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Particles with very large velocities in spinorial mechanics

A. Proca

Institut Henri Poincaré – Paris

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Translated by D. H. Delphenich

Abstract. – Upon taking the spinorial mechanics of a material point as a point of departure, the author studies the case where the three-dimensional velocity of the latter is equal to that of light. That condition is naturally realizable only if the rest mass is zero. In this case, one obtains the description of an entire class of particles, among which, one finds the photon, and for which one describes some of their important characteristics.

1. – By means of a simple passage to the limit, all of relativistic mechanics can provide the description of properties of a particle that moves with the velocity of light, and which, consequently, presents some characteristics that are very close to those of a photon. This is true, for example, for the spinorial mechanics of particles that we have developed, moreover ([1]).

Note that it does not suffice to make the velocity of a particle tend to c in order to automatically obtain the description of a photon, and this is true for several reasons, namely:

A priori, one must have *several* types of particles that move with the velocity of light, but which are different in other properties (for example, related to their spin), and the comparison is flawed because the information that we possess on the photon is expressed in terms of undulatory quantities (electromagnetic waves) that they are attached to, while the mechanical description that we imagine appeals to corpuscular notions. Any reconciliation of these two viewpoints necessarily supposes the introduction of new hypotheses.

Nonetheless, the study of the limiting case is interesting in itself, and we shall sketch it out below. As in the cited article, we confine ourselves to the "classical" form of the theory, not only in order to leave open any question of interpretation, but also, and above all, in order to obtain results that are independent of quantization, and which therefore submit to a test of our fundamental hypothesis.

In order to facilitate the discussion, in the following paragraph we will summarize the results that were already acquired.

([1]) Jour. Phys. **15** (1954), 65.

2. – Notations, fundamental equations, etc. – The fundamental hypothesis consists of assuming that the motion of a point is essentially described by spinorial variables, instead of by vectors x^ρ, as in classical relativistic mechanics.

For ease of formalism, we adopt the technique that is employed in the Dirac theory of the electron. We let ξ denote a matrix with one column of elements ξ_1, ξ_2, ξ_3, ξ_4 that transforms like the Ψ of the Dirac theory, and which we call a *spinor,* to abbreviate. The adjoint matrix will be defined by:

$$\xi^+ = i\tilde{\xi}^*\gamma^4,$$

where $\tilde{\xi}^*$ is the transposed and conjugated matrix of ξ, and γ_1, γ_2, γ_3, γ_4 are 4×4 matrices that satisfy:

$$\gamma^\rho \gamma^\sigma + \gamma^\sigma \gamma^\rho = 2\delta^{\rho\sigma}.$$

In general, the results are independent of the representation of the γ^ρ; when we need it, we shall choose the usual representation:

$$\gamma^1 = \begin{pmatrix} \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & -i & \cdot \\ \cdot & i & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} \cdot & \cdot & -i & \cdot \\ \cdot & \cdot & \cdot & i \\ i & \cdot & \cdot & \cdot \\ \cdot & -i & \cdot & \cdot \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{pmatrix}.$$

The spinors ξ, ξ^+ define the particle and its motion. The equations that they obey (see below) are established by starting with a Lagrangian that is chosen by reasons of simplicity and invariance. The same is true in the Dirac theory, and since the simple elements are the same in the two cases, it is not surprising that some of the results of the two theories are formally identical.

By starting with the spinors ξ, ξ^+, one may calculate the quantities that characterize the particle in spacetime. The link between the spinorial space of the ξ and spacetime is given by the relation:

(1) $$\frac{dx^\rho}{d\tau} = \xi^+ \gamma^\rho \xi,$$

where τ is the invariant parameter with respect to which one defines the motion.

Let λ_ρ be the quantity of motion vector, which is timelike. Define the invariant:

(2) $$\lambda = m_0\, c = \sqrt{-\lambda_\rho \lambda^\rho},$$

m_0 being the "rest mass" of the particle.

With $x^4 = ict = ix^0$ and $\gamma^5 = \gamma^1\,\gamma^2\,\gamma^3\,\gamma^4$, we set the real quadratic spacetime quantities equal to:

$$\Omega_1 = -\,i\,\xi^+\xi, \qquad \Omega_2 = \xi^+\gamma^5\xi,$$

$$u^\rho = \xi^+\gamma^\rho\xi, \qquad w^\rho = i\,\xi^+\,i\,\gamma^5\gamma^\rho\xi,$$

$$m^{\rho\sigma} = i\,\xi^+\frac{i}{2}(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho)\,\xi.$$

A certain number of algebraic identities exist between these quantities, among which we will need the following ones:

$$(3)\qquad \begin{cases} w_q u_r - w_r u_q = -m_{qr}\Omega_2 - i m_{p4}\Omega_1, \\ -i w_p u_4 + i w_4 u_p = -m_{qr}\Omega_1 + i m_{p4}\Omega_2, \end{cases} \qquad (p,\,q,\,r = 1,\,2,\,3),$$

$$(4)\qquad \begin{cases} (m_{23})^2 + (m_{21})^2 + (m_{12})^2 = -(u_4)^2 - (w_4)^2 - \Omega_2^2, \\ (m_{14})^2 + (m_{24})^2 + (m_{34})^2 = +(u_4)^2 + (w_4)^2 + \Omega_1^2, \\ \sum (m_{23})(m_{14}) = i\Omega_1\Omega_2, \end{cases}$$

$$(5)\qquad \begin{cases} \sum (m_{23})u_1 = +\Omega_1 w_4, \\ \sum (m_{14})u_1 = -\Omega_2 w_4. \end{cases}$$

Finally, the relations between the proper time ds of the particle and $d\tau$ is given by (*loc. cit.*, eq. (5)):

$$(6)\qquad -\,c^2\,ds^2 = \sum (dx^\rho)^2 = -\,(\Omega_1^2 + \Omega_2^2)\,d\tau^2\,.$$

This being the case, the fundamental equations, in the absence of a field, are written:

$$(7)\qquad \begin{cases} \dfrac{d\xi}{d\tau} = -\lambda_\rho\gamma^\rho\xi, & \dfrac{d\xi^+}{d\tau} = +\lambda_\rho\xi^+\gamma^\rho, \\ \dfrac{dx^\rho}{d\tau} = \xi^+\gamma^\rho\xi, & \dfrac{d\lambda_\rho}{d\tau} = 0. \end{cases}$$

a) *For* $\lambda \neq 0$, the general solution is:

$$(8)\qquad \xi = a\,\exp[i\lambda t] + b\,\exp[-i\lambda t], \qquad \xi^+ = b^+\exp[i\lambda t] + a^+\exp[-i\lambda t],$$

where λ_ρ, a, b are constants that satisfy:

(9)
$$\begin{cases} (i\lambda+\lambda_\rho\gamma^\rho)a=0, & (-i\lambda+\lambda_\rho\gamma^\rho)b=0, \\ a^+(i\lambda+\lambda_\rho\gamma^\rho)=0, & b^+(-i\lambda+\lambda_\rho\gamma^\rho)=0, \\ \lambda^2+\lambda_\rho\lambda^\rho=0. \end{cases}$$

If one takes four *arbitrary* complex constants and arranges them into a column φ then one can write:

(10)
$$\begin{cases} a=(-i\lambda+\lambda_\rho\gamma^\rho)\varphi, & b=(i\lambda+\lambda_\rho\gamma^\rho)\varphi, \\ a^+=\varphi^+(-i\lambda+\lambda_\rho\gamma^\rho), & b^+=\varphi^+(i\lambda+\lambda_\rho\gamma^\rho). \end{cases}$$

b) *For* $\lambda = 0$, one has the general solution as a function of four arbitrary constants that are arranged into the form of a column g:

(11)
$$\xi = (1-\lambda_\rho\,\gamma^\rho\cdot\tau)\,g, \qquad \xi^+ = g^+(1+\lambda_\rho\,\gamma^\rho\cdot\tau).$$

3. – Particles that move with the velocity of light. – We impose upon the preceding particle the single condition that its three-dimensional velocity be equal to that of light. One can see immediately that this condition entails another one: that the *rest mass of the particle* – m_0, and therefore $\lambda = m_0c$ – *must be zero.*

We examine this question in detail, since we will need some results that correspond to several other studies. One has (6):

$$\sum (dx)^2 = -\,(\Omega_1^2+\Omega_2^2)\,d\tau^2,$$

so if the velocity is equal to c then one must have:

(12)
$$\boxed{\Omega_1=-i\xi^+\xi=0, \qquad \Omega_2=\xi^+\gamma^5\xi=0.}$$

As functions of the four components λ of the quantity of motion and four arbitrary constants that contain the solution (10), these conditions translate into:

(13)
$$\lambda^2\varphi^+\varphi = 0, \qquad \lambda^2\varphi^+\gamma^5\varphi = 0$$

and

$$\lambda\cdot\lambda_\rho(\varphi^+\gamma^5\gamma^\rho\varphi) = 0.$$

Two cases are possible:

a) $\varphi^+\varphi = \varphi^+\gamma^5\varphi = \lambda_\rho\varphi^+\gamma^5\gamma^\rho\varphi = 0$, with $\lambda \neq 0$,

and

b) $\lambda = \sqrt{-\lambda_\rho\lambda^\rho} = m_0c = 0.$

Consider case *a*). Since $\varphi^+\varphi = \varphi^+\gamma^5\varphi = 0$, the identities (3) that were mentioned permit us to write the proportionality as:

$$\frac{\varphi^{+}\gamma^{\rho}\varphi}{\varphi^{+}\gamma^{5}\gamma^{\rho}\varphi} = \frac{\varphi^{+}\gamma^{\sigma}\varphi}{\varphi^{+}\gamma^{5}\gamma^{\sigma}\varphi} = \ldots$$

One therefore has not only $\lambda_\rho \varphi^{+}\gamma^{5}\gamma^{\rho}\varphi = 0$, but also:

$$\lambda_\rho \varphi^{+}\gamma^{\rho}\varphi = 0.$$

On the other hand:

$$\sum \lambda_\rho^2 = -\lambda^2 \qquad \text{and} \qquad \sum (\varphi^{+}\gamma^{\rho}\varphi)^2 = -(\Omega_1^2 + \Omega_2^2) = 0.$$

These relations permit us to write a Lagrange identity from which one deduces the proportionality:

$$\frac{\varphi^{+}\gamma^{\rho}\varphi}{\lambda_\rho} = \frac{\varphi^{+}\gamma^{\sigma}\varphi}{\lambda_\sigma} = \ldots$$

Consequently, since $\sum (\varphi^{+}\gamma^{\rho}\varphi)^2 = 0$, the sum:

$$\sum \lambda_\rho^2 = -\lambda^2 = 0$$

will likewise be zero.

Therefore, case *a*) reduces to case *b*), and one can conclude that in spinorial mechanics only particles of zero rest mass can attain the velocity of light. *"Zero rest mass" can say nothing more than this:* One will always have the following relation between the components of the quantity of motion $\lambda_k = p_k$ and the energy $\lambda_4 = i\,\lambda_0 = i(w/c)$ in that case:

$$p_1^2 + p_2^2 + p_3^2 = \left(\frac{w}{c}\right)^2.$$

4. – Particles of zero rest mass. – We remark, however, that if $\lambda = 0$ then it is the relation (11), and not (10), that is appropriate, and to which one must apply our conditions $\Omega_1 = \Omega_2 = 0$. From (11), one deduces that:

$$\Omega_1 = -i\,\xi^{+}\xi = -i\,g^{+}g, \qquad \Omega_2 = \xi^{+}\gamma^{5}\xi = g^{+}\gamma^{5}g - 2\lambda_\rho \cdot g^{+}\gamma^{5}\gamma^{\rho}g = 0.$$

Therefore, our conditions amount to restricting the generality of the constant g by imposing the following conditions upon it:

$$g^{+}g = 0, \qquad g^{+}\gamma^{5}g = 0, \qquad \text{and} \qquad \lambda_\rho \cdot g^{+}\gamma^{5}\gamma^{\rho}g = 0. \tag{14}$$

One satisfies these conditions (14) by taking g to be the solutions of the equation:

$$\lambda_\rho \gamma^{\rho} g = 0, \tag{15}$$

as one easily convinces oneself.

[We remark that these equations can be written in a form that presents an analogy with that of the case $\lambda \neq 0$, namely:

$$(16) \qquad \begin{cases} \lambda_0 g = \lambda_K \alpha^K g, \\ \lambda_0 \tilde{g}^* = \lambda_K \tilde{g}^* \alpha^K, \end{cases}$$

where $\tilde{g}^*$ is the transposed and conjugated matrix to g, $i\lambda_0 = \lambda_4$, and α^K are the Dirac matrices:

$$\alpha^K = i\,\gamma^4 \gamma^K \qquad (K = 1, 2, 3) \qquad \alpha^4 = \gamma^4].$$

With this solution, and taking (11) into account, one has $\xi = g$.

All of the ξ, *and therefore all of the spacetime quantities of a particle of velocity c* (among which one must include the photon), *are constants.* This solution is deduced from the general solution, which is valid for any λ:

$$\xi = a \exp [i\lambda\tau] + b \exp [-i\lambda\tau],$$

upon making $\lambda = 0$. In this case, one can take $g = a + b$, and one has:

$$\lambda_\rho \gamma^\rho g = 0.$$

We nonetheless remark that the g that satisfy the equations:

$$\lambda_\rho \gamma^\rho g = 0$$

depend upon only *two* arbitrary complex constants, instead of four, as in the case of $\lambda \neq 0$. Since two of the four components g_1, g_2 are arbitrary, the other two are deduced, for example, by the operation:

$$(17) \qquad \begin{vmatrix} g_3 \\ g_4 \end{vmatrix} = S \begin{vmatrix} g_1 \\ g_2 \end{vmatrix},$$

with $S = (1/\lambda_0)\,(\lambda_1\sigma^1 + \lambda_2\sigma^2 + \lambda_3\sigma^3)$, where the σ are the 2×2 Pauli matrices; the existence of the relation:

$$\lambda_0^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

introduces two signs.

5. – Spatial disposition of various measurable quantities that are attached to the particle. – In the space of the various spacetime quantities, consider:

$$\xi^+ A\,\xi = g^+ A\,g$$

that are attached to a particle of zero rest mass.

By virtue of (14) and paragraph **3**, we may say that:

a) The velocity $u^\rho = g^+ \gamma^\rho g$ is proportional to λ_ρ and to the vector:

$$w^\rho = i\, g^+\, i\gamma^5 \gamma^\rho g,$$

and the vectors u^K, λ^K, and w^K have the same support in space.

b) Consider the spin of the particle (*loc. cit.*, § 11):

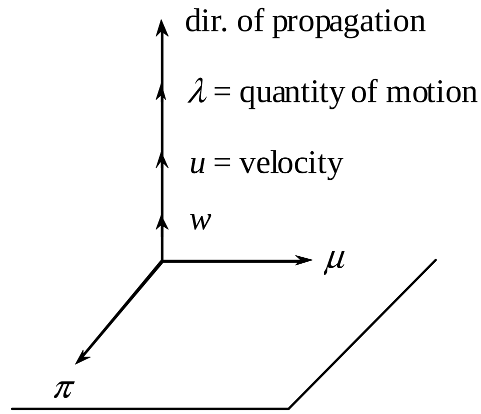
$$\tfrac{1}{2} m_{\rho\sigma} = -\tfrac{1}{2}\xi^+ \cdot \tfrac{1}{2}(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho) \cdot \xi,$$

which is an anti-symmetric tensor, and, as is customary, separate the "spatial part" from the "temporal part," which we call:

$$(18) \qquad \mu_i = -\tfrac{1}{2} m_{iK}, \qquad i\pi_K = \tfrac{1}{2} m_{K4}, \qquad (i, j, K = 1, 2, 3),$$

respectively. Since $g^+g = g^+\gamma^5 g = 0$, the identities (4) and (5) give:

$$(19) \qquad \begin{cases} \sum \mu_K^2 = \sum \pi^2, \quad \sum \mu_K \pi_K = 0, \\ \sum u_K \mu_K = \sum u_K \pi_K = 0. \end{cases}$$



The two "vectors" π and μ are equal, perpendicular to each other, and perpendicular to the direction of the velocity.

One can thus draw the associated figure of the vectors that accompany a particle that has the velocity of light in its motion, a figure that reproduces the general disposition of the electromagnetic field of a photon.

6. – Simple expressions. – We do not restrict the generality if we choose a frame such that the propagation takes place along the z axis, so $u_1 = u_2 = 0$, which entails that $\lambda_1 = \lambda_2 = 0$.

