const.},$$

i.e., one must have:

(31)
$$\frac{\partial Q}{\partial a} = \text{const.}$$

for all *x* as soon as one replaces *y*, *z*, *π*, *κ* with any solutions of the canonical system (5) that include the parameter *a*. However, it follows from this that *Q* itself must also remain constant along the integral curves of the system (5), so $Q(x, y, z, \pi, \kappa) = \text{const.}$ must be an integral of the canonical system. Conversely, when *Q* is an integral of (5), (30) will represent a solution of the **Jacobi** equations ([1]). Knowing an integral of the canonical system (5) and knowing an infinitesimal transformation of the system into itself are therefore equivalent to each other.

Finally, as a corollary to that, one gets **Poisson**'s theorem: If we have two integrals $Q_1(x, y, z, \pi, \kappa) = \text{const.}$ and $Q_2(x, y, z, \pi, \kappa) = \text{const.}$ then:

$$\mathfrak{p}_1 = \frac{\partial Q_1}{\partial y}, \qquad \mathfrak{y}_1 = -\frac{\partial Q_1}{\partial \pi} \; ; \quad \mathfrak{k}_1 = \frac{\partial Q_1}{\partial z}, \qquad \mathfrak{z}_1 = -\frac{\partial Q_1}{\partial \kappa},$$
$$\mathfrak{p}_2 = \frac{\partial Q_2}{\partial y}, \qquad \mathfrak{y}_2 = -\frac{\partial Q_2}{\partial \pi} \; ; \quad \mathfrak{k}_2 = \frac{\partial Q_2}{\partial z}, \qquad \mathfrak{z}_2 = -\frac{\partial Q_2}{\partial \kappa}$$

will be two systems of solutions to the **Jacobi** equations, and we will find the new integral:

(32)
$$\left(\frac{\partial Q_1}{\partial y}\frac{\partial Q_2}{\partial \pi} - \frac{\partial Q_1}{\partial \pi}\frac{\partial Q_2}{\partial y}\right) + \left(\frac{\partial Q_1}{\partial z}\frac{\partial Q_2}{\partial \kappa} - \frac{\partial Q_1}{\partial \kappa}\frac{\partial Q_2}{\partial z}\right) = \text{const.}$$

from them.

([1]) **H. Poincaré**, *Les méthodes Nouvelles de la mécanique celeste*, t. 1, pp. 168.

§ 3.

Integrating a given first-order partial differential equation by means of an extremal integral.

Hamilton's process that was described in the first section for integrating the **Euler-Lagrange** equations by reducing them to the integration of his two partial differential equations, which **Jacobi** then further developed, can generally be regarded as something that only complicates the problem. That is because in the theory of first-order partial differential equations, one prefers to think that, conversely, it is a great advance when the integration of the partial differential equations can be reduced to the integration of a system of ordinary differential equations. Now, **Jacobi** achieved precisely that goal of the theory of partial differential equations when he inverted **Hamilton**'s line of reasoning and succeeded in integrating a given first-order partial differential equation (which might not include the function itself):

$$\frac{\partial V}{\partial x} = H\left(x, y, z, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) \tag{1}$$

in such a way that it would be regarded as the **Hamilton-Jacobi** differential equation of a variational problem that is determined by it and demanded that this variational problem should be solved by a direct study of the **Euler-Lagrange** equations in the sense of § **2**.

If we would now like to expound upon that conception of things then we cannot couple it with the usual interpretation of an integral of equation (1) as a hypersurface (M_3) in four-dimensional (x, y, z, V)-space. Rather, it would seem preferable to interpret such a solution as a covering of the three-dimensional (x, y, z)-space.

According to **Lagrange**, in the general theory of first-order partial differential equations, one asks what the complete (general, resp.) integral would be. It is easy to choose extremal fields such that the value of the extremal integral (**Hilbert**'s independence integral, resp.) exhibits such a covering of space that corresponds to the complete, or also general, integral in the **Lagrange** sense.

With the function H (x, y, z, π, κ) that is given in the differential equation (1), we can next derive the integrand in the extremal integral $f(y', z', y, z, x)$ by setting:

$$y' = -\frac{\partial H}{\partial \pi}, \qquad z' = -\frac{\partial H}{\partial \kappa}, \tag{2}$$

solving those equations for π and κ, and then getting:

$$f(y', z', y, z, x) = -\pi\frac{\partial H}{\partial \pi} - \kappa\frac{\partial H}{\partial \kappa} + H \tag{3}$$

by substituting the calculated values. We can then find the variational problem of the **Euler-Lagrange** equations that belongs to the function (3). In what follows, we shall assume that its integral curves, viz., the extremals, are determined as in § **2**.

The value of the extremal integral that is extended from an arbitrary fixed point P_0 (x_0, y_0, z_0) to any point P (x, y, z) of the region considered:

$$V(x, y, z\,;\, x_0, y_0, z_0) = \int_{x_0, y_0, z_0}^{x, y, z} E\; f(y', z', y, z, x)\, dx \tag{4}$$

gives such a covering of space that must be referred to as a complete integral. That is because the constants x_0, y_0, z_0 can be determined in such a way that this integral includes a given "element" x_1, y_1, z_1, V_1, $\left(\frac{\partial V}{\partial x}\right)_1$, $\left(\frac{\partial V}{\partial y}\right)_1$, $\left(\frac{\partial V}{\partial z}\right)_1$ that satisfies the partial differential equation. (Moreover, the value of V_1 does not matter at all, since an integral of the differential equation (1) is determined only up to a constant.) Namely, if we lay an extremal through the point x_1, y_1, z_1 with a direction $(y')_1$, $(z')_1$ such that:

$$\left(\frac{\partial V}{\partial y}\right)_1 = (f_{y'})_1\,, \qquad \left(\frac{\partial V}{\partial z}\right)_1 = (f_{z'})_1$$

then we need only to choose the point P_0 along that extremal such that we will have:

$$\int_{x_0, y_0, z_0}^{x_1, y_1, z_1} E\; f(y', z', y, z, x)\, dx = V_1$$

in order to have fulfilled all conditions, since $\left(\frac{\partial V}{\partial y}\right)_1$ takes the correct value automatically as a result of the partial differential equation (1) ([1]).

Now in order to represent an arbitrarily-given complete integral $V(x, y, z, a, b)$ as a covering by means of the extremal integral, we next give fixed values to the constants a and b such that we will obtain a certain particular integral $V(x, y, z)$. As was just explained, we can construct an associated complete integral $V(x, y, z\,;\, x_0, y_0, z_0)$ for every element of that particular integral. The set of all those complete integrals will imply the particular integral from which started in such a way that x_0, y_0, z_0 will be certain functions of x, y, z :

$$x_0 = x_0(x, y, z)\,, \qquad y_0 = y_0(x, y, z)\,, \qquad z_0 = z_0(x, y, z)\,. \tag{5}$$

Now since that particular integral, as well as each of the complete integrals that are constructed in that way, satisfy the **Hamilton-Jacobi** equation (1), the functions (5) must satisfy the *total* linear differential equation:

([1]) Cf., on this, **J. Hadamard**, *Calcul des variations*, t. 1, pp. 160-163.

$$\frac{\partial V_0}{\partial x_0}\,dx_0 + \frac{\partial V_0}{\partial y_0}\,dy_0 + \frac{\partial V_0}{\partial z_0}\,dz_0 = 0\,,$$

or

(6) $$(f - y'\,f_{y'} - z'\,f_{z'})_0\,dx_0 + (f_{y'})_0\,dy_0 + (f_{z'})_0\,dx_0 = 0\,.$$

The points x_0, y_0, z_0 will then fill up a surface that intersects the family of extremals transversally; the family of extremals then define a general independence field. If we now let the parameters a and b vary arbitrarily in the complete integral V (x, y, z, a, b) then the surface that is determined by equation (6) will likewise vary with those parameters. We will then get a two-parameter family of initial surfaces, and therefore also a two-parameter family of independence fields, as we already introduced them in § **1**, and that covering will yield the most general complete integral.

With **Lagrange**, we can go from the complete integral to the general one in such a way that we select a one-parameter family from the two-parameter family of transversal surfaces and construct their envelope. In that way, we will get a certain new surface to which one of the transversal families of extremals belongs. The covering of space that is thereby determined corresponds to the general integral, so it will contain an arbitrary function that will come into play when we select a one-parameter family from the two-parameter family of transversal surfaces.

Therefore, the integration of the partial differential equation (1) is, in fact, reduced to the integration of a system of ordinary differential equations, namely, to the **Euler-Lagrange** equations:

(7) $$\begin{cases} \dfrac{d}{dx}(f_{y'}) - f_y = 0, \\ \dfrac{d}{dx}(f_{z'}) - f_z = 0, \end{cases}$$

which make it possible for one to know the family of extremals. The extremals are nothing but the so-called *characteristics* of the partial differential equation, which is a fact that also emerges formally and immediately when we replace equations (7) with the associated canonical system:

(8) $$\begin{cases} \dfrac{d\pi}{dx} = \dfrac{\partial H}{\partial y}, & \dfrac{dy}{dx} = -\dfrac{\partial H}{\partial \pi}, \\ \dfrac{d\kappa}{dx} = \dfrac{\partial H}{\partial z}, & \dfrac{dz}{dx} = -\dfrac{\partial H}{\partial \kappa}, \end{cases}$$

with which, we have achieved precisely the form of the equations for characteristics that **Cauchy** gave. All properties of the characteristics, or as one says more intuitively, the "characteristic strips," must also emerge clearly from our extremals, except that we must correspondingly replace our interpretation of the integral of (1) as the "tangent plane to the hypersurface" with the "gradient of the covering." Any integral curve of (7), along with its direction coefficients, will determine that gradient, since we have:

$$\frac{\partial V}{\partial y} = f_{y'}, \qquad \frac{\partial V}{\partial z} = f_{z'},$$

and $\partial V / \partial x$ is obtained from the given partial differential equation (1). If two different space-coverings have an element in common, i.e., they have the same value and the same gradient at some point in space, then they will have an element in common all along the entire characteristic that runs through that point, because the two associated independence fields must obviously have the entire extremal in common, since such a thing is established uniquely by its starting point and initial direction.

Finally, we shall address the determination of a two-parameter family of characteristics in such a way that the associated integral of (1) reduces to a given function $V = \varphi(y, z)$ for $x = x_0$. In our way of looking at things, that "**Cauchy** problem" takes the form of giving a covering of space that assumes the given value $V(x_0, y, z) = \varphi(y, z)$ in the plane $x = x_0$. In order to solve that, we must lay extremals in space that start from the individual points of the plane $x = x_0$ and whose directions y', z' are determined in such a way that we will have:

$$f_{y'} = \frac{\partial \varphi}{\partial y}, \qquad f_{z'} = \frac{\partial \varphi}{\partial z}. \tag{9}$$

We will then have:

$$V(x, y, z) = \varphi(y_0, z_0) + \int_{x_0, y_0, z_0}^{x, y, z} E\; f(y', z', y, z, x)\, dx \tag{10}$$

(in which y_0 and z_0 are functions of x, y, z), which is a function that will certainly reduce to $\varphi(y, z)$ for $x = x_0$. We must now show that the function $V(x, y, z)$ satisfies the partial differential equation (1). If we calculate its partial derivatives then when we consider the fact that we must substitute:

$$\frac{\partial x_0}{\partial y_0} = 0, \qquad \frac{\partial x_0}{\partial z_0} = 0, \qquad \text{as well as} \qquad (f_{y'})_0 = \frac{\partial \varphi}{\partial y_0}, \quad (f_{z'})_0 = \frac{\partial \varphi}{\partial z_0},$$

in those equations here, from equations (17*) in subsection 5) (pp. 9), we will have:

$$\left\{\begin{aligned}
\frac{\partial V}{\partial x} &= \frac{\partial \varphi}{\partial y_0}\frac{\partial y_0}{\partial x} + \frac{\partial \varphi}{\partial z_0}\frac{\partial z_0}{\partial x} + (f - y' f_{y'} - z' f_{z'}) - \left(\frac{\partial \varphi}{\partial y_0}\frac{\partial y_0}{\partial x} + \frac{\partial \varphi}{\partial z_0}\frac{\partial z_0}{\partial x}\right) = f - y' f_{y'} - z' f_{z'}, \\
\frac{\partial V}{\partial y} &= \frac{\partial \varphi}{\partial y_0}\frac{\partial y_0}{\partial y} + \frac{\partial \varphi}{\partial z_0}\frac{\partial z_0}{\partial y} + f_{y'} - \left(\frac{\partial \varphi}{\partial y_0}\frac{\partial y_0}{\partial y} + \frac{\partial \varphi}{\partial z_0}\frac{\partial z_0}{\partial y}\right) = f_{y'}, \\
\frac{\partial V}{\partial x} &= \frac{\partial \varphi}{\partial y_0}\frac{\partial y_0}{\partial z} + \frac{\partial \varphi}{\partial z_0}\frac{\partial z_0}{\partial z} + f_{z'} - \left(\frac{\partial \varphi}{\partial y_0}\frac{\partial y_0}{\partial z} + \frac{\partial \varphi}{\partial z_0}\frac{\partial z_0}{\partial z}\right) = f_{z'}.
\end{aligned}\right. \tag{11}$$

Since the very meaning of the function f in equation (3) implies that it is precisely the partial differential equation (1) that will arise upon eliminating y' and z' from those three equations, we will see that (10) is, in fact, the solution to the **Cauchy** problem. With that, we have simultaneously shown that the extremal field that is constructed from the initial conditions (9) is an independence field. **A. Mayer** ([1]) first constructed a general independence field by construction in that way, moreover.

([1]) **A. Mayer**, "Über den **Hilbert**schen Unabhangigkeitssatz," Math. Ann. **62** (1906), pp. 341, *et seq.*

PART TWO

The variational problem for the double integrals.

Chapter One.

The variational problem with one unknown function.

§ 1.

The variational problem and the equation between the partial functional derivatives of the extremal integral.

1) The variational problem and the Euler-Lagrange equation.

The basic variational problem here reads:

Determine the unknown function $z = z(x, y)$ in the double integral of a given function $f(z_x, z_y, z, x, y)$:

$$(1) \qquad I = \iint_{(S)} f(z_x, z_y, z, x, y)\, dx\, dy \qquad \left(z = z(x, y),\ z_x = \frac{\partial z}{\partial x},\ z_y = \frac{\partial z}{\partial y} \right)$$

such that the double integral will be an extremum. In that way, the double integral is thought to be extended over a given region S in (x, y)-plane with the (analytic, free of double points) boundary curve C. Let the values of the function z (viz., the *boundary values*) be initially given as fixed on that curve C as functions of the arc-length s :

$$(2) \qquad z = z(s) \qquad \text{on } C .$$

The calculus of variations then yields a first condition for the occurrence of an extremum in the form of demanding that the function $z(x, y)$ must satisfy the **Euler-Lagrange** equation:

$$(3) \qquad \frac{\partial}{\partial x}(f_{z_x}) + \frac{\partial}{\partial y}(f_{z_y}) - f_z = 0 \qquad \left(f_{z_x} = \frac{\partial f}{\partial z_x},\ f_{z_y} = \frac{\partial f}{\partial z_y},\ f_z = \frac{\partial f}{\partial z} \right)$$

We shall assume (although this has probably not been proved at this level of generality) that in general there is one and only one surface $z = z(x, y)$ that satisfies equation (3) and goes through the given space curve (2), and in that way contains no singular points in the interior of the region S.

Here, the calculus of variations also calls any surface that succeeds in satisfying equation (3) an extremal, such that we can formulate our assumption as follows: One and only one extremal goes through every space curve in the region of space considered.

2) The extremal integral as a line function. The line functions and their functional derivatives.

Conversely, we now imagine that a closed space curve L is given in a suitable region of space and that an extremal is laid through it. We calculate the value of the integral (1) when it is extended over the region over the extremal that is bounded by L (which will be suggested by the symbol E in the formulas). That "extremal integral" will have a well-defined value for every given space curve, so it is a *line function:*

$$(4) \qquad I\,[L\,|] = \int E \int f(z_x, z_y, z, x, y)\,dx\,dy\,.$$

Volterra (1), who introduced such line functions into analysis, considered the "simple" line function as a simplest special case of a line function. We will arrive at its definition in the following way: We imagine that two space curves L_1 and L_2 are given that have a curve segment l in common. If we suppress the segment l then a new closed space curve L_3 (which might also be analytic, etc.) will arise from L_1 and L_2 . Now, if a line function $S\,[L|]$ assumes the values $S\,[L_1|]$, $S\,[L_2|]$, $S\,[L_3|]$ for the three curves L_1 , L_2 , L_3 , resp., then **Volterra** called it "simple" when the relation:

$$(5) \qquad S\,[L_3|] = S\,[L_1|] + S\,[L_2|]$$

existed. The extremal integral $I\,[L|]$ is not a simple line function, in general, since the value of the extremal integral depends upon the extremal, and they are entirely independent of each other due to the arbitrariness in the curve segment l. Meanwhile, one can give an example of a simple line function. One imagines that three functions of position in space $A\,(x, y, z)$, $B\,(x, y, z)$, and $C\,(x, y, z)$ are given and that they satisfy the condition:

$$(6) \qquad \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0\,.$$

From **Gauss**'s theorem, when the surface integral:

$$(6^*) \qquad \iint A\,dy\,dz + B\,dz\,dx + C\,dx\,dy$$

is taken over an arbitrary integration surface, it will be independent of that integration surface, so it initially represents a line function. However, that line function is simple because due to the independence of the value of the integral on the integration surface, we can deform an arbitrary integration surface that is laid through L_3 in such a way that it goes through the curve L_1, as well as the curve L_2, without changing the value of the integral. We will then, in fact, have:

$$S\,[L_3|] = S\,[L_1|] + S\,[L_2|]\,.$$

Volterra had also proved that, conversely, any simple line function must be represented by means of three functions of position in the given way as an integral that is independent of the surface.

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(1) **V. Volterra**, "Sur une généralisation de la théorie des fonctions d'une variable imaginaire," Acta math. **12** (1889), 233-252.

Volterra adapted, above all, the concept of the derivative to line functions as a consistent expansion of the concept of the derivatives of an ordinary function of several variables. His line of reasoning was the following: We focus on a well-defined point on the space curve L and vary the space curve in a certain neighborhood of that point in such a way that we subject each of its points to a certain displacement δx that is parallel to the x-axis. The line function will once more have a well-defined value $S[L'|]$ for the space curve L' thus-varied. **Volterra** defined the expression:

(7) $$\frac{S[L'|]-S[L|]}{\int \delta x\, d\sigma},$$

in which the denominator is the area of the surface patch that is bounded by both L' and L, while σ is the arc-length along L. The boundary value that this expression tends towards when δx, as well as the length σ, become infinitely small in any way in the neighborhood of the point in question, at which δx was non-zero, shall be independent of the type of passage to the limit. **Volterra** then called it the "functional derivative with respect to the x-axis": S_x . He defined functional derivatives along the y-axis and the z-axis – viz., S_y (S_z, resp.) – analogously. If we interpret those three functional derivatives as components of a vector then every point on the space curve will be assigned a certain vector (S_x , S_y , S_z) by **Volterra**'s operation. If we vary the curve L by displacing each of its points by the line segment δx, δy, δz then, as **Volterra** showed, the variation of a line function S [L|] will be given by an integral over L in the following way:

(8) $$\delta S\,[L|] = \int_L (S_x\, \delta x + S_y\, \delta y + S_z\, \delta z)\, d\sigma\,.$$

For a displacement of the space curve L into itself, one must obviously have $\delta S = 0$, so it will then follow that:

(9) $$S_x\frac{dx}{d\sigma} + S_y\frac{dy}{d\sigma} + S_z\frac{dz}{d\sigma} = 0\,,$$

i.e., the vector S_x , S_y , S_z will be perpendicular to the curve L.

We can then replace the vector S_x , S_y , S_z with the vector product of the unit vector $\frac{dx}{d\sigma}$, $\frac{dy}{d\sigma}$, $\frac{dz}{d\sigma}$ and another vector S_{yz} , S_{zx} , S_{xy} , such that:

(10) $$\left\{\begin{aligned} S_x &= \frac{dz}{d\sigma}S_{zx} - \frac{dy}{d\sigma}S_{xy}\,, \\ S_y &= \frac{dx}{d\sigma}S_{xy} - \frac{dz}{d\sigma}S_{yz}\,, \\ S_z &= \frac{dy}{d\sigma}S_{yz} - \frac{dx}{d\sigma}S_{zx}\,. \end{aligned}\right.$$

Only the component of the vector S_{yz} , S_{zx} , S_{xy} , whose components we would like to call *functional derivatives with respect to the coordinate planes*, that is normal to the space curve is well-defined, while its tangential components are still arbitrary and can be established at will. Upon introducing that new vector, equation (8) will go to:

$$(11) \qquad \delta S\,[L|] = \int_L \left\{ S_{yz}\left(\frac{dy}{d\sigma}\delta z - \frac{dz}{d\sigma}\delta y\right) + S_{zx}\left(\frac{dz}{d\sigma}\delta x - \frac{dx}{d\sigma}\delta z\right) + S_{xy}\left(\frac{dx}{d\sigma}\delta y - \frac{dy}{d\sigma}\delta x\right)\right\} d\sigma\,.$$

There are many advantages to the fact that with the **Volterra** representation, which is distinguished by its symmetry, the components of the vector are not independent in the first representation, and they are not determined completely in the second. **Hadamard** ([1]), and in conjunction with him, **P. Lévy** ([2]), gave up the complete symmetry in order to avoid that flaw. They imagined that a space curve L was always given by its projection onto the (x, y)-plane, i.e., a plane curve C, and then assigned it the ordinates $z = z\,(s)$ that belonged to each of its points, in which s means the arc-length along C. A line function will then be a function of the plane curve C and the function $z\,(s)$:

$$(12) \qquad S\,[L|] = S^*\,[C, z\,(s)|]\,.$$

The appearance of the auxiliary condition (8) above will be avoided by always performing the variation of the plane curve C in the (x, y)-plane perpendicular to that curve, so the variation of the space curve L will then be characterized when one is given two displacement components $\delta n\,(s)$ and $\delta z\,(s)$. If the associated functional derivatives of $S\,[L|]$ are equal to S_n^* (S_z^*, resp.) then the expression for the variation of S^* will read:

$$(13) \qquad \delta S^* = \int_C (S_n^*\,\delta n + S_z^*\,\delta z)\,ds\,.$$

If compare formulas (11) and (13), while observing that:

$$(14) \qquad \begin{cases} \delta x = \delta n \dfrac{dx}{dn} = -\,\delta n \dfrac{dy}{ds}\,, \\ \delta y = \delta n \dfrac{dy}{dn} = \quad \delta n \dfrac{dx}{ds}\,, \\ d\sigma = ds \cdot \sqrt{1 + \left(\dfrac{dz}{ds}\right)^2}\,, \end{cases}$$

then we will find that:

([1]) **J. Hadamard**, "Mémoire sur le problème d'analyse rélatif à l'équilibre des plaques élastiques encastrées," Paris, Mém. sav. étrang. **23** (1908).

([2]) **P. Lévy**, "Sur les équations intégro-différentielles définissant des fonctions de lignes," Paris, Thesis, 1911; "Sur l'intégration des équations aux dérivées fonctionelles partielles," Palermo circ. mat. Rend. **37** (1914), 113-168.

$$S_n^* \,\delta n + S_z^* \,\delta z = S_{yz}\left(\frac{dy}{ds}\,\delta z - \frac{dz}{ds}\frac{dx}{ds}\,\delta n\right) + S_{zx}\left(-\frac{dz}{ds}\frac{dy}{ds}\,\delta n - \frac{dx}{ds}\,\delta z\right) + S_{xy}\cdot\delta n\,.$$

With that, the connection between the **Hadamard-Lévy** functional derivatives and the **Volterra** derivatives with respect to the coordinate places is given by the two relations:

$$\text{(15)} \qquad \left\{\begin{aligned} S_n^* &= S_{xy} - \frac{dz}{ds}\left(S_{yz}\frac{dx}{ds} + S_{zx}\frac{dy}{ds}\right), \\ S_z^* &= S_{yz}\frac{dy}{ds} - S_{zx}\frac{dx}{ds}\,. \end{aligned}\right.$$

3) The limit formula in the calculus of variations and the functional derivatives of the extremal integral.

In order to now arrive at the functional derivatives of the extremal integral that was introduced in equation (4), we can start from the "limit formula" for the double integral. **Gauss** ([1]) first exhibited the principle for the derivative of such a limit formula in an example, and **Poisson** ([2]) then developed it in general. We shall derive it in the following way:

We imagine that a certain space curve has been given and lay an extremal through it. Along with that, we consider a space curve that is close to the given one, once more lay an extremal through it (for the sake of simplicity), and then imagine that the points of the two extremal surface patches are in one-to-one correspondence with each other in an arbitrary way, but that the points of the boundary of one correspond to points of the boundary of the other. The connecting line segment between two corresponding points has the components δx, δy, δz , such that a point x, y, z on the starting extremal is associated with the point $x + \delta x$, $y + \delta y$, $z + \delta z$ on the neighboring extremal. In order to exhibit the limit formula, it will then be convenient to consider x, y, and therefore z, as well, to be functions of two parameters u and v. With that, δx, δy, δz, and therefore the coordinates of the neighboring extremal surface patch will take the form of functions of u and v. In that way, depending upon the way that the relation between both surface patches is presented, u and v will run through the same domain of values both times depending upon whether one is dealing with the starting extremal or the neighboring one.

In order to introduce the parameters u and v into the integral (1), with **Hadamard** ([3]), we now form the two-rowed determinants:

$$\text{(16)} \qquad \lambda_{(yz)} = \begin{vmatrix} \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}, \qquad \lambda_{(zx)} = \begin{vmatrix} \dfrac{\partial z}{\partial u} & \dfrac{\partial x}{\partial u} \\ \dfrac{\partial z}{\partial v} & \dfrac{\partial x}{\partial v} \end{vmatrix}, \qquad \lambda_{(xy)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix},$$

([1]) **C. F. Gauss**, "Principia generalia theoriae figurae fluidorum in statu aequilibrii," (1830), *Werke* **5**, pp. 60

([2]) **S. D. Poisson**, Paris, Mém. de l'acad. royale d. sc. **12** (1832), pp. 290.

([3]) Cf., **M. Fréchet**, *loc. cit.* (on pp. 1 of this treatise), p. 188.

so we will have:

(17) $$z_x = \frac{\lambda_{(zy)}}{\lambda_{(xy)}}, \qquad z_y = \frac{\lambda_{(xz)}}{\lambda_{(xy)}},$$

and therefore:

(18) $$f(z_x, z_y, z, x, y) = f\left(\frac{\lambda_{(zy)}}{\lambda_{(xy)}}, \frac{\lambda_{(xz)}}{\lambda_{(xy)}}, z, x, y\right) = \frac{1}{\lambda_{(xy)}}(\lambda_{(yz)}, \lambda_{(xz)}, \lambda_{(xy)}, z, x, y),$$

where F is homogeneous of degree one in the three determinants. The integral (1) will then take the form:

(19) $$I = \iint F(\lambda_{(zy)}, \lambda_{(xz)}, \lambda_{(xy)}, z, x, y)\, du\, dv.$$

If we would now like to calculate the variation of this integral (19) under the transition from an initial extremal to a neighboring one then that will greatly simply the integral (1) by the fact that the domain of integration of the u, v will not change under that variation. We then get:

(20) $$\delta I = \iint \{F_{\lambda_{(zy)}}\delta\lambda_{(xy)} + F_{\lambda_{(xz)}}\delta\lambda_{(xz)} + F_{\lambda_{(xy)}}\delta\lambda_{(xy)} + F_z\,\delta z + F_x\,\delta x + F_y\,\delta y\}\, du\, dv.$$

If we partially integrate the first three terms in that and consider the fact that the double integral drops out of the resulting expression (since the initial surface was an extremal) then the relation (20) will go to:

(21) $$\delta I = \int_L \left\{F_{\lambda_{(zy)}}\left(\delta z\,\frac{dy}{d\sigma} - \delta y\,\frac{dz}{d\sigma}\right) + F_{\lambda_{(xz)}}\left(\delta x\,\frac{dz}{d\sigma} - \delta z\,\frac{dx}{d\sigma}\right) + F_{\lambda_{(xy)}}\left(\delta x\,\frac{dy}{d\sigma} - \delta y\,\frac{dx}{d\sigma}\right)\right\} d\sigma,$$

in which the integral over the given boundary L extends over the piece of the extremal that we started from. Finally, if we would like to once more return to x and y as independent variables then we would first get:

(22) $$f_{z_x} = F_{\lambda_{(zy)}}, \qquad f_{z_y} = F_{\lambda_{(xz)}}$$

by differentiating (18) with respect to $\lambda_{(zy)}$ [$\lambda_{(xz)}$, resp.]. Furthermore, from the homogeneity property of F, one will have:

(23) $$F_{\lambda_{(xy)}} = \frac{F}{\lambda_{(xy)}} - F_{\lambda_{(xz)}}\frac{\lambda_{(xz)}}{\lambda_{(xy)}} - F_{\lambda_{(yz)}}\frac{\lambda_{(yz)}}{\lambda_{(xy)}} = f - z_x\, f_{z_x} - z_y\, f_{z_y}.$$

(21) will then go to the "boundary formula":

$$(24)\quad \delta I = \int_L \left\{ (f - z_x\, f_{z_x} - z_y\, f_{z_y}) \left(\delta x \frac{dy}{d\sigma} - \delta y \frac{dx}{d\sigma} \right) + f_{z_x} \left(\delta z \frac{dy}{d\sigma} - \delta y \frac{dz}{d\sigma} \right) + f_{z_y} \left(\delta x \frac{dz}{d\sigma} - \delta z \frac{dx}{d\sigma} \right) \right\} d\sigma \,.$$

Upon comparing that with formula (11), we can infer from that boundary formula that the **Volterra** derivatives with respect to the coordinate planes are:

$$(25)\qquad I_{yz} = f_{z_x}, \qquad I_{zx} = f_{z_y}, \qquad I_{xy} = z_x\, f_{z_x} + z_y\, f_{z_y} - f \,,$$

for the extremal integral $I\,[L]$, and in that way we can impose the indeterminacy that appears (cf., pp. 20) in a certain way. We will soon return to its formal expression.

Moreover, that convention comes about in such a way that the functional derivatives of the **Hilbert** independent integral, which represents a simple line function, will become merely functions of position (cf., pp. 51).

From the relations (15), the (**Hadamard-Lévy**) derivatives of the extremal in the form $I^*[C, z(s)]$ that is analogous to (12) will read:

$$(26)\qquad \begin{cases} I_n^* = z_x\, f_{z_x} + z_y\, f_{z_y} - f - \dfrac{dz}{ds}\left(f_{z_x} \dfrac{dx}{ds} + f_{z_y} \dfrac{dy}{ds} \right), \\[2mm] I_n^* = f_{z_x} \dfrac{dy}{ds} - f_{z_y} \dfrac{dx}{ds}. \end{cases}$$

One sees that here it is convenient to replace the coordinates x and y with the natural coordinates along the curve C. Namely, if we introduce the natural derivatives:

$$(27)\qquad \begin{cases} \dfrac{dz}{ds} = \quad z_x \dfrac{dx}{ds} + z_y \dfrac{dy}{ds}, \\[2mm] \dfrac{dz}{dn} = - z_x \dfrac{dy}{ds} + z_y \dfrac{dx}{ds} \end{cases}$$

into f in place of z_x and z_y, by which, it might go to $\bar{f}\,(z_s, z_n, z, x, y, x_s)$ ([1]), then it will follow immediately that:

$$(28)\qquad \begin{cases} \dfrac{\delta \bar{f}}{\delta z_s} = \quad f_{z_x} \dfrac{dx}{ds} + f_{z_y} \dfrac{dy}{ds}, \\[2mm] \dfrac{\delta \bar{f}}{\delta z_n} = - f_{z_x} \dfrac{dy}{ds} + f_{z_y} \dfrac{dx}{ds}. \end{cases}$$

([1]) x, y, x_s determine the line element of the curve C along which the natural derivatives of z are constructed. That line element is also determined by the curve C and the arc-length s that belongs to the point in question. However, the notation $\bar{f}$ (z_s, z_n, z, C, s) can give rise to the misunderstanding that $\bar{f}$ is regarded as a line function.

Since one has:

$$z_x\, f_{z_x} + z_y\, f_{z_y} = z_s\, \bar{f}_{z_s} + z_n\, \bar{f}_{z_n}, \tag{29}$$

moreover, the relations (26) will take the form:

$$\begin{cases} I_n^* = \quad z_n\, \bar{f}_{z_n} - \bar{f}, \\ I_z^* = -\, \bar{f}_{z_n}, \end{cases} \tag{30}$$

and the variation of the extremal integral itself will ultimately be ([1]):

$$\delta I^*[C, z(s)] = \int_C [(z_n\, \bar{f}_{z_n} - \bar{f})\,\delta n - \bar{f}_{z_n}\,\delta z]\, ds\ . \tag{31}$$

The disadvantage of the derivatives (25) in comparison to the derivatives (30) that they are not mutually independent emerges clearly here. That is because if the space curve L [C, z (s)] that determines the line function is given then we will likewise know that:

$$\frac{dz}{ds} = z_x \frac{dx}{ds} + z_y \frac{dy}{ds}\ ,$$

and therefore, a relation between the partial derivatives of the extremals that go through L.

4) The equation for the partial functional derivatives of the extremal integral.

We can infer from equations (25) [(30), resp.] that the integration of the **Euler-Lagrange** equations for given boundary values will be achieved, in principle, when the extremal integral $I[L|]$ is known as a function of the boundary curve L. That is because we can then construct the functional derivatives of I [L|] for each point on the boundary curve, then calculate the direction cosines of the tangent plane to the extremal surface that goes through L for each point of the boundary curve, and thus construct the extremal by a method that goes back to **Cauchy** (by a power series). We know *a priori* that this must imply a surface that is free of singularities in the interior of the region that is bounded by the curve L.

In order to derive some advantage from that argument, it is necessary for us to find the extremal integral I [L|] by a direct determination, which does not assume any knowledge of the extremals that we indeed required in order to be able to define I [L|]. Such a direct approach can be given by a generalization of **Hamilton**'s train of thought on pp. 5. If we solve the first two equations (25):

([1]) The signs deviate from the corresponding formulas of the first section by the fact that the direction of the *interior* normal is chosen to be positive, while previously dx was counted as positive when it pointed *outward* from the integration interval (cf., pp. 5).

$$f_{z_x} = I_{yz}\,, \qquad f_{z_y} = I_{zx}$$

for z_x and z_y and introduce the function:

(32) $$H\,(z, y, y, I_{yz}, I_{zx}) = z_x\, f_{z_x} + z_y\, f_{z_y} - f$$

by substituting the calculated values of z_x and z_y then from the third of equations (25), that will imply the equation:

(33) $$I_{xy} = H\,(z, x, y, I_{yz}, I_{zx})$$

between the partial functional derivatives of the extremal integral $I\,[L|]$.

It follows immediately from:

$$H\,(z, x, y, I_{yz}, I_{zx}) = z_x \cdot \pi + z_x \cdot \kappa - f\,(z_x, z_y, z, x, y)$$

that

$$z_x = \frac{\partial H}{\partial \pi}\,, \qquad z_y = \frac{\partial H}{\partial \kappa}\,.$$

For the functional derivatives, we will then get the further condition equations for each point of L:

(33*) $$\frac{dz}{d\sigma} = \frac{\partial H}{\partial \pi}\cdot\frac{dx}{d\sigma} + \frac{\partial H}{\partial \kappa}\cdot\frac{dy}{d\sigma}\,,$$

in which one substitutes $\pi = I_{yz}$, $\kappa = I_{zx}$. That is the analytical statement of the condition that we subjected the **Volterra** functional derivatives to in (25) above in order to eliminate the indeterminacy that is present in them ([1]).

On the other hand, if we take the **Hadamard-Lévy** approach and consider that the derivative dz / ds is known along the curve $L = [C, z\,(s)]$, as well as $z\,(s)$, then we will need only to eliminate dz / dn from equations (30) in order to obtain a partial functional differential equation for the extremal integral. If we then calculate z_n from the equation:

$$-\,\bar{f}_{z_n} = I_z^*\,,$$

and if we introduce the function:

(34) $$K\,(I_z^*, z_s, z, x, y, x_s) = z_n\,\bar{f}_{z_n} - \bar{f}\,,$$

while substituting the calculated values, then, from the first equation in (30), the partial functional differential equation will now read:

([1]) The basic meaning of that condition is discussed in **M. Fréchet**, *loc. cit.* (pp. 1), pp. 194.

(35) $$I_n^* = K(I_z^*, z_s, z, x, y, x_s) \;.$$

In order to now investigate the question of the extent to which the extremal integral is determined by equations (33) [(35, resp.], we would like to do something different from the determination of the extremal integral I [L|] by the boundary curve on pp. 30. Somewhat vaguely, one cares to say that the general integral of a second-order partial differential equation like (3) depends upon two arbitrary functions ([1]). In order to allow two such functions to enter into the definition of the extremal integral (4), instead of doing what we have done up to now, we would like to imagine that a space curve L is given, along with its two L_1 and L_2, which we would like to assume do not intersection [and similarly for their projections onto the (x, y)-plane]. From the theorems that were assumed on pp. 29 on the solubility of the boundary-value problem for the **Euler-Lagrange** equation, one and only one extremal is determined by those two space curves L_1 and L_2 that is free of singularities ([2]) in the interior of the region that is bounded by the two curves L_1 and L_2 . The extremal integral would then seem to be a function of the two space curves L_1 and L_2 , which collectively define the boundary line L of the doubly-connected region. If will then fulfill equation (33) [equation (35), resp.], along with its functional derivatives, for each point of both boundary curves.

It will become immediately clear from this that not just any line function that satisfies equation (33) [(35), resp.] can play the role of the extremal integral in the solution of the boundary-value problem for the **Euler-Lagrange** equation (3) (in the older sense of pp. 29) for a curve L. That is because if we imagine that a fixed space curve L_0 is given then an extremal will likewise be determined by L_0 and L, and the associated value of the extremal integral, which also appears as a line function of L, along with its functional derivatives, will fulfill equation (33) [(35), resp.]. However, the tangent planes along L to that second extremal have nothing at all to do with the tangent planes to the extremal that solved the boundary-value problem for the curve L in the older sense, namely, in such a way that the region that is enclosed by L is simply connected or singularity-free. The complication will be that of selecting the extremal integral precisely from the set of solutions to the partial functional differential equation.

We can get around that complication in a way that is analogous to what **Jacobi** did in the problem of the simple integral by going from the two **Hamilton** equations to the **Hamilton-Jacobi** equation. We would like to fix a certain curve L_0 in the region of space in question and think of all closed space curves L in the region as being connected to L_0 by an extremal. The value of the extremal integral:

(36) $$I = \int_{L_0}^{L} E \int f(z_x, z_y, z, x, y)\, dx\, dy \,,$$

which extends from L_0 to L, associates each space curve L with a certain numerical value. It thus defines a line function. It satisfies equation (33) [(35), resp.], and indeed when we regard the initial

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([1]) Cf., **É. Goursat**, *Leçons sur l'integration des équations aux dérivées partielles du second ordre,* t. 1, Paris (1896), pp. 31.

([2]) Cf., an example in **É. Picard**, *Traité d'analyse*, t.2., 2nd ed., Paris (1905), pp. 86.

curve L_0 as arbitrary, it will represent a complete integral of that equation, which agrees with the definition that was given by **P. Lévy** ([1]). Knowing that complete integral will lead immediately to the integration of the **Euler-Lagrange** equation (3), because if we have constructed the derivative $I_{z_0}^*$ along $L_0 = [C_0, z_0 (s_0)]$ then we will know the normal derivative dz / dn for the extremal along L_0 . If we set:

$$(37) \qquad I_{z_0}^* = \zeta(s_0) ,$$

in which ζ is an arbitrary function, then we will have found the general extremals of the **Euler-Lagrange** equation in so doing. Of course, the solution to the boundary-value problem for two given curves $[C_0, z_0 (s_0)]$ and $[C, z (s)]$ will demand that $\zeta(s_0)$ is determined along C_0 such that the extremal goes through $L = [C, z (s)]$.

5) The "function field" of the extremals and the partial functional differential equation.

The tool that we just employed in order to simplify the integration of the partial functional differential equation was that of selecting those members of the set of all extremals of the given variational problem that depend upon *two* arbitrary functions that go through a fixed curve L_0, which defines a set that depends upon only *one* arbitrary function. However, we can also imagine a set of extremals that depend upon an arbitrary function in a more general way.

We bound a certain region in space in such a way that we let x and y vary in an annular surface in the (x, y)-plane that is bounded by two closed curves C_0' and C_1', while the inequality $z_1 < z < z_0$ might exist for z. In that region, we would like to imagine a set of extremals that are given such that one and only one extremal goes through any closed curve L that lies in the region and does not contract to a point. That property is analogous to the one that serves as the definition of an extremal field for a family of extremals that depend upon only one parameter. Here, we would like to call a set of extremals that possess the required property with the similar name of "function field," which is a name that should express the idea that one curve is necessary for one to characterize the individual extremals in the set.

We can define a line function on such a function field with the help of the value of the extremal integral: We need only to imagine that an initial curve $L_0 = [C_0, z_0 (s_0)]$ is given along each extremal in the function field along which the extremal integral is to be extended. The two determining pieces C_0 and $z_0 (s_0)$ that belong to L_0 are constrained to each other, since L_0 should always lie on the extremal that is determined by L, such that the variation δz_0 of z_0 is determined by the variation δn_0 of C_0 . In so doing, it will always be assumed that L_0 varies continuously when L moves continuously in the function field in any way.

We would now like to examine when the line function thus-defined:

$$(38) \qquad W[L|] = \int_{L_0}^{L} E \int f(z_x, z_y, z, x, y)\, dx\, dy$$

([1]) **P. Lévy**, *loc. cit.* (pp. 1 ([4])), pp. 156, *et seq.*, as well as § **3** of this chapter.

satisfies the partial functional differential equation (35). In order to determine the functional derivatives of W, we would like to apply the boundary formula from the variational calculus (31). In so doing, we should observe that the boundary of the surface patch on the extremal consists of the two curves L_0 and L. In the integration in formula (31), which extends over the entire boundary, the curve C [C_0, resp.] that belongs to L and L_0 is traversed in the opposite sense. If we would prefer that C_0 is traversed in the same sense as C then the sign of the associated integral must be changed, and will then get:

(39) $$\delta W = \int_C \{(z_n\,\bar f_{z_n} - \bar f)\,\delta n - \bar f_{z_n}\,\delta z\}\,ds - \int_{C_0} \{(z_n\,\bar f_{z_n} - \bar f)\,\delta n_0 - \bar f_{z_n}\,\delta z_0\}\,ds_0 \;.$$

In that way, δz_0 , and as a result, δn_0, as well, will be determined when one is given δz and δn along L, since [C_0, z_0 (s_0)] is indeed a line function of L. The variations δz_0 and δn_0 will possess the following form:

(40) $$\begin{cases} \delta z_0(s_0) = \int\limits_C \{\delta z_0'(s_0, s)\,\delta n + \delta z_0''(s_0, s)\,\delta z\}\,ds\,, \\ \delta n_0(s_0) = \int\limits_C \{\delta n_0'(s_0, s)\,\delta n + \delta n_0''(s_0, s)\,\delta z\}\,ds\,. \end{cases}$$

When we introduce those expressions into (39), that will give the values:

(41) $$\begin{cases} W_n^* = z_n\,\bar f_{z_n} - \bar f - \int\limits_{C_0} \{(z_n\,\bar f_{z_n} - \bar f)\,\delta n_0' - \bar f_{z_n}\,\delta z_0'\}\,ds_0\,, \\ W_z^* = -\bar f_{z_n} \qquad - \int\limits_{C_0} \{(z_n\,\bar f_{z_n} - \bar f)\,\delta n_0'' - \bar f_{z_n}\,\delta n_0''\}\,ds_0 \end{cases}$$

to the functional derivatives of $W\,[L|] = W^*[C, z\,(s)]$.

One can see from this that the value of the extremal integral (38) will yield a solution to equation (35), and thus to equation (33), as well, only when the integral that appears in (41) vanishes for all variations δn_0, δz_0 . Those associated variations must always fulfill the condition:

(42) $$(z_n\,\bar f_{z_n} - \bar f)\Big|^{C_0} \delta n_0 - \bar f_{z_n}\Big|^{C_0} \delta z_0 = 0$$

then. If we revert back to the coordinates x and y in that relation by means of equations (14) and (27), (28) then it will take the form:

(42*) $$\begin{aligned} &\frac{dy}{ds_0}\Big[(f - z_x\,f_{z_x})^{L_0}\,\delta x_0 - (z_y\,f_{z_x})^{L_0}\,\delta y_0 + (f_{z_x})^{L_0}\,\delta z_0\Big] \\ -&\frac{dx}{ds_0}\Big[-(z_x\,f_{z_y})^{L_0}\,\delta x_0 + (f - z_y\,f_{z_y})^{L_0}\,\delta y_0 + (f_{z_y})^{L_0}\,\delta z_0\Big] \;= 0\,, \end{aligned}$$

and since the curve L_0 can be taken arbitrarily along the extremal in question, that will imply the following two equations:

$$(43)\qquad \begin{cases} (f - z_x\, f_{z_x})^{L_0}\, \delta x_0 \qquad - (z_y\, f_{z_x})^{L_0}\, \delta y_0 + (f_{z_x})^{L_0}\, \delta z_0 = 0, \\ \quad -(z_x\, f_{z_y})^{L_0}\, \delta x_0 + (f - z_y\, f_{z_y})^{L_0}\, \delta y_0 + (f_{z_y})^{L_0}\, \delta z_0 = 0. \end{cases}$$

If we solve that system then that will imply the following condition for the components of the displacement along L_0 :

$$(43^*)\qquad \delta x_0 : \delta y_0 : \delta z_0 = (f_{z_x})^{L_0} : (f_{z_y})^{L_0} : (z_x\, f_{z_x} + z_y\, f_{z_y} - f)^{L_0}.$$

If a parametric field (so a one-parameter family of surfaces) were selected from the function field then the partial derivatives of the extremals z_x, z_y in those equations (43^*) would be functions of the position $p\,(x, y, z)$, $q\,(x, y, z)$ in the region of space considered. The equations would then determine a two-parameter family of curves that cuts through the family of surfaces and fills up the region of space simply and without gaps, just like the one-parameter family of field extremals. We would like to call the curves of that two-parameter family the "transversals" of the field.

Just as a family of transversals can be determined for each parametric field from equations (43^*), every such field is also an "independence field," i.e., the **Hilbert** independence theorem will be true for it. The **Hilbert** integral:

$$I^* = \iint \{ f(p, q, z, x, y) + (z_x - p)\, f_p + (z_y - q)\, f_q \}\, dx\, dy$$

is independent of the integration surface, so it represents a line function (like the value of the extremal integral), and is, in particular, a simple line function. Its functional derivatives (according to **Volterra**):

$$I^*_{yz} = f_p\,, \qquad I^*_{zx} = f_q\,, \qquad I^*_{xy} = p\, f_p + q\, f_q - f\,,$$

which are merely functions of position, satisfy the partial functional differential equation (according to **Volterra**) (33). The independent integral then represents a solution to that differential equation.

If we consider a parametric field from which we arrive at the independent integral to belong to a family of fields then the slope functions p and q will depend upon the parameters of the family (e.g., the parameter a), in addition to x, y, z. Therefore, the independent integral itself is also a line function that depends upon the parameter a. Differentiating I^* with respect to a will yield the relation:

$$\frac{\partial I^*}{\partial a} = \iint \left\{ (z_x - p)\, \frac{\partial f_p}{\partial a} + (z_y - q)\, \frac{\partial f_q}{\partial a} \right\} dx\, dy\,,$$

from which, it will emerge that:

$$\frac{\partial I^*}{\partial a} = \text{const.} = b$$

along the individual extremals of the field.

That theorem is identical to the generalization of **Jacobi**'s theorem for simple integrals that given by **M. Fréchet** (*loc. cit.*, pp. 196, *et seq.*). However, that formal adaptation is of no help in the solution to the boundary-value problem for the **Euler-Lagrange** equation (3). (Confer no. **6** on that).

6) *The function fields and the boundary-value problem for the Euler-Lagrange equation.*

One can easily construct a function field that satisfies the condition (42). In order to do that, we need only to imagine that a surface is given over a certain doubly-connected region in the (x, y)-plane. A certain curve L_0 on the surface belongs to each closed curve C_0 in the given region (which cannot be contracted to a point), and dz / ds and $\delta z / \delta n$, which indeed refer to the surface, have well-defined values at each point of L_0 . The normal derivative z_n for an extremal can then be calculated for each point of L_0 from equation (42), so the extremal itself can then be constructed. The set of all extremals that are obtained in that way does, in fact, represent a function field, and the value of the extremal integral that belongs to each space curve L will define an integral of the partial functional differential equation.

Naturally, a solution to the boundary-value problem for the **Euler-Lagrange** equation cannot be achieved for a fixed function field, since *only one* arbitrary function is available in it. In order to address the boundary-value problem, we must imagine that a set of function fields is given that depends upon yet another arbitrary function. We can perhaps imagine that, instead of the surface that we just used, we have a set of such surfaces, whose individual members are always characterized when we are given a certain function $\alpha(t)$ of the parameter t, where the parameter t might vary within a certain interval, say from 0 to 1. For each choice of the function $\alpha(t)$ in the interval $0 \leq t \leq 1$. The value of the extremal integral is then a functional that depends upon the function $\alpha(t)$, in addition to the curve L :

$$I = I[L, \alpha(\underset{0}{\overset{1}{t}})|] . \tag{44}$$

If we vary the function $\alpha(t)$ then the initial curve L_0 of the individual extremal in the function field will go to a new curve L_0'. In that way, the displacements of its points will be given by:

$$\delta n_0(s_0) = \int_0^1 \delta n_0^*(s_0, t)\,\delta\alpha(t) , \qquad \delta z_0(s_0) = \int_0^1 \delta z_0^*(s_0, t)\,\delta\alpha(t) .$$

As a result, from the boundary formula for the calculus of variation, when we fix the curve L, the functional derivative of I with respect to $\alpha(t)$ will be given by:

$$I_\alpha = \int_{C_0} \{(z_n \bar{f}_{z_n} - \bar{f})\,\delta n_0^* - \bar{f}_{z_n}\,\delta z_0^*\}\,ds_0 .$$

Therefore, that derivative will have the same value for all curves L that lie on the same extremals of the function field, so it will be a function of only t for the individual extremals I_α :

(45) $$I_\alpha\,[L, \alpha\,(\overset{1}{\underset{0}{t}})\,|\,;\,t] = c\,(t)\,.$$

In order to now solve the boundary-value problem for the **Euler-Lagrange** (3) for two given curves L_1 and L_2, we first imagine that the values of the extremal integral that belong to the two curves have been established for fixed $\alpha\,(t)$: viz., $I_1 = I\,[L_1, \alpha\,(t)\,|\,]$ and $I_2 = I\,[L_2, \alpha\,(t)\,|\,]$. For the functional derivatives with respect to α for the same parameter t, we will get from (45) that:

$$(I_1)_\alpha = c_1\,(t)\,, \qquad\qquad (I_2)_\alpha = c_2\,(t)\,.$$

Should both curves lie on one and the same extremal, then one would need to have $c_1\,(t) = c_2\,(t)$, so:

$$(I_1)_\alpha - (I_2)_\alpha = 0\,,$$

from which the function $\alpha\,(t)$ is determined. The boundary-value problem is then solved by that, because when we compute the functional derivatives of the extremal integral that are defined with the calculated function $\alpha\,(t)$ along L_1 or L_2, we will get an equation for the normal derivative of the desired extremal.

§ 2.

Integration theory of the Euler-Lagrange equation and the associated canonical system.

The **Euler-Lagrange** equation:

(1) $$\frac{\partial}{\partial x}(f_{z_x}) + \frac{\partial}{\partial y}(f_{z_y}) - f_z = 0\,,$$

which arises from the variational problem:

(2) $$I = \iint f\,(z_x, z_y, z, x, y)\,dx\,dy = \text{extrem.},$$

is a second-order partial differential equation that is distinguished by the fact that it includes the second derivatives of the unknown function $z\,(x, y)$ only linearly.

1) The associated first canonical system.

In the previous section, we first established the arbitrary functions that appear in the general solution of (1) by prescribing the extremals though which a given space curve went and demanded that the surface patch on the extremal that was bounded by that space curve was free of singularities. We then generalized the simplest form of the boundary-value problem by giving two space curves and demanded that an extremal would be determined in such a way that it went through both space curves and had no singular points in the interior of the doubly-connected region that was bounded by both space curves.

By contrast, in the spirit of the general theory of integration for partial differential equations, one seeks to establish a certain integral surface by prescribing only that it should go through a given space curve, but it should possess a given plane as the tangent plane at each point of that space curve, so it should contact a given developable surface along the space curve. Analytically, that geometrically-formulated condition means that one knows the value of $z = z(s_0)$ (from which, we will simultaneously know the derivative dz / ds_0 along C_0) along a given closed curve C_0 in the (x, y)-plane, as well as the normal derivative dz / dn of the extremal. **Cauchy** showed that an integral surface is generally determined uniquely by those prescribed initial values and has further worked out how it could actually be exhibited by a power series as an extension of the process that one refers to as the "*calcul des limites*" for ordinary differential equations ([1]).

One might then imagine that in order for one to work through that problem, it would be convenient to resolve the second-order partial differential equation into a system of first-order partial differential equations in a manner that is analogous to how one replaced the **Euler-Lagrange** equations with the canonical system in § **2** of the previous chapter. If we would like to formally adapt the line of reasoning there to the present problem ([2]) then we would have to apply the **Legendre** transformation to the present problem, and therefore introduce new variables π and κ by the relations:

$$\pi = f_{z_x} , \qquad \kappa = f_{z_y} . \tag{3}$$

If one calculates z_x and z_y from those two equations:

$$z_x = p(\pi, \kappa, z, x, y) , \quad z_y = q(\pi, \kappa, z, x, y) \tag{3*}$$

and introduces the new function:

$$H(\pi, \kappa, z, x, y) = p\, f_p + q\, f_q - f , \tag{4}$$

which will coincide with equation (32) of § **1** when one substitutes those values, then one will have the relations:

$$\frac{\partial H}{\partial \pi} = p , \qquad \frac{\partial H}{\partial \kappa} = q , \qquad \frac{\partial H}{\partial z} = -\frac{\partial f}{\partial z} ,$$

and the **Euler-Lagrange** equation (1) will then go to the system:

([1]) **É. Goursat**, *loc. cit.* (pp. 38), t. 1, pp. 24-28.

([2]) **V. Volterra**, "Sulla equazioni differenziali che provengono da questioni di calcolo delle variazioni," Roma, Acc. Lincei Rend. (4) 6^1 (1890), pp. 43.

(5) $$\begin{cases} z_x = \dfrac{\partial H}{\partial \pi}, \quad z_y = \dfrac{\partial H}{\partial \kappa}, \\ \pi_x + \kappa_y = -\dfrac{\partial H}{\partial z}, \end{cases}$$

which we (with **Volterra**) would like to call the *first canonical system*.

That is equivalent to the **Euler-Lagrange** equation (1), as one easily convinces oneself. However, in the question of the existence of an integral and fixing the arbitrariness that is inherent to it, which one can resolve with the help of the method that was given by **Cauchy** and **S. von Kowalewski**, it can be shown, as one would expect, that one cannot give the values of z, π, and κ arbitrarily along a curve C_0 in the (x, y)-plane. One can, perhaps, prescribe only z and one of the other two functions π or κ, or also a certain combination of both of the initial values. However, insofar as z, π, and κ cannot be given arbitrarily, all of the advantage that the introduction of the canonical system brought with it in the previous chapter, namely, the clarity of the geometrical representation of the solutions, will be lost. There, a well-defined extremal started from each point in space. By contrast, if one would like to interpret the solutions of the system (5) as surfaces (M_2) in a five-dimensional space of (x, y, z, π, κ) here then an extremal would not go through every curve in space, but only through those curves that satisfy a certain condition.

2) The second form of the canonical system.

We can get around the difficulty that consists of the fact that z (s) is known along with dz / ds on a closed curve C in the (x, y)-plane, which is obviously precisely the same one that was present in the previous section for the functional derivatives of a line function (according to **Volterra**), in a way that is similar to what we did with the latter. In order to construct the integral surface from the curve C_0 on which z and dz / dn are given as functions of arc-length, perhaps up to a second closed curve C in the (x, y)-plane, which might be encircled by the curve C_0 (to have a concrete case in mind), we would like to imagine that the region between the two curves C_0 and C is filled up simply and without gaps by a one-parameter family of curves C_α that includes C_0 and C. Furthermore, we construct the family of orthogonal trajectories to that family of curves C_α, which likewise fills up the region between C_0 and C simply and without gaps, and whose parameter we would like to denote by β.

If we introduce the parameters α and β of those two families of curves as coordinates in the (x, y)-plane:

(6) $$x = x(\alpha, \beta), \quad y = y(\alpha, \beta)$$

then z will also go to a function of α and β:

(6*) $$z = z(\alpha, \beta),$$

and we will have:

(7) $$f(z_x, z_y, z, x, y) = \bar{\bar{f}}(z_\alpha, z_\beta, z, \alpha, \beta),$$

such that the variational problem (2) now reads:

(8) $$I = \iint \bar{\bar{f}}(z_\alpha, z_\beta, z, \alpha, \beta)\,\Delta\, d\beta\, d\alpha = \text{extrem.}$$

The functional determinant Δ of the x, y with respect to α, β in that will be:

(9) $$\Delta = \sqrt{(x_\alpha^2 + y_\alpha^2)(x_\beta^2 + y_\beta^2)},$$

due to the orthogonality of the new coordinate lines.

The **Euler-Lagrange** equation (1) will then take on the form:

$$\frac{\partial}{\partial\alpha}(\bar{\bar{f}}_{z_\alpha} \cdot \Delta) + \frac{\partial}{\partial\beta}(\bar{\bar{f}}_{z_\beta} \cdot \Delta) - \bar{\bar{f}}_z \cdot \Delta = 0$$

in the new coordinates, or after substituting Δ from (9):

(10) $$\left\{\begin{aligned} &\frac{1}{\sqrt{x_\alpha^2 + y_\alpha^2}} \cdot \frac{\partial}{\partial\alpha}\left(\bar{\bar{f}}_{z_\alpha}\sqrt{x_\alpha^2 + y_\alpha^2}\right) + \frac{1}{\sqrt{x_\beta^2 + y_\beta^2}} \cdot \frac{\partial}{\partial\beta}\left(\bar{\bar{f}}_{z_\beta}\sqrt{x_\beta^2 + y_\beta^2}\right) \\ &+ \frac{\bar{\bar{f}}_{z_\alpha}}{\sqrt{x_\beta^2 + y_\beta^2}} \cdot \frac{\partial}{\partial\alpha}\left(\sqrt{x_\beta^2 + y_\beta^2}\right) + \frac{\bar{\bar{f}}_{z_\beta}}{\sqrt{x_\alpha^2 + y_\alpha^2}} \cdot \frac{\partial}{\partial\beta}\left(\sqrt{x_\alpha^2 + y_\alpha^2}\right) - f_z \quad = 0. \end{aligned}\right.$$

With the use of the orthogonality property of the parameter lines:

(11) $$x_\alpha \cdot x_\beta + y_\alpha \cdot y_\beta = 0,$$

one immediately verifies that:

(12) $$\left\{\begin{aligned} \frac{\partial}{\partial\alpha}\left(\sqrt{x_\beta^2 + y_\beta^2}\right) &= \sqrt{x_\alpha^2 + y_\alpha^2} \cdot \frac{y_\beta\, x_{\beta\beta} - x_\beta\, y_{\beta\beta}}{x_\beta^2 + y_\beta^2} &&= -\sqrt{(x_\alpha^2 + y_\alpha^2)(x_\beta^2 + y_\beta^2)} \cdot \frac{1}{\rho_\alpha}, \\ \frac{\partial}{\partial\beta}\left(\sqrt{x_\alpha^2 + y_\alpha^2}\right) &= \sqrt{x_\beta^2 + y_\beta^2} \cdot \frac{y_\alpha\, x_{\alpha\alpha} - x_\alpha\, y_{\alpha\alpha}}{x_\beta^2 + y_\beta^2} &&= +\sqrt{(x_\alpha^2 + y_\alpha^2)(x_\beta^2 + y_\beta^2)} \cdot \frac{1}{\rho_\beta}, \end{aligned}\right.$$

in which ρ_α is the radius of curvature of the curve C_α at the point in question, and ρ_β is that of the associated orthogonal trajectory. The **Euler-Lagrange** equation (10) then takes on the form:

(13)
$$\frac{1}{\sqrt{x_\alpha^2+y_\alpha^2}}\cdot\frac{\partial}{\partial\alpha}\left(\bar{\bar{f}}_{z_\alpha}\sqrt{x_\beta^2+y_\beta^2}\right)+\frac{1}{\sqrt{x_\beta^2+y_\beta^2}}\cdot\frac{\partial}{\partial\beta}\left(\bar{\bar{f}}_{z_\beta}\sqrt{x_\beta^2+y_\beta^2}\right)$$
$$-\frac{1}{\rho_\alpha}\cdot\bar{\bar{f}}_{z_\alpha}\sqrt{x_\alpha^2+y_\alpha^2}+\frac{1}{\rho_\beta}\cdot\bar{\bar{f}}_{z_\beta}\sqrt{x_\beta^2+y_\beta^2}-\bar{\bar{f}}_z=0.$$

Let us introduce the natural coordinates of the arc-length along the parameter lines into this relation! If the arc-length along the curve C_α is denoted by s, and the one along the orthogonal trajectories is denoted by n then we will have:

(14)
$$ds=\sqrt{x_\beta^2+y_\beta^2}\,d\beta\,,\qquad dn=\sqrt{x_\alpha^2+y_\alpha^2}\,d\alpha\,,$$

and therefore:

(15)
$$z_s=\frac{z_\beta}{\sqrt{x_\beta^2+y_\beta^2}}\,,\qquad z_n=\frac{z_\alpha}{\sqrt{x_\alpha^2+y_\alpha^2}}\,,$$

as well as:

(16)
$$\bar{\bar{f}}_{z_s}=\bar{\bar{f}}_{z_\beta}\sqrt{x_\beta^2+y_\beta^2}\,,\qquad \bar{\bar{f}}_{z_n}=\bar{\bar{f}}_{z_\alpha}\sqrt{x_\alpha^2+y_\alpha^2}\,.$$

Upon introducing those new quantities, (13) will be put into the form:

(17)
$$\frac{\partial}{\partial n}(\bar{\bar{f}}_{z_n})+\frac{\partial}{\partial s}(\bar{\bar{f}}_{z_s})-\frac{1}{\rho_s}\bar{\bar{f}}_{z_n}+\frac{1}{\rho_n}\bar{\bar{f}}_{z_s}-\bar{\bar{f}}_z=0\,.$$

In addition, we can now easily free ourselves of the arbitrariness in the choice of the curve parameter C_α again. If we consider any curve C in the (x, y)-plane and construct a neighboring curve C' to it, by measuring out the increment δn along C, which is a function of the arc-length s, then we can always consider those two curves to belong to a family of curves C_α, like the one that we just employed. It follows immediately from this that under the transition from C to C':

(18)
$$\delta z=z_n\,\delta n\,,\qquad\text{as well as}\qquad \delta\bar{\bar{f}}_{z_n}=\frac{\delta\bar{\bar{f}}_{z_n}}{\delta n}\delta n\,.$$

On the other hand, the radius of curvature of the orthogonal trajectory of the family that was introduced, which includes C and C', must be:

(19)
$$\frac{1}{\rho_n}=\frac{\delta\Theta}{\delta n}\,,$$

in which $\delta\Theta$ is the angle between the normals at two associated points of C and C'. That angle is identical to the angle that the tangents to C and C' define at the point in question, so:

(20) $$\delta\Theta = \frac{d\,\delta n}{ds} = \delta n',$$

which will then imply that:

(19*) $$\frac{1}{\rho_n} = \frac{\delta n'}{\delta n}.$$

If we write $\bar{f}(z_n, z_s, z, x, y, x_s)$ for $\bar{\bar{f}}$ with the use of natural coordinates in the manner of pp. 35 and denote the curvature of the curve C by $k = 1 / \rho_s$ then (17) will go to:

(21) $$\delta(\bar{f}_{z_n}) = -\frac{d}{ds}(\bar{f}_{z_n}\,\delta n) + (k\,\bar{f}_{z_n} + \bar{f}_z)\,\delta n.$$

From there, we will now move on to the canonical system by introducing the derivative of $\bar{f}$ with respect to z_n:

(22) $$-\frac{\partial\bar{f}}{\partial z_n} = \varphi$$

as a new unknown. We use that equation to calculate z_n as a function of φ, z, z_s, and s along C and introduce the new function:

(23) $$K(\varphi, z, z_s, x, y, x_s) = z_n\,\bar{f}_{z_n} - \bar{f},$$

by replacing the calculated values of z_n, in agreement with equation (35) of § **1**. For every point of the curve C, we will then have:

$$\frac{\partial K}{\partial\varphi} = -z_n, \qquad \frac{\partial K}{\partial z} = -\frac{\partial\bar{f}}{\partial z}, \qquad \frac{\partial K}{\partial z_s} = -\frac{\partial\bar{f}}{\partial z_s},$$

so we will find from equations (18) and (21) that:

(24) $$\begin{cases} \delta z = -\dfrac{\partial K}{\partial\varphi}\,\delta n, \\ \delta\varphi = -\dfrac{d}{ds}(K_{z_s}\cdot\delta n) + (k\cdot\varphi + K_z)\,\delta n, \end{cases}$$

with which, we have arrived at the system of equations that belongs to the **Euler-Lagrange** equation (1), and which we would like to call the *second canonical system*. Formally, it is the **Euler-Lagrange** system that belongs to the variational problem:

$$I = -\int\left\{\int[z_n\cdot\varphi + K(\varphi, z_s, z, x, y, x_s)]\,ds\right\}\delta n\ ,$$

because that will read:

$$z_n + \frac{\partial K}{\partial\varphi} = 0\ ,$$

$$\frac{d}{ds}\left(\frac{\partial K}{\partial z_s}\,\delta n\,ds\right) + \frac{\delta}{\delta s}(\varphi\,\delta n\,ds) - \frac{\partial K}{\partial z}\,\delta n\,ds = 0\ ,$$

and will go to (24) when we consider the fact that:

$$\delta\,ds = -\,k\cdot ds\ \delta n$$

in the second equation.

