Projective geometry and spacetime structure

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Affine geometry

- In affine geometry the basic objects are points in a space A^n on which translations (i.e., $\mathbb{R}^n$) act transitively and effectively.
    - Hence, there is a unique vector in $\mathbb{R}^n$ that takes any given point of the space to any other point.
- Lines are defined by the orbit of a point under translation by all scalar multiples of a vector in $\mathbb{R}^n$.
- Choosing a frame in $\mathbb{R}^n$ and a point in A^n defines a coordinate system for A^n.
- The key concept in affine geometry is the possibility of there being *parallel* lines.
    - Viz., non-intersecting lines that lie in a common plane.
    - Such lines can be made to coincide by a translation.
    - Similarly, parallelism is preserved by translations.

Projective geometry

- In projective geometry, points in a projective space are lines in an affine space of one higher dimension.

    - Hence, lines in a projective space are planes in an affine space, etc.

        - More generally, curves correspond to ruled surfaces

        - This has the consequence that the equations of geodesics in projective differential geometry are PDE’s for a family of surfaces, not ODE’s for a family of curves.

- It is better to think of an *n*-dimensional projective space as being obtained by compactifying an *n*-dimensional affine space by the addition of a *hyperplane at infinity*, not by projection from an *n*+1-dimensional vector space.

  - Consequently, projective geometry amounts to affine geometry plus the asymptotic behavior of geometry at infinity.

  - There are no longer any parallel lines, since all lines intersect in projective space, if only at infinity.

  - The key concept is not the parallelism of subspaces, but their *incidence*, i.e., the geometric character of their intersection.

Projective coordinates and transformations

- One can define either homogeneous or inhomogeneous coordinates on $\mathbb{R}P^n$.
  - One uses n +1 homogeneous coordinates $(x^0, x^1, \ldots, x^n)$ to specify a line in $\mathbb{R}^n - \{0\}$.
  - Since this is one coordinate too many, one uses n inhomogeneous coordinates $(X^1, \ldots, X^n)$ for the points of $\mathbb{R}P^n$, where, if $x^0 \neq 0$:

$$X^i = \left(\frac{x^1}{x^0}, \cdots, \frac{x^1}{x^0} \right), \qquad i = 1, \ldots, n$$

- The fundamental transformations of a projective space must preserve the incidence of subspaces.
  - Hence, they must take points to points, lines to lines, etc., and preserve intersections.
- Such transformations – called *projectivities* or *homographies* – can be represented either by linear transformations of the homogeneous coordinates or linear fractional transformations of the inhomogeneous coordinates.
  - Either way, the basic group is isomorphic to $SL(n+1; \mathbb{R})$.

Plücker-Klein correspondence

- A k-plane Π_k in $\mathbb{R}^n$ can be represented non-uniquely by a decomposable k-vector on $\mathbb{R}^n$.

- One simply chooses a frame $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ for Π_k and forms the k-vector $\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_k$.

- Any other k-frame $\{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$ for Π_k will be uniquely related to $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ by an invertible matrix $\mathbf{f}_i = A_i^j \mathbf{e}_j$, which makes:

$$\mathbf{f}_1 \wedge \ldