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On the Dirac equation

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Abstract. – The author shows that the wave function that is defined by the Dirac equation is not a quantity with four scalar components, but one with sixteen. When one then addresses the problem in full generality, most of the objections that one presently makes about the Dirac equation vanish. The concept of ψ as a semi-vector then becomes useless; the equation is written in a form that is invariant under the Lorentz transformations and is perfectly symmetric, moreover.

In order to obtain the most general results, the author uses hypercomplex numbers. He writes the four-dimensional current by means of the new ψ and attaches the number of components it has to the "degrees of freedom" of the electron. He shows that these 16 components are susceptible to an immediate physical interpretation and he specifies the meaning that one gives to these quantities in terms of probabilities.

Finally, in the course of these considerations, two notions present themselves very naturally: that of the "fifth dimension" and that of the "tensorial or hypercomplex probability." The author then rapidly examines their essential traits.

1. Introduction. – The substitution of the relativistic Dirac equation for that of Schrödinger marks an extremely important step forward in wave mechanics. However, the study of that equation further presents some gaps that singularly restrict its scope. The objections that one makes show clearly that, following the expression of Darwin, "several things have passed through the net;" they do not, however, permit one to specify what element has escaped the analysis, nor to see how one may discover it.

The considerations that follow have the goal of pointing out and filling in one of these gaps. Their point of departure is the banal observation that first comes to light when one begins the study of the Dirac equation. It concerns the passage from that unique system to a system of four partial differential equations that are deduced from it, and which, to abbreviate, we call the *Darwin equations*.

Let α_μ^{ik} be the elements of the matrices with four rows and four columns α_μ (μ = 1, 2, 3, 4) that appear in the Dirac equation $H\psi = 0$. Dirac, Darwin, and their school of all those of who occupied themselves with that equation have admitted that when one develops that equation, one must write:

$$\alpha_\mu \, \psi = \sum_{k=1}^{4} \alpha_\mu^{ik} \psi_k \, . \tag{1}$$

for the action of the α_μ on ψ. This amounts to defining four wave functions ψ_1, ψ_2, ψ_3, ψ_4. One then supposes that ψ is expressed by means of these four functions like a matrix that reduces to just one column of elements ψ_1, ψ_2, ψ_3, ψ_4.

Now, *there is no justification for this last assumption;* it is a gratuitous hypothesis that restricts the generality and introduces pointless complications. Indeed, the Hamiltonian H, which is the sum of four matrices, is itself a matrix with four rows and four columns. The equation $H\psi = 0$ signifies that one must find a quantity that gives zero when multiplied by such a matrix. It is obvious that, in general, *ψ must be a matrix with four rows and four columns* ([1]). It is not surprising that one finds asymmetric results and details of one assumes from the outset, and with no necessity, that three of these columns have their elements equal to zero. Moreover, it is obvious that upon multiplying H times ψ and annulling the elements of the matrix product the 16 resulting equations are identical in groups of four; these are the equations that are produced by the composition of a given line of H with the columns of ψ. ψ is thus a matrix whose last three columns are identical to the first one, but which *are non-zero* in any case, as one presently assumes.

The correct results that the Dirac equation leads to may be obtained only if one eliminates the limiting hypothesis. We shall attempt to do that in the following paragraphs by working through the idea that we just sketched out qualitatively.

2. The Dirac equation. In order to account for the effects that were attributed to the rotating electron in the theory of spectra, and to satisfy the demands of relativity, Dirac was compelled to replace the relativistic equations of Gordon and Klein with the first order equation:

$$(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \alpha_4 mc)\ \psi = 0 \tag{2}$$

or:

$$p_0 = -\frac{h}{2\pi i}\frac{\partial}{c\,\partial t} + \frac{e}{c}A_0\,,$$

$$p_r = +\frac{h}{2\pi i}\frac{\partial}{\partial x_r} + \frac{e}{c}A_r \qquad (r = 1, 2, 3).$$

A_0 is the scalar potential, A_1, A_2, A_3 define the vector potential, and $-e$ is the charge of the electron.

The α_μ are operators that commute with the p_i, the x_i, and t, and which satisfy, in addition, the conditions:

$$\alpha_\mu\,\alpha_\nu + \alpha_\nu\,\alpha_\mu = 0, \qquad \alpha_\mu^2 = 1 (\mu, \nu = 1, 2, 3, 4). \tag{3}$$

They can be put into the form of matrices with four rows and four columns whose elements are $(\alpha_\mu)^{ik}$; Dirac pointed out a possible form for these matrices.

The α_μ do not operate on the x, y, z, t; ψ must therefore contain a new variable ζ on which the α_μ do operate, namely, $\psi(x, y, z, t;\ \zeta)$. Most often, one writes that variable with

([1]) If it has more, since a certain number of its elements will be arbitrary, then the number of unknowns will exceed the number of equations that must determine them.

an index ψ_ζ. If the α_μ are expressed in the form of matrices then one defines the manner by which they operate on ψ by:

$$\alpha_\mu\ \psi_k = \sum (\alpha_\mu)^{ik} \psi_k\,. \tag{4}$$

This definition succeeds in specifying the meaning of equation (2). With the Dirac matrices, the unique equation (2) is equivalent, by (4), to the system of equations in four unknown functions ψ_1, ψ_2, ψ_3, ψ_4 that were written for the first time by Darwin:

$$\left.\begin{aligned}
(p_0 + mc)\psi_1 + (p_1 - ip_2)\psi_4 + p_3\psi_3 &= 0,\\
(p_0 + mc)\psi_2 + (p_1 + ip_2)\psi_3 - p_3\psi_4 &= 0,\\
(p_0 - mc)\psi_3 + (p_1 - ip_2)\psi_2 + p_4\psi_1 &= 0,\\
(p_0 - mc)\psi_4 + (p_1 + ip_2)\psi_1 - p_3\psi_2 &= 0.
\end{aligned}\right\} \tag{5}$$

Dirac decomposed the α into factors by using the six matrices with four rows and four columns σ_1, σ_2, σ_3, and ρ_1, ρ_2, ρ_3 :

$$\alpha_1 = \rho_1\ \sigma_1, \qquad \alpha_2 = \rho_1\ \sigma_2\,, \qquad \alpha_3 = \rho_1\ \sigma_3, \qquad \alpha_4 = \rho_3\,. \tag{6}$$

If the "vector" (σ_1, σ_2, σ_3) is denoted by $\bar{S}$ and (p_1, p_2 , p_3), by $\bar{P}$ then the symbolic equation (2) may be further written:

$$[p_0 + \rho_1(\bar{S}\,, \bar{P}\,) + \rho_3\, mc]\ \psi = 0, \tag{7}$$

when $(\bar{S}\,, \bar{P}\,)$ represents the scalar product of the vectors $\bar{S}$ and $\bar{P}$.