We can also regard the system (24) as a system of two total functional differential equations, since its form and its integration are essentially determined, on the one hand, by the curve C that we started with, and on the other, by being given δn, which mediates the transition to the neighboring curve.

If we interpret the solutions to the system in a four-dimensional (x, y, z, φ)-space then a certain surface (M_2) will go through every closed space curve (M_1) of the R_4 that is given by way of:

$$C,\ z\,(s),\ \varphi\,(s),$$

which will satisfy the system (24) and thus represent an extremal to our variational problem. Of course, for the actual exhibition of the extremals of the system, we must once more convert it back to a differential system by giving a certain family of curves C_α in the (x, y)-plane and thus replacing the variation that is determined by δn with a differentiation with respect to α. At each point of a curve C, $z\,(s)$, $\varphi\,(s)$, once we know dz / ds and $d\varphi / ds$, the system (24) will yield $\delta\varphi / \delta n$ and $\delta z / \delta n$, and with that, we will know the tangent planes to the developable surface along the curve. It is clear from this that the extremal can be achieved by a direct generalization of the **Cauchy-Lipschitz** process, as well as by adapting the method of successive approximations for ordinary differential equations to the general problem that we have here.

3) Adapting the concept of a "relative invariant."

Even in the present simple problem, it would be appropriate to show how the concepts whose development proved to be appropriate to the theory of the simple integral can be adapted to the problem of double integrals. Of course, its full meaning will first become apparent in problems with several independent functions.

In order to next adapt the concept of relative invariant, we couple it with the boundary formula, as we found it in equation (24) of the previous section, and consider the expression:

$$(25) \qquad \int_{(\Gamma)} \left\{ (f - z_x\, f_{z_x} - z_x\, f_{z_x}) \left(\delta x \frac{dy}{d\sigma} - \delta y \frac{dx}{d\sigma} \right) + f_{z_x} \left(\delta z \frac{dy}{d\sigma} - \delta y \frac{dz}{d\sigma} \right) + f_{z_y} \left(\delta x \frac{dy}{d\sigma} - \delta z \frac{dx}{d\sigma} \right) \right\} d\sigma ,$$

in which the integral extends over a closed curve Γ that lies completely on an extremal, and the derivatives z_x, z_y are the direction coefficients of the extremal that goes through Γ, while δx, δy, δz mean arbitrary increments.

If we now imagine that the extremal in question belongs to a one-parameter family of extremals that defines a field in the ordinary sense of the word then it will be a parametric field in the terminology that was chosen here. We would like to lay an arbitrary tubular surface through the closed curve Γ that cuts each extremal of the field along one and only one closed curve. It is clear from the property of the field that we will get a family of curves Γ_α on the tubular surface that fills up that surface simply and without gaps. We can then introduce the parameter α, together with the arc-length σ of the curves on the tubular surface as coordinates. If we then integrate the expression (25) for α from α_1 to α_2 then we will get a double integral that extends over the region of the tubular surface that is bounded by Γ_{α_1} and Γ_{α_2}:

$$(26) \qquad \begin{aligned} &\int_{\alpha_1}^{\alpha_2} d\alpha \int_{\Gamma_\alpha} \left\{ f - z_x\, f_{z_x} - z_y\, f_{z_y}) \left(\frac{\delta x}{\delta\alpha} \frac{dy}{d\sigma} - \frac{\delta y}{\delta\alpha} \frac{dx}{d\sigma} \right) \right. \\ &\quad \left. + f_{z_x} \left(\frac{\delta z}{\delta\alpha} \frac{dy}{d\sigma} - \frac{\delta y}{\delta\alpha} \frac{dz}{d\sigma} \right) + f_{z_y} \left(\frac{\delta x}{\delta\alpha} \frac{dz}{d\sigma} - \frac{\delta z}{\delta\alpha} \frac{dx}{d\sigma} \right) \right\} d\sigma . \end{aligned}$$

If we refer the tubular surface to the coordinates x and y then that double integral will go to:

$$(26^*) \qquad \iint \left\{ f + \left(\frac{\delta z}{\delta x} - z_x \right) f_{z_x} + \left(\frac{\delta z}{\delta y} - z_y \right) f_{z_y} \right\} dx\, dy .$$

The double integral extends over the region on the tubular surface that belongs to the region in the (x, y)-plane, and $\delta z / \delta x$, $\delta z / \delta y$ are the direction coefficients of the tubular surface.

We lay a second tubular surface through the two curves Γ_{α_1} and Γ_{α_2} that define a closed surface of the type of a torus together with the first one. The second surface will be cut by the extremal family of the field along curves Γ'_α that also covers it simply and without gaps as a one-parameter family, and in that way, Γ_{α_1} and Γ_{α_2} would coincide with Γ'_{α_1} [Γ'_{α_2}, resp.], and the curves Γ_α and Γ'_α are associated with each other by the two field extremals that go through them.

The extremal integral I [cf., § **1**, eq. (4)] that extends over the piece of the field extremal that is bounded by Γ_α and Γ'_α has a well-defined value that is a function of only α. For its derivative, (25) will imply that:

$$\begin{aligned}\frac{\delta I}{\delta\alpha} = \int_{\Gamma_\alpha}\Big\{&(f - z_x\, f_{z_x} - z_y\, f_{z_y})\left(\frac{\delta x}{\delta\alpha}\frac{dy}{d\sigma} - \frac{\delta y}{\delta\alpha}\frac{dx}{d\sigma}\right)\\ &+ f_{z_x}\left(\frac{\delta z}{\delta\alpha}\frac{dy}{d\sigma} - \frac{\delta y}{\delta\alpha}\frac{dz}{d\sigma}\right) + f_{z_y}\left(\frac{\delta x}{\delta\alpha}\frac{dz}{d\sigma} - \frac{\delta z}{\delta\alpha}\frac{dx}{d\sigma}\right)\Big\}\,d\sigma\\ - \int_{\Gamma'_\alpha}\Big\{&(f - z_x\, f_{z_x} - z_y\, f_{z_y})\left(\frac{\delta x}{\delta\alpha}\frac{dy}{d\sigma} - \frac{\delta y}{\delta\alpha}\frac{dx}{d\sigma}\right)\\ &+ f_{z_x}\left(\frac{\delta z}{\delta\alpha}\frac{dy}{d\sigma} - \frac{\delta y}{\delta\alpha}\frac{dz}{d\sigma}\right) + f_{z_y}\left(\frac{\delta x}{\delta\alpha}\frac{dz}{d\sigma} - \frac{\delta z}{\delta\alpha}\frac{dx}{d\sigma}\right)\Big\}\,d\sigma,\end{aligned}$$

in which the integration is performed in the same sense for both curves. If we integrate that equation for α from α_1 to α_2 and consider that I_{α_1}, as well as I_{α_2}, vanishes then we will get:

$$\int_{\alpha_1}^{\alpha_2} d\alpha \int_{\Gamma_\alpha}\left\{(f - z_x\, f_{z_x} - z_y\, f_{z_y})\left(\frac{\delta x}{\delta\alpha}\frac{dy}{d\sigma} - \frac{\delta y}{\delta\alpha}\frac{dx}{d\sigma}\right) + \cdots\right\} d\sigma = \int_{\alpha_1}^{\alpha_2} d\alpha \int_{\Gamma'_\alpha}\{(f - \cdots\}\,d\sigma. \tag{27}$$

We can express that fact by saying that we call the integral:

$$\int_\Gamma\left\{(f - z_x\, f_{z_x} - z_y\, f_{z_y})\left(\frac{\delta x}{\delta\alpha}\frac{dy}{d\sigma} - \frac{\delta y}{\delta\alpha}\frac{dx}{d\sigma}\right) + f_{z_x}\left(\frac{\delta z}{\delta\alpha}\frac{dy}{d\sigma} - \frac{\delta y}{\delta\alpha}\frac{dz}{d\sigma}\right) + f_{z_y}\left(\frac{\delta x}{\delta\alpha}\frac{dz}{d\sigma} - \frac{\delta z}{\delta\alpha}\frac{dx}{d\sigma}\right)\right\} d\sigma$$

the element of a relative integral invariant, so that is then identical to saying that the **Hilbert** integral (26*) for this simple problem for any parametric field of extremals will vanish when it is extended over a closed surface, in such a way that the **Hilbert** independent theorem will be true for any parametric field.

4) The Jacobi equation.

In the previous section, the transition from a relative integral invariant to an absolute one led to certain properties of the **Jacobi** equation that is associated with the **Euler-Lagrange** equations. For the present simple problem, it is meaningless to speak of an absolute integral invariant, but nonetheless we would also like to direct our attention to the **Jacobi** equation that belongs to (1).

It arises, in its own right, as the **Euler-Lagrange** equation of the second variation of the integral (2):

$$I = \iint \Phi(\mathfrak{z}_x, \mathfrak{z}_y, z, x, y)\,dx\,dy, \tag{28}$$

where

$$2\Phi = f_{z_x z_x}\,\mathfrak{z}_x^2 + 2 f_{z_x z_y}\,\mathfrak{z}_x\,\mathfrak{z}_x + f_{z_y z_y}\,\mathfrak{z}_y^2 + 2 f_{z_x z}\,\mathfrak{z}_x\,\mathfrak{z} + 2 f_{z_y z}\,\mathfrak{z}_y\,\mathfrak{z} + f_{zz}\,\mathfrak{z}^2, \tag{28*}$$

and one imagines that the values of its coefficients for z and its derivatives have be replaced with the values on an arbitrary extremal. It reads:

(29) $$\frac{\partial}{\partial x}(\Phi_{\mathfrak{z}_x}) + \frac{\partial}{\partial y}(\Phi_{\mathfrak{z}_y}) - \Phi_{\mathfrak{z}} = 0 .$$

We can prove a theorem for its integration that is similar to the theorem of **Jacobi** that was mentioned on pp. 19. An extremal of equation (1) will be established when the values of z, as well as dz / dn, are given on any curve C_0 in the (x, y)-plane as functions of the arc-length s_0 :

(30) $$z = z_0 (s_0) , \qquad z_n = z_{n_0}(s_0) .$$

The functional values $z (x, y)$ on the extremals that belong to an arbitrary point in the (x, y)-plane appear to depend upon the two functions (30). If we vary them then we will get the variation of z in the form:

(31) $$\delta z = \int_{C_0} \left\{ z_{z_0}(x, y; C_0, z_0, z_{n_0}, s_0)\, \delta z_0 + z_{z_{n_0}}(x, y; C_0, z_0, z_{n_0}, s_0)\, \delta z_{n_0} \right\} ds_0 ,$$

in which z_{z_0} and $z_{z_{n_0}}$ are the functional derivatives of z with respect to the two functions (30) at the point s_0 on C_0 . Now, since introducing the extremal $z (x, y, z_0, z_{z_0})$ into equation (1) will make that equation into an identity, the variation of that identity will show that this δz will satisfy the **Jacobi** equation (29), under sufficient assumptions, when the starting extremal $z (x, y ; C_0, z_0, z_{z_0})$ is substituted for z and its derivatives in its coefficients. Now, since δz_0 and $\delta z_{z_{n_0}}$ are arbitrary functions of the parameters s_0 , due to the linear character of the **Jacobi** equation, the functional derivatives are also solutions:

(32) $$\mathfrak{z}^{(1)} = z_{z_0}(x, y; C_0, z_0, z_{n_0}, s_0), \qquad \mathfrak{z}^{(2)} = z_{z_{n_0}}(x, y; C_0, z_0, z_{n_0}, s_0) ,$$

and indeed solutions that include one arbitrary parameter s_0 .

As a further point, we would like to stress that the **Jacobi** equation (29) is self-adjoint. That is because if we set the left-hand side of (29) equal to $L(\mathfrak{z})$ then we will immediately have:

$$\mathfrak{z}^{(1)} \cdot L(\mathfrak{z}^{(2)}) = \frac{\partial}{\partial x}\left(\mathfrak{z}^{(1)}\, \Phi_{\mathfrak{z}_x^{(2)}}\right) + \frac{\partial}{\partial y}\left(\mathfrak{z}^{(1)}\, \Phi_{\mathfrak{z}_y^{(2)}}\right) - (\Phi_{\mathfrak{z}_x^{(2)}}\, \mathfrak{z}_x^{(1)} + \Phi_{\mathfrak{z}_y^{(2)}}\, \mathfrak{z}_y^{(1)} + \Phi_{\mathfrak{z}^{(2)}}\, \mathfrak{z}^{(1)}) ,$$

and it will follow from this that when $\mathfrak{z}^{(1)}$ and $\mathfrak{z}^{(2)}$ are two solutions of $L(\mathfrak{z}) = 0$:

$$\frac{\partial}{\partial x}\left(\mathfrak{z}^{(1)}\, \Phi_{\mathfrak{z}_x^{(2)}} - \mathfrak{z}^{(2)}\, \Phi_{\mathfrak{z}_x^{(1)}}\right) + \frac{\partial}{\partial y}\left(\mathfrak{z}^{(1)}\, \Phi_{\mathfrak{z}_y^{(2)}} - \mathfrak{z}^{(2)}\, \Phi_{\mathfrak{z}_y^{(1)}}\right) = 0 .$$

By integrating that expression over a doubly-connected region in the (x, y)-plane that is bounded by the two curves C_1 and C_2 :

$$\int_{C_1}\left\{\mathfrak{z}^{(1)}\left(\Phi_{\mathfrak{z}_x^{(2)}}\frac{dy}{ds}-\Phi_{\mathfrak{z}_x^{(1)}}\frac{dx}{ds}\right)-\mathfrak{z}^{(2)}\left(\Phi_{\mathfrak{z}_y^{(2)}}\frac{dy}{ds}-\Phi_{\mathfrak{z}_y^{(1)}}\frac{dx}{ds}\right)\right\}ds$$

$$=\int_{C_2}\left\{\mathfrak{z}^{(1)}\left(\Phi_{\mathfrak{z}_x^{(2)}}\frac{dy}{ds}-\Phi_{\mathfrak{z}_x^{(1)}}\frac{dx}{ds}\right)-\mathfrak{z}^{(2)}\left(\Phi_{\mathfrak{z}_y^{(2)}}\frac{dy}{ds}-\Phi_{\mathfrak{z}_y^{(1)}}\frac{dx}{ds}\right)\right\}ds \ .$$

If we also introduce the coordinates α and β of pp. 45 in the (x, y)-plane here in order to arrive at the second canonical system that belongs to equation (29) then the function Φ will take the form:

$$2\bar{\bar{\Phi}} = \bar{\bar{f}}_{z_\alpha z_\alpha}\,\mathfrak{z}_\alpha^2 + 2\,\bar{\bar{f}}_{z_\alpha z_\beta}\,\mathfrak{z}_\alpha\,\mathfrak{z}_\beta + \bar{\bar{f}}_{z_\beta z_\beta}\,\mathfrak{z}_\beta^2 + 2\,\bar{\bar{f}}_{z_\alpha z}\,\mathfrak{z}_\alpha\,\mathfrak{z} + 2\,\bar{\bar{f}}_{z_\beta z}\,\mathfrak{z}_\beta\,\mathfrak{z} + \bar{\bar{f}}_{zz}\,\mathfrak{z}^2,$$

and after introducing the natural coordinates n and s, when we set:

$$\mathfrak{f} = -\,\bar{\Phi}_{\mathfrak{z}_n}$$

and introduce the function K in place of the function $\bar{f}$ for the coefficients, we will get the associated canonical function $X = \mathfrak{z}_n\,\bar{\Phi}_{\mathfrak{z}_n} - \bar{\Phi}$ in the form of:

$$2X\,(\mathfrak{f}, \mathfrak{z}_s, \mathfrak{z}, x, y, x_s)$$

(34)
$$= \frac{\partial^2 K}{\partial\varphi^2}\,\mathfrak{f}^2 + 2\,\frac{\partial^2 K}{\partial\varphi\,\partial z_s}\,\mathfrak{f}\cdot\mathfrak{z}_s + \frac{\partial^2 K}{\partial z_s^2}\,\mathfrak{z}_s^2 + 2\,\frac{\partial^2 K}{\partial\varphi\,\partial z}\,\mathfrak{f}\cdot\mathfrak{z} + 2\,\frac{\partial^2 K}{\partial z_s\,\partial z}\,\mathfrak{z}_s\cdot\mathfrak{z} + \frac{\partial^2 K}{\partial z^2}\,\mathfrak{z}^2.$$

That implies that the second canonical system that is associated with the **Jacobi** equation (29) is:

(35)
$$\begin{cases} \delta\mathfrak{z} = -\left(\dfrac{\partial^2 K}{\partial\varphi^2}\,\mathfrak{f} + \dfrac{\partial^2 K}{\partial\varphi\,\partial z_s}\,\mathfrak{z}_s + \dfrac{\partial^2 K}{\partial\varphi\,\partial z}\,\mathfrak{z}\right)\delta n, \\[2ex] \delta\mathfrak{f} = -\dfrac{d}{ds}\left[\left(\dfrac{\partial^2 K}{\partial\varphi\,\partial z_s}\,\mathfrak{f} + \dfrac{\partial^2 K}{\partial z_s^2}\,\mathfrak{z}_s + \dfrac{\partial^2 K}{\partial z_s\,\partial z}\,\mathfrak{z}\right)\delta n\right] + \left(k\cdot\mathfrak{f} + \dfrac{\partial^2 K}{\partial\varphi\,\partial z}\,\mathfrak{f} + \dfrac{\partial^2 K}{\partial z_s\,\partial z}\,\mathfrak{z}_s + \dfrac{\partial^2 K}{\partial z^2}\,\mathfrak{z}\right)\delta n, \end{cases}$$

which will then represent the **Jacobi** system that is associated with the system (24) in its own right, as one can also derive immediately from (24) by variation.

Naturally, this system, like equation (29), is self-adjoint, because we have:

$$\delta\,[(\mathfrak{z}^{(1)}\,\mathfrak{f}^{(2)} - \mathfrak{z}^{(2)}\,\mathfrak{f}^{(1)})\,ds] = [\mathfrak{z}^{(1)}\,(\delta\,\mathfrak{f}^{(2)} - k\,\mathfrak{f}^{(2)}\,\delta n) + \mathfrak{f}^{(2)}\,\delta\,\mathfrak{z}^{(1)} - \mathfrak{z}^{(2)}\,(\delta\,\mathfrak{f}^{(1)} - k\,\mathfrak{f}^{(1)}\,\delta n) - \mathfrak{f}^{(1)}\,\delta\,\mathfrak{z}^{(2)}]\,ds$$

$$= -\,\frac{d}{ds}\left\{\delta n\left[\frac{\partial^2 K}{\partial\varphi\,\partial z_s}\,(\mathfrak{f}^{(2)}\,\mathfrak{z}^{(1)} - \mathfrak{f}^{(1)}\,\mathfrak{z}^{(2)}) + \frac{\partial^2 K}{\partial z_s^2}\,(\mathfrak{z}_s^{(1)}\,\mathfrak{z}^{(1)} - \mathfrak{z}_s^{(1)}\,\mathfrak{z}^{(2)})\right]\right\}\cdot ds$$

for two systems of solutions $\mathfrak{z}^{(1)}$, $\mathfrak{f}^{(1)}$ [$\mathfrak{z}^{(2)}$, $\mathfrak{f}^{(2)}$, resp.] of (35), which is a relation that follows immediately from the fact that the integral over a closed curve C will be:

(36) $$\int_C (\mathfrak{z}^{(1)}\,\mathfrak{f}^{(2)} - \mathfrak{z}^{(2)}\,\mathfrak{f}^{(1)})\,ds = \text{const.}$$

That result agrees precisely with (33).

5) The concept of a "contact transformation" of a line function.

If we lay any surface through a closed space curve $\Gamma_1 = (C_1, z_1(s_1))$ and introduce a one-parameter family of curves into it that fills it up simply and without gaps then we can associate Γ_1 with a certain curve Γ_2 in the family. Since the surface can be laid through Γ_1 arbitrarily and the family of curves can likewise be chosen arbitrarily on it, we can obviously associate Γ_1 with an arbitrary curve Γ_2.

We would like to specialize that association by demanding of the surface that mediates the transition from Γ_1 to Γ_2 that it should be itself an extremal of a given variational problem, say, the variational problem that belongs to the integral (2) itself. We would like to call such a transformation of the curve $\Gamma_1 = (C_1, z_1(s_1))$ to the curve $\Gamma_2 = (C_2, z_2(s_2))$ a "contact transformation," and indeed on the following grounds: When the extremal integral I is taken over the surface patch on the extremal in question that is bounded by the two curves Γ_1 and Γ_2, it will be a function of the two curves Γ_1 and Γ_2, which collectively define the boundary line of the extremal surface patch: $I[\Gamma_1, \Gamma_2 \mid]$. If we fix the two curves C_1 and C_2 in the (x, y)-plane for the following consideration then I will seem to be a functional of the two functions $z_1(s_1)$ and $z_2(s_2)$:

(37) $$I = I[z_1(s_1), z_2(s_2)],$$

and when we also set $(-\bar{f}_{z_n}) = \varphi$, from equation (31) in the previous section, the basic formula of the calculus of variations will imply:

(38) $$\delta I = \int_{C_2} \varphi_2\,\delta z_2\,ds_2 - \int_{C_1} \varphi_1\,\delta z_1\,ds_1 .$$

That equation is obviously the analogue of equation (23) on pp. 20 and gives us the basis for why we call the transformation a contact transformation, in analogy with the terminology that was used there. The line function I characterizes the contact transformation. From (37), the associated extremal is therefore thought of as being determined by the two boundary curves Γ_1 and Γ_2. Instead of that, we can also fix them in such a way that we also give $\varphi_1(s_1)$, in addition to $z_1(s_1)$, and in that way we will get I as a functional of the latter two functions:

(39) $$I = I[z_1(s_1), z_2(s_2)] = \bar{I}[z_1(s_1), z_2(s_2)] .$$

We will then speak of an "infinitesimal" contact transformation when the curves Γ_1 and Γ_2, and therefore C_1 and C_2, are infinitely close to each other. We would then like to denote the transition from the curve C_1 to C_2 with the notation δn that we have applied up to now. In order to avoid confusion in formula (38), might replace δ with the symbol d':

(38*) $$d' I = \int_{C_2} \varphi_2 \, d' z_2 \, ds_2 - \int_{C_1} \varphi_1 \, d' z_1 \, ds_1 .$$

If we let C_1 and C_2 move close together then we will get:

(40) $$d' I = - \int_C \delta(\varphi \, d' z \, ds) = - \int_C [\delta\varphi \, d' z + \varphi \, d'(\delta z) - k \, \varphi \, d'z \, \delta n] \, ds ,$$

in which k is the curvature of the curve C. Now, for that case, one has:

$$I = \int_C \overline{f}\,(z_n, z_s, z, x, y, x_s)\,\delta n \, ds ,$$

and therefore:

(41) $$d' I = \int_C (d' \, \overline{f}) \, \delta n \, ds .$$

Setting (40) equal to (41) will then yield:

$$\begin{aligned} - \int_C [\delta\varphi \, d' z + \varphi \, d'(\delta z) - k \, \varphi \, d'z \, \delta n] \, ds &= \int_C \left\{ d' \, \overline{f} \cdot \delta n + d'(\varphi \, \delta z) \right\} ds \\ &= \int_C d' \left(\overline{f} + \varphi \frac{\delta z}{\delta n} \right) \delta n \; ds = - \int_C d' K \, \delta n \, ds , \end{aligned}$$

when one recalls equation (23). Therefore:

$$\int_C [(\delta\varphi - k \, \varphi \, \delta n) \, d' z - \delta z \, d'\varphi] \, ds = \int_C \left[\frac{\partial K}{\partial \varphi} d'\varphi + \frac{\partial K}{\partial z_s} d' z_s + \frac{\partial K}{\partial z} d' z \right] \delta n \, ds ,$$

or after partially integrating the second term on the right:

(42) $$\begin{aligned} &\int_C [(\delta\varphi - k \, \varphi \, \delta n) \, d' z - \delta z \, d'\varphi] \, ds \\ &= \int_C \left\{ \frac{\partial K}{\partial \varphi} \delta n \, d'\varphi - \left[\frac{d}{ds} \left(\frac{\partial K}{\partial z_s} \delta n \right) - \frac{\partial K}{\partial z} \delta n \right] d' z \right\} ds . \end{aligned}$$

Upon comparing the coefficients in that and considering the arbitrariness of $d'z$ and $d'\varphi$, we will conclude that:

(43)
$$\begin{cases} \delta z = -\dfrac{\partial K}{\partial \varphi}\,\delta n, \\[2ex] \delta\varphi = -\dfrac{d}{ds}\left(\dfrac{\partial K}{\partial z_s}\,\delta n\right) + \left(k\,\varphi + \dfrac{\partial K}{\partial z}\right)\delta n. \end{cases}$$

We then obtain the second canonical system, i.e., the differential equation of the extremals of the variational problem, as the defining equations of an infinitesimal contact transformation.

Whereas the finite contact transformations were characterized by the line function *I*, it is the ordinary function *f* (the function *K* that it determines, resp.) that is characteristic of the infinitesimal transformation. By their very nature, those two expressions *I* and *f* do not seen as consistent with each other as the corresponding functions *I* and *H* (*f*, resp.) for ordinary contact transformations.

6) The transformation of the set of all extremals into itself.

That difference will become noticeable when we ask what the analogue of the transformation of the canonical system might be. An extremal is established uniquely by a curve in R_4 [C, $z(s)$, $\varphi(s)$], so it will determine two functions $z_0(s_0)$ and $\varphi_0(s_0)$ on a hypersurface $C = C_0$ as "initial values." We would like to permute the initial functions on C_0 in such a way that an arbitrary pair of functions $z_0(s_0)$, $\varphi_0(s_0)$ will be associated with another pair of functions by a transformation group with the one parameter α. Therefore, on an arbitrary curve C, the functions $z(s)$ and $\varphi(s)$, which belong to any extremal, will go to other functions that belong to the transformed extremal under the transformation. In that way, every point with the arc-length s along C will be assigned certain variations of the functions $z(s)$, $\varphi(s)$, namely, $\delta z(s)$, $\delta\varphi(s)$. Naturally, those variations depend upon all values of the functions $z(s)$ and $\varphi(s)$ along C, and will perhaps possess the form:

(44)
$$\begin{cases} \delta z = \mathfrak{a}[C, z(s), \varphi(s)\,|\,; s]\,\delta\alpha, \\ \delta\varphi = \mathfrak{m}[C, z(s), \varphi(s)\,|\,; s]\,\delta\alpha, \end{cases}$$

which will then represent the analytical expression for the transformation.

The condition on the functionals $\mathfrak{a}$ and $\mathfrak{m}$ then follows that they must be solutions:

(45)
$$\mathfrak{z} = \mathfrak{a}\,[C, z(s), \varphi(s)\,|\,; s], \qquad \mathfrak{f} = \mathfrak{m}\,[C, z(s), \varphi(s)\,|\,; s]$$

to the **Jacobi** equations (35). In order to further establish that, we must observe that the expression:

$$\int_C \varphi\, d'z\, ds$$

must remain the element of a relative integral invariant under the transformation, so we must have:

(46)
$$\delta \int_C \varphi\, d'z\, ds \;=\; d'W\, \delta\alpha\,,$$

in which W is understood to mean a line function $W\,[C, z\,(s), \varphi\,(s) \mid\,]$. We will then have:

$$\int_C [\mathfrak{m}\, d'z + \varphi\, d'\mathfrak{a}]\, ds \;=\; d'W$$

or

(47)
$$\int_C [\mathfrak{m}\, d'z - \mathfrak{a}\, d'\varphi]\, ds \;=\; d'\left[W - \int_C \varphi \cdot \mathfrak{a}\, ds\right].$$

If we then set:

(48)
$$W - \int_C \varphi \cdot \mathfrak{a}\, ds \;=\; Q\,[C, z\,(s), \varphi\,(s) \mid\,]\,,$$

and if Q_s and Q_φ are the partial functional derivatives of Q at the point s :

(49)
$$Q_s = Q_s\,[C, z\,(s), \varphi\,(s) \mid ; s]\,, \quad Q_\varphi = Q_\varphi\,[C, z\,(s), \varphi\,(s) \mid ; s]\,,$$

then it will follow from:

$$\int_C [\mathfrak{m}\, d'z - \mathfrak{a}\, d'\varphi]\, ds \;=\; \int_C [Q_s\, d'z + Q_\varphi\, d'\varphi]\, ds$$

that:

(50)
$$\mathrm{m} = Q_s\,, \qquad \mathfrak{a} = -\,Q_\varphi\,.$$

If we now have any extremal $z\,(C, s)$, $\varphi\,(C, s)$ [1] then we can always regard them as belonging to a family of extremals. If α is the parameter of such a family then:

$$\mathfrak{z} = \frac{\partial z}{\partial a}\,, \qquad \mathfrak{f} = \frac{\partial \varphi}{\partial a}$$

will be a solution of the **Jacobi** equation, and therefore from (36) the relation will exist that:

$$\int_C \left[Q_s \frac{\partial z}{\partial a} + Q_\varphi \frac{\partial \varphi}{\partial a}\right] ds \;=\; \text{const.},$$

i.e.:

$$\frac{\partial Q}{\partial a} = \text{const.}$$

[1] The curve C and the arc-length s together determine the point considered in the (x, y)-plane directly, so only the variables x and y will enter here.

Thus, the condition that:

$$Q\,[C, z\,(s), \varphi\,(s)\mid\,] = \text{const.} \tag{51}$$

must also be fulfilled for every extremal $z\,(C, s)$, $\varphi\,(C, s)$, which is fact that we express by saying that we shall call the line function Q an *integral* of the canonical system (24).

In order to invert that argument, we show the following: If there are two expressions $A\,(C, s)$, $B\,(C, s)$ such that the relation:

$$\int_C [A\cdot\mathfrak{z} + B\cdot\mathfrak{f}]\,ds = \text{const.} \tag{52}$$

is true for the solutions $\mathfrak{z}\,(C, s)$, $\mathfrak{f}\,(C, s)$ of the **Jacobi** equations (35) then:

$$\mathfrak{z} = B\,(C, s), \qquad \mathfrak{f} = -A\,(C, s) \tag{53}$$

will be a solution of the **Jacobi** equations. That is because we have:

$$\begin{aligned}
0 &= \delta\int_C [A\,\mathfrak{z} + B\,\mathfrak{f}]\,ds \\
&= \int_C [\delta A\cdot\mathfrak{z} + A\,\delta\mathfrak{z} + \delta B\,\mathfrak{f} + B\,\delta\mathfrak{f} - (A\,\mathfrak{z} + B\,\mathfrak{f})\,k\,\delta n]\,ds \\
&= \int_C \left\{(\delta A - k\,A\,\delta n)\,\mathfrak{z} + (\delta B - k\,B\,\delta n)\,\mathfrak{f} - A\left(\frac{\partial^2 K}{\partial\varphi^2}\,\mathfrak{f} + \frac{\partial^2 K}{\partial\varphi\,\partial z_s}\,\mathfrak{z}_s + \frac{\partial^2 K}{\partial\varphi\,\partial z}\,\mathfrak{z}\right)\delta n\right. \\
&+ \left. B\left[-\frac{d}{ds}\left(\left(\frac{\partial^2 K}{\partial z_s\,\partial\varphi}\,\mathfrak{f} + \frac{\partial^2 K}{\partial z_s^2}\,\mathfrak{z}_s + \frac{\partial^2 K}{\partial z_s\,\partial\varphi}\,\mathfrak{z}\right)\delta n\right) + \left(k\,\mathfrak{f} + \frac{\partial^2 K}{\partial\varphi\,\partial z}\,\mathfrak{f} + \frac{\partial^2 K}{\partial z_s\,\partial z}\,\mathfrak{z}_s + \frac{\partial^2 K}{\partial z^2}\,\mathfrak{z}\right)\delta n\right]\right\} ds
\end{aligned}$$

or after partially-integrating some of the terms:

$$\begin{aligned}
\int_C \left\{\mathfrak{z}\left[\delta A - \frac{d}{ds}\left(\left(-\frac{\partial^2 K}{\partial z_s\,\partial\varphi}\,A + \frac{\partial^2 K}{\partial z_s^2}\,B_s + \frac{\partial^2 K}{\partial z_s\,\partial\varphi}\,B\right)\delta n\right) + \left(-k\,A - \frac{\partial^2 K}{\partial z\,\partial\varphi}\,A + \frac{\partial^2 K}{\partial z\,\partial z_s}\,B_s + \frac{\partial^2 K}{\partial z^2}\,B\right)\delta n\right] ds\right. \\
\left. + \;\mathfrak{f}\left[\delta B - \left(\frac{\partial^2 K}{\partial\varphi^2}\,A - \frac{\partial^2 K}{\partial\varphi\,\partial z_s}\,B_s - \frac{\partial^2 K}{\partial\varphi\,\partial z}\,B\right)\delta n\right]\right\} ds = 0\,.
\end{aligned}$$

From that formula, we conclude that the integrand of (54) must vanish as a result of the arbitrariness of δn on C, and then further deduce that the following equations must be true:

$$\delta B = -\left(-\frac{\partial^2 K}{\partial \varphi^2}A + \frac{\partial^2 K}{\partial \varphi\,\partial z_s}B_s + \frac{\partial^2 K}{\partial \varphi\,\partial z}B\right)\delta n,$$

$$\delta A = -\frac{d}{ds}\left(\left(-\frac{\partial^2 K}{\partial z_s\,\partial \varphi}A + \frac{\partial^2 K}{\partial z_s^2}B_s + \frac{\partial^2 K}{\partial z_s\,\partial \varphi}B\right)\delta n\right) + \left(-k\,A - \frac{\partial^2 K}{\partial z\,\partial \varphi}A + \frac{\partial^2 K}{\partial z\,\partial z_s}B_s + \frac{\partial^2 K}{\partial z^2}B\right)\delta n,$$

which then give equations (35), and the assertion above is proved.

If we now have an integral of the canonical equations (24), in the sense that was defined above:

$$Q\,[C, z\,(s),\ \varphi\,(s)] = \text{const.}$$

then the relation:

$$\int_C \{Q_z\,[C, z\,(s), \varphi\,(s)\,|;\, s] + Q_\varphi\,[C, z\,(s), \varphi\,(s)\,|;\, s]\,\mathfrak{f}(s)\}\,ds = \text{const.}$$

will obviously be true for the **Jacobi** equations (35), and therefore:

(56) $$\mathfrak{z} = Q_\varphi\,, \qquad \mathfrak{f} = -\,Q_z$$

will be a solution of the **Jacobi** equations, as we just proved.

With that, we ultimately have the direct analogue of **Poisson**'s theorem, because from two integrals of the canonical equations:

$$Q^{(1)}\,[C, z\,(s), \varphi\,(s)\,|\,] = \text{const.}, \qquad Q^{(2)}\,[C, z\,(s), \varphi\,(s)\,|\,] = \text{const.},$$

we will get two systems of solutions to **Jacobi**'s equations:

$$\mathfrak{z}^{(1)} = Q_\varphi^{(1)}\,, \qquad \mathfrak{f}^{(1)} = -\,Q_z^{(1)}\,,$$
$$\mathfrak{z}^{(2)} = Q_\varphi^{(2)}\,, \qquad \mathfrak{f}^{(2)} = -\,Q_z^{(2)}\,,$$

and from that, according to (36):

(57) $$\int_C \{Q_z^{(2)}\,Q_\varphi^{(1)} - Q_\varphi^{(2)}\,Q_z^{(1)}\}\,ds$$

will be a new integral.

§ 3.

Integrating partial functional differential equations by means of extremal integrals.

The train of thought that was presented in the first section can also be inverted for the problem that was treated in this chapter, and the integration of certain partial functional differential equations can be achieved in such a way that one reduces that problem to the solution of a variational problem in realm of double integrals, i.e., one solves the **Euler-Lagrange** equations (cf., § **2**) directly.

Of course, that way of thinking would accomplish very much less here, insofar as the partial functional differential equation that appears in § **1** is not by any means the most general of its kind, while in § **3** of the first section the basis for the argument would cease to be meaningful, namely, that the *most general* first-order partial differential equation could be regarded as the **Hamilton-Jacobi** equation of a variational problem. Nonetheless, it would seem that this inversion would not be meaningless here for the special class of partial functional differential equations from the systematic standpoint.

The problem to be solved here reads: One knows of a line function $V\,[L\mid\,]$ that is defined for all closed curves of a region of space in question that a relation exists between its **Volterra** derivatives with respect to the coordinate planes for each point of an individual curve:

$$V_{xy} = H\,(x, y, z, V_{yz}, V_{zx})\,, \tag{1}$$

while the indeterminacy in the **Volterra** derivatives is established by the condition:

$$\frac{dz}{d\sigma} = \frac{\partial H}{\partial \pi}\frac{dx}{d\sigma} + \frac{\partial H}{\partial \kappa}\frac{dy}{d\sigma} \qquad (\pi = V_{yz},\ \kappa = V_{zx}). \tag{1*}$$

Determine that line function.

In order to solve that problem, we will exhibit the integrand of a variational problem as a function H by setting:

$$z_x = \frac{\partial H}{\partial \pi}\,, \qquad z_y = \frac{\partial H}{\partial \kappa}\,,$$

which is possible, due to (1*), and after eliminating π and κ, defining the function:

$$f\,(z_x, z_y, z, x, y) = \pi\,\frac{\partial H}{\partial \pi} + \kappa\,\frac{\partial H}{\partial \kappa} - H\,.$$

Just as in § **1**, that function f belongs to a function $\overline{f}\,(z_s, z_n, z, x, y, x_s)$, and from that we can define a function $K\,(\varphi, z_s, z, x, y, x_s)$ as the one that is associated with H, such that the equation:

$$V_n^* = K\,(V_s^*, z_s, z, x, y, x_s) \tag{2}$$

will enter in, along with (1), with no auxiliary condition ([1]).

We will obtain a complete integral of (1) or (2) when we extend the extremal integral from a fixed curve L_0 along all curves L of the region of space considered:

$$V[L, L_0 \,|\,] = \int_{L_0}^{L} E \int f(z_x, z_y, z, x, y)\, dx\, dy\,, \tag{3}$$

and in that way, the curve L_0 can again be retroactively regarded as arbitrary. In fact, among all of the line functions, we can easily find one of them that includes a given element $L = (C, z(s))$, V, V_z^*, and V_n^*, which must naturally satisfy equation (2). We need only to determine the function $z_n(s)$ along L from the relation:

$$\frac{\partial \bar{f}}{\partial z_n} = -\,V_z^*\,, \tag{4}$$

and lay the extremal through L that possesses the normal derivative z_n . We must then choose the curve L_0 along that extremal in such a way that the value of the extremal integral (3) takes the required value V precisely. The derivative V_n^* takes the correct value by itself since the element fulfills the differential equation (2). The line function (3) then, in fact, represents a complete integral since it apparently depends upon the arbitrary space curve L_0, so on *two* functions of one variable, such that the complete integral of (2) will depends upon *only one* arbitrary function of one variable.

In order to represent an arbitrarily-given integral $V[L \,|\,]$ with the help of a function field of extremals, we imagine that we have determined the totality of curves $L_0 = (C_0, z_0(s_0))$ for which we have:

$$V[L \,|\,] = 0\,.$$

If we assume that it is possible to determine an implicitly-given line function ([2]) then a certain function $z_0(s_0)$ will be given on any curve C_0 . We then lay the extremals whose normal derivatives are determined from the relation (4) through all curves L_0 . The function field thus-constructed then implies a covering of space by curves by way of the extremal integral that coincides with the given integral $V[L \,|\,]$ because the functional derivatives satisfy the given equation (2) since:

$$\int_{C_0} \{(z_n \bar{f}_{z_n} - \bar{f})\,\delta n_0 - \bar{f}_{z_n}\,\delta z_0\}\, ds_0 = (\delta V)_{L_0} = 0\,,$$

and the integral reduces to the given integral $V[L \,|\,]$ for the set of all curves L_0 ([3]).

([1]) Meanwhile, K is not the most general function of its kind. Indeed, each K belongs to a $\bar{f}$, but the latter must satisfy the condition that the derivative x_s must drop out under the substitution of z_x and z_y for z_s and z_n , so $z_s \bar{f}_{z_s} + z_n \bar{f}_{z_n}$ must be independent of x_s .

([2]) Cf., **V. Volterra**, *Leçons sur les fonctions de lignes*, Paris (1913), Chap. 4.

([3]) For the unique determinacy of the integral, cf., **P. Lévy**, *loc. cit.* (pp. 1), pp. 149, *et seq.*

If the given integral were to depend upon an arbitrary function of one variable $\alpha(\overset{1}{\underset{0}{t}})$, in addition to L :

(5) $$V = V[L,\ \alpha(\overset{1}{\underset{0}{t}})\ |\ ],$$

so it is a complete integral, then the set of all curves L_0 would also depend upon that function. We then get a set of function fields that depend upon an arbitrary function $\alpha(\overset{1}{\underset{0}{t}})$ and would represent the most general complete integral of (2).

The solution of the partial functional differential equation (2) is then reduced to the integration of the **Euler-Lagrange** equation:

(6) $$\frac{\partial}{\partial x}(f_{z_x}) + \frac{\partial}{\partial y}(f_{z_y}) - f_z = 0 .$$

We will refer to the extremals that it determined as *characteristics* of equation (2). They are identical to the "charactéristiques de première espèce" that **P. Lévy** introduced, which are also obtained formally when we replace equation (6) with the associated second canonical system ([1]):

(7) $$\begin{cases} \delta z = -\dfrac{\partial K}{\partial \varphi}\,\delta n, \\[2ex] \delta z = -\dfrac{d}{ds}(K_{z_s}\cdot \delta n) + (k\,\varphi + K_z)\,\delta n. \end{cases}$$

As an essential property of the characteristics, we have the fact that they are fixed uniquely by an element of an integral of (2) since we know of that element of the extremal, not only the curve L through which it goes, but also the normal derivative along that curve. There will then be two integrals of (2) with an element in common that are both common and is common to the whole characteristic that it determines.

Finally, we would like to treat a problem that is a direct generalization of the "**Cauchy** problem" for ordinary first-order partial differential equations. Let a closed curve C_0 be given in the (x, y)-plane, and then the cylinder with C_0 as its base whose generators are parallel to the z-axis. Moreover, a certain line function $\Phi(z_0(s_0))$ might be given that possesses a certain value for all closed curves $z_0(s_0)$ on the cylinder that cannot be contracted to a point. Determine the integral of equation (2) that assumes the value Φ precisely for the curves L on the surface of the cylinder.

In order to solve that problem, we define the functional derivative Φ_z for every function $z_0(s_0)$, which is then a certain function of the arc-length s_0, and then determine the function $z_n(s_0)$ from the equation:

([1]) **P. Lévy**, *loc. cit.*, pp. 152.

(4*) $$\left(\frac{\partial \bar{f}}{\partial z_n}\right)_0 = -\Phi_0 .$$

If we then lay the extremal (or as we can also say, the characteristic) through each curve z_0 (s_0) whose normal derivative is equal to z_n (s_0) then we will have the required integral in the expression:

(8) $$V[L\,|\,] = \Phi[L\,|\,] + \int_{L_0}^{L} E \int f(z_x, z_y, z, x, y)\, dx\, dy .$$

That is because it reduces to $\Phi\,[L_0\,|\,]$ for $L = L_0$, but we further have that its variation is:

$$\delta V = \int_{C_0} \Phi_{z_0}\, \delta z_0\, ds_0 + \int_C \{(z_n\, \bar{f}_{z_n} - \bar{f})\, \delta n - \bar{f}_{z_n}\, \delta z\}\, ds + \int_{C_0} (\bar{f}_{z_n})_0\, \delta z_0\, ds_0 .$$

Thus, if we recall (4*) then:

$$V_n = z_n\, \bar{f}_{z_n} - \bar{f} , \qquad V_z = -\bar{f}_{z_n} .$$

However, those expressions satisfy equation (2), since it emerges from them precisely by eliminating z_n, as was explained in the first section.

CHAPTER TWO

The variational problem with two unknown functions.

§ 1.

The variational problem and the partial functional differential equation.

