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Point mechanics

By A. PROCA
Institut Henri Poincaré, Paris

Translated by D. H. Delphenich

Summary. – Currently, classical mechanics does not take advantage of all the resources that are placed at its disposal by the principle of special relativity. The author thus proposes a new form for mechanics that reduces to the preceding one in a particular case in which the power of description seems more extensive.

The basic hypothesis consists of assuming that the fundamental equations involve, not only the x^k, but also the new variables ξ, that have a spinorial character, and starting from which, one may calculate the world quantities. In other words, the motion of a material point is determined, not only in space-time, but in a certain subordinate spinor space that is linked to the universe by the intermediary of the space of velocities.

One writes the equations of motion and their simple first integrals. The material particle thus defined possesses spin. Its motion does not generally obey the law of inertia; it is composed of the superposition of a displacement with constant velocity and a "Schrödinger zitterbewegung." The latter can be absent, and in that particular case the point moves with constant velocity, as in classical mechanics. One then studies the motion of a charged particle in an electromagnetic field, and one recovers the expression for the Lorentz force, as well as the effect of an electromagnetic moment, which has the same form as in classical electrodynamics.

1. The recent experimental results concerning the various fundamental particles, as well as the grave difficulties that one encounters in the theory in its present form, make a complete revision of it indispensable.

The new demands are quite varied, and it seems difficult at the present time to establish a coherent formalism that allows one to solve at this time, for example, the problem of the self-energy and that of the quantization of mass. For some time, we shall be further obliged to seek partial solutions that resolve this or that particular problem before we can begin a completely satisfactory general synthesis. Now, it is clear that we will arrive at this synthesis all the more easily when our instruments of investigation are more perfect. It does not therefore seem devoid of interest to examine these instruments before all else – in other words, the mechanics of material points – in order to see if they might not be ameliorated, and in a completely general fashion, without reference to any difficulty or particular problem.

2. By "mechanics," we naturally intend this to mean quantum mechanics. Meanwhile, that proceeds from classical mechanics, and it is obvious that the imperfections of the latter have repercussion in quantum mechanics itself. The correct program of work is thus comprised of submitting classical mechanics to a critical examination, as indicated above, in the hopes of ameliorating it, if necessary, and then passing to quantum mechanics by the usual process of quantization. It is this path that we shall attempt to follow.

3. In order to analyze the present situation, we remark that when one neglects the effects of gravitation and when one takes inertia into account by means of a mass coefficient the classical mechanics of fundamental particles is dominated entirely by the principle of special relativity. In other words, it must satisfy the demands of relativity.

Now, some of these demands are of such a nature that one looks for reasons for them not to hold, which is basically equivalent to the partial rejection of the principle of relativity. We are thus confronted with the following alternative: One either assumes all of the consequences of that principle or rejects them. The latter attitude is untenable in the present state of science, so we are led to a new order of accepting *all* of the consequences that follow from the application of that principle of relativity.

The example that is most striking in that state of affairs is the existence, as required by relativity, of characteristics that translate into negative "energies" or "masses." In classical theory, the rest mass of a corpuscle with a momentum p_μ ($\mu = 1, 2, 3, 4$) can be either $m_0 = + \sqrt{-\sum_{\rho=1}^{4} p_\rho^2}$ or $- m_0$. It is true that sometimes the double sign is not a nuisance and that we can silently pass over the case of $m_0 < 0$. It is not less true that this rule is not general. Indeed, on the contrary, the double sign of m_0 is essentially related to the quadratic character of the invariant of relativity. If we are to assume the validity of special relativity then we must therefore account for the possibility of a negative sign, and, if possible, abandon the physical interpretation in a satisfactory manner.

4. Another remark that relates to the desired amelioration of mechanics concerns the possibility of classically describing a particle that possesses a non-zero spin.

One often hears that spin is a quantum effect; the only argument that appeals to this thesis is the fact that one has not succeeded in describing a particle with spin in classical mechanics. Various attempts have been made in that sense ([1]), but they have not always arrived at results that were complete or indisputable.

In quantum mechanics, spin appears when one takes relativity into account. It seems very probable that it is likewise true in classical mechanics and that the origin of spin must be sought for in the fundamental rotation of relativity, rather than in the discontinuities of quantization. Be that as it may, the description of spin necessitates the introduction of new variables, the choice of which is not easy; fortunately, a reasonable solution to this problem can be provided by the observation in the following paragraph.

5. The condition of covariance with respect to Lorentz transformations permits us to characterize the quantities that are capable of describing physical phenomena (e.g., scalars, vectors, etc.) and to likewise establish, in some way, a hierarchy that is based upon their increasing complexity. Among these quantities, one finds not only tensors, but also spinors. One may likewise say that in the hierarchy above the spinors of rank one

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([1]) *See*, in particular: M. MATHISSON, Acta Phys. Polonica **6** (1937), 163 and J. WEYSSENHOFF, *ibid.* **9** (1947), 1. Likewise *see* L. de Broglie, *La théorie des particules de spin* ½, Gauthier-Villars, Paris, 1952, where one finds a discussion of the WEYSSENHOFF theory. *See* the critique of the latter by C. Møller, Comm. Dublin Institute A, no. 5 and Ann. Inst. H. Poincaré **11** (1949), 251. Finally, we cite M. H. L. PRYCE, Proc. Roy. Soc. A **195** (1948), 62; HÖNL, Naturwiss. **26** (1938), 408; HÖNL and PAPAPETROU, Z. PHYS. **112** (1939), 512 and **114** (1939), 478; PAPAPETROU, Proc. Roy. Soc. **209** (1951), 248 and 260.

are "simpler" than the most elementary tensor – namely, the vector; indeed, one can form a vector by starting with spinors and using rational operations, while the converse is not true.

Now, for obvious reasons, no classical theory appeals to spinors ([2]). It is true that no *measurable* physical quantity can be represented by a spinor, by reason of the double-valued nature of the latter, but this eventuality is not the only one that is possible. If one assumes that the measurable physical quantities in spacetime are exclusively tensors then it is indeed difficult to not assume that the introduction of spinors into the present theory presents marked advantages, by reason of the possibilities that they offer of utilizing all of the resources of the Lorentz group. The following paragraphs are destined to show the conclusions that one can infer from that supposition.

6. Basic hypotheses. – The minimum number of spinors that are compatible with the invariance with respect to the Lorentz group that we postulate is two. We let ξ denote the set of these spinors, which are grouped as in the ψ of the Dirac theory, and we set, as is customary, $\xi^+ = i\xi^*\gamma^4$ = adjoint of ξ.

Our fundamental hypotheses are then stated in the following fashion:

I. *Any physical phenomenon, and in particular, the motion of a point, is described by means of the spinors ξ and ξ^+.*

Since experimental results refer to measurements in spacetime, it is indispensible to complete I by pointing out in what fashion this spacetime is related to the spinorial spaces of ξ and ξ^+. To that effect, we shall assume that:

II. *If τ is an invariant parameter that describes motion and γ^ρ are* 4×4 *matrices such that $\gamma^r\gamma^s + \gamma^s\gamma^r = 2\delta^{rs}$ then one sets:*

$$\boxed{\frac{dx^\rho}{d\tau} = \xi^+\gamma^\rho\xi} \qquad (\rho = 1, 2, 3, 4).$$

This choice is arbitrary, but natural. The right-hand side is the only real time-like vector that is formed linearly from ξ and ξ^+, and which one can identify with a world-vector that characterizes the motion.

Therefore, from I, the basic equations of the theory relate to ξ, ξ^+, and their derivatives; phenomena, and in particular, the motion of a point, are no longer determined in the spacetime of x^ρ, but in the subordinate spinorial spaces of ξ, ξ^+ that are coupled to the velocity. We may consider these spaces to be the fundamental elements in the description of phenomena. Accordingly, one can say that the fundamental

([2]) Nevertheless, *see* F. Klein's *theory of the gyroscope*, which seems to be the first one to utilize spinorial quantities in a problem of classical mechanics, as well as the recent paper of Bopp. For the school of that author, as well as that of Darboux (*Leçons sur la théorie générale des surfaces*), which is, moreover, the first citation, spinorial quantities are introduced as auxiliary variables in the study of rotations; the fundamental equations are not modified.

transformations are not the ones that concern the vectors x^ρ, but the ones that define the fundamental spinors, and which result from Lorentz transformations.

The link between the spinorial spaces and ordinary spacetime x^ρ is given by II. The passage from one to the other can be defined by making a curve in ξ, ξ^+ that is given in the parametric form $\xi_r = f_r(\tau)$ correspond to the world-line that is defined parametrically by the differential equations $dx^\rho = \xi^+(t)\ \gamma^\rho\ \xi(t)\ d\tau$. Being given ξ and ξ^+ does not permit one to calculate the coordinates themselves, but only their derivatives. The coordinates remain independent of ξ.

7. Notations. – ξ is a matrix with one column and complex elements ξ_1, ξ_2, ξ_3, ξ_4 that have the same variance as the ψ in the Dirac theory of the electron. The adjoint matrix ξ^+ can be written as a function of the transposed and conjugated matrix $\bar{\xi}^*$:

$$\xi^+ = i\bar{\xi}^*\gamma^4. \tag{1}$$

The γ^ρ (ρ = 1, 2, 3, 4) are the 4×4 von Neumann matrices (such that the quadratic invariant is $\xi^+\xi$) and $\gamma^5 = \gamma^1\ \gamma^2\ \gamma^3\ \gamma^4$. We adopt the following notations (which are those of Pauli) for the 16 quadratic covariants that are formed by starting with the products of the γ^ρ:

$$\left.\begin{array}{lll} \text{Invariant.....................} & \Omega_1 & = -i\xi^+\xi, \\ \text{World vector..............} & u^\rho & = \xi^+\gamma^\rho\xi, \\ \text{Anti-symmetric tensor} & m_{\rho\sigma} & = i\xi^+\dfrac{i}{2}(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho)\xi, \\ \text{Pseudo-vector............} & w^\rho & = i\xi^+ i\gamma^5\gamma^\rho\xi, \\ \text{Pseudo-invariant........} & \Omega_2 & = \xi^+\gamma^5\xi. \end{array}\right\} \tag{2}$$