The quantities that are attached to the photon have the following values as functions of two arbitrary complex quantities ξ_1 and ξ_2 :

velocity (described with respect to τ):

$$u_1 = 0, \qquad u_2 = 0, \qquad u_3 = \pm 2\,(\xi_1^*\xi_1 + \xi_2^*\xi_2), \qquad u_4 = 2i\,(\xi_1^*\xi_1 + \xi_2^*\xi_2),$$

velocity in ordinary space:

$$V_1 = 0, \qquad V_2 = 0, \qquad V_3 = \pm c.$$

Vector w, which is proportional to the velocity:

$$w_1 = 0, \qquad w_2 = 0, \qquad w_3 = -2(\xi_1^*\xi_1 - \xi_2^*\xi_2), \qquad w_4 = \mp 2i(\xi_1^*\xi_1 - \xi_2^*\xi_2).$$

Components of the spin:

$$\tfrac{1}{2}m_{23} = (\xi_1^*\xi_2 + \xi_2^*\xi_1), \qquad \tfrac{1}{2}m_{31} = (\xi_1^*\xi_2 - \xi_2^*\xi_1), \qquad \tfrac{1}{2}m_{12} = 0,$$

$$\tfrac{1}{2}m_{14} = \mp(1/i)(\xi_1^*\xi_2 - \xi_2^*\xi_1), \qquad \tfrac{1}{2}m_{24} = \pm(\xi_1^*\xi_2 + \xi_2^*\xi_1), \qquad \tfrac{1}{2}m_{34} = 0,$$

so, with the notation (18):

$$\mu_1 = \xi_1^*\xi_2 + \xi_2^*\xi_1, \qquad \mu_2 = (1/i)(\xi_1^*\xi_2 - \xi_2^*\xi_1), \qquad \mu_3 = 0,$$

$$\pi_1 = \mp(1/i)(\xi_1^*\xi_2 - \xi_2^*\xi_1), \qquad \pi_2 = \pm(\xi_1^*\xi_2 + \xi_2^*\xi_1), \qquad \pi_3 = 0.$$

The corresponding lengths are:

$$\sum \mu_K^2 = \sum \pi_K^2 = 4\xi_1^* \cdot \xi_1 \cdot \xi_2^*\xi_2 .$$

7. – Some remarks.

a) In a general fashion, the basic equations provide the ξ only up to a factor, and a normalization is necessary. For example, one can set, as is customary:

$$\xi_1^*\xi_2 + \xi_2^*\xi_1 = 1.$$

b) Similarly, a corpuscle that has the velocity of light still remains characterized by the vector w_K . One easily sees that a particularly interesting case is the one in which:

$$w_K = 0 \qquad (K = 1, 2, 3),$$

for the following reason: One has, *in general* (i.e., for arbitrary velocity):

$$u_\rho\, m^{\rho K} = -\, w_K\, \Omega_2,$$

so if $w_K = 0$ then $u_\rho\, m^{\rho K} = 0$. Consider the frame in which the particle is at rest: $u_1 = u_2 = u_3 = 0$. Therefore, $m^{4K} = 0$ in that frame, as is well-known. Hence, a particle with $w^K = 0$

is such that its spin reduces to its spatial components, and thus, to just a "vector," in the rest system.

It is not impossible that this case is realized in nature exclusively, so the theory with $w^K \neq 0$ gives the general ideal case. Be that as it may, we maintain that $w^K = 0$ represents a particularly interesting case, and we have likewise imagined that this is true for the limiting case that is of interest to us here, and in which one can no longer speak of the "rest system."

In this case, $w^K = 0$ signifies that $\xi_1^* \xi_1 = \xi_2^* \xi_2$, so the normalization condition becomes $\xi_1^* \xi_1 = \frac{1}{2}$, and one has simply:

$$\sum \mu_K^2 = \sum \pi_K^2 = 1.$$

c) In this same case, the values of the spacetime quantities are deduced from ξ_1, ξ_2 – i.e., ultimately from two angles φ_1 and φ_2, which represent the phases of ξ_1, ξ_2.

The vectors $\boldsymbol{\mu}$ and $\boldsymbol{\pi}$ are perpendicular to each other, and their collective azimuth around the direction of motion – as measured by, for example, the angle that $\boldsymbol{\mu}$ makes with the *Ox* axis – is equal to $\varphi_2 - \varphi_1$.

One thus defines a sort of *direction of polarization*, as for photons, but which does not depend upon a wave.

We have called the tensor with the components $\frac{1}{2} m^{\rho\sigma}$ the "spin." Its length is zero: $\sum m_{\rho\sigma} m^{\rho\sigma} = 0$. *The length of the spin of particles with velocity c is zero.*

Among these particles, one must likewise include the photon. However, a comparison at this point is not possible, considering that the study of the photon has been made only as a function of the electromagnetic field that constitutes light, and that we have not further studied these fields in spinorial mechanics.

One must examine how that kind of mechanics is normally attached to the fields of these particles, which we shall do in the following article.

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