The problem in the previous chapter shall now be generalized by assuming that we are starting with not one, but two, unknown functions in the variational problem. The treatment of the problem that was posed generally follows a parallel course to what was done in the previous section and can then be achieved more concisely here. By contrast, at some points, the more general character of the present problem will emerge conspicuously.

1) The variational problem and the Euler-Lagrange equations.

The basic variational problem now reads: Determine two unknown functions $z(x, y)$ and $t(x, y)$ such that the integral:

$$I = \iint_S f(z_x, z_y, z, t, t_x, t_y, t, x, y)\,dx\,dy \tag{1}$$

is an extremum. In so doing, let the values of the two functions z and t be initially fixed on the boundary curve C of the domain of integration S in the (x, y)-plane:

$$\left.\begin{aligned} z &= z(s), \\ t &= t(s), \end{aligned}\right\} \quad \text{on } C . \tag{2}$$

The first condition for the occurrence of an extremum is that the functions $z(x, y)$ and $t(x, y)$ must satisfy the two **Euler-Lagrange** equations:

$$\left\{\begin{aligned} \frac{\partial}{\partial x}(f_{z_x}) + \frac{\partial}{\partial y}(f_{z_y}) - f_z &= 0, \\ \frac{\partial}{\partial x}(f_{t_x}) + \frac{\partial}{\partial y}(f_{t_y}) - f_t &= 0. \end{aligned}\right. \tag{3}$$

We would like to interpret the four variables that we are dealing with here as coordinates of the points in a four-dimensional (linear) space (R_4) and call the one-dimensional manifolds in that space "curves," the two-dimensional ones "surfaces," and the three-dimensional ones "hypersurfaces."

A space curve in R_4 through which the desired surface $z = z(x, y)$, $t = t(x, y)$ [which is again called an *extremal*, as a solution to the differential equations (3)] must go is given by equations (2). Here, as well, we assume that in general there will be one and only one extremal that goes through a given space curve $(C, z(s), t(s)) = L$ and is singularity-free in the interior of the region of the (x, y)-plane that is bounded by C.

2) The extremal integral as a line function and its functional derivatives.

If we now imagine that conversely a closed space curve L is given in a certain region of R_4 then it will determine a unique extremal. The extremal integral:

(4) $$I[L\,|\,] = \int E \int f(z_x, z_y, z, t, t_x, t_y, t, x, y)\,dx\,dy$$

will then be a line function on R_4 . We can once more deduce the functional derivatives from the boundary formula of the calculus of variations them, and we shall proceed to their derivation directly.

We vary the space curve L that determined the value of the extremal integral. In that way, we will get a neighboring space curve L' through which we likewise lay an extremal. We define a one-to-one correspondence between those two extremal surface patches such that points on the boundary will again correspond to boundary points. The line that connects corresponding points of the two extremal surface patches represents a "four-vector" of the first kind ([1]) in R_4 whose components we would like to denote by δx, δy, δz, δt. If we again introduce x, y as functions of two parameters u and v, with which z and t, as well as the coordinates of the points of the neighboring extremal $x + \delta x$, $y + \delta y$, $z + \delta z$, $t + \delta t$, will also be functions of those parameters, then the two extremal surface patches are constructed over the same region in the (u, v)-plane. The variation of the region of the independent variables will then be avoided in the variation of the extremal integral.

If we again denote the two-rowed functional determinants of the variables x, y, z, t with respect to u and v by $\lambda_{(xy)}$, $\lambda_{(zx)}$, $\lambda_{(yz)}$, $\lambda_{(tx)}$, $\lambda_{(yt)}$ then we will have:

$$z_x = \frac{\lambda_{(zy)}}{\lambda_{(xy)}}, \qquad z_y = \frac{\lambda_{(xz)}}{\lambda_{(xy)}}, \qquad t_x = \frac{\lambda_{(ty)}}{\lambda_{(xy)}}, \qquad t_y = \frac{\lambda_{(xt)}}{\lambda_{(xy)}},$$

and we will further have ([2]):

([1]) Vector calculus in four-dimensional space is developed thoroughly in **A. Sommerfeld**, "Zur Relativitätstheorie, I. Vierdimensional Vektoralgebra," Ann. Phys. (Leipzig) (4) **32** (1910), pp. 749; "II. Vierdimensionale Vektoranalysis," *ibid.* **33** (1910), pp. 649.

([2]) It should be noted that the determinant $\lambda_{(zt)}$ does not appear in the subdeterminants λ, nor does it enter into F. One will then have $F_{\lambda_{(zt)}} \equiv 0$.

$$f(z_x, z_y, z, t_x, t_y, t, x, y) = \frac{1}{\lambda_{(xy)}} (\lambda_{(zy)}, \lambda_{(xz)}, \lambda_{(ty)}, \lambda_{(xt)}, \lambda_{(xy)}, z, t, x, y) ,$$

in which F is again homogeneous of degree one in the subdeterminants λ. As a result, the integral (1) will take the form:

$$I = \iint F(\lambda_{(zy)}, \lambda_{(xz)}, \lambda_{(ty)}, \lambda_{(xt)}, \lambda_{(xy)}, z, t, x, y)\, du\, dv \ ,$$

and we will get:

$$\delta I = \iint [F_{\lambda_{(zy)}}\, \delta\lambda_{(zy)} + F_{\lambda_{(xy)}}\, \delta\lambda_{(xz)} + F_{\lambda_{(ty)}}\, \delta\lambda_{(ty)} + F_{\lambda_{(xt)}}\, \delta\lambda_{(xt)} + F_{\lambda_{(xy)}}\, \delta\lambda_{(xy)} \\ + F_z\, \delta z + F_t\, \delta t + F_x\, \delta x + F_y\, \delta y]\, du\, dv$$

for the variation of I. Upon partial integration and recalling the **Euler-Lagrange** equations, that will imply:

$$\delta I = \int_L \left\{ F_{\lambda_{(zy)}} \left(\delta z \frac{dy}{d\sigma} - \delta y \frac{dz}{d\sigma} \right) + F_{\lambda_{(xz)}} \left(\delta x \frac{dz}{d\sigma} - \delta z \frac{dx}{d\sigma} \right) + F_{\lambda_{(zy)}} \left(\delta t \frac{dy}{d\sigma} - \delta y \frac{dt}{d\sigma} \right) \right. \\ \left. + F_{\lambda_{(xt)}} \left(\delta x \frac{dt}{d\sigma} - \delta t \frac{dx}{d\sigma} \right) + F_{\lambda_{(xy)}} \left(\delta x \frac{dy}{d\sigma} - \delta y \frac{dx}{d\sigma} \right) \right\} d\sigma .$$

By returning to the original quantities, we will then get the relation:

(5)
$$\delta I = \int_L \left\{ (f - z_x\, f_{z_x} - z_y\, f_{z_y} - t_x\, f_{t_x} - t_y\, f_{t_y}) \left(\delta x \frac{dy}{d\sigma} - \delta y \frac{dx}{d\sigma} \right) \right. \\ \left. + f_{z_x} \left(\delta z \frac{dy}{d\sigma} - \delta y \frac{dz}{d\sigma} \right) + f_{z_y} \left(\delta x \frac{dz}{d\sigma} - \delta z \frac{dx}{d\sigma} \right) + f_{t_x} \left(\delta t \frac{dy}{d\sigma} - \delta y \frac{dt}{d\sigma} \right) + f_{t_y} \left(\delta x \frac{dt}{d\sigma} - \delta t \frac{dx}{d\sigma} \right) \right\} d\sigma$$

as the boundary formula of the calculus of variations.

In analogy with what we did in the previous section, we define the functional derivatives of a line function $S[L\,|\,]$ on R_4 to be the components of a four-vector S_x, S_y, S_z, S_t . The variation of the line function $S[L\,|\,]$ has the form:

(6)
$$\delta S = \int_L (S_x\, \delta x + S_y\, \delta y + S_z\, \delta z + S_t\, \delta t)\, d\sigma ,$$

from which it emerges that we have:

(7)
$$S_x \frac{dx}{d\sigma} + S_y \frac{dy}{d\sigma} + S_z \frac{dz}{d\sigma} + S_t \frac{dt}{d\sigma} = 0 ,$$

i.e., that the four-vector S_x, S_y, S_z, S_t must lie on the normal hyperplane to the space curve L at the location σ in question.

It follows from (7) that the four-vector S_x, S_y, S_z, S_t can be regarded as a vector product that is constructed from the four-vector $dx / d\sigma$, $dy / d\sigma$, $dz / d\sigma$, $dt / d\sigma$ and a six-vector:

$$\{ S_{yx}, S_{zx}, S_{yz}, S_{tx}, S_{yt}, S_{zt}\}$$

in the following way:

$$(8) \qquad \begin{cases} S_x = & & S_{yx}\dfrac{dx}{d\sigma} + & S_{zx}\dfrac{dz}{d\sigma} + & S_{tx}\dfrac{dt}{d\sigma} \\ S_y = & S_{xy}\dfrac{dx}{d\sigma} + & & S_{zy}\dfrac{dz}{d\sigma} + & S_{ty}\dfrac{dt}{d\sigma} \\ S_z = & S_{xz}\dfrac{dx}{d\sigma} + & S_{yz}\dfrac{dy}{d\sigma} + & & S_{tz}\dfrac{dt}{d\sigma} \\ S_t = & S_{xt}\dfrac{dx}{d\sigma} + & S_{yt}\dfrac{dy}{d\sigma} + & S_{zt}\dfrac{dz}{d\sigma} & \end{cases} \qquad (S_{\alpha\beta} = - S_{\beta\alpha}).$$

Naturally, the components of the six-vector are not determined uniquely by equations (8), but the general solution when S'_{yx}, … is a special solution will be:

$$(8^*) \qquad S_{yz} = S'_{yx} + c\frac{dt}{d\sigma} - d\frac{dz}{d\sigma}, \qquad \text{etc.,}$$

in which the additional terms are determined from the two-rowed determinants in the matrix:

$$\left\| \begin{matrix} a & b & c & d \\ \dfrac{dx}{d\sigma} & \dfrac{dy}{d\sigma} & \dfrac{dz}{d\sigma} & \dfrac{dt}{d\sigma} \end{matrix} \right\|,$$

where we understand a, b, c, d to mean arbitrary constants (for each point σ of L). If we introduce the expressions (8) for the components of the four-vector into equation (6) then it will take the form:

$$(9) \qquad \begin{aligned} \delta S = \int_L \Big\{ & S_{yx}\left(\delta x\frac{dy}{d\sigma} - \delta y\frac{dx}{d\sigma}\right) + S_{yz}\left(\delta z\frac{dy}{d\sigma} - \delta y\frac{dz}{d\sigma}\right) + S_{zx}\left(\delta x\frac{dz}{d\sigma} - \delta z\frac{dx}{d\sigma}\right) \\ & + S_{yt}\left(\delta t\frac{dy}{d\sigma} - \delta y\frac{dt}{d\sigma}\right) + S_{tx}\left(\delta x\frac{dt}{d\sigma} - \delta t\frac{dx}{d\sigma}\right) + S_{zt}\left(\delta t\frac{dz}{d\sigma} - \delta z\frac{dt}{d\sigma}\right)\Big\} d\sigma. \end{aligned}$$

Upon comparing the two formulas (5) and (9), we will get:

$$(10) \qquad I_{xy} = z_x\, f_{z_x} + z_y\, f_{z_y} + t_x\, f_{t_x} + t_y\, f_{t_y} - f, \quad I_{yz} = f_{z_x}, \quad I_{zx} = f_{z_y}, \quad I_{yt} = f_{t_x}, \quad I_{tx} = f_{t_y},$$

for the extremal integral as the (**Volterra**) derivatives with respect to the coordinate planes, while I_{st} will prove to be identically zero. In that way, a certain degree of arbitrariness is imposed upon the four of them, and in particular, one has set $a = 0$, $b = 0$.

If we would like to avoid the indeterminacy that is present these (**Volterra**) derivatives from the outset then we must appeal to the **Hadamard-Lévy** picture. In place of the curve L, we will then have to choose its projection C onto the (x, y)-plane to be the integration path for the curve integral in formula (5). When we replace δx and δy with δn using formula (14) on pp. 32 and introduce the natural derivatives $\frac{dz}{ds}$, $\frac{dt}{ds}$ $\left(\frac{dz}{dn}, \frac{dt}{dn}, \text{ resp.}\right)$ of the functions $z(x, y)$ and $t(x, y)$ as an extension of equations (27) and (28) on pp. 35 and 36, those formulas will take the form:

$$\delta I^* = \int_C \{(z_n\,\overline{f}_{z_n} + t_n\,\overline{f}_{t_n} - \overline{f})\,\delta n - \overline{f}_{z_n}\delta z - \overline{f}_{t_n}\delta t\}\,ds\,. \tag{11}$$

We then get the functional derivatives (according to **Hadamard-Lévy**) of the extremal integral in the form of:

$$\left\{\begin{aligned} I_n^* &= z_n\,\overline{f}_{z_n} + t_n\,\overline{f}_{t_n} - \overline{f}, \\ I_n^* &= -\overline{f}_{z_n}, \\ I_t^* &= -\overline{f}_{z_t}. \end{aligned}\right. \tag{12}$$

They are expressed in terms of the (**Volterra**) derivatives with respect to the coordinate planes, which would emerge from the derivative, in the following way:

$$\begin{aligned} I_n^* &= I_{xy} - \frac{dz}{ds}\left(I_{yz}\frac{dx}{ds} + I_{zx}\frac{dy}{ds}\right) - \frac{dt}{ds}\left(I_{yt}\frac{dx}{ds} + I_{tx}\frac{dy}{ds}\right), \\ I_z^* &= I_{yz}\frac{dy}{ds} - I_{zx}\frac{dx}{ds}, \\ I_t^* &= I_{yt}\frac{dy}{ds} - I_{tx}\frac{dx}{ds}. \end{aligned} \tag{13}$$

3) The equation for the partial functional derivatives of the extremal integral.

In the same way as in the previous section, knowing the extremal integral as a line function on R_4 will also achieve the integration of the system (3) of **Euler-Lagrange** equations here. In particular, for a given space curve L, we can give the extremal that goes through it and includes no singular points in the region bounded by L. That will then come down to being able to calculate the extremal integral directly, and we will find the Ansatz for that in a way that is analogous to what we did in the previous section. That shows that the extremal integral $I[L\mid]$ will satisfy a partial functional differential equation.

If we use the (**Volterra**) functional derivatives as a basis then, from the relations (10), we can calculate the four partial derivatives z_x, z_y, t_x, t_y from:

$$f_{z_x} = \pi\,, \qquad f_{z_y} = \kappa\,, \qquad f_{t_x} = \psi\,, \qquad f_{t_y} = \chi\,,$$

and upon eliminating those derivatives, we can define the function:

(14) $$H\,(z, t, x, y, \pi, \kappa, \psi, \chi) = z_x\, f_{z_x} + z_y\, f_{z_y} + t_x\, f_{t_x} + t_y\, f_{t_y} - f\,.$$

We will then have:

$$\frac{\partial H}{\partial \pi} = z_x\,, \qquad \frac{\partial H}{\partial \kappa} = z_y\,, \qquad \frac{\partial H}{\partial \psi} = t_x\,, \qquad \frac{\partial H}{\partial \chi} = t_y\,,$$

and when we now set:

$$\pi = I_{yz}\,, \qquad \kappa = I_{zx}\,, \qquad \psi = I_{yt}\,, \qquad \chi = I_{tx}\,,$$

from the first of the relations (10) between the partial functional derivatives of the extremal integrals, the following equation will exist:

(15) $$I_{xy} = H\,(z, t, x, y, I_{yz}, I_{zx}, I_{yt}, I_{tx})\,,$$

whereby the two quantities c and d in the functional derivatives (according to **Volterra**), which are still arbitrary, are fixed by the auxiliary conditions:

(15*) $$\left\{\begin{aligned} \frac{dz}{d\sigma} &= \frac{\partial H}{\partial \pi}\frac{dx}{d\sigma} + \frac{\partial H}{\partial \kappa}\frac{dy}{d\sigma}\,, \\ \frac{dt}{d\sigma} &= \frac{\partial H}{\partial \psi}\frac{dx}{d\sigma} + \frac{\partial H}{\partial \chi}\frac{dy}{d\sigma}\,. \end{aligned}\right.$$

If we appeal to the (**Hadamard-Lévy**) derivatives then we must calculate the normal derivatives z_n and t_n from the relations:

$$I_z^* = -\,\bar{f}_{z_n}\,, \qquad I_t^* = -\,\bar{f}_{t_n}\,,$$

and construct the function:

(16) $$K\,(I_z^*, I_t^*, z, z_s, t, t_s, x, y, x_s) = z_n\,\bar{f}_{z_n} + t_n\,\bar{f}_{t_n} - \bar{f}$$

by substituting the calculated expressions then the partial functional differential equation:

(17) $$I_n^* = K\,(I_z^*, I_t^*, z, z_s, t, t_s, x, y, x_s)$$

will emerge.

When we adapt the argument that posed in the previous section for the equations that were analogous to (15) and (17) to the present equations, that will imply that the extremal integral (when we imagine it as being determined by the space curve L, as we have up to now) is not the only integral of that partial functional differential equation, and the complication will arise of selecting precisely that special solution from among the set of all solutions.

In order to get that complication out of the way, we would like to consider two curves L_0 and L and assume that they are connected by an extremal that is in the interior of a (doubly-connected) region without singularities and bounded by those curves. The extremal integral:

$$I\,[L, L_0\,|\,] = \int_{L_0}^{L} E \int f(z_x, z_y, z, t_x, t_y, t, x, y)\,dx\,dy \tag{18}$$

that is extended over that double-connected region satisfies equations (15) [(17), resp.] for each point of the two curves L_0 and L, along with its functional derivatives.

If we keep the curve L_0 fixed and vary L then the set of all extremals that can be drawn from L_0 to all curves L will define a function field. The extremal integral (18) exhibits a covering of space by curves in the function field that defined an integral of the partial functional differential equation. If we also vary the curve L_0 then the function field and the associated curve covering will also vary, such that we will get a set of integrals that depends upon an arbitrary curve in R_4 . As we will explain more precisely later on (§ **3**), the function field will define a complete integral of equation (15) [(17), resp.] once L_0 is regarded as variable.

We can now solve the boundary-value problem for the **Euler-Lagrange** equations (3) by means of such a complete integral when we imagine that two curves L_0 and L are given as the boundary and choose the given curve L_0 to be the initial curve of the line function (18). From the boundary formula of the calculus of variations, the functional derivatives of the extremal integral I will always remain the same at each point of L_0 as long as the curve L remains on one and the same extremal of the function field. If we then form those derivatives with respect to the functions z and t and set them equal to new arbitrary functions ζ and τ:

$$I_z^* = \zeta(s_0)\,, \qquad I_t^* = \tau(s_0) \tag{19}$$

then every choice of those functions $\zeta(s_0)$ and $\tau(s_0)$ will correspond to a well-defined extremal. Should the extremal go through the given curve L, in particular, then the curve L would have to be introduced into the derivatives (19) of I, and the functions ζ and τ could then be determined from equations (19). With that, we will know the normal derivatives of the desired extremal on L_0, and we can then construct it from L_0 .

5) [sic] *The function field of the extremals and the partial functional differential equation.*

Instead of the special function field that consists of all extremals that start from a given curve L_0 in R_4, we can imagine that we are given a more general function field that is subject to only the condition that one and only one extremal of the function field should go through each curve L in the region considered. If we imagine that an "initial curve" L_0 is determined on each extremal of the function field then the extremal integral:

(20) $$W[L\,|\,] = \int_{L_0}^{L} E \int f(z_x, z_y, z, t_x, t_y, t, x, y)\, dx\, dy$$

will produce a covering of the space by curves.

In order to look for the condition for the line function $W[L\,|\,] = W[C, z(s), t(s)]$ thus-defined to satisfy the partial functional differential equation (17), we form the variation:

(21) $$\begin{aligned}\delta W = &\int_C \{(z_n\,\overline{f}_{z_n} + t_n\,\overline{f}_{t_n} - \overline{f})\,\delta n - \overline{f}_{z_n}\delta z - \overline{f}_{t_n}\delta t\}\,ds \\ &- \int_{C_0} \{(z_n\,\overline{f}_{z_n} + t_n\,\overline{f}_{t_n} - \overline{f})\,\delta n_0 - \overline{f}_{z_n}\delta z_0 - \overline{f}_{t_n}\delta t_0\}\,ds_0,\end{aligned}$$

in which $L_0 = [C_0, z_0(s_0), t_0(s_0)]$, and the variation of L_0 is determined by that of L, so the relations will exist:

(22) $$\begin{cases}\delta z_0(s_0) = \int_C \{\delta z_0'(s_0, s)\,\delta n + \delta z_0''(s_0, s)\,\delta z + \delta z_0'''(s_0, s)\,\delta t\}\,ds, \\ \delta t_0(s_0) = \int_C \{\delta t_0'(s_0, s)\,\delta n + \delta t_0''(s_0, s)\,\delta z + \delta t_0'''(s_0, s)\,\delta t\}\,ds, \\ \delta n_0(s_0) = \int_C \{\delta n_0'(s_0, s)\,\delta n + \delta n_0''(s_0, s)\,\delta z + \delta n_0'''(s_0, s)\,\delta t\}\,ds.\end{cases}$$

One then gets the functional derivatives of the line function $W[L\,|\,]$ from (21) as:

$$\begin{aligned}W_n^* &= z_n\,\overline{f}_{z_n} + t_n\,\overline{f}_{t_n} - \overline{f} - \int_{C_0} \{(z_n\,\overline{f}_{z_n} + t_n\,\overline{f}_{t_n} - \overline{f})\,\delta n_0' - \overline{f}_{z_n}\delta z_0' - \overline{f}_{t_n}\delta t_0'\}\,ds_0, \\ W_z^* &= -\overline{f}_{z_n} \qquad\qquad\qquad - \int_{C_0} \{(z_n\,\overline{f}_{z_n} + t_n\,\overline{f}_{t_n} - \overline{f})\,\delta n_0'' - \overline{f}_{z_n}\delta z_0'' - \overline{f}_{t_n}\delta t_0''\}\,ds_0, \\ W_t^* &= -\overline{f}_{t_n} \qquad\qquad\qquad - \int_{C_0} \{(z_n\,\overline{f}_{z_n} + t_n\,\overline{f}_{t_n} - \overline{f})\,\delta n_0''' - \overline{f}_{z_n}\delta z_0''' - \overline{f}_{t_n}\delta t_0'''\}\,ds_0.\end{aligned}$$

Therefore, if the line function $W[L\,|\,]$ is to satisfy the partial functional differential equation (17) then the integral in those formulas must vanish. That will happen when the curve L_0 satisfies the condition:

(23) $$(z_n\,\overline{f}_{z_n} + t_n\,\overline{f}_{t_n} - \overline{f})^{C_0}\,\delta n_0 - (\overline{f}_{z_n})^{C_0}\,\delta z_0 - (\overline{f}_{t_n})^{C_0}\,\delta t_0 = 0.$$

When we revert to the coordinates x and y, that will take the form:

$$\frac{dy}{ds_0}\Big[(f - z_x\,f_{z_x} - t_x\,f_{t_x})^{L_0}\,\delta x_0 - (z_y\,f_{z_x} + t_x\,f_{t_x})^{L_0}\,\delta y_0 + (f_{z_x})^{L_0}\,\delta z_0 + (f_{t_x})^{L_0}\,\delta t_0\Big]$$

$$- \frac{dx}{ds_0}\Big[-(z_x\, f_{z_y} + t_x\, f_{t_y})^{L_0}\,\delta x_0 + (f - z_y\, f_{z_y} - t_y\, f_{t_y})^{L_0}\,\delta y_0 + (f_{z_y})^{L_0}\,\delta z_0 + (f_{t_y})^{L_0}\,\delta t_0\Big] = 0\,.$$

Due to the arbitrariness in the curve L_0, it will then follow from this that one has the two equations for the directions of advance ([1]):

$$(23^*)\qquad \begin{cases} (f - z_x\, f_{z_x} - t_x\, f_{t_x})^{L_0}\,\delta x_0 - (z_y\, f_{z_x} + t_x\, f_{t_x})^{L_0}\,\delta y_0 + (f_{z_x})^{L_0}\,\delta z_0 + (f_{t_x})^{L_0}\,\delta t_0 = 0, \\ -(z_x\, f_{z_y} + t_x\, f_{t_y})^{L_0}\,\delta x_0 + (f - z_y\, f_{z_y} - t_y\, f_{t_y})^{L_0}\,\delta y_0 + (f_{z_y})^{L_0}\,\delta z_0 + (f_{t_y})^{L_0}\,\delta t_0 = 0. \end{cases}$$

6) The function fields and the boundary-value problem for the Euler-Lagrange equations.