These 16 terms are not independent, but are coupled by nine well-known quadratic relations ([3]). One has, for example:

$$\left.\begin{array}{c} \sum (u^\rho)^2 = -\sum (w^\rho)^2 = -(\Omega_1^2 + \Omega_2^2), \\ \dfrac{1}{2}\sum (m_{\rho\sigma})^2 = \Omega_1^2 - \Omega_2^2, \\ u^\rho m_{\rho\sigma} + \Omega_2 w_\sigma = 0, \quad \cdots \end{array}\right\} \tag{3}$$

In spacetime, we shall use the coordinates:

$$x^k \quad \text{and} \quad x^4 = ict = ix^0,$$

c being the velocity of light.

([3]) We cite only the work of O. COSTA DE BEUREGARD, Thesis, Paris, 1943; KOFINK, Annalen der Physik **38** (1940), 421; G. PETIAU, J. math. pures et appl. **25** (1946), 335.

We call the real parameter, with respect to which one considers the variation of the various quantities, τ. By hypothesis, one has:

$$\frac{dx^\rho}{d\tau} = \xi^+ \gamma^\rho \xi. \tag{II}$$

We let *ds* refer to the "proper time" element of the particle in ordinary spacetime, which is defined by:

$$-c^2\, ds^2 = \sum_{\rho=1}^{4} (dx^\rho)^2\,. \tag{4}$$

By virtue of (II) and (3), one will then have:

$$-c^2 \frac{ds^2}{d\tau^2} = \sum (u^\rho)^2 = -(\Omega_1^2 + \Omega_2^2)\,,$$

namely:

$$d\tau = \frac{\pm c}{\sqrt{\Omega_1^2 + \Omega_2^2}}\, ds\,. \tag{5}$$

On then deduces the expression for the *world-velocity* of the particle from this:

$$v^\rho = \frac{dx^\rho}{ds} = \frac{\pm c(\xi^+\gamma^\rho\xi)}{\sqrt{\Omega_1^2 + \Omega_2^2}} \qquad \text{and} \qquad \sum (v^\rho)^2 + c^2 = 0. \tag{6}$$

8. Lagrangian in the absence of a field. – Now consider the free motion of a given point.

The corresponding Lagrangian will depend essentially upon ξ. For reasons of relativistic variance, its simplest form will be the one that we are already familiar with:

$$L = \frac{1}{2}\left(\frac{d\xi^+}{d\tau}\xi - \xi^+ \frac{d\xi}{d\tau}\right). \tag{7}$$

If we consider the analogy with the other known case of wave mechanics then we will be tempted to add a second term to this first one that has the form $ik\xi^+\xi$. Later on (§ 19), we shall, however, verify that this term is naturally included in the interaction terms with an exterior field. There is therefore no reason to introduce it in the description of the behavior of the particle in the absence of any interaction.

By contrast, other terms are introduced into the Lagrangian by reason of the conditions that must apply to its variation. Indeed, from our fundamental hypothesis II, the quadratic covariant $\xi^+\gamma^\rho\xi$ must always be equal to $dx^\rho / d\tau = \dot{x}^\rho$. We are in the presence of a variation with constraint conditions. We use the process of multipliers and consider four unknown functions λ_ρ of τ that form the components of a vector.

The Lagrangian in the absence of a field will then be:

$$L = \frac{1}{2}\left(\frac{d\xi^+}{d\tau}\xi - \xi^+\frac{d\xi}{d\tau}\right) + \lambda_\rho(\dot{x}^\rho - \xi^+\gamma^\rho\xi), \tag{8}$$

and the variation will be applied to ξ^+, ξ, and x.

Moreover, no matter what the more or less valid reasons were that led us to the form (8), this Lagrangian constitutes a new hypothesis that we must add to those of paragraph 6.

9. Equations of the free material point. Quantity of motion. – The fundamental equations that are deduced from (8) are:

$$\frac{d\xi}{d\tau} = -\lambda_\rho\gamma^\rho\xi, \qquad \frac{d\xi^+}{d\tau} = +\lambda_\rho\,\xi^+\gamma^\rho, \tag{9}$$

$$\frac{dx^\rho}{d\tau} = \xi^+\gamma^\rho\xi, \qquad \frac{d\lambda_\rho}{d\tau} = 0. \tag{10}$$

The last of (10) are nothing but:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\rho}\right) = \frac{\partial L}{\partial x^\rho},$$

which introduces $\partial L/\partial\dot{x}^\rho$ as the conjugate momentum to x – in other words, the "quantity of motion." In the new theory, the *quantity of motion thus has the Lagrange multipliers for its components*, which one uses in the variation.

In the absence of a field, from (10), this *quantity of motion is a constant vector* ([4]) that is not necessarily proportional to the world-velocity, as in the classical theory.

This is true whenever the particle possesses spin, and this difference between velocity and the quantity of motion must become very interesting from the experimental standpoint. The behavior of such a corpuscle can be different from the behavior that we know of under conveniently chosen conditions.

By assuming that the constant vector λ is likewise timelike – i.e., $\lambda_\rho\,\lambda^\rho < 0$ – we can define the invariant:

$$\left.\begin{aligned} \lambda = m_0 c = \sqrt{-\lambda_\rho\lambda^\rho}, \\ \lambda^2 + \lambda_\rho\lambda^\rho = 0; \end{aligned}\right\} \tag{11}$$

([4]) We note that here this amounts to constancy with respect to the parameter τ. In most cases, this is equivalent to a "world constancy" $d\lambda_\rho / ds = 0$, due to the proportionality (5) between $d\tau$ and the proper time ds.

m_0 represents the *rest mass* of the particle, which appears here as an integration constant.

10. Spacetime quantities for the free motion. Velocity. – A being an arbitrary 4×4 matrix, one has the following general relation from equations (9), in which $[A, B] = AB - BA$:

$$\frac{d}{d\tau}(\xi^+ A\xi) = \lambda_\rho\, \xi^+ [\gamma^\rho, A]\, \xi, \tag{12}$$

which gives rise to a series of interesting conclusions.

For example, by setting $A = 1$, one has a first integral:

$$\frac{d\Omega_1}{d\tau} = 0. \tag{13}$$

With $A = \gamma^5$ and $\gamma^5\gamma^\rho$, one finds:

$$\frac{d\Omega_2}{d\tau} = 2\,\lambda_\rho w^\rho \tag{14}$$

and

$$\frac{dw^\rho}{d\tau} = 2\,\lambda^\rho\,\Omega_2\,, \tag{15}$$

which entails that:

$$\frac{d^2\Omega_2}{d\tau^2} + 4\lambda^2\,\Omega_2 = 0. \tag{16}$$

One likewise has another first integral:

$$(\lambda_\rho\lambda_\sigma + \lambda^2\,\delta_{\rho\sigma})\,w^\rho = \text{const.}, \tag{17}$$

and as a consequence also:

$$\frac{d^2 w_\sigma}{d\tau^2} + 4\,\lambda^2\,w_\sigma = \text{const.} \tag{18}$$

Upon setting $A = w^\sigma$, one obtains the derivative of the velocity, and therefore the *law of motion, properly speaking.* One has:

$$\boxed{\frac{du^\sigma}{d\tau} = -2\lambda_\rho m^{\rho\sigma}.} \tag{19}$$

One deduces another first integral from this:

$$\lambda\, u^\rho = \text{const.} \tag{20}$$

Upon taking (23) into account, which we will establish later on, we may finally write:

$$\boxed{\frac{d^2u^\sigma}{d\tau^2} + 4\lambda^2 u^\sigma = \text{const.} = -4\lambda^\sigma(\lambda_\rho u^\rho),} \tag{21}$$

from which, one deduces that the general expression for the velocity will be:

$$u^\sigma = \frac{dx^\sigma}{d\tau} = A^\sigma e^{2i\lambda\tau} + B^\sigma e^{-2i\lambda\tau} + C^\sigma, \tag{22}$$

with

$$C^\sigma = -\frac{\lambda^\sigma}{\lambda^2}(\lambda_\rho u^\rho) = \text{const.}$$

One reads off the behavior of a material point in the absence of a field from this formula. One sees that in the particular case where the initial conditions are such that $A^\sigma = B^\sigma = 0$ *the point is displaced with a constant velocity, as in classical mechanics.* However, in general, the velocity oscillates with a frequency $2\lambda = 2m_0c$ in a very complicated manner, and especially the world-velocity, which likewise contains another oscillating term Ω_2 . The details concerning these conditions and the expression for the coefficients are given in paragraph 12.

11. Spin. – Upon setting:

$$A = \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu),$$

one finds:

$$\lambda_\mu u_\nu - \lambda_\nu u_\mu + \tfrac{1}{2}\frac{dm_{\mu\nu}}{d\tau} = 0, \tag{23}$$

or

$$\lambda_\mu dx_\nu - \lambda_\nu dx_\mu + \tfrac{1}{2} dm_{\mu\nu} = 0, \tag{24}$$

or furthermore:

$$x_\mu \lambda_\nu - x_\nu \lambda_\mu - \tfrac{1}{2} m_{\mu\nu} = \text{const.} \tag{25}$$

We may thus assert that the free particle possesses *spin* that is given by $\tfrac{1}{2}m_{\mu\nu}$, *or, more generally, by:*

$$\text{Spin} = \tfrac{1}{2}m_{\mu\nu} + \text{const.}, \tag{26}$$

since the defining equation (24) is, in reality, fixed only up to an arbitrary constant.

Taking (23) into account, one finds the first integral:

$$\lambda_\mu m_{\nu\rho} + \lambda_\nu m_{\nu\mu} + \lambda_\rho m_{\mu\nu} = \text{const.} \tag{27}$$

Upon differentiating (23), one finds:

$$\begin{aligned}\frac{d^2m_{\mu\nu}}{d\tau^2} &= 4\lambda\rho\,(\lambda_\mu m_{\rho\nu} - \lambda_\nu m_{\mu\rho}) \\ &= -4\lambda\rho\,(\text{const.} - \lambda_\rho m_{\mu\nu}) \\ &= \text{const.} - 4\lambda^2 m_{\mu\nu},\end{aligned}$$

so

$$\frac{d^2 m_{\mu\nu}}{d\tau^2} + 4\lambda^2\, m_{\mu\nu} = \text{const.} = -\,4\lambda\rho\,(\textstyle\sum \lambda_\mu\, m_{\nu\rho}). \tag{28}$$

The definition of spin depends upon the choice of arbitrary constant. One can take it to be zero, in which case, the spin is equal to $\frac{1}{2} m_{\mu\nu}$. With that hypothesis, a classical particle (which will be studied later on in paragraph 14) will have a spin whose components will all be *constant*, and not necessarily zero, insofar as one does not confuse these constants with the ones on the right-hand side. We remark that these two possibilities are equivalent: From (23), if $m_{\mu\nu}$ = const. then one has λ_μ proportional to the velocity, and the particle is indeed a classical material point.

If one intends to define a "spin" absolutely whose components are null for the particular case of a classical particle then one must choose the constant in (26) in such a fashion that the solution to (28) is purely oscillatory. In this case, the "spin" may be written in a simple form, namely:

$$\begin{aligned} S_{\mu\nu} &= \tfrac{1}{2} m_{\mu\nu} + \text{const.} \\ &= \frac{\lambda_\rho}{2\lambda^2}(\lambda_\mu\, m_{\nu\rho} - \lambda_\nu\, m_{\mu\rho}), \end{aligned}$$

or furthermore:

$$\begin{aligned} S_{\mu\nu} &= \frac{1}{4\lambda^2}\frac{d}{d\tau}(\lambda_\mu\, u_\nu - \lambda_\nu\, u_\mu), \\ &= \frac{1}{4\lambda^2}\frac{d^2}{d\tau^2}(\lambda_\mu\, x_\nu - \lambda_\nu\, x_\mu). \end{aligned}$$

One easily verifies, with the help of (21), that (23) is satisfied by this value of $S_{\mu\nu}$; on the other hand, in the classical case $\lambda_\mu\, u_\nu - \lambda_\nu\, u_\mu = 0$, so one immediately has $S_{\mu\nu} = 0$ ([5]).

12. General solution. "Potentials." – From equations (9) and (10), one deduces:

$$\frac{d^2\xi}{d\tau^2} + \lambda^2\xi = 0, \qquad \frac{d^2\xi^+}{d\tau^2} + \lambda^2\xi^+ = 0, \qquad \frac{d\lambda_\rho}{d\tau} = 0.$$

The general solution is thus written:

$$\xi = a\, e^{i\lambda\tau} + b\, e^{-i\lambda\tau}, \qquad \xi^+ = b^+\, e^{i\lambda\tau} + a^+ e^{-i\lambda\tau}, \tag{29}$$

where $\lambda_\rho\,\lambda^\rho + \lambda^2 = 0$ and λ_ρ, a, b are constants that satisfy the equations:

([5]) This choice can be guided by considerations that relate to interaction or quantization; for example, *see* paragraph 17. Moreover, it is the expression $\frac{1}{2} m_{\mu\nu}$ (constant = zero) that possesses the correct proper values when one quantizes it by means of the usual commutation relations.

$$\left.\begin{array}{ll}(i\lambda+\lambda_\rho\gamma^\rho)a=0, & (-i\lambda+\lambda_\rho\gamma^\rho)b=0,\\ a^+(i\lambda+\lambda_\rho\gamma^\rho)=0, & b^+(-i\lambda+\lambda_\rho\gamma^\rho)=0.\end{array}\right\} \tag{30}$$

One may write the solution to these equations (30) in a convenient form by remarking that the two *a* components and the two *b* components – therefore, four, in all – are arbitrary in the solution. One then considers four arbitrary complex constants φ_1, φ_2, φ_3, φ_4 that are grouped into one column φ, and which we call a "potential." ([6]). The solution to equations (30) is then written simply:

$$\left.\begin{array}{ll}a=(-i\lambda+\lambda_\rho\lambda^\rho)\varphi, & b=(i\lambda+\lambda_\rho\lambda^\rho)\varphi,\\ a^+=\varphi^+(-i\lambda+\lambda_\rho\lambda^\rho), & b^+=\varphi^+(i\lambda+\lambda_\rho\lambda^\rho).\end{array}\right\} \tag{31}$$

The general form of the ξ as functions of τ is given by (29), with (31), or again by:

$$\left.\begin{array}{l}\xi=2(\lambda\sin\lambda\tau+\lambda_\rho\gamma^\rho\cos\lambda\tau)\varphi,\\ \xi^+=\varphi^+\cdot 2(-\lambda\sin\lambda\tau+\lambda_\rho\gamma^\rho\cos\lambda\tau).\end{array}\right\} \tag{32}$$

One sees the two signs of λ appear in the general solutions, as required by relativity.

The choice of arbitrary constants *a* and *b*, or λ_ρ, corresponds to the various "initial conditions" that one envisions. Note that these "initial conditions" are more numerous than in classical theory and likewise involve certain characteristics of the particle itself, and not only the elements of motion. Upon choosing them conveniently, one will obtain some interesting special cases, such as, for example, the one in which ξ contains only the mass with one sign, which is examined in the next paragraph.

13. Harmonic case. Classical particle. – This case, which we will call the "harmonic case," is realized when, for example, $a = 0$ [a value that indeed satisfies the conditions (30)]. One then has simply:

$$\xi = b\,e^{-i\lambda\tau}, \qquad \xi^+ = b^+\,e^{i\lambda\tau}. \tag{33}$$

We examine how this presents itself in terms of motion in spacetime.

We first remark that all of the real quantities in that space that have the form $\xi^+A\xi$ are constants.

$\Omega_1 = -\,ib^+b$ is a constant in every case, but the second group of equations (31) also gives:

$$\Omega_1 = b^+\,\gamma^5\,b = 0. \tag{34}$$

From (6), it then results that the *world-velocity of the particle is constant:*

([6]) This process has been applied to the theory of the Dirac electron by B. Kwal.

$$v^\sigma = \frac{c(b^+\gamma^\sigma b)}{\Omega_1} = \frac{ic(b^+\gamma^\sigma b)}{b^+ b}, \tag{35}$$

and therefore *the particle can be assimilated into a classical material point.*

Upon multiplying the two equations (30) by γ^ρ, b, and b^+ and adding them, one obtains:

$$-i\,\lambda\, b^+\gamma^\rho\, b + \gamma^\rho\, b^+ b = 0,$$

so, upon comparing this with (35) and (11), one gets:

$$\boxed{\lambda^\sigma = m_0 v^\sigma.} \tag{36}$$

Therefore, in this case the motion is proportional to the velocity, which again justifies, at the same time, the assimilation of m_0 to the rest mass and of λ_σ to the quantity of motion.

The components of $m_{\mu\nu}$ are constants, so those of $S_{\mu\nu}$ are zero, by virtue of (36).

The harmonic case thus indeed describes a classical particle. Conversely, if the spin is constant or zero and one simultaneously has $\Omega_2 = 0$ then one will be dealing with a classical particle. Indeed, one then has $m_{\mu\nu}$ = const., so, from (23), λ_μ is proportional to u_μ . Since, in general [*see* (3)]:

$$u^\rho\, m_{\rho\nu} = -\,\Omega_2\, w_\nu = 0,$$

one then deduces that $\lambda^\rho\, m_{\rho\nu} = 0$, so, from (19), $du_\nu / d\tau = 0$; the particle thus moves with constant velocity.

The case of the classical particle – i.e., the one in which the motion conforms to the *principle of inertia* – is thus only a special case of free motion. In the general case, this principle is no longer valid in its classical form, but it is easy to see what replaces it.

We finally remark that spin appears when one passes from the harmonic case to the general case – i.e., *when one introduces the mass* – m_0 *in the expression for* ξ, *along with* + m_0 . This is the mechanism by which special relativity introduces spin, and this mechanism has nothing to with the analogy of a rotating body; what is essential is not the rotation of a positive mass, but the oscillation between + m_0 and – m_0, which is the true classical counterpart to the "Schrödinger zitterbewegung." This is also the reason for the fact that the appearance of spin does not entail an augmentation of the energy of rotation, which is inevitably produced in the case of a body in rotation.

The mechanism above is exactly the one by which spin in introduced into the wave mechanics of the electron, where it has its known successes ([7]).

14. Classical particles at rest. – The manner by which spinors intervene in the theory becomes clearer when one considers the case of a classical particle at rest. In this

([7]) Hönl and Papapetrou (*loc. cit.*) have proposed a model of this type, but they have treated it as a mechanism in spacetime that is subject to the laws of classical mechanics. In the present theory, the "nagative mass" appears in the same way that the "negative frequencies" appear in the theory of radiation.

case, since $\lambda_\rho = m_0\, v_\rho$, the condition $v_k = 0$ translates into $\lambda_k = 0$ (k = 1, 2, 3) and thus entails that either $i\lambda = \lambda_4$ or $i\lambda = -\lambda_4$. For each of these two cases, the world-quantities of the particle (at rest) depend upon only one spinor: viz., (φ_1, φ_2) or (φ_3, φ_4), resp.. For example, if $i\lambda = \lambda_4$ then one has:

$$\left.\begin{array}{c} i\Omega_1 = u_4 = -4\lambda^2 i(\varphi_1^*\varphi_1 + \varphi_2^*\varphi_2); \\ \\ m_{23} = -w_1 = -4\lambda^2(\varphi_1^*\varphi_2 + \varphi_2^*\varphi_1), \\ m_{31} = -w_2 = +4\lambda^2 i(\varphi_1^*\varphi_2 - \varphi_2^*\varphi_1), \\ m_{12} = -w_3 = -4\lambda^2(\varphi_1^*\varphi_1 - \varphi_2^*\varphi_2); \\ \\ m_{14} = m_{24} = m_{34} = 0. \end{array}\right\} \tag{37}$$

In this case, "spin" reduces to its spatial components, which have the same direction as w_r, moreover.

15. Interactions. Particles in a field. – The laws of motion in the case of interaction are deduced from the Lagrangian (8), when it is augmented by an interaction term L_{int} , which is a function of ξ, ξ^+, and x^k . In order for the theory to be plausible, it is necessary that one must choose the latter term in such a fashion as to recover the classical formulas for the known cases; for example, for the case of the motion of a charge in an electromagnetic field.

Consider the manner by which the interaction term L_{int} depends on the ξ and ξ^+. The simplest hypothesis obviously consists of assuming a bilinear dependency. One thus writes:

$$L_{\text{int}} = \xi\, O\, \xi^+, \tag{38}$$

where O is an invariant operator that acts on the ξ, and is a function of the x^k, but independent of the ξ, ξ^+, and is such that L_{int} = real. Its most general form will thus be:

$$O = \sum_{\alpha=1}^{16} A_\alpha \gamma\cdots = A_0 + A_\rho\, \gamma^\rho + \frac{i}{2} A_{\rho\sigma}(\gamma^\rho \gamma^\sigma - \gamma^\sigma \gamma^\rho) + \dots \tag{39}$$

where $\gamma\dots$ are the 16 independent products of the γ^ρ and the A_0, A_1, … are 16 ordinary functions of the x^ρ that present the variances and reality conditions that are required in order to make $\xi\, O\, \xi^+$ invariant and real.

This type of interaction is therefore written in terms of at most 16 ordinary functions. The equations of motion in this case will be:

$$\left.\begin{aligned}\frac{d\xi^+}{d\tau} &= \lambda_\rho \xi^+ \gamma^\rho - \xi^+ O, & \frac{d\xi}{d\tau} &= -\lambda_\rho \gamma^\rho \xi + O\xi, \\ \frac{dx^\rho}{d\tau} &= \xi^+ \gamma^\rho \xi, & \frac{d\lambda_\rho}{d\tau} &= \xi^+ \frac{\partial O}{\partial x_\rho} \xi.\end{aligned}\right\} \tag{40}$$

We remark that $-i\xi^+\xi = \Omega_1$ always stays constant in the general case.

The variation of the *velocity* will be given by:

$$\frac{d}{d\tau}(\xi^+ \gamma^\sigma \xi) = \frac{du^\sigma}{d\tau} = -2\,\lambda_\rho\, m^{\rho\sigma} + \xi^+(\gamma^\sigma O - O\gamma^\sigma)\,x. \tag{41}$$

The *world-force* will be deduced by differentiating the quantity of motion that is given by the last equation in (40) with respect to *ds*.

16. Electromagnetic field. – The interaction of a charge e with an electromagnetic field $h_{\rho\sigma} = \frac{\partial A_\rho}{\partial x^\sigma} - \frac{\partial A_\sigma}{\partial x^\rho}$ with a potential A_ρ is of the preceding type.

First, define the action of the field on the particle and take:

$$O = e\,A_\rho\,\gamma^\rho. \tag{42}$$

The equations are written:

$$\left.\begin{aligned}\frac{d\xi}{d\tau} &= -(\lambda_\rho - eA_\rho)\gamma^\rho \xi, \quad \frac{d\xi^+}{d\tau} = -\xi^+(\lambda_\rho - eA_\rho)\gamma^\rho, \\ &\frac{d(\lambda_\rho - eA_\rho)}{d\tau} = eh_{\sigma\rho}u^\sigma.\end{aligned}\right\} \tag{43}$$

These are deduced from the equations of a free particle by replacing, as is customary, the momenta λ_ρ with $\lambda_\rho - e\,A_\rho$. The last equation gives:

$$\frac{d(\lambda_\rho - eA_\rho)}{ds} = e\,h_{\sigma\rho}\,\frac{dx^\sigma}{ds} = h_{\sigma\rho}\,e\,v^\sigma = h_{\sigma\rho}\,j^\sigma$$
$$= \text{Lorentz force}, \tag{44}$$

upon calling $j^\sigma = ev^\sigma$ the current. We remark that since the particle is, by hypothesis, a dimensionless geometric point, the current will be a convection current ev^σ uniquely. Likewise, if the particle possesses an electromagnetic moment $M_{\mu\nu}$ then its contribution $\partial M_{\mu\nu} / \partial x^r$ to the current will be zero, by reason of the fact that $M_{\mu\nu}$ is not distributed over space.

17. Electromagnetic moment. – Now, assume that the charged particle has a spin, and due to this fact, it possesses an electromagnetic moment that is proportional to it. This is not obvious, *a priori*, since the association with a rotating body is not valid. The interaction considered in the preceding paragraph then leads to the Lorentz force, which represents the totality of the action of the field on the particle, but introduces no coupling between the field and the electromagnetic moment, which shows that it is incomplete. Some new terms ([8]) must appear in the expressions for the force and the couple that are exerted on the particle.

We assume that the electromagnetic moment is proportional to the spin of the particle, and we make the hypothesis that one has:

$$\mu_{\rho\sigma} = \frac{c\kappa}{\sqrt{\Omega_1^2 + \Omega_2^2}} \times \text{spin} \qquad (\kappa = \text{const.}). \tag{45}$$

The multiplicative factor is constant only when $\Omega_2 = 0$, which entails that $u^\rho\, m_{\rho\sigma} = 0$, so it is only in the *case where the spin has no temporal component in the rest system* that it reduces to a tri-dimensional vector.

Likewise, suppose that one takes the additive constant that appears in the spin to be zero, which will then be simply equal to $\frac{1}{2} m_{\rho\sigma}$. We then define the interaction by means of an operator:

$$O = e\, A_\rho\, \gamma^\sigma - (\gamma^\rho \gamma^\sigma - \gamma^\sigma\, \gamma^\rho)\, h_{\rho\sigma}\,, \tag{46}$$

where A_ρ is the field potential, $h_{\rho\sigma}$ is the corresponding field, and κ is a convenient constant. One deduces from this that the Lorentz force of the preceding paragraph is augmented by the associated supplementary term. Indeed, one has:

$$\boxed{\frac{d(\lambda_\rho - eA_\rho)}{ds} = h_{\rho\sigma}\, j^\sigma + \tfrac{1}{2}\mu_{\alpha\beta} \frac{\partial h_{\alpha\beta}}{\partial x^\rho}.} \tag{47}$$

18. Couple. – The other equations may be written:

$$\left.\begin{aligned} \frac{d\xi^+}{d\tau} &= \xi^+(\lambda_\rho - eA_\rho)\gamma^\rho + \frac{\kappa}{4} h_{\rho\sigma}\xi^+(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho), \\ \frac{d\xi}{d\tau} &= -(\lambda_\rho - eA_\rho)\gamma^\rho\xi - \frac{\kappa}{4} h_{\rho\sigma}(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho)\xi. \end{aligned}\right\} \tag{48}$$

With the preceding notations, one deduces from this that:

$$\frac{d}{ds}\left(\tfrac{1}{2} m_{\alpha\beta}\right) = (\lambda_\alpha - e\, A_\alpha)\, v_\beta - (\lambda_\beta - e\, A_\beta)\, v_\alpha + (\mu_{\alpha\beta}\, h_{\sigma\alpha} - \mu_{\alpha\alpha}\, h_{\sigma\beta}), \tag{49}$$

([8]) Which have already been considered in relativistic form by various authors.

which involves a supplementary term that was already encountered by other authors.

One may then recover the results of the interaction of an electron in an electromagnetic field under an interaction of the bilinear type $\xi^+ O \xi$, with O equal to (46), and which is also written:

$$O = (\gamma_\rho \partial^\rho + k)\,(\gamma_\sigma A^\sigma), \qquad \partial^\rho = \frac{\partial}{\partial x^\rho}. \tag{50}$$

19. We remark that the operator above does not contain any term that is independent of the γ^ρ (any more than the terms in γ^5). The simplest bilinear interaction is defined by:

$$O = ik, \tag{51}$$

where k is a real constant. The equations are then:

$$\frac{d\xi}{d\tau} = (-\lambda_\rho \gamma^\rho + k)\,\xi,$$
$$\frac{d\xi^+}{d\tau} = \xi^+ (\lambda_\rho \gamma^\rho - k), \qquad \frac{d\lambda_\rho}{d\tau} = 0. \tag{52}$$

If ξ_{free} is the known solution to the case of the free particle $k = 0$ that was studied previously then the solution to (52) is written:

$$\xi = e^{i\kappa\tau}\,\xi_{\text{free}}. \tag{53}$$

This type of interaction, to which we have already alluded in paragraph 8, acts on the ξ, and does not affect the real quantities that are measured in spacetime.

The case (51) when k is a function of the x^ρ seems to be more complicated.

Finally, up to the present, we have considered only interaction terms that are bilinear in ξ and ξ^+. It is clear that other cases are possible, and that the form of L_{int} as a function of ξ and ξ^+ will thus permit one to establish a criterion for classifying the interactions in increasing order of complexity. One will thus obtain, in particular, nonlinear classical mechanics, which is not devoid of interest.

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