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Non-relativistic theory of particles with integer spin

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Abstract. – The author shows that a theory with three components that each satisfy a Schrödinger equation is capable of describing, in the Newtonian approximation, the behavior of a particle with integer spin and positive kinetic energy. He deduces that theory as the first approximation to a relativistic theory that was proposed in a previous article. One studies the various quantities that are attached to the particle and their relationships.

The theory must be applied to particles of the same type as *heavy electrons with small velocities.*

1. Introduction. – In the Journal de Physique **7** (1938), pp. 347, we developed a relativistic theory of electrons that were characterized by an energy that remains positive. That article did not study the problem of the proper values of the spin. Now, the essence of this problem – namely, the question of the integer or half-integer values – may be solved very conveniently by looking for the *Newtonian approximation* to the proposed theory.

Moreover, this Newtonian approximation is interesting in itself by its simplicity and convenience.

It must be applied to the case of *slow, heavy electrons* if one is to succeed in proving that such particles exist.

2. Fundamental quantities. – For the notations and the fundamental results, one must refer to the cited article. In that article, the absolute value of the various quantities – e.g., current, energy, etc. – did not interest us. It is convenient to rewrite them in order to establish the correct values of the constant factors. In this case, one has, for the densities of:

The Lagrange function:

$$L = \frac{h^2}{2m}\left(\frac{F_{rs}G_{rs}}{2} + k^2\psi_r^*\psi_r\right). \tag{2.1}$$

Current:

$$j_s = \frac{ieh}{2mc}(\psi_r^* G_{rs} - \psi_r F_{rs}). \tag{2.2}$$

Energy-quantity of motion tensor:

$$T_{r\rho} = \frac{h^2}{2m}[F_{rs}\, G_{\rho s} + F_{rs}\, G_{\rho s} + k^2(\psi_r^*\psi_\rho + \psi_\rho^*\psi_r) - L\delta_{r\rho}]. \tag{2.3}$$

Electromagnetic moment:

$$m_{rs} = \frac{ieh}{2mc}(\psi_r^* \psi_s - \psi_s^* \psi_r)\,. \tag{2.4}$$

Spin of the particle:

$$\mathfrak{M}_{ab} = \frac{h}{2\varepsilon}(\psi_b\, F_{4a} - \psi_a\, F_{4b} + \psi_b^* G_{4a} - \psi_a^* G_{4b}). \tag{2.5}$$

3. Preliminaries. Passage from the Gordon equation to the Schrödinger equation. – The problem posed consists of finding what the equations and expressions of the preceding paragraph become in the Newtonian approximation; i.e., when one supposes that $c \to \infty$ or $1/c \to 0$.

In order to see how one might develop this process of approximations, and above all, to exhibit certain characteristic details, we first apply it to the Gordon equation; one thus comes back to the well-known Schrödinger theory.

Therefore, take the Gordon equation, and to simplify, consider the case of the absence of a field (for the formulas, cf., for example, PAULI and WEISSKOPF, Helvetica Physica Acta **7** (1934), 709) . The equation is written:

$$\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \Delta\psi + \frac{m^2 c^2}{h^2}\psi = 0. \tag{3.1}$$

If we separate, as one usually does, a term in mc^2 by setting:

$$\psi(x, y, z, t) = u(x, y, z, t)\; e^{-\frac{imc^2}{h}t} \tag{3.2}$$

then (3.1) becomes:

$$-\frac{2im}{h}\frac{\partial u}{\partial t} = \Delta u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\,. \tag{3.3}$$

This equation is solved by successive approximations; the first one is:

$$-\frac{2im}{h}\frac{\partial u}{\partial t} = \Delta u, \tag{3.4}$$

which is precisely the classical Schrödinger equation:

$$ih\,\frac{\partial u}{\partial t} = Hu, \quad \text{with} \quad H = (-\,ih)^2\,\Delta 2m. \tag{3.5}$$