There are always function fields that fulfill the condition (23). We will get one when we imagine that a hypersurface is given inside of the space R_4 that lies over the region of the (x, y)-plane that is bounded by two curves C_1 and C_2 and construct all extremals in such a way that they are "transversal" to that hypersurface, i.e., in the following way: We draw all closed curves on the hypersurface that cannot be contracted to a point on it. Let $L = (C, z(s), t(s))$ be any of those curves. We will then know the derivatives dz / ds and dt / ds on it immediately. The hyperplane at a point in question whose trace in the (x, y)-plane is normal to the projection C cuts out a two-dimensional surface from the given hypersurfaces on which $\delta z / \delta n$ (for constant t) and $\delta t / \delta n$ (for constant z) are determined. When we then lay an extremal through L such that the normal derivatives $z_n(s)$, $t_n(s)$ are calculated from the equations:

$$(24)\qquad \begin{cases} (z_n\, \bar f_{z_n} + t_n\, \bar f_{t_n} - \bar f)\,\delta n - \bar f_{z_n}\,\delta z = 0, \\ (z_n\, \bar f_{z_n} + t_n\, \bar f_{t_n} - \bar f)\,\delta n - \bar f_{t_n}\,\delta z = 0, \end{cases}$$

the condition (23) will be fulfilled for every extremal on the initial curve. Hence, the curve covering through the extremal integral in the function field that is constructed will define a solution to the partial functional differential equation (17).

In order to arrive at a solution to the boundary-value problem for the **Euler-Lagrange** equations, we must have a complete integral of equation (17) at our disposal. We will obtain such

---

([1]) If we were to select a parametric field from the function field then the coefficients of the increments in these equations would be functions of position, i.e., functions of x, y, z, t. The two equations would then represent two total linear differential equations for the two functions x and y of the variables z and t. In order for them to be completely integrable and determine a two-parameter family of curves, namely, the *transversals* of the field:

$$x = x(z, t, \lambda, \mu)\,, \qquad\qquad y = y(z, t, \lambda, \mu)$$

(where λ and μ are two parameters), two integrability conditions that are easy to give must be fulfilled. That suggests the question of whether those two integrability conditions, together with the differential equations for the slope functions of the field (which are also derived immediately from the **Euler-Lagrange** equations here), say that the parametric field is an "independence field," as was true in the first section. Meanwhile, certain terms appear in the integrability conditions that make it seem that the answer to that question is not very simple.

a thing in a function field that depends upon two arbitrary functions $\alpha(u)$ and $\beta(u)$ in a certain interval (say, from 0 to 1), e.g., that would be fulfilled in the example above when the hypersurface depends upon those two functions.

In general, let:

$$I = I[L, \alpha(\overset{1}{\underset{0}{u}}), \beta(\overset{1}{\underset{0}{u}})|] \tag{25}$$

be a curve covering of a function field that represents a complete integral of (17). Naturally, the extremals, as well as the initial curves L_0, will then vary with $\alpha(u)$ and $\beta(u)$. If we fix a certain curve L then we will get the variation of $I[L, \alpha(\overset{1}{\underset{0}{u}}), \beta(\overset{1}{\underset{0}{u}})|]$ with $\alpha(u)$ and $\beta(u)$ from the boundary formula of the calculus of variations:

$$\begin{aligned} \delta I &= \int_0^1 \{I_\alpha\,\delta\alpha + I_\beta\,\delta\beta\}\,du \\ &= \int_{C_0} \{(z_n\,\bar{f}_{z_n} + t_n\,\bar{f}_{t_n} - \bar{f})\,\delta n_0^* - \bar{f}_{z_n}\delta z_0^* - \bar{f}_{t_n}\,\delta t_0^*\}\,ds_0\,, \end{aligned} \tag{26}$$

in which:

$$\begin{cases} \delta n_0^* = \int_0^1 [\delta n_0^{(\alpha)}(s_0, u)\cdot\delta\alpha + \delta n_0^{(\beta)}(s_0, u)\cdot\delta\beta]\,du\,, \\ \delta z_0^* = \int_0^1 [\delta z_0^{(\alpha)}(s_0, u)\cdot\delta\alpha + \delta z_0^{(\beta)}(s_0, u)\cdot\delta\beta]\,du\,, \\ \delta t_0^* = \int_0^1 [\delta t_0^{(\alpha)}(s_0, u)\cdot\delta\alpha + \delta t_0^{(\beta)}(s_0, u)\cdot\delta\beta]\,du\,, \end{cases} \tag{27}$$

such that the functional derivatives of I with respect to α and β will be:

$$\begin{cases} I_\alpha = \int_{C_0} \{(z_n\,\bar{f}_{z_n} + t_n\,\bar{f}_{t_n} - \bar{f})\,\delta n_0^{(\alpha)} - \bar{f}_{z_n}\delta z_0^{(\alpha)} - \bar{f}_{t_n}\,\delta t_0^{(\alpha)}\}\,ds_0\,, \\ I_\beta = \int_{C_0} \{(z_n\,\bar{f}_{z_n} + t_n\,\bar{f}_{t_n} - \bar{f})\,\delta n_0^{(\beta)} - \bar{f}_{z_n}\delta z_0^{(\beta)} - \bar{f}_{t_n}\,\delta t_0^{(\beta)}\}\,ds_0\,. \end{cases} \tag{28}$$

Those derivatives will have the same value for all curves L that lie on the same extremal as long as the abscissa u with respect to which the functional derivatives are taken is fixed. The extremals are then characterized by the equations:

$$I_\alpha[L, \alpha(\overset{1}{\underset{0}{u}}), \beta(\overset{1}{\underset{0}{u}})|] = c^{(\alpha)}(u)\,, \qquad I_\beta[L, \alpha(\overset{1}{\underset{0}{u}}), \beta(\overset{1}{\underset{0}{u}})|] = c^{(\beta)}(u)\,. \tag{29}$$

The boundary-value problem for two given curves L_1 and L_2 is the dealt with in the older way on pp. 43. For one of the function fields, we form the two values:

$$I_1 = I[L_1, \alpha(\overset{1}{\underset{0}{u}}), \beta(\overset{1}{\underset{0}{u}})|] \qquad \text{and} \qquad I_2 = I[L_2, \alpha(\overset{1}{\underset{0}{u}}), \beta(\overset{1}{\underset{0}{u}})|] \ .$$

Should the two curves L_1 and L_2 lie on the same extremal, then the equations:

$$(I_1)_\alpha - (I_2)_\alpha = 0 \ , \qquad (I_1)_\beta - (I_2)_\beta = 0 \tag{30}$$

would have to exist for all values of u in the interval from 0 to 1. From that, we would calculate the functions $\alpha(u)$ and $\beta(u)$ in the interval from 0 to 1, and we would have then determined the function field in which the extremal that goes through L_1 simultaneously goes through L_2 .

§ 2.

The Euler-Lagrange equations and the associated canonical system.

1) The Euler-Lagrange equations and the canonical system.

If we would now like to treat the **Euler-Lagrange** equations:

$$\begin{aligned} &\frac{\partial}{\partial x}(f_{z_x}) + \frac{\partial}{\partial y}(f_{z_y}) + f_z = 0, \\ &\frac{\partial}{\partial x}(f_{t_x}) + \frac{\partial}{\partial y}(f_{t_y}) - f_t = 0 \end{aligned} \tag{1}$$

from the standpoint of the *general* theory of integration then the boundary-value problem that was the center of focus in § **1** will be replaced with the problem of determining an extremum such that not only are the values of the functions given:

$$z = z(s) \ , \qquad t = t = t(s) \tag{2}$$

along a given curve C, but they must also possess prescribed normal derivatives:

$$\frac{dz}{dn} = z_n(s) \ , \quad \frac{dt}{dn} = t_n(s) \ . \tag{2*}$$

Such an extremal can be constructed by **Cauchy**'s process *des calcul des limites*, because from (2) and (2*), the four first-order partial derivatives z_x, z_y, t_x, t_y are known along C, in addition to z and t.

By applying the **Legendre** transformation:

(3) $$\pi = f_{z_x}\,, \qquad \kappa = f_{z_y}\,, \qquad \psi = f_{t_x}\,, \qquad \chi = f_{t_y}\,,$$

we can convert the system (1) of two second-order partial differential equations into the following canonical system of six first-order partial differential equations:

(4) $$\begin{cases} z_x = \dfrac{\partial H}{\partial \pi}\,, \quad z_y = \dfrac{\partial H}{\partial \kappa}\,, \\ \pi_x + \kappa_y = -\dfrac{\partial H}{\partial z}\,, \\ t_x = \dfrac{\partial H}{\partial \psi}\,, \quad t_y = \dfrac{\partial H}{\partial \chi}\,, \\ \psi_x + \chi_y = -\dfrac{\partial H}{\partial t}\,, \end{cases}$$

in which we have introduced:

$$H(z, t, x, y, \pi, \kappa, \psi, \chi) = z_x\, f_{z_x} + z_y\, f_{z_y} + t_x\, f_{t_x} + t_y\, f_{t_y} - f\,,$$

in agreement with equation (14) on pp. 69.

We again have the complication with this *first canonical system* that the values of all six unknown functions cannot be given arbitrarily on a plane curve C, but they must satisfy two integrability conditions.

In order to avoid those auxiliary conditions, we introduce a *second canonical system*. In order to do that, we appeal to the natural derivatives z_s, z_n, t_s, t_n of z and t on C, under which, f will go to the function $\bar{f}(z_s, z_n, z, t_s, t_n, t, x, y, x_s)$.

If eliminate the derivatives z_n and t_n by means of:

$$\varphi = -\,\bar{f}_{z_n}\,, \qquad \omega = -\,\bar{f}_{t_n}$$

and introduce the new function:

$$K(\varphi, \omega, z, z_s, t, t_s, x, y, x_s) = z_n\, \bar{f}_{z_n} + t_n\, \bar{f}_{t_n} - \bar{f}\,,$$

in agreement with equation (15) on pp. 69, then we will get the second canonical system:

$$
(5) \qquad \begin{cases}
\delta z = -\dfrac{\partial K}{\partial \varphi}\,\delta n\,, \\[2mm]
\delta \varphi = -\dfrac{d}{ds}(K_{z_s}\,\delta n) + (k\,\varphi + K_s)\,\delta n\,, \\[2mm]
\delta t = -\dfrac{\partial K}{\partial \omega}\,\delta n\,, \\[2mm]
\delta \omega = -\dfrac{d}{ds}(K_{t_s}\,\delta n) + (k\,\omega + K_t)\,\delta n\,.
\end{cases}
$$

That system can be regarded as a system of four total functional differential equations, so the extremals can be determined by the **Cauchy-Lipschitz** process or the method of successive approximation in that way of looking at things.

2) The characteristic "relative invariant" and the independence field.

The following considerations are again initially connected with the **Euler-Lagrange** equations themselves. We draw a closed curve L and consider the integral that extends over L:

$$
\delta R\,[L\,|\,] = \int_L \Big\{ (f - z_x\, f_{z_x} - z_y\, f_{z_y} - t_x\, f_{t_x} - t_y\, f_{t_y}) \left(\delta x \frac{dy}{d\sigma} - \delta y \frac{dx}{d\sigma} \right)
$$
$$
+\; f_{z_x}\left(\delta z \frac{dy}{d\sigma} - \delta y \frac{dz}{d\sigma} \right) + f_{z_y}\left(\delta x \frac{dz}{d\sigma} - \delta z \frac{dx}{d\sigma} \right) + f_{t_x}\left(\delta t \frac{dy}{d\sigma} - \delta y \frac{dt}{d\sigma} \right) + f_{t_x}\left(\delta x \frac{dt}{d\sigma} - \delta t \frac{dx}{d\sigma} \right) \Big\}\, d\sigma\,,
$$

which is coupled with the boundary formula (5) on pp. 66, and in which the δx, δy, δz, δt initially mean arbitrary increments.

We then imagine that parametric field of extremals is given:

$$
(7) \qquad z = z\,(x, y, a, b)\,, \qquad t = t\,(x, y, a, b)
$$

and intersect that field with a certain (tubular) hypersurface. The extremals of the field cut the hypersurface in a certain curve, and we would like to think that the hypersurface is determined such that all intersection curves are closed curves. (That assumption shall just express the addition of the word "tubular.") From the fundamental property of the field, the hypersurface of intersection curves will be filled up by extremals simply and without gaps. In that way, the curves of the two two-parameter families of curves are in one-to-one correspondence with each other by way of the extremal of the field that connects two associated curves.

On the first hypersurface, we imagine that a one-parameter family of curves has been selected from the two-parameter family that defines a closed surface (M_2) in the (three-dimensional) hypersurface. Let the parameter for that one-parameter family be α. On the second hypersurface, that one-parameter will be associated with a one-parameter family of curves that likewise defines a closed surface.

Two associated curves L_1 (α) and L_2 (α) of the two families of curves bound a certain surface patch on the field extremals that connect them, so the associated value of the extremal integral takes the form of a function of α:

$$I(\alpha) = \int_{L_1}^{L_2} E \int_{(\alpha)}^{(\alpha)} f(z_x, z_y, z, t_x, t_y, t, x, y)\, dx\, dy \qquad \begin{bmatrix} z = z(x, y, a(\alpha), b(\alpha)) \\ t = t(x, y, a(\alpha), b(\alpha)) \end{bmatrix}.$$

From the boundary formula in the calculus of variations, its derivative with respect to α is:

$$\text{(8)} \qquad \frac{\delta I}{\delta\alpha} = \frac{\delta R[L_2(\alpha)\,|]}{\delta\alpha} - \frac{\delta R[L_1(\alpha)\,|]}{\delta\alpha}.$$

If we integrate that equation over the closed surface of the curve L (α) then:

$$\oint \frac{\delta I}{\delta\alpha}\,\delta\alpha = 0$$

and that will imply that:

$$\oint \frac{\delta R[L_2(\alpha)\,|]}{\delta\alpha}\,\delta\alpha = \oint \frac{\delta R[L_1(\alpha)\,|]}{\delta\alpha}\,\delta\alpha,$$

i.e., the following integral is constant:

$$\oint \delta\alpha \int_{L(\alpha)} \left\{ (f - z_x f_{z_x} - z_y f_{z_y} - t_x f_{t_x} - t_y f_{t_y}) \left(\frac{\delta x}{\delta\alpha}\frac{dy}{d\sigma} - \frac{\delta y}{\delta\alpha}\frac{dx}{d\sigma} \right) + f_{z_x} \left(\frac{\delta z}{\delta\alpha}\frac{dy}{d\sigma} - \frac{\delta y}{\delta\alpha}\frac{dz}{d\sigma} \right) \right.$$

$$\left. + f_{z_y} \left(\frac{\delta x}{\delta\alpha}\frac{dz}{d\sigma} - \frac{\delta z}{\delta\alpha}\frac{dx}{d\sigma} \right) + f_{t_x} \left(\frac{\delta t}{\delta\alpha}\frac{dy}{d\sigma} - \frac{\delta y}{\delta\alpha}\frac{dt}{d\sigma} \right) + f_{t_y} \left(\frac{\delta x}{\delta\alpha}\frac{dt}{d\sigma} - \frac{\delta t}{\delta\alpha}\frac{dx}{d\sigma} \right) \right\} d\sigma = \text{const.},$$

no matter how the tubular hypersurface might lie in the field. We express that fact by saying that we call the integral (6) the element of a "relative integral invariant."

When we select a one-parameter family L (a) from that two-parameter family of intersection curves of the field with the hypersurface then we will, at the same time, select a one-parameter family of extremals from the field (7) that fills up a certain three-dimensional (curved) space simply and without gaps. Naturally, that three-dimensional space includes the two-dimensional surface T that is defined by the curves L (α). If we define the curved three-dimensional space in ordinary linear three-dimensional space then the surface T will be either mapped as a closed surface again, so a type of toral surface, or its map will be a non-closed (i.e., bounded) surface. In the former case, the constant in (9) is equal to zero, as was explained on pp. 50, while in the latter case, it will generally be non-zero.

If we introduce x and y as integration variables in formula (9) then we will get:

(10) $$\iint \left\{ f + \left(\frac{\delta z}{\delta x} - z_x \right) f_{z_x} + \left(\frac{\delta z}{\delta y} - z_y \right) f_{z_y} + \left(\frac{\delta t}{\delta x} - t_x \right) f_{t_x} + \left(\frac{\delta t}{\delta y} - t_y \right) f_{t_y} \right\} dx\,dy = \text{const.},$$

and the **Hilbert** integral that appears on the left extends over a closed surface. We then find that the field in question (7) is an independence field if and only if the constant in the formula (9) is also equal to zero for all closed surfaces of the second kind.

It further emerges from the argument above that we can, with no loss of generality, choose the surface T that carries the curves L (α) in the relative integral invariant (9) in the argument above such that it lies on a "cylindrical hypersurface C," i.e., on a hypersurface that consists of the planes:

$$x = \text{const.}, \qquad y = \text{const.}$$

that might lie through the individual points of a closed curve C in the (x, y)-plane. For such a cylindrical hypersurface C, formula (9) will assume the simpler form:

(11) $$\int_{\bigcirc} \delta\alpha \int_C \left\{ \left(f_{z_x} \frac{dy}{ds} - f_{z_y} \frac{dx}{ds} \right) \frac{\delta z}{\delta\alpha} + \left(f_{t_x} \frac{dy}{ds} - f_{t_y} \frac{dx}{ds} \right) \frac{\delta t}{\delta\alpha} \right\} ds = \text{const.}$$

The given field will be an independence field if and only if the constant in that formula is equal to zero for any one-parameter family of curves L (α) that defined a closed surface on the cylindrical hypersurface C.

3) The associated "absolute integral invariants" and the "Jacobi equations."

If we write formula (11) in the form:

$$\int_C ds \int_{\bigcirc} \left\{ \left(f_{z_x} \frac{\delta z}{\delta\alpha} + f_{z_y} \frac{\delta t}{\delta\alpha} \right) \frac{dy}{ds} - \left(f_{t_x} \frac{\delta z}{\delta\alpha} + f_{t_y} \frac{\delta t}{\delta\alpha} \right) \frac{dx}{ds} \right\} \delta\alpha = \text{const.}$$

then we can convert the inner integral, which represents a curve integral over the curve $a = a$ (α), $b = b$ (α) in the plane s = const. on the cylindrical hypersurface, into a double integral using **Stokes**'s theorem:

$$\int_C ds \iint \left\{ \left[\frac{\partial}{\partial b} \left(f_{z_x} \frac{\partial z}{\partial a} + f_{t_x} \frac{\partial t}{\partial a} \right) - \frac{\partial}{\partial a} \left(f_{t_x} \frac{\partial z}{\partial b} + f_{t_y} \frac{\partial t}{\partial b} \right) \right] \frac{dy}{ds} \right.$$
$$\left. - \left[\frac{\partial}{\partial b} \left(f_{z_y} \frac{\partial z}{\partial a} + f_{t_y} \frac{\partial t}{\partial a} \right) - \frac{\partial}{\partial a} \left(f_{z_y} \frac{\partial z}{\partial b} + f_{t_y} \frac{\partial t}{\partial b} \right) \right] \frac{dx}{ds} \right\} da\,db = \text{const.}$$

and thus obtain an "absolute integral invariant." Therefore, the element of that invariant:

(12)
$$\int_C ds\left\{\left[\frac{\partial z}{\partial a}\frac{\partial f_{z_x}}{\partial b}+\frac{\partial t}{\partial a}\frac{\partial f_{t_x}}{\partial b}-\frac{\partial z}{\partial b}\frac{\partial f_{z_x}}{\partial a}-\frac{\partial t}{\partial b}\frac{\partial f_{t_x}}{\partial a}\right]\frac{dy}{ds}\right.$$
$$\left.-\left[\frac{\partial z}{\partial a}\frac{\partial f_{z_y}}{\partial b}+\frac{\partial t}{\partial a}\frac{\partial f_{t_y}}{\partial b}-\frac{\partial z}{\partial b}\frac{\partial f_{z_y}}{\partial a}-\frac{\partial t}{\partial b}\frac{\partial f_{t_y}}{\partial a}\right]\frac{dx}{ds}\right\}ds$$

will be constant, no matter how the curve C might be chosen, as long as we remain on a certain extremal of the field.