3. Objections. – From the outset, two objections were made to this equation, or to the ones that were derived from it, which are very important because they bring to light an imperfection of the theory, and, as we will confirm later on, some inadmissible restrictions from the physical viewpoint. In the first place, one can criticize the complete lack of symmetry in its form. The introduction of relativity generally increases the symmetry since the coordinates and time are treated in the same fashion; here, we have scarcely arrived at that result. Next, the equation thus obtained is not written in a tensorial form, while, in reality, it is not altered by a Lorentz transformation; although it is basically invariant, its form is not. The set of four scalars (ψ_1, ψ_2, ψ_3, ψ_4) *is not a vector*: It obeys different transformation laws that have not been encountered in physics up to the present.

With the goal of clarifying this question, various authors have sought to continue, as far as possible, the study of either the Hamiltonian $H = p_0 + \rho_1(\bar{S}\,, \bar{P}\,) + \rho_3\ mc$ or the magnitude ψ. Among those who studied H, Eddington proposed to give a symmetric form to the fundamental equation, and he arrived at it by starting with a system of matrices that were different from that of Dirac. However, the general solution of the symmetrization of H was given by Schouten in a paper that we shall examine later on.

Some other authors have studied, more specifically, the analogy with the Maxwell equations, and have proposed to replace the four functions ψ_1, ψ_2, ψ_3, ψ_4 with either eight other ones that are defined by a different system of equations or by just two functions ψ_1, ψ_2 that must suffice. Finally, another category of authors have taken the Dirac equations for their basis and sought to directly study the new quantity ψ, to which one has likewise given a new name: the "semi-vector."

As interesting as these parallel considerations are, it does not seem that they must provide the complete solution to the problem. This seems to be much simpler: We seek to show that all of the difficulties disappear when one takes care to envision the problem in full generality. The imperfections that one encounters are produced simply by an unjustified specialization of the givens. Furthermore, the general treatment offers us the advantage of making the physical interpretation of the calculations as simple as possible, which is extremely desirable in this chapter of theoretical physics.

We shall thus recall the study of Dirac in the beginning by following his personal line of reasoning; to simply, we first assume that the (electromagnetic) field is zero.

4. General relativistic equation. – Let the energy equation in special relativity be:

$$- p_0^2 + p_1^2 + p_2^2 + p_3^2 + m_0^2 c^2 = 0, \tag{8}$$

where m_0 is the proper mass, p_0 is the energy, and p_1, p_2, p_3 are the momenta of the electron. Set $p_4 = i\, p_0$ and:

$$t_k = p_k = \frac{h}{2\pi i}\frac{\partial}{\partial x_k} \qquad \begin{pmatrix} k = 1, 2, 3, 4, \\ x_4 = ict, \end{pmatrix} \tag{9}$$

$$t_5 = m_0\, c.$$

(8) becomes:

$$\sum_{i=1}^{5} t_i^2 = 0. \tag{10}$$

The Schrödinger equation that is deduced from equation (10) by the usual process is not admissible because it is of second order; according to Dirac, it must be of first order. In order to obtain such an equation, Dirac decomposed (10) into two factors of first degree. The most symmetric way of doing this obviously consists in setting:

$$\sum_1^5 t_i^2 \equiv F^2 \equiv \left(\sum_1^5 E_i t_i\right)\left(\sum_1^5 E_i t_i\right). \tag{11}$$

The equation that will replace the Schrödinger equation will then be simply:

$$F\psi = 0,$$

or

$$(E_1 t_1 + E_2 t_2 + E_3 t_3 + E_4 t_4 + E_5\, m_0\, c)\ \psi = 0, \tag{12}$$

the E_i being defined by the identity (11). Upon equating the coefficients of the two sides, one sees that the E_i must satisfy the conditions:

$$\left.\begin{aligned} E_i^2 &= 1, \\ E_i\, E_k &= -\, E_k\, E_i\, . \end{aligned}\right\} \qquad (i, k = 1, 2, 3, 4, 5). \tag{13}$$

No other restriction is imposed upon them. From (13), the E_i are numbers whose multiplication is not commutative; *they are thus hypercomplex numbers, in general*. This simple observation, which nonetheless has considerable generality, will give us the key to solving the problem.

We shall thus commence by studying the particular system of complex numbers in which F is written by taking, in addition, the simplifying hypothesis that this system has an associative multiplication ([1]).

5. The quadri-quaternions. – In order to determine a system of complex numbers, it suffices to know its order n and a basis $e_1, e_2, \ldots, e_n$; i.e., n numbers in the system that are linearly independent. Any number of the system may then be put into the form:

$$x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n\, ,$$

the x_i being ordinary numbers.

The determination of the simplest system in which one can write the Dirac equation was done by Schouten in a very-little-known note ([2]). First, observe that the equation $E_i F \psi = 0$, which is equivalent to (12), is a sum of five terms. One of them has the coefficient 1, by virtue of (13), and the other four are numbers of the system ε_1, ε_2, ε_3, ε_4, which verify, in addition, the relations:

$$\begin{aligned} \varepsilon_i^2 &= 1, \\ \varepsilon_i \varepsilon_r + \varepsilon_r \varepsilon_i &= 0. \end{aligned} \tag{14}$$

One then concludes that *four* numbers ε_1, ε_2, ε_3, ε_4 (and the principal unity 1) suffice to write this equation.

Thus, consider four numbers ε that are constrained only to verify the conditions (14). The product of two of them is again a number of the system, so the numbers:

$$1, \quad \varepsilon_1\varepsilon_2, \quad \varepsilon_2\varepsilon_3, \quad \varepsilon_3\varepsilon_1, \quad \varepsilon_1\varepsilon_4, \quad \varepsilon_2\varepsilon_4, \quad \varepsilon_3\varepsilon_4$$

likewise belong to it. Upon repeating the same argument, one sees that the same is true for:

([1]) For the study of such numbers, see, for example, the excellent article of CARTAN in the French edition of the *Encyclopédie des Sciences mathématiques*, t. 1, vol. 1, fasc. 3, Gauthier-Villars and B. G. Teubner, 1908, or also DICKSON, *Algebren und ihre Zahlentheorie*, Zürich, 1927.

([2]) Proc, Amst., **32** (1929), pp. 105.

$$\varepsilon_1\varepsilon_2\varepsilon_3, \quad \varepsilon_2\varepsilon_3\varepsilon_4, \quad \varepsilon_3\varepsilon_4\varepsilon_2, \quad \varepsilon_4\varepsilon_1\varepsilon_2,$$

and

$$\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4,$$

and that these numbers are linearly independent, moreover. However, if one attempts to pursue this process further, one is stopped: Any product that is composed of two numbers of the preceding list reproduces another one that has already been found. We are thus in the presence of the maximum number of linearly independent numbers of the system. This maximum number – viz., $n = 16$ – is therefore the *order* of the system; the numbers above may be chosen to be unity; they constitute a *basis*. *There are* 16 *units*, namely:

$$\begin{gathered} 1; \quad \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4; \\ \varepsilon_1\varepsilon_2, \quad \varepsilon_2\varepsilon_3, \quad \varepsilon_3\varepsilon_1, \quad \varepsilon_1\varepsilon_4, \quad \varepsilon_2\varepsilon_4, \quad \varepsilon_3\varepsilon_4; \\ \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4; \qquad \varepsilon_1\varepsilon_2\varepsilon_3, \quad \varepsilon_2\varepsilon_3\varepsilon_4, \quad \varepsilon_3\varepsilon_4\varepsilon_2, \quad \varepsilon_4\varepsilon_1\varepsilon_2, \end{gathered}$$

which we denote by:

$$\varepsilon_0 = 1, \ \varepsilon_1, \ldots, \ \varepsilon_{12} = \varepsilon_1\varepsilon_2, \ldots \quad \varepsilon_{123} = \varepsilon_1\varepsilon_2\varepsilon_3, \quad \varepsilon_{1234} = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4,$$

or also sometimes, to simplify, by e_i (i = 1, 2, …, 15, 16).

Such systems have already been imagined by the mathematicians; they are the *quadri-quaternions*, a system with associative multiplication with sixteen units and a principal unity ([1]).

The remark of Schouten consists of observing that when one is given such a system that is defined by ε_1, ε_2, ε_3, ε_4, one may find at most *five numbers* E_i of the system that satisfy:

$$\left.\begin{aligned} E_i^2 &= 0, \\ E_i\,E_k + E_k\,E_i &= 0. \end{aligned}\right\} \quad (i, k = 1, 2, 3, 4, 5), \tag{16}$$

Indeed, it will suffice to take:

$$\begin{aligned} E_k &= \varepsilon_k \qquad (k = 1, 2, 3, 4), \\ E_5 &= \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4. \end{aligned} \tag{17}$$

Such a system of numbers will be called an *orthogonal system*.

Now, the general equation (12) has a linear and homogeneous form whose coefficients satisfy (16). *The Hamiltonian F is therefore a quadri-quaternion such that eleven of its components are zero, while the other ones correspond to an orthogonal system* ([2]).

Summarizing the results that we have obtained up to now, we can say: The Hamiltonian considered by Dirac consists of a linear and homogeneous form whose

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([1]) See, for example, COMBEBIAC, *Bull. Soc. Math. France* **30** (1920), pp. 1; W.-K. CLIFFORD, *Mathematical Papers*, *London* (1882), pp. 266, 397; LIPSCHITZ, *Bull. Sc. math.* **11** (1887), pp. 115; *Journal math. pures appl.* **2** (1886), pp. 373 and 439.

([2]) Compare this to the results of Eddington, Proc. Roy, Soc. A**121** (1928), pp. 524.

coefficients are hypercomplex numbers that form an orthogonal system in the system of quadri-quaternions $e_1, e_2, \ldots, e_{16}$.

6. The wave function ψ. – Our goal is the study of the wave function ψ. Having posed the problem as we just did, the solution is immediate. ψ is determined by the equation:

$$F\,\psi = 0,$$

where F is a quadri-quaternion. *The general solution will thus be a quadri-quaternion of the form:*

$$\psi_1\, e_1 + \psi_2\, e_2 + \ldots + \psi_{16}\, e_{16},$$

where the numbers ψ_i, which are *ordinary numbers,* will have arbitrary – but not necessarily zero – values.

The first conclusion that one may deduce from this assertion is that the ψ *defined by the Dirac equation has sixteen components,* and not four, as one usually assumes. This equation is thus equivalent, not to four differential equations, but to 16: One obtains them by taking the product:

$$F\,\psi = (e_1\, t_1 + e_2\, t_2 + e_3\, t_3 + e_4\, t_4 + e_5\, t_5)(\,\psi_1\, e_1 + \psi_2\, e_2 + \ldots + \psi_{16}\, e_{16}) = 0,$$

taking into account the rules of multiplication, and annulling the coefficients of the 16 units $e_1, e_2, \ldots, e_{16}$.

One may introduce a notation that simplifies the calculation enormously. Any unit e_k is a product of numbers $1, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$. We may thus write:

$$\begin{aligned}\psi = \psi_0 &+ (\psi_1\, \varepsilon_1 + \psi_2\, \varepsilon_2 + \psi_3\, \varepsilon_3 + \psi_4\, \varepsilon_4) \\ &+ (\psi_{12}\, \varepsilon_{12} + \psi_{23}\, \varepsilon_{23} + \psi_{31}\, \varepsilon_{31} + \psi_{14}\, \varepsilon_{14} + \psi_{24}\, \varepsilon_{24} + \psi_{34}\, \varepsilon_{34}) \\ &+ (\psi_{123}\, \varepsilon_{123} + \psi_{234}\, \varepsilon_{234} + \psi_{341}\, \varepsilon_{341} + \psi_{412}\, \varepsilon_{412}) + \psi_{1234}\, \varepsilon_{1234}\,. \end{aligned} \tag{19}$$

The index of each ψ_i is equal to the number that is defined by the indices of the units that correspond to it, taken in the same order: e.g., the coefficient of $\varepsilon_{123} = \varepsilon_1\, \varepsilon_2\, \varepsilon_3$ will thus be ψ_{123}, and so on. In addition, the coefficient of the principal unit $\varepsilon_0 = 1$ will be denoted by ψ_0.

It is, moreover, easy to write the equation directly if one adopts the notation (18). We commence with an arbitrary term of F – for example, $e_1\, t \equiv \varepsilon_1\, t_1$ – and multiply it by a likewise arbitrary term of ψ_1 – for example, $\psi_{14}\, \varepsilon_1\varepsilon_4 = \psi_{14}\, \varepsilon_{14}$. ψ_{14} will thus be the coefficient of $\varepsilon_1\, \varepsilon_2\, \varepsilon_3 = \varepsilon_4$ in $F\psi$. We thus attempt to find out what the other coefficients of e_4 are in $F\psi$. The following term in the Dirac equation is $\varepsilon_2\, t_2$, so the term $\psi_k\, \varepsilon_k$ that it corresponds to must be such that $\varepsilon_2\varepsilon_k = \varepsilon_4$; therefore $\varepsilon_k = \varepsilon_2\, \varepsilon_4$, $k = 24$, and it will consequently be ψ_{24}. When the same process is applied to all of the terms, there will be five of them. One has, for the equation:

$$t_1\;\psi_{14} + t_2\;\psi_{24} + t_3\;\psi_{34} + t_4\;\psi_0 + t_3\;\psi_{123} = 0,$$

or

$$\frac{\partial \psi_{14}}{\partial x_1}+\frac{\partial \psi_{24}}{\partial x_2}+\frac{\partial \psi_{34}}{\partial x_3}+\frac{\partial \psi_{0}}{\partial x_4}+\frac{2\pi i}{h}m_0 c\,\psi_{123}=0, \qquad (19)$$

and 15 other analogous equations, which one writes down immediately.

The general solution of the Dirac equation thus involves 16 functions, instead of four. One thus understands the sense in which the interpretation that one is presently given is limiting. One can better understand this difference with the aid of an observation of Schouten ([1]): The system S of complex numbers that we envisioned above is basically the product of two systems of quaternions. In other words, a basis for this system may be obtained in the following manner: If one is given two systems of quaternions that have 1, λ_1, λ_2, λ_3, λ_4 and 1, μ_1, μ_2, μ_3, μ_4 for their units then the sixteen numbers 1, λ_i, μ_i, $\lambda_i\,\mu_k = \mu_k\,\lambda_i$ (i, k = 1, 2, 3) may be considered to be the units of the system S ([2]). It is this peculiarity that justifies the name "quadri-quaternions." Any number in this system may then be written in the form:

$$N_0 + N_1\lambda_1 + N_2\lambda_2 + N_3\lambda_3\,, \qquad (20)$$

the components N_i being themselves quaternions.

Schouten observed that Dirac, by employing the matrices σ and ρ, basically used this particular system of units. Similarly, without specifying the matrix nature of the coefficients, the ψ may thus be written in the form (20) with four components, but these components ψ_1, ψ_2, ψ_3, ψ_4 *are themselves quaternions*. Considering them to be scalars thus constitutes an error that might lead to incomplete results that are valid only in certain special cases.

We thus now know the form of the general solution of the symbolic equation of Dirac. The first thing to do will thus be the systematic examination of the results that were obtained up to now when one replaces the usual four components with the 16 ψ_k. However, before entering into such a task, we can point out some results of a general nature that are independent of the solution equations (19). These results are destined to show that the process employed, which is the only logically admissible one, possesses some other advantages, in addition, among which, one has that of facilitating the physical interpretation of the quantities ψ_k.

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7. Lorentz transformations. – The general relativistic equation (12), $F\psi = 0$, is invariant under Lorentz transformations, just like the Dirac equation itself, but moreover, it is written *in an invariant form*. This is an extremely satisfying characteristic, since the

([1]) SCHOUTEN, *loc. cit.*, pp. 106.
([2]) For example, by taking:

$$E_1 = \lambda_1\,\mu_3\,, \qquad E_2 = \lambda_2\,\mu_3\,, \qquad E_3 = \lambda_3\,\mu_3\,, \qquad E_4 = -\,i\mu_1, \qquad E_1 = -\,i\mu_2\,.$$

theory is essentially based, as Darwin remarked ([1]), on the invariance of form of the equations. We thus set $x_4 = ict$ and consider the Lorentz transformations:

$$x_4' = \sum_{l=1}^{k} O_{kl} x_l \qquad (l = 1, 2, 3, 4), \tag{21}$$

with

$$\sum_{l=1}^{k} O_{kl} O_{il} = \sum_{l=1}^{k} O_{kl} O_{li} = \delta_{ik} ,$$

equations to which we add a fifth one that expresses the invariance of the proper mass, and which is not contained in the preceding ones:

$$m_0 = \text{constant}. \tag{21′}$$

Since the p (or the t) transform like a vector, the equation $F\psi = 0$ becomes:

$$\left\{\sum_{i=1}^{5}\left(\sum_{l=1}^{4} O_{li} t_l'\right) E_i\right\}\psi' = \left\{\sum_{l=1}^{4}\left(\sum_{i=1}^{4} O_{li} E_i\right) t_i' + m_0 c\, E_5\right\}\psi' = 0,$$

which has the original form:

$$\left(\sum_{l=1}^{5} E_l' t_l'\right)\psi' = 0$$

if

$$E_l' = \sum_{1}^{4} O_{li} E_i \qquad (i = 1, 2, 3, 4), \tag{22}$$

$$E_5' = E_5, \tag{23}$$

and the new equation is always a Dirac equation, because one has:

$$E_j'^2 = 1,$$
$$E_i' E_j' + E_j' E_i' = 0 \qquad (i, j + 1, 2, 3, 4, 5),$$

as one verifies immediately.

The relation (23) is a consequence of (22), because one has:

$$E_5' = E_1' E_2' E_3' E_4' = E_1\, E_2\, E_3\, E_4 = E_5 .$$

([1]) DARWIN, *loc. cit.*, pp. 657. Darwin observed that in order to assure this formal invariance of the equations it is necessary to introduce 16 quantities into the calculations, which he declined to define, because it seemed absurd to him that an electron needed so many elements in order for it to be defined. We nonetheless confirm that this situation, far from being a mathematical complication, brings us closer to the physical side of the question.

Now consider the transformation law for the ψ under the hypothesis of the formal invariance of the equation $F\,\psi = 0$. The law (22) for the $E_1 = \varepsilon_1$, $E_2 = \varepsilon_2$, $E_3 = \varepsilon_3$, $E_4 = \varepsilon_4$ provides us with the transformation law of all the units $\mathbf{e}_k$, and consequently, all the components ψ_k of:

$$\psi = \psi_0 + \psi_1\,\varepsilon_1 + \ldots + \psi_{1234}\,\varepsilon_1\,\varepsilon_2\,\varepsilon_3\,\varepsilon_4\,.$$

One easily sees that E_1, E_2, E_3, E_4 transform according to (22), and thus, like the components of a vector. The same is true for:

$$E_2\,E_3\,E_4\,, \qquad E_3\,E_4\,E_1\,, \qquad E_4\,E_1\,E_2\,, \qquad E_1\,E_2\,E_3\,,$$

which one assures by writing:

$$-E_5\,E_1, \qquad E_5\,E_2, \qquad -E_5\,E_3, \qquad E_5\,E_4,$$

and taking into account that, from (23), E_5 is an invariant. It follows from this that the ψ_1, ψ_2, ψ_3, ψ_4, on the one hand, and the ψ_{123}, ψ_{234}, ψ_{341}, ψ_{412}, on the other, can represent the components of two vectors that we denote by P and R, respectively. In the same fashion, one accounts for the fact that (ψ_{12}, ψ_{23}, ψ_{31}, ψ_{14}, ψ_{24}, ψ_{34}) are the components of an anti-symmetric tensor M of second rank. Moreover, 1 and E_5 are invariants; ψ_0 and ψ_{1234} are thus scalars that we denote by m and λ, respectively.

The ψ of Dirac is therefore a mathematical entity *that is composed of two scalars m and λ, two vectors P and R, and an anti-symmetric tensor M of second rank.* This is not what one has called a *semi-vector*, but a quantity whose transformation laws are the ordinary laws of tensor calculus. One immediately sees what the general type of quantities of this genre is if one observes that the numbers of components of its constituents, when conveniently arranged, reproduce the coefficients of a binomial development:

$$1,\ 4,\ 6,\ 4,\ 1.$$

The general solution of the fundamental equation of quantum mechanics thus introduces 16 components, in place of the four of Darwin; this complication is nonetheless redeemed by a considerable advantage. Indeed, the important point is that *one can give an immediate physical interpretation to these* 16 *quantities,* which it does not have, and cannot have, in the Dirac theory with four components.

As one confirms later on, the two vectors P and R are attached to the momentum quadri-vector (p_1, p_2, p_3, E/c) and the radius quadri-vector (x, y, z, ct). M is attached to the tensor that is defined by the magnetic moment (components 12, 23, 31), and the electric moment (components 14, 24, 34) of the electron. The invariant m will be attached to the rest mass, and finally λ will be linked with the fundamental de Broglie wave length.

8. The degrees of freedom of the electron. – One can reasonably think that, *a priori*, the elements that are attached to the *independent* units e_k, like the ψ_k, refer to the *characteristic* quantities of the electron. The preceding analysis shows us that we have

16 units e_k . This fact leads us to examine the following problem: *Independently of the Dirac equation, and likewise of the theory of quanta*, how many independent quantities do we need in order to determine an electron? How many possible independent variations, or furthermore, *how many degrees of freedom does an electron possess*?

A simple calculation, which has never been done because the problem was never posed, gives us 16 quantities for the result: Namely, four coordinates, four momenta, three components for the magnetic moment, three for the electric moment (which *Frenkel* introduced without the aid of quantum theory), the mass, and the de Broglie wavelength, or, more precisely, the mass and the coordinate that is conjugate to it ([1]). One therefore justifies this number – viz., 16 – of degrees of freedom *physically*, a fundamental result on which Eddington ([2]) based the calculation of the value of the elementary charge *e*. One can easily deduce from this that when a system of two electrons is present one will have a number of degrees of freedom that is equal to $16^2 - C_{16}^2 = 256 - 120 = 136$. However, whereas for the Eddington school these numbers result from abstract mathematical considerations (this number is the number of certain abstract "rotations" in a matrix space), here they appear with an immediate physical interpretation. This intuitive calculation is confirmed by the fact that the Dirac equation automatically introduces, with no supplementary hypothesis, sixteen probabilities that correspond to each of these quantities, and this agrees with the transformation law that relativity imposes upon them. This does not seem to be a simple coincidence.

We shall now examine how one can be led to justify the interpretation that we have given for the quantities that correspond to the ψ_k . The basic principle of the method is the same one that permits us to attribute, for example, the character of an angular momentum for the electron to the σ of Dirac. Rigorously, this justification can be done, up to a certain point, by means of concepts that are known and universally accepted. Nonetheless, it seems to us that in this fashion one greatly and unnecessarily limits the generality of the proofs. We shall thus treat the problem in a manner that seems logically and physically correct to us. This will lead us to use a notion that we have introduced previously ([3]), with the name of "the fifth dimension." It will always be possible for the reader to assume that the introduction of this "fifth dimension" in what follows is only a mathematical artifice; the results will not be altered. We nevertheless think that it presents a certain interest and physical significance that is much more profound than that.

9. The fifth dimension. – Consider a material point of *variable mass* that is in motion. In classical analytical mechanics, one describes motion in a complete fashion by giving the values of the coordinates, the momenta, the energy, and finally, the mass, or a quantity that is proportional to it, *mc*. We have shown ([3]) that this manner of proceeding is a sin against symmetry and generality. The quantities envisioned can be arranged into a table such as:

([1]) See *Comptes Rendus* **186** (1928), pp. 739 and **186** (1928); the question of this conjugate coordinate will be reprised in § 9.

([2]) Proc. Roy. Soc. **A122** (1929), pp. 358.

([3]) *Comptes Rendus* **186** (1928), pp. 739.

$$\left.\begin{matrix} p_x, & p_y, & p_z, & E/c, & mc, \\ x, & y, & z, & t, & ?, \end{matrix}\right\} \tag{24}$$

in such a fashion that they correspond pair-wise. Each momentum then corresponds to a coordinate to which it is canonically conjugate.

It is indubitable that the mass must be arranged in the category of momenta; indeed, one obtains a conservation law, as one does for the other momenta. However, analytical mechanics makes the additional arbitrary hypothesis that *the variable that is canonically conjugate to the mass does not exist, or rather, that it is always equal to zero.* We must eliminate this hidden hypothesis if we would like to study the problem of motion in full generality; *this is, above all, desirable in a relativistic theory*. As we have shown, the conjugate variable to mass represents the de Broglie wavelength λ. If one considers – as one always does – that the rest mass is well-defined and equal to m_0 then the value of λ is indeterminate of the form $u\,\lambda_0$, λ_0 being the fundamental de Broglie wavelength, $\lambda_0 = h / m_0c$, and n is an arbitrary integer.

Quantum mechanics, just like classical mechanics, sets $\lambda \equiv 0$; however, the commutation relations $pq - qp = (h/2\pi i)\,1$ lead to an absurdity in this case, namely, that one can consider the mass to be an element that is susceptible to variation.

In summation, the description of the motion that analytical mechanics envisions is incomplete; one must introduce the coordinate λ that is conjugate to the mass. The universe in which phenomena take place is not space-time *x*, *y*, *z*, *ct*, but a space-time that is completed by a fifth dimension λ that characterizes *matter* in some way. The repugnance by which one employs this fifth dimension seem to us completely analogous to what one felt, before Minkowski, when considering the fundamental element of space-time; i.e., the adjunction of time to space into an intimate fusion of *space-time*. Yet, it seems absurd to us (although this is not an argument) that the *structural* element that one must start with in order to describe phenomena is *uniquely* space and time. This amounts to reducing the matter to space and time. It seems much more satisfactory to introduce λ from the outset, since it is introduced by itself, as well, and to start with a universe that is not a space-time, but a *space-time-matter*. We may hope to establish a coherent, and above all complete, theory of phenomena (which will necessarily be a "field" theory) only by starting with a geometric element that is defined in the complete five-dimensional universe. It is particularly interesting to remark that the new theory of Einstein is not in contradiction with this demand, since it is independent of the number of dimensions of the continuum envisioned.

Regardless of these general considerations, we return to the Dirac equation. It is a linear combination of momenta p_0, p_1, p_2, p_3, *and* $m_0\,c$. The quantity m_0c thus intervenes in a symmetric fashion, and *plays the same role as the momenta* p_k . In order to pass to the Dirac equation, one must set:

$$p_k = t_k = \frac{h}{2\pi i}\frac{\partial}{\partial x_k} \qquad (k = 1, 2, 3, 4).$$

In order to increase the symmetry, we may also set:

$$m_0 c = t_5 = \frac{h}{2\pi i}\frac{\partial}{\partial x_5}, \tag{25}$$

x_5 representing a fifth coordinate. Eddington, justifiably pursuing the goal of attaining a higher degree of symmetry, was led to write the formal equation (25); however, he is to be defended for attributing the role of a fifth dimension to x_5 ([1]). His argument is based on the form of the Dirac equation for two electrons; it is not presently valid, since that equation has since been recognized to be incorrect, and in particular, by Eddington himself, who replaced it with an equation of Gaunt ([2]), which is itself subject to criticism.

On the contrary, for us the appearance of $m_0\, c$ in the Dirac equation and the symmetric role that it plays is a confirmation of the hypotheses that were previously presented. We believe that we can legitimately set:

$$t_5 = \frac{h}{2\pi i}\frac{\partial}{\partial x_5},$$

t_5 being the fifth dimension, because this relation signifies something more that a simple calculation device. We may, moreover, specify that t_5, when applied to ψ gives $m_0 c\,\psi$; however, this does not signify that if one applies it to another function one will always recover $m_0 c$ as a factor. In particular, $t_5\,(x_5\;\psi) \neq x_5\; t_5 = m_0 c\; x_5\;\psi$.

10. Intepretation of the quantities that are attached to the e_i . – First consider the ψ_{12}, ψ_{23}, ψ_{31}, which are the coefficients of the E_1E_2, E_2E_3, E_3E_1, respectively.

Let $F = \sum_1^5 E_i t_i$, and look for the first integrals of the motion; i.e., the expressions that satisfy:

$$FX - XF = 0.$$

Eddington observed that $x_1\, t_2 - x_2\, t_1$ is not a first integral, but, by comparison:

$$M_{12} = x_1\, t_2 - x_2\, t_1 + \frac{1}{2}\frac{h}{2\pi i} E_1\, E_2$$

is one. Indeed, it is easy to verify that M_{12} commutes with F.

One interprets this in the following fashion: In ordinary mechanics, $x_1\, t_2 - x_2\, t_1$ will be the moment of impulse, which is an integral of the motion. In the new mechanics, in order to have a first integral, one must add $\frac{1}{2}\frac{h}{2\pi i} E_1 E_2$. This quantity (and consequently,

([1]) Proc. Roy. Soc. A**121** (1928), pp. 524.
([2]) *Ibidem*, A**122** (1929), pp. 513; see also G. BREIT, Phys. Rev. **34** (1929), pp. 553.

also $\frac{1}{2}\frac{h}{2\pi i}E_2E_3$, $\frac{1}{2}\frac{h}{2\pi i}E_3E_1$) is therefore of the same nature as the preceding one; in fact, it is the angular momentum of the electron, corresponding to its magnetic moment.

In an analogous fashion, the terms in:

$$E_1E_4, \quad E_2E_4, \quad E_3E_4 ,$$

(relative to the coordinate x_4, and therefore, to time) correspond to the electric moment of the electron. The set of these six components:

$$(12), \quad (23), \quad (31); \qquad\qquad (14), \quad (24), \quad (34)$$

form an anti-symmetric tensor of second rank that represents the total moment of the electron, which is the "six-component vector" of Frenkel.

The combinations E_1E_2, … that characterize them where they appear are the components of this moment. It is therefore entirely natural to suppose that the scalars ψ_{12}, ψ_{23}, …, which are the coefficients of E_1E_2, E_2E_3, … in

$$\psi = \psi_0 + \ldots + \psi_{12}\, E_1E_2 + \ldots,$$

are the quantities that are attached to each of the components of the total moment tensor of the electron. To abbreviate, we say that ψ_{12}, for example, is the "probability" of the 12 component of the magnetic moment of the electron. In the following paragraph, we will examine the interpretation of these probabilities ψ_k in a more precise fashion in the language of ordinary probability.

Now consider the ψ_{234}, ψ_{341}, ψ_{412}, ψ_{123} . They are the coefficients of $E_2\, E_3\, E_4$, … that one may also write as:

$$-E_5\,E_1, \qquad E_5\,E_2, \qquad -E_5\,E_3, \qquad E_5\,E_4\,.$$

The same calculation as before shows us that:

$$M_{k5} = x_k\, t_5 - x_5\, t_k + \frac{1}{2}\frac{h}{2\pi i}E_5\,E_k$$

is a first integral.

The $\frac{1}{2}\frac{h}{2\pi i}E_5\,E_1$ or $\frac{1}{2}\frac{h}{2\pi i}E_2\,E_3\,E_4$ is therefore of the same nature as the moment $x_1\, t_5 - x_5\, t_1$, or simply the moment $x_1\, t_5$. Now, the latter operator (if M is applied to ψ) reduces to $m_0c\, x_1$. Consequently, the probability ψ_{234} that is attached to $E_2\, E_3\, E_4$ may be considered to be simply attached to the coordinate x, since m_0 remains constant. Thus:

$$\psi_{234},\ \psi_{341},\ \psi_{412},\ \psi_{123}$$

are the probabilities that are attached to the four coordinates x, y, z, ct.

On the other hand, in the expression for the Hamiltonian the coefficients of E_1, E_2, E_3, E_4, E_5 are the quantities of motion, the energy, and the mass, respectively. It is therefore natural to further suppose here that the ψ – viz., the coefficients of the E_1, … – will be attached to probabilities:

ψ_1, ψ_2, ψ_3	to the components of the quantity of motion,
ψ_k,	to the energy, or to E/c,
$\psi_5 = \psi_{1234}$,	to the mass, or to $m_0\,c$.

Finally, as for the coefficient ψ_0 of the principal unit, there can be no doubt in that regard: The complete system that we have introduced shows quite well that this probability is attached to the coordinate λ that is conjugate to the mass, namely, the de Broglie wavelength.

We therefore have "probabilities" for each of the components of the quantities that characterize an electron. The following table summarizes this:

ψ_1, ψ_2, ψ_3:	quantity of motion	ψ_4:	energy,	ψ_{1234}:	mass,
ψ_{234}, ψ_{341}, ψ_{412}:	coordinates x, y, z,	ψ_{123}:	time,	ψ_0:	de Broglie wavelength,
ψ_{12}, ψ_{23}, ψ_{31}:	magnetic moment,	ψ_{14}, ψ_{24}, ψ_{34}:	electric moment.		

The introduction of 16 components, far from being a complication, simplifies the physical interpretation of the general problem.

11. Nature of the ψ_k. Hypercomplex probabilities. We now examine the nature of the "probabilities" ψ_1, ψ_2, … a little more closely. In the first place, it is obvious that the same interpretation, such as it is, must be true for the ψ_k, as well for $\psi = \psi_0 + \psi_1 E_1 + \ldots$ Today, one assumes that the ψ that is given by the wave equation is a probability, or, if one prefers, a quantity that allows one to calculate the probabilities uniquely if one starts with it ([1]). However, what physical sense must we give to that assertion in the case of the ψ_k?

Consider the Schrödinger equation. $|\psi|^2\,dq$ will be the probability of the presence of the electron in the volume element dq and $\int |\psi|^2\,dq = 1$. In order to obtain the probability of an energy state n, we must develop the general solution in a series of characteristic functions that correspond to the energy:

$$\psi = \sum_n c_n \psi_n(q)\, e^{-\frac{2\pi i}{h} E_n t},$$

([1]) Here, we recover the same situation as in the original theory of Schrödinger: The ψ_k are ordinary numbers, but they may be complex numbers. This will not create any difficulty if one adopts the concept of imaginary probability that was proposed previously [J. Phys. **10** (1929), pp. 12]. In the contrary case, one must specify that it is the modulus of ψ that will give this probability.

or into integrals, if one is dealing with a continuous spectrum. $| c_n |^2$ will then be the probability for the atom to be found in the n^{th} quantum state, the functions ψ_n being normalized. If we would like to obtain the probability for the system to have a moment that is found between p and $p + dp$, we only have to develop the solution in the corresponding proper functions. In general, when we have the general solution ψ the probability that a variable r is found between r and $r + dr$ will be given by the coefficients of the development of ψ in corresponding proper functions. The precise manner by which this may be done is given by the Dirac transformation theory; Darwin has recently insisted on the physical significance of this process ([1]).

The rational interpretation of the ψ_k emerges immediately. In the usual case, we must develop *the same* function ψ into a Fourier series (or integral) in no particular variable. However, in the present case, where:

$$\psi = \psi_0 + \psi_1 E_1 + \ldots,$$

we develop the ψ_k *that corresponds to that variable itself*. If we would like to obtain the probability for the atom to be found in the energy state n then we would have to develop the function ψ_4 in a series; however, if we seek the probability for the coordinate x to be found between x and $x + dx$ then we no longer develop ψ_4, but the function that corresponds to x, viz., ψ_{234} . Each element that determines an electron thus has its proper probability function ψ_k .

The foregoing also indicates what one must say when one confirms that ψ is a probability. That probability is a global element: It gives information about *all* of the characteristic quantities of the electron.

In a certain sense, ψ contains all of the probabilities concerning the electron. Now, ψ is a hypercomplex number. It is also a set of sixteen components, among which one may find vectors, a tensor, etc. One may then speak of *tensorial* or *hypercomplex probabilities*; these terms are introduced with no ambiguity. One does not encounter them in the classical calculus of probabilities because they never appear in the problems that one usually studies. However, it is perfectly legitimate to associate several continuous probabilities that satisfy certain conditions for them to be a "tensor" and to study the tensorial calculus of probabilities for itself. This has not been done, and even if it were done, it would remain a simple mental exercise, because one did not, *a priori*, see any immediate class of problems that might suggest this calculus since one could not easily imagine any problems that would use this concept.

Things are different now. The undulatory mechanics shows us the class of problems in which these probabilities appear. At the same time, the preceding developments bring to light the interest that a calculus of this type might present for the implementation of the new concepts of quantum mechanics. It is for this reason that it has not been completely pointless to exhibit this problem in a specific fashion for the mathematicians.

([1]) DARWIN, "La théorie ondulatoire de la matiére," Annales de l'Institut Henri Poincaré **1** (1930), pp. 25.

12. Components of the quadri-dimensional current. – Another very important question can be treated without entering into the direct study of equations (19), viz., that of the current and continuity equation that it satisfies.

In the case of the Schrödinger equation, or that of Dirac, the current is calculated by starting with ψ and its conjugate $\bar{\psi}$. It thus seems that it is necessary to generalize the definition of imaginary conjugate in our case. However, it is not in this manner that we shall proceed, since another method lends itself to this generalization better. Indeed, the current is calculated, as usual, by starting with ψ, which is solution of the given equation, and ψ^+, which is a solution of an *adjoint equation* that is easily derived from the preceding one.

Then consider the fundamental equation (12):

$$\frac{h}{2\pi i}\left(E_1\frac{\partial\psi}{\partial x_1}+E_2\frac{\partial\psi}{\partial x_2}+E_3\frac{\partial\psi}{\partial x_3}+E_4\frac{\partial\psi}{\partial x_4}\right)+m_0c\,E_5\,\psi=0. \tag{12}$$

The adjoint equation will be:

$$-\frac{h}{2\pi i}\left[\frac{\partial(\psi^+E_1)}{\partial x_1}+\cdots\right]+m_0c\;\psi^+\,E_5=0,$$

or

$$-\frac{h}{2\pi i}\left[\frac{\partial\psi^+}{\partial x_1}E_1+\cdots+\frac{\partial\psi^+}{\partial x_4}E_4\right]+m_0c\;\psi^+\,E_5=0. \tag{24}$$

Multiplying the first one by ψ^+ on the left and the second one by ψ on the right, and taking the difference gives:

$$\frac{h}{2\pi i}\left[\left(\psi^+E_1\frac{\partial\psi}{\partial x_1}+\frac{\partial\psi^+}{\partial x_1}E_1\psi\right)+\cdots+\left(\psi^+E_4\frac{\partial\psi}{\partial x_4}+\frac{\partial\psi^+}{\partial x_4}E_4\psi\right)\right]=0,$$

or furthermore:

$$\frac{\partial}{\partial x}(c\psi^+E_1\psi)+\frac{\partial}{\partial y}(c\psi^+E_2\psi)+\frac{\partial}{\partial z}(c\psi^+E_3\psi)-\frac{\partial}{\partial t}(\psi^+iE_4\psi)=0.$$

This is the continuity equation, which shows that one may take the components of the quadri-dimensional current to have the expressions:

$$j_x=c\,\psi^+E_1\,\psi,\quad j_y=c\,\psi^+E_2\,\psi,\quad j_z=c\,\psi^+E_3\,\psi,\quad \rho=-\,\psi^+(i\,E_4)\,\psi. \tag{26}$$

These have the same form as the classical expressions; all of the results that were established before must thus persist.

Observe, moreover, that the usual continuity equation has only four terms since *the mass is constant*. If this were not true then one would have to write an equation in five terms and the current would be a current with five components, where the fifth one would represent a sort of matter flux. These considerations are attached to the ones that were presented in paragraph 9; we do not insist upon them.

When studying the relativistic equation in full generality, one can therefore hope to have more complete results and agree with physical reality more closely for certain considerations that are drifting much too far away. The first problem that these general considerations poses consists in studying, in detail, how the introduction of the 16 components modifies the solution to the problems that were solved before and in seeing if other problems are not susceptible to a simple treatment. It is clear that certain results that were originally obtained by a process of approximation remain the same as before; by contrast, other problems must receive a completely altered solution.