In order to obtain the *current*, we first substitute (3.2) in the relativistic expression:

$$s_\nu = \frac{ieh}{2mc}\left(\frac{\partial \psi^*}{\partial x^\nu}\psi - \frac{\partial \psi}{\partial x^\nu}\psi^*\right). \tag{3.6}$$

This gives the *charge density* ($\nu = 4$) as:

$$\rho = -s_4, \qquad \rho = \frac{-ieh}{2mc}\left(\frac{\partial u^*}{\partial x^\nu}u - \frac{\partial u}{\partial x^\nu}u^* + \frac{2imc^2}{h}u^* u\right), \tag{3.7}$$

which, upon neglecting the terms in $1/c^2$, indeed provides the *charge* density of a Schrödinger particle:

$$\rho = e\,u^* u.$$

The *current* is likewise the same:

$$s_a = \frac{ieh}{2mc}\left(\frac{\partial u^*}{\partial x^\nu}u - \frac{\partial u}{\partial x^\nu}u^*\right). \tag{3.8}$$

The *energy-quantity of motion* tensor is:

$$\left.\begin{aligned} T_{\mu\nu} &= -\frac{h^2}{2m}\left(\frac{\partial \psi^*}{\partial x^\mu}\frac{\partial \psi}{\partial x^\nu} + \frac{\partial \psi^*}{\partial x^\nu}\frac{\partial \psi}{\partial x^\mu}\right) - L\delta^{\mu\nu} \\ &\text{with} \\ L &= -\frac{h^2}{2m}\sum_{\mu=1}^{4}\frac{\partial \psi^*}{\partial x^\mu}\frac{\partial \psi}{\partial x^\mu} - \frac{mc^2}{2}\psi^*\psi. \end{aligned}\right\} \tag{3.9}$$

Upon substituting this into (3.2), one obtains the *quantity of motion:*

$$P_a = \frac{iT_{a4}}{c}$$

$$P_a = \frac{iT_{a4}}{c} = -\frac{h^2}{2mc^2}\left(\frac{\partial \psi^*}{\partial x^a}\frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\frac{\partial \psi}{\partial x^a}\right), \qquad (a = 1, 2, 3), \tag{3.10}$$

namely:

$$P_a = \frac{-ih}{2}\left(u^*\frac{\partial u}{\partial x^a} - \frac{\partial u^*}{\partial x^a}u\right) + \frac{1}{c^2}\left(\frac{-h^2}{2mc^2}\right)\left(\frac{\partial u^*}{\partial x^a}\frac{\partial u}{\partial t} + \frac{\partial u^*}{\partial t}\frac{\partial u}{\partial x^a}\right), \tag{3.11}$$

or finally, in the approximation that we assumed:

$$P_a = \frac{-ih}{2}\left(u^*\frac{\partial u}{\partial x^a} - \frac{\partial u^*}{\partial x^a}u\right). \tag{3.12}$$

The mean of this density is then:

$$G_a = \int P_a dV = -\, ih \int u^* \frac{\partial u}{\partial x^a} dV\,, \tag{3.13}$$

i.e., it is equal to the mean of the operator $-\,ih\,\partial / \partial x^a$, as in the Schrödinger theory.

The calculation of *energy* requires special attention. In the Schrödinger theory, the energy is calculated from the mean of the operator $ih\,\partial / \partial t$:

$$E = \frac{ih}{2}\int\left(u^* \frac{\partial u}{\partial t} - \frac{\partial u^*}{\partial t} u\right) dV\,. \tag{3.14}$$

In the Gordon theory, the general expression is:

$$E = \frac{h^2}{2m}\int\left[\frac{1}{c^2}\frac{\partial \psi^*}{\partial t}\frac{\partial \psi}{\partial t} + \sum_{a=1}^{3}\frac{\partial \psi^*}{\partial x^a}\frac{\partial \psi}{\partial a^a} + \frac{m^2c^2}{h^2}\psi^*\psi\right] dV\,. \tag{3.15}$$

We seek the expression that this becomes in the assumed approximation. Upon substituting it into (3.2), one finds:

$$E = mc^2\int u^* u\, dV + \frac{h^2}{2m}\int\left[\frac{1}{c^2}\frac{\partial u^*}{\partial t}\frac{\partial u}{\partial t} + \sum_{a=1}^{3}\frac{\partial u^*}{\partial x^a}\frac{\partial u}{\partial x^a} + \frac{im}{c}\left(u^*\frac{\partial u}{\partial t} - \frac{\partial u^*}{\partial t}u\right)\right] dV\,, \tag{3.16}$$

so, by virtue of the fundamental equations:

$$E = mc^2\int u^* u\, dV + ih\int\left(u^*\frac{\partial u}{\partial t} - \frac{\partial u^*}{\partial t}u\right) dV + \frac{h^2}{2mc^2}\int\frac{\partial u^*}{\partial t}\frac{\partial u}{\partial t} dV\,. \tag{3.17}$$

This expression contains three terms. When $c \to \infty$, the first one becomes infinite, the second is unchanged, and the third one becomes negligible. The first term represents a proper energy that, by the definition, the chosen approximation does not take into account. If we agree to eliminate it, what remains is:

$$\text{Energy} = ih\int\left(u^*\frac{\partial u}{\partial t} - \frac{\partial u^*}{\partial t}u\right) dV\,, \tag{3.18}$$

which has twice the value that is calculated in the Schrödinger theory.

It is easy to see what the origin of the factor 2 might be. Pauli and Weisskopf, on pp. 720 of the cited paper, have analyzed the structure of the general expression (3.15) for the energy and have shown that this energy is composed of:

a) An infinite term,

b) A term that represents the energy of a particle of charge $+\,e$ and momentum $\vec{p}$,

c) A term that represents the energy of a particle of charge $-\,e$ and current $-\,\vec{p}$.

Now, when we pass (in the absence of charge) to the Schrödinger approximation, after suppressing the infinite energy, the expression E continues to represent a sum of two terms, each of which, however, now refers to *the same* particle. In effect, this approximation no longer admits both of the particles $+ e$ *and* $- e$, considering that the charge density is (3.7) $\rho = e\, u^* u$.

The energy thus calculated represents the energy of positive and negative particles that are based in just one $\rho = e\, u^* u$; it is thus equal to two times their individual energy. On the other hand, in the Schrödinger theory, properly speaking, one calculates the energy of just one particle, and not two juxtaposed particles.

The approximation process must take this situation into account. For the correct calculation of the energy, one must then:

a) *Suppress the infinite term.*

b) *Divide the result by* 2.

One then comes back to the classical result. We encounter analogous situations in the calculations that follow.

4. General equations of a particle with positive energy. – We now use the approximation process just described in order to deduce the *non-relativistic equations* for particles with spin from our general relativistic equations.

Set:

$$\psi_r = e^{i\varepsilon k x_4} \cdot u_r \qquad (r = 1, 2, 3, 4). \tag{4.1}$$

Since:

$$x_4 = ict, \qquad \varepsilon^2 = -1, \qquad k = mc / h,$$

one will have:

$$i\varepsilon k\, x_4 = -\frac{imc^2}{h} t\,.$$

In general, one can write:

$$\left.\begin{aligned} G_{rs} &= e^{i\varepsilon k x_4}[g_{rs} + i\varepsilon k(\delta_{r4} \cdot u_s - \delta_{s4} \cdot u_r)], \\ F_{rs} &= e^{-i\varepsilon k x_4}[f_{rs} - i\varepsilon k(\delta_{r4} \cdot u_s^* - \delta_{s4} \cdot u_r^*)], \end{aligned}\right\} \tag{4.2}$$

where one has set:

$$\left.\begin{aligned} f_{rs} &= (\partial_r + iA_r)u_s^* - (\partial_s + iA_s)u_r^*, \\ g_{rs} &= (\partial_r - iA_r)u_s - (\partial_s - iA_s)u_r. \end{aligned}\right\} \tag{4.3}$$

The *fundamental equations* are written (see the cited article):

$$\left.\begin{aligned} (\partial_r - iA_r)G_{ab} + (\partial_4 - iA_4)G_{4b} &= k^2\psi_b, \\ (\partial_a - iA_a)G_{a4} &= k^2\psi_4, \end{aligned}\right\} \qquad (a, b = 1, 2, 3). \tag{4.4}$$

From now on, we reserve the letters a, b for indices that vary from 1 to 3.

Upon substituting (4.1) and (4.2) into (4.4), one obtains:

$$(\partial_a - iA_a)\, g_{ab} + (\partial_4 - iA_4)\, g_{4b} + i\varepsilon k\, [g_{4b} + (\partial_4 - iA_4)\, u_b] = 0, \tag{4.5}$$

$$(\partial_a - iA_a)\, g_{a4} \qquad\qquad + i\varepsilon k\, (\partial_a - iA_a)\, u_a = k^2 u_4 . \tag{4.6}$$

All of this is true with no approximation.

The form of equation (4.6) now permits us to go into a solution by successive approximations in powers of $1/k$ – i.e., in $1/c$. Indeed, one may write it:

$$u_4 = -\frac{1}{i\varepsilon k}(\partial_a - iA_a)\, u_a - \frac{1}{k^2}(\partial_a - iA_a)\,(\partial_4 - iA_4)\, u_a + \frac{1}{k^2}(\partial_a - iA_a)^2\, u_4 , \tag{4.7}$$

and the suppression of the last two terms will give us the first approximation to u_4 .

In the arguments that follow, we will have to make $1/c$ go to zero. We remark that this procedure *must not be applied* to the c that enters into the expression for A_a:

$$A_a = \frac{e}{hc}\Phi_a . \tag{4.8}$$

Indeed, that c is found uniquely by converting the value of e, as measured in electrostatic units, to electromagnetic units. In other words, we make *the velocity of light* tend to ∞. Now, c does not represent that velocity in (4.8), but, in fact, the *ratio of two types of electric units*, and the passage $1/c \to 0$ does not affect that. It then follows that the expression $(\partial_a - iA_a)\psi$ is of order zero in comparison to $1/c$ (if ψ itself is of order zero). By contrast, the c in $(\partial_4 - iA_4)\psi$ represents the velocity of light for both $\partial_4\psi$ and the *scalar* potential A_4. Therefore, $(\partial_a - iA_a)\psi$ is of order $1/c$ under the same hypotheses.

We return to equation (4.7). Suppress the last term and neglect the second one, which, as we have seen, is of order $1/c^3$ if the u_a are of order zero. What remains is:

$$u_4 = -\frac{1}{i\varepsilon k}\,(\partial_a - iA_a)\, u_a . \tag{4.8}$$

Therefore, if the u_a (a = 1, 2, 3) are of order zero in this theory then u_4 will be of first order. In the Dirac theory, the same relation exists between *two* of the four components of the wave function and the two others. These facts are in agreement with the difference that exists between the two theories from standpoint of representations of the Lorentz group – viz., spinorial in the one case and quaternionic in the other.

(4.8) permits us to eliminate u_4 from the fundamental equations (4.5). The theory that results from this will be a *theory with three components* u_b (b = 1, 2, 3).

The result of the substitution is written:

$$\begin{gathered} 2iek\,(\partial_4 - iA_4)\, u_b + (\partial_a - iA_a)^2\, u_b \\ + [(\partial_b - iA_b)\,(\partial_a - iA_a) - (\partial_a - iA_a)\,(\partial_b - iA_b)]\, u_a + (\partial_4 - iA_4)\, g_{4b} = 0. \end{gathered} \tag{4.9}$$

Keep just the terms of order zero (which amounts to neglecting the last term); one may write resulting equations in the form:

$$\boxed{2i\varepsilon k(\partial_4 - iA_4)u_b + (\partial_a - iA_a)^2 u_b + \frac{ie}{hc} H_{ab} u_a = 0 \qquad (b = 1, 2, 3),} \tag{4.10}$$

or furthermore:

$$\boxed{ih\frac{\partial u_b}{\partial t} = \frac{1}{2m}(-h^2)(\partial_a - iA_a)^2 u_b - hc\varepsilon A_4 u_b - \frac{ieh}{2mc} H_{ab} u_a .} \tag{4.11}$$

If one wishes to introduce a Hamiltonian then one might consider the wave function u to be a matrix with just one column of elements u_1, u_2, u_3 . Let $\mathbf{m}_x$, $\mathbf{m}_y$, $\mathbf{m}_z$ denote the following matrices:

$$\mathbf{m}_x = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{vmatrix}, \qquad \mathbf{m}_y = \begin{vmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{vmatrix}, \qquad \mathbf{m}_z = \begin{vmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \tag{4.12}$$

and let:

$$H_{23} = H_x, \qquad H_{31} = H_y, \qquad H_{12} = H_z$$

be the of the external electromagnetic field. Equation (4.11) is then written:

$$ih\frac{\partial u}{\partial t} = \mathbf{H}\, u, \tag{4.13}$$

with

$$\mathbf{H} = \left[\frac{1}{2m}(-h^2)(\partial_a - iA_a)^2 - hc\varepsilon A_4\right] \cdot \mathbf{1} + \frac{eh}{2mc}(\mathbf{m}_x H_x + \mathbf{m}_y H_y + \mathbf{m}_z H_z). \tag{4.14}$$

In the expression for this Hamiltonian one recognizes one part that is due to the kinetic energy of the particle, a second one that represents the potential energy, and finally, a term that provides the energy of a particle that is endowed with a magnetic moment in an external field.

In the approximation that we stopped at (only terms of order zero), everything else is zero.

5. Case of the absence of a field. Equation and fundamental relations. – A free particle ($A_s = 0$) will be described by a wave function with three components u_1, u_2, u_3 , which satisfies the three equations that are deduced from (4.10) or (4.11):

$$2i\varepsilon k\, \partial_4 u_b + u_b = 0,$$

namely:

$$ih\,\frac{\partial u_b}{\partial t} + \frac{h^2}{2m}\Delta u_b = 0 \qquad (b = 1, 2, 3), \tag{5.1}$$

which is identical to the Schrödinger equation for each of the components.

The values of the various quantities that are attached to that particle are derived from the general expression in paragraph 2, while also taking into account that:

$$u_4 = -\frac{1}{i\varepsilon k}\partial_a u_a\,; \tag{5.2}$$

we keep the terms up to order $1/c$, inclusive.

Note that we obtain a new theory by a process of approximation, which one may evaluate without having to account for its origin.

We further remark that during the passage from the relativistic theory to the approximate theory one must take into account, not only equations, but also boundary conditions. We thus assume that, by hypothesis, *the values of* u_1, u_2, u_3, *and also* u_4 (which corresponds to ψ_4) *are annulled at infinity.* Thus, our particle is defined by three components that are zero at infinity, and whose divergence $\partial_a u_a$ is likewise zero at the boundary.

From this, one immediately deduces two fundamental relations:

a) Equations (5.1) provide the continuity equations by a known process.

$$\frac{\partial}{\partial t}(u_b^* u_b) + \frac{ih}{2m}\frac{\partial}{\partial x^a}(u_b \cdot \partial_a u_b^* - u_b^* \cdot \partial_a u_b) = 0, \tag{5.3}$$

from which, one concludes, with the preceding hypothesis, that:

$$\int u_b^* u_b \cdot dV = \text{constant.} \tag{5.4}$$

(5.3) is the continuity equation for electricity.

b) One has, by virtue of the fundamental equations (5.1) and (5.2):

$$\partial_4 u_4 = -\frac{1}{i\varepsilon k}\,\Delta u_4\,. \tag{5.5}$$

Moreover:

$$\partial_4(u_4^* u_4) = u_4^*\left(-\frac{1}{i\varepsilon k}\right)\Delta u_4 + u_4\left(\frac{1}{i\varepsilon k}\right)\Delta u_4^* = \frac{1}{i\varepsilon k}\partial_a(u_4 \cdot \partial_a u_4^* - u_4^* \cdot \partial_a u_4)\,, \tag{5.6}$$

from which, one deduces, upon taking into account that u_4 is zero at the boundary:

$$\int u_4^* u_4 \cdot dV = \text{constant.} \tag{5.7}$$

6. Charge and current. – Taking (4.1) and (4.3) into account, we write the general expression for the *charge* density:

$$j_4 = \varepsilon j_0 = \frac{ieh}{2mc}(\psi_a^* G_{a4} - \psi_a F_{a4}) \tag{6.1}$$

in the form:

$$j_0 = \frac{ieh}{2mc}(u_a g_{a4} - u_a f_{a4}) + e u_a^* u_a \ . \tag{6.2}$$

From the fundamental formula (5.2), u_4 is of order $1/c$, just like $\partial_4 u_a$; therefore, g_{a4}, f_{a4} are of order $1/c$. It then follows that the first term of the preceding sum (5.2) is of order $1/c^2$, and therefore negligible. In the first approximation, the charge density of the particle is given by:

$$\rho = e u_a^* u_a , \tag{6.3}$$

and the charge, by $e \int u_a^* u_a \, dV$, in perfect analogy with the classical result of Schrödinger.

The *current* density:

$$j_b = \frac{ieh}{2mc}(\psi_a^* G_{ab} - \psi_a F_{ab}) + \frac{ieh}{2mc}(\psi_4^* G_{4b} - \psi_4 F_{4b}) , \quad (b \neq 4) \tag{6.4}$$

is written by substituting (4.1):

$$j_b = \frac{ieh}{2mc}[u_a^* g_{ab} - u_a f_{ab} + i\varepsilon k(u_4^* u_b + u_b^* u_4)] + \frac{ieh}{2mc}(u_4^* g_{4b} - u_4 f_{4b}) .$$

The last term is now of order $1/c^2$. Moreover, upon taking into account the fundamental equations one may write for the first part:

$$j_b = \frac{ieh}{2mc}[u_a \cdot \partial_b u_a^* - u_a^* \cdot \partial_b u_a) + \partial_a \left[\frac{ieh}{2mc}(u_a^* u_b - u_b^* u_a) \right]. \tag{6.5}$$

The current is thus of order $1/c$, and the charge is of order 0. The density (6.5) is composed of a first part that corresponds to the conduction current and a second one that is *due to the magnetic moment* of the particle, as we will confirm in the following paragraph.

The current itself is given by:

$$J_b = \frac{ieh}{2mc} \int (u_a \cdot \partial_b u_a^* - u_a^* \cdot \partial_b u_a) \, dV . \tag{6.6}$$

We have already verified the existence of a conservation law (5.3); it persists for the two types of currents.

7. Electromagnetic moment. – The six components of the relativistic electromagnetic moment (2.4):

$$m_{rs} = \frac{ieh}{2mc}(\psi_r^*\psi_s - \psi_s^*\psi_r),$$

separate into two types: The ones that that do not contain the index 4, and which represent the *magnetic moment* density:

$$\mu_{ab} = \frac{ieh}{2mc}(u_a^*u_b - u_b^*u_a), \tag{7.1}$$

and the other ones that correspond to the *electric moment:*

$$\frac{ieh}{2mc}(u_4^*u_b - u_b^*u_4). \tag{7.2}$$

The latter are of order $1/c^2$, so they are negligible, in our approximation.

Therefore, the particle under study *possesses a magnetic moment*, but not a proper electric moment. The magnetic moment must give a contribution to the current; however, we have seen that such a contribution exists (formula (6.5)).

8. Spin. – The expression (2.5) for the spin:

$$M_{ab} = \frac{h}{2\varepsilon}(\psi_b F_{4a} + \psi_a F_{b4} + \psi_b^* G_{4a} + \psi_a^* G_{4b}) \tag{2.5}$$

is written:

$$M_{ab} = \frac{h}{2\varepsilon}(u_b f_{4a} + u_a f_{b4} + u_b^* g_{4a} + u_a^* g_{4b}) + ih(u_b^*u_a - u_a^*u_b),$$

which, upon retaining only the terms of order zero, reduces to:

$$M_{ab} = ih(u_b^*u_a - u_a^*u_b). \tag{8.1}$$

9. Quantity of motion. – A component p_b of the density of the quantity of motion is written:

$$p_b = \frac{1}{\varepsilon c} T_{b4}, \tag{9.1}$$

T_{b4} being given by (2.3), so:

$$p_b = \frac{h^2}{2\varepsilon mc}[F_{bs}G_{4s} + F_{4s}G_{bs} + k^2(\psi_b^*\psi_4 + \psi_4^*\psi_b)]. \tag{9.2}$$

Upon substituting (2.1), one finds that:

$$p_b = \frac{h^2}{2\varepsilon mc}[g_{4a}f_{ba} + f_{4a}g_{ba} + i\varepsilon k(u_a f_{ba} - u_a^* g_{ba}) + k^2(u_a^* u_b - u_4^* u_b)]. \tag{9.3}$$

The first parenthesis is of order $1/c^2$, and thus negligible; by virtue of (5.2), the remaining ones become:

$$p_b = \frac{-ih}{2}(u_a^* \cdot \partial_b u_a - u_a \partial_b u_a^*) + \frac{1}{2}\cdot \partial_a[ih(u_a^* u_b - u_b^* u_a)]. \tag{9.4}$$

We remark that this density is *proportional to the current.*

We further remark that it is composed of two terms: a first term that is, properly speaking, the quantity of translational motion and a second one that corresponds to the spin. The existence of this latter term is paramount. In addition, it is obvious that it is not usually pointed out in the research on quantum mechanics that relate to this subject. To take an intuitive example, it is obvious that in a continuous set of points the quantity of motion of each of them will be different according to whether that set, when considered as a totality, does or does not possess a rotational motion.

The total quantity of motion, by contrast, is not altered by the presence of spin; it is equal to:

$$G_b = \int p_b\, dV = \frac{-ih}{2}\int (u_a^* \cdot \partial_b u_a - u_a \cdot \partial_b u_a^*)\, dV \tag{9.5}$$

$$= \int u_a^* \left(-ih\frac{\partial}{\partial x^b}\right) u_a\, dV\,. \tag{9.6}$$

If we know the value of the density of the quantity of motion then we may look for its moment:

$$M_{mn} = \int (x_m\, p_n - x_n\, p_m)\, dV \qquad (m, n = 1, 2, 3), \tag{9.7}$$

and thus confirm the presence of a proper moment; i.e., a spin.

We write the density (9.4) in the form:

$$p_m = p'_m + \frac{ih}{2}\partial_a(u_a^* u_b - u_b^* u_a), \tag{9.8}$$

in which p'_m denotes the density of the translational quantity of motion. (9.7) is then written:

$$M_{mn} = \int (x_m p'_n - x_n p'_m)\, dV + \frac{ih}{2}\int [x_m \cdot \partial_a(u_a^* u_n - u_n^* u_a) - x_n \cdot \partial_a(u_a^* u_m - u_m^* u_a)]\, dV\,. \tag{9.9}$$

The first integral is the orbital moment of the quantity of motion; *the second one is not zero.* Indeed, all of the terms are null, except for the ones with $a = m$ in one case and $a = n$ in the other. Finally:

$$M_{mn} = \int (x_m p'_n - x_n p'_m)\, dV + ih \int (u_n^* u_n - u_m^* u_n)\, dV . \tag{9.10}$$

The particle thus possesses a spin that is equal to:

$$ih \int (u_n^* u_n - u_m^* u_n)\, dV , \tag{9.11}$$

a value that is identical to the one in § 8.

10. Energy. – The energy density is written:

$$E = \frac{h^2}{2m}\left[\frac{F_{ab} G_{ab}}{2} - F_{a4} G_{a4} + k^2 (\psi_a^* \psi_a - \psi_4^* \psi_4)\right], \tag{10.1}$$

or furthermore:

$$E = \frac{h^2}{2m}\left[\frac{f_{ab} g_{ab}}{2} + i\varepsilon k (u_a f_{a4} - u_a^* g_{a4}) + k^2 u_4^* u_4\right] + \frac{h^2}{2m}(- f_{a4} g_{a4} + 2k^2 u_a^* u_a) . \tag{10.2}$$

Taking into account the fundamental equations, this expression is written:

$$E = \frac{h^2}{2m}\left\{\partial_a^2 \left(\frac{u_b^* u_b}{2}\right) - \partial_{ab}\left(\frac{u_a^* u_b + u_b^* u_a}{2}\right) + 2i\varepsilon k\, \partial_a (u_a^* u_b - u_b^* u_a) \right.$$
$$\left. -2i\varepsilon k\, (u_a^* \cdot \partial_4 u_a - u_b^* \cdot \partial_4 u_a) - f_{a4} g_{a4} + 2k^2 u_a^* u_a + 2k^2 u_4^* u_4 \right\} \tag{10.3}$$

After discarding the term $- f_{a4}\, g_{a4}$, which is of order $1/c^2$, the energy itself will be:

$$\int E\, dV = ih \int (u_a^* \cdot \partial_t u_a - u_a \cdot \partial_t u_a^*)\, dV + 2mc^2 \int u_a^* u_a\, dV + 2mc^2 \int u_4^* u_4\, dV . \tag{10.4}$$

If we agree to proceed as in the case of the Gordon equation, we must suppress the term:

$$2mc^2 \int u_a^* u_a\, dV ,$$

which becomes infinite with c, and divide the rest by 2; this gives:

$$\text{Energy} = \frac{ih}{2} \int (u_a^* \cdot \partial_t u_a - u_a \cdot \partial_t u_a^*)\, dV + 2mc^2 \int u_4^* u_4\, dV .$$

To that approximation, the energy is therefore the mean of $\partial / \partial t$, plus another term. Now, this term is therefore a constant, by virtue of paragraph 5, formula (5.7), a constant that will be interesting to study.

Finally, we remark that if the energy is positive in the *relativistic* theory, even in the presence of a field (cf., the cited article or formula (2.3)), then this is no longer true in the present approximation. The energy of a *free* particle always remains positive, as it must, but by virtue of (4.11) or (4.14) the total energy in a field can also be negative by the effect of the potential energy term (as in the Schrödinger theory) or by the effect of the spin energy.

10. Conclusion. – The preceding analysis shows that one can describe a particle with positive energy and integer spin by a function with three components that each satisfy a Schrödinger equation. The fact that the spin must be an integer is manifested in relativistic theory by the vectorial character of the wave function and in the Newtonian approximation by the statement that the ratio of the magnetic moment (7.1) is, moreover, obtained immediately from the one that one has established, and the spin (8.1) or (9.11) is equal to $e / 2mc$. The conclusion is that there exists a proportionality, not only between the current itself and the quantity of motion, but also between the corresponding densities (cf., § 9); therefore, the theory does not apply to ordinary electrons. One might hope that it is applicable to the particles of the heavy electron type.

Finally, it is interesting to see the manner in which the spin is introduced in these vectorial theories, and the importance that the *densities* acquire; the translation of the preceding theory into the language of operators is extremely instructive in that regard.

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