The relation (12) expresses a property of the **Jacobi** equations that belongs to the **Euler-Lagrange** equations (1). If Φ ($\mathfrak{z}_x$, $\mathfrak{z}_y$, $\mathfrak{z}$, $\mathfrak{t}_x$, $\mathfrak{t}_x$, $\mathfrak{t}$, x, y), as the second variation of the function f, is the following quadratic form with 21 terms:

(13)
$$2\Phi = f_{z_x z_x}\,\mathfrak{z}_x^2 + 2 f_{z_x z_y}\,\mathfrak{z}_x\,\mathfrak{z}_y + f_{z_y z_y}\,\mathfrak{z}_y^2 + f_{t_x t_x}\,\mathfrak{t}_x^2 + \cdots + f_{zz}\,\mathfrak{z}^2 + 2 f_{zt}\,\mathfrak{z}\cdot\mathfrak{t} + f_{tt}\,\mathfrak{t}^2$$

then the **Jacobi** equations will read:

(14)
$$\begin{cases} \dfrac{\partial}{\partial x}(\Phi_{\mathfrak{z}_x}) + \dfrac{\partial}{\partial y}(\Phi_{\mathfrak{z}_y}) - \Phi_{\mathfrak{z}} = 0, \\[2ex] \dfrac{\partial}{\partial x}(\Phi_{\mathfrak{t}_x}) + \dfrac{\partial}{\partial y}(\Phi_{\mathfrak{t}_y}) - \Phi_{\mathfrak{t}} = 0. \end{cases}$$

We will immediately get two systems of solutions to the **Jacobi** equations (14) from the derivatives of the functions (7) that define the field in question with respect to the parameters:

(15)
$$\mathfrak{z}^{(1)} = \frac{\partial z}{\partial a}, \qquad \mathfrak{t}^{(1)} = \frac{\partial t}{\partial a}; \qquad \mathfrak{z}^{(2)} = \frac{\partial z}{\partial b}, \qquad \mathfrak{t}^{(2)} = \frac{\partial t}{\partial b}.$$

One further sees directly that:

$$\frac{\partial f_{z_x}}{\partial a} = \Phi_{\mathfrak{z}_x^{(1)}}, \quad \frac{\partial f_{z_x}}{\partial b} = \Phi_{\mathfrak{z}_x^{(2)}}, \quad \text{etc.}$$

From equation (12), the following relation exists between two systems of solutions of the **Jacobi** equations (14):

(16)
$$\int_C \left\{ (\mathfrak{z}^{(1)}\,\Phi_{\mathfrak{z}_x^{(2)}} + \mathfrak{t}^{(1)}\,\Phi_{\mathfrak{t}_x^{(2)}} - \mathfrak{z}^{(2)}\,\Phi_{\mathfrak{z}_x^{(1)}} - \mathfrak{t}^{(2)}\,\Phi_{\mathfrak{t}_x^{(1)}})\frac{dy}{ds}\right.$$
$$\left. -\,(\mathfrak{z}^{(1)}\,\Phi_{\mathfrak{z}_y^{(2)}} + \mathfrak{t}^{(1)}\,\Phi_{\mathfrak{t}_y^{(2)}} - \mathfrak{z}^{(2)}\,\Phi_{\mathfrak{z}_y^{(1)}} - \mathfrak{t}^{(2)}\,\Phi_{\mathfrak{t}_y^{(1)}})\frac{dx}{ds}\right\} ds = \text{const.}$$

i.e., the **Jacobi** system (14) is self-adjoint.

The solutions (15) of the **Jacobi** equations that were just employed are only a special case of more general solutions that can be obtained as solutions to the **Euler-Lagrange** equations that do not depend upon parameters, but on arbitrary functions. If we have a system of solutions of equations (1) that depend upon two arbitrary functions $z_0(s_0)$, $t_0(s_0)$ on a fixed curve C_0 in the (x, y)-plane:

$$z = Z(x, y; C_0, z_0(s_0), t_0(s_0)), \qquad t = T(x, y; C_0, z_0(s_0), t_0(s_0)) \tag{17}$$

then the variations:

$$\begin{cases} \delta z = \int_{C_0} (Z_{z_0}\ \delta z_0 + Z_{t_0}\ \delta t_0)\, ds_0 , \\ \delta t = \int_{C_0} (T_{z_0}\ \delta z_0 + T_{t_0}\ \delta t_0)\, ds_0 \end{cases} \tag{18}$$

will represent solutions of the **Jacobi** equations, and due to their linear character, the functional derivatives will also be themselves solutions. The solutions (15) are included in (18), and they will be produced when we select a two-parameter family like (7) from the set of functions (17) that defines a parametric field.

It still remains for us to give the canonical form equations of (14). If we introduce the natural derivatives along the curve then the quadratic form (13) will take the form:

$$2\bar{\Phi} = \bar{f}_{z_n z_n}\, \mathfrak{z}_n^2 + 2\bar{f}_{z_n z_s}\, \mathfrak{z}_n\, \mathfrak{z}_s + \bar{f}_{z_n z_s}\, \mathfrak{z}_s^2 + \cdots + \bar{f}_{zz}\, \mathfrak{z}^2 + 2\bar{f}_{zt}\, \mathfrak{z} \cdot \mathfrak{t} + \bar{f}_{tt}\, \mathfrak{t}^2 . \tag{19}$$

If we then introduce the new variables:

(21)[*sic*]
$$\mathfrak{f} = -\bar{\Phi}_{\mathfrak{z}_n} , \qquad \mathfrak{w} = -\bar{\Phi}_{\mathfrak{t}_n}$$

and define the canonical function:

$$\begin{aligned} &2X(\mathfrak{f}, \mathfrak{w}, \mathfrak{z}_s, \mathfrak{z}, \mathfrak{t}_s, \mathfrak{t}, x, y, x_s) \\ &= \frac{\partial^2 K}{\partial \varphi^2}\, \mathfrak{f}^2 + 2\frac{\partial^2 K}{\partial \varphi\, \partial z_s}\, \mathfrak{f}\, \mathfrak{z}_s + \frac{\partial^2 K}{\partial z_s^2}\, \mathfrak{z}_s^2 + \frac{\partial^2 K}{\partial \omega^2}\, \mathfrak{w}^2 + \cdots + \frac{\partial^2 K}{\partial z^2}\, \mathfrak{z}^2 + 2\frac{\partial^2 K}{\partial z\, \partial t}\, \mathfrak{z}\, \mathfrak{t} + \frac{\partial^2 K}{\partial t^2}\, \mathfrak{t}^2 \end{aligned} \tag{21}$$

by eliminating $\mathfrak{z}_n$ and $\mathfrak{t}_n$, which is likewise a quadratic form with 21 terms, then we will get:

(22)
$$\begin{cases} \delta\mathfrak{z} = -\dfrac{\partial X}{\partial \mathfrak{f}}\,\delta n, \\ \delta\mathfrak{f} = -\dfrac{d}{ds}\left(\dfrac{\partial X}{\partial \mathfrak{z}_s}\,\delta n\right) + \left(k \cdot \mathfrak{f} + \dfrac{\partial X}{\partial \mathfrak{z}}\right)\delta n, \\ \delta\mathfrak{t} = -\dfrac{\partial X}{\partial \mathfrak{w}}\,\delta n, \\ \delta\mathfrak{w} = -\dfrac{d}{ds}\left(\dfrac{\partial X}{\partial \mathfrak{t}_s}\,\delta n\right) + \left(k \cdot \mathfrak{w} + \dfrac{\partial X}{\partial \mathfrak{t}}\right)\delta n \end{cases}$$

as a second canonical form of the **Jacobi** equations (14). That system is self-adjoint, i.e., one has:

(23)
$$\int_C \{\mathfrak{z}^{(1)}\mathfrak{f}^{(2)} - \mathfrak{z}^{(2)}\mathfrak{f}^{(1)} + \mathfrak{t}^{(1)}\mathfrak{w}^{(2)} - \mathfrak{t}^{(2)}\mathfrak{w}^{(1)}\}\,ds = \text{const.}$$

for any two solutions of the system (22).

4) The canonical system as a contact transformation. The transformation of the set of all extremals into itself.

We can regard the canonical system (5) as an infinitesimal contact transformation in the realm of line functions in the same way as in the previous section for the variational problem with one dependent variable, and the extremal integral plays the role of the characteristic line function in that. Here, we have the following relation for the extremals:

(24)
$$d'I = \int_{C_2} (\varphi_2\, d'z_2 + \omega_2\, d't_2)\, ds_2 - \int_{C_1} (\varphi_1\, d'z_1 + \omega_1\, d't_1)\, ds_1,$$

from the boundary formula for the calculus of variations, in which one has:

(25)
$$d'I = \int \underset{C_1}{\overset{C_2}{E}} \int d'f\,(z_x, z_y, z, t_x, t_y, t, x, y)\, dx\, dy\,.$$

Upon contracting the curves C_1 and C_2, it will follow from this that:

(26)
$$-\int_C \delta\,(\varphi\, d'z + \omega\, d't)\, ds = \int_C d'f\; \delta n\, ds\,,$$

and by a conversion that is analogous to the one on pp. 54, we will find that:

(27)
$$\int_C \{\delta\varphi\, d'z - \delta z\, d'\varphi + \delta\omega\, d't - \delta t\, d'\omega - k\,(\varphi\, d'z + \omega\, d't)\}\, ds = \int_C d'K\; \delta n\, ds\,,$$

such that we will obtain precisely the second canonical system (5) by comparing the coefficients in (27).

If we further look for the transformation group that permutes the individual extremals of the set of extremals amongst themselves in the same way as on pp. 56 then we will get infinitesimal displacements for the individual points of a curve $[C, z(s), \varphi(s), t(s), \omega(s)]$ that lies on a certain extremal in the form of:

$$(28)\qquad \begin{cases} \delta z = \mathfrak{a}[C, z(s), \varphi(s), t(s), \omega(s) \,|; s]\,\delta\alpha, \\ \delta t = \mathfrak{b}[C, z(s), \varphi(s), t(s), \omega(s) \,|; s]\,\delta\alpha, \\ \delta\varphi = \mathfrak{m}[C, z(s), \varphi(s), t(s), \omega(s) \,|; s]\,\delta\alpha, \\ \delta\omega = \mathfrak{n}[C, z(s), \varphi(s), t(s), \omega(s) \,|; s]\,\delta\alpha, \end{cases}$$

in which:

$$(29)\qquad \mathfrak{z} = \mathfrak{a}, \quad \mathfrak{f} = \mathfrak{m}, \quad \mathfrak{t} = \mathfrak{b}, \quad \mathfrak{w} = \mathfrak{n}$$

must be solutions of the **Jacobi** equations (22). It will follow from the fact that the relative integral invariant whose elements is:

$$\int_C (\varphi\, d'z + \omega\, d't)\, ds$$

must remain such a thing under the transformation that the functional on the right-hand side of (28) must be derivable from a line function:

$$Q\,[C, z(s), \varphi(s), t(s), \omega(s) \,|\,]$$

as its functional derivatives:

$$(30)\qquad \begin{cases} \mathfrak{a} = Q_\varphi[C, z(s), \varphi(s), t(s), \omega(s) \,|; s], \\ \mathfrak{b} = Q_\omega[C, z(s), \varphi(s), t(s), \omega(s) \,|; s], \\ \mathfrak{m} = -Q_z[C, z(s), \varphi(s), t(s), \omega(s) \,|; s], \\ \mathfrak{n} = -Q_t[C, z(s), \varphi(s), t(s), \omega(s) \,|; s]. \end{cases}$$

The two conditions (29) and (30) collectively say that one must have:

$$(31)\qquad Q\,[C, z(s), \varphi(s), t(s), \omega(s) \,|\,] = \text{const.}$$

for a variation of the curve $[C, z(s), \varphi(s), t(s), \omega(s) \,|\,]$ on the same extremal, i.e., that the line function Q must be an integral of the canonical system (5), with the terminology on pp. 58, and that entire argument can also be inverted.

Finally, as an analogue of **Poisson** theorem, one again has the theorem that a new integral can be derived from two integrals of the canonical system:

(32)
$$\begin{cases} Q^{(1)}[C, z(s), \varphi(s), t(s), \omega(s)|] = \text{const.}, \\ Q^{(2)}[C, z(s), \varphi(s), t(s), \omega(s)|] = \text{const.} \end{cases}$$

That is because for the systems of solutions of the **Jacobi** equations:

$$\begin{aligned} &\mathfrak{z}^{(1)} = Q_\varphi^{(1)}, \quad \mathfrak{f}^{(1)} = -\,Q_z^{(1)}, \quad \mathfrak{t}^{(1)} = Q_\omega^{(1)}, \quad \mathfrak{w}^{(1)} = -\,Q_t^{(1)}, \\ &\mathfrak{z}^{(2)} = Q_\varphi^{(2)}, \quad \mathfrak{f}^{(2)} = -\,Q_z^{(2)}, \quad \mathfrak{t}^{(2)} = Q_\omega^{(2)}, \quad \mathfrak{w}^{(2)} = -\,Q_t^{(2)}, \end{aligned}$$

which one gets from (32), (23) will imply the relation:

(33)
$$\int_C \{Q_\varphi^{(1)} Q_z^{(2)} - Q_\varphi^{(2)} Q_z^{(1)} + Q_\omega^{(1)} Q_t^{(2)} - Q_\omega^{(2)} Q_t^{(1)}\} ds = \text{const.}$$

§ 3.

Integrating partial functional differential equations by means of an extremal integral.

Here as well we would like to briefly go into the inversion of the argument in § **1** as part of a systematic treatment of the subject, so the reduction of the integration of a partial functional differential equation to the solution of an associated variational problem by a direct study of its **Euler-Lagrange** equations.

One then deals with the determination of a line function $V[L|]$ from the equation:

(1)
$$V_{xy} = H(x, y, z, t, V_{yz}, V_{zx}, V_{yt}, V_{tx})$$

when the auxiliary conditions:

(1*)
$$\begin{cases} \dfrac{dz}{d\sigma} = \dfrac{\partial H}{\partial \pi}\dfrac{dx}{d\sigma} + \dfrac{\partial H}{\partial \kappa}\dfrac{dy}{d\sigma} & [\pi = V_{yz}, \quad \kappa = V_{zx} \\ \dfrac{dt}{d\sigma} = \dfrac{\partial H}{\partial \psi}\dfrac{dx}{d\sigma} + \dfrac{\partial H}{\partial \chi}\dfrac{dy}{d\sigma} & \psi = V_{yt}, \quad \chi = V_{tx}] \end{cases}$$

fix the functional derivatives completely. In place of it, we can also eliminate the auxiliary conditions for the functional derivatives and use the equation:

(2)
$$V_n^* = K(V_z^*, V_t^*, z, z_s, t, t_s, x, y, x_s)$$

as our basis, although the function K in that must satisfy certain conditions.

In analogy to what was done on pp. 60, we can use the **Legendre** transformation on H (K, resp.) to determine a function $f(z_x, z_y, z, t_x, t_y, t, x, y)$ that we will then choose to be the integrand in a variational problem. The associated extremal integral will represent a complete integral of (1) [(2), resp.] for an arbitrary initial curve L_0, which is analogous to what was said on pp. 61.

The fact that *every* integral $V[L \mid]$ of those equations can be represented by the value of the extremal integral in a function field follows from the fact that an integral is determined uniquely when we know its value on a cylindrical surface that belongs to a curve C_0 in the (x, y)-plane:

$$V[C_0, z(s), t(s) \mid] = \Phi[z(s), t(s) \mid] . \tag{4}$$

Namely, if we form the functional derivatives of Φ with respect to z and t, determine the functions z_n and t_n from the equations:

$$-\bar{f}_{z_n} = \Phi_z , \qquad -\bar{f}_{t_n} = \Phi_t , \tag{5}$$

and lay the extremal of the variational problem that belongs to the function (2) through each curve $z(s)$, $t(s)$ that belongs to C_0 , and whose normal derivatives are the functions $z_n(s)$ and $t_n(s)$ that were just determined, then we will have a well-defined function field, and the curve covering of that field will provide us with the desired solution to the partial functional differential equation (1) in the form:

$$V[L \mid] = \Phi[(C_0), z(s), t(s) \mid] + \int \overset{L_1}{\underset{L_9}{E}} \int f(z_x, z_y, z, t_x, t_y, t, x, y)\, dx\, dy . \tag{6}$$

That line function does, in fact, reduce to the desired functions (4) for $C = C_0$, and will satisfy equation (1), which would follow from the boundary formula for the calculus of variations, in a manner that is analogous to what was done on pp. 72. It will then follow immediately from the uniqueness of an integral that any given integral of (1) can always be represented as a curve covering of a function field. At the same time, the process shows that an integral that still includes arbitrary functions can be represented by a family of function fields that correspondingly vary with those arbitrary functions. We can also obtain a "general" integral from such a "complete" integral that depends upon two arbitrary functions by assuming that an arbitrary functional relationship exists between the two arbitrary functions, and then construct the "envelope" ([1]) of the family of function fields that then depends upon an arbitrary function. The integration of equation (1) is the reduced to the determination of the extremals of the associated variational problem, namely, the "characteristics" of the given equation.

In conclusion, we would like to derive yet another property of the function fields that represent integrals of the partial functional differential equations (1) that is, in a certain sense, an analogue of the theorem of **A. Mayer** that was mentioned on pp. 18.

If we select a parametric field:

$$z = z(x, y, z, b) , \qquad t = t(x, y, a, b) \tag{3}$$

([1]) **P. Lévy**, *loc. cit.* (pp. 1), pp. 156.

from the function field then a line function that is defined in the function field will go to a function of the plane curve C and the two parameters a and b.

The line function (4) that is considered for a fixed curve C will become a function of the two parameters Φ (a, b). If the function field is determined in the way that was given on the previous page then we will have:

$$\text{(4)} \qquad \begin{aligned} \frac{\partial \Phi}{\partial a} &= \int_{C_0} \left\{ \left(f_{z_x} \frac{\partial z}{\partial a} + f_{t_x} \frac{\partial t}{\partial a} \right) \frac{dy}{ds} - \left(f_{z_y} \frac{\partial z}{\partial a} + f_{t_y} \frac{\partial t}{\partial a} \right) \frac{dx}{ds} \right\} ds, \\ \frac{\partial \Phi}{\partial b} &= \int_{C_0} \left\{ \left(f_{z_x} \frac{\partial z}{\partial b} + f_{t_x} \frac{\partial t}{\partial b} \right) \frac{dy}{ds} - \left(f_{z_y} \frac{\partial z}{\partial b} + f_{t_y} \frac{\partial t}{\partial b} \right) \frac{dx}{ds} \right\} ds \end{aligned}$$

in the parametric field that we now consider. It then follows that:

$$\text{(5)} \qquad \begin{aligned} \int_{C_0} \Bigg\{ &\left[\frac{\partial}{\partial b} \left(f_{z_x} \frac{\partial z}{\partial a} + f_{t_x} \frac{\partial t}{\partial a} \right) - \frac{\partial}{\partial a} \left(f_{z_x} \frac{\partial z}{\partial a} + f_{t_x} \frac{\partial t}{\partial a} \right) \right] \frac{dy}{ds} \\ &- \left[\frac{\partial}{\partial b} \left(f_{z_y} \frac{\partial z}{\partial a} + f_{t_y} \frac{\partial t}{\partial a} \right) - \frac{\partial}{\partial a} \left(f_{z_y} \frac{\partial z}{\partial a} + f_{t_y} \frac{\partial t}{\partial a} \right) \right] \frac{dx}{ds} \Bigg\} ds = 0, \end{aligned}$$

which is a relation that says that it will follow from the argument on pp. 79 that the parametric field is an "independence field." The function fields that lead to integrals of equation (1) then possess the property that every parametric field that is selected from them is an independence field.

Curriculum Vita

I, Heinrich Friedrich Wilhelm Georg Prange, evangelical denomination, was born in Hannover on 1 January 1885 as the son of the merchant Georg Prange. From Easter 1891 on, I attended the preschool of what was, at the time, the “Höherem Bürgerschule I” (now the Oberrealschule) in Hannover, and since Easter 1894, the humanistic Gymnasium “Lyceum II” (now Königliche Goethe-Gymnasium) in Hannover, which I left on Easter 1903 with a testimony of maturity. After that, I studied mathematics from Easter 1903 to Michaelmas 1903 in Göttingen, from Michaelmas 1904 until then in 1905 in Munich, and from Michaelmas 1905 until then in 1906 at Göttingen again. My teachers were:

in Göttingen: Baumann, Hilbert, Klein, Liebisch, Minkowski (†), Mollwo, G. E. Müller, Peter, Riecke, Runge, Schilling, Schultheß, Schwarzschild, Voigt, Wiechert, Zermelo.

in Munich: von Baeyer, von Braunmühl (†), Doehlemann, Lindemann, Lipps (†), Pringsheim, Röntgen, E. von Weber.

An outbreak of lung disease in the Summer of 1906 compelled me to give up on studies completely. An attempt to take them up again in 1908 immediately led to a new breakdown in my health. My recovery happened slowly at that time. In Fall 1910, I had again recuperated to the extent that I could be employed as an assistant in the exercises of Herrn Prof. C. H. Müller at the technischen Hochschule in Hannover. At the same time, I once more began to address mathematics, and in June 1912, I passed the test for a teaching assistant in the higher schools before the test committee in Göttingen, whereupon I entered the technischen Hochschule in Hannover as an assistant in mathematics on Michaelmas 1912.

I owe all of my teachers my most heartfelt thanks, especially the unforgettable H. Minkowski, as well as Herrn Geheimrat Klein and Herrn Geheimrat Hilbert, and especially to the latter for the fact that he undertook the presentation of this work.

Furthermore, thanks are due to Herrn Geheimrat Kiepert and Prof. C. H. Müller in Hannover for their friendly cooperation. In 1910, I entered into a personal relationship with Prof. Müller that was very beneficial to me. At that time, he had directed me to some exercises that increased my capacity to work, which contributed significantly to the fact that I gradually overcame my sickness completely and could engage in autonomous work. He had also followed the creation of this work with interest and gave me worthwhile advice on its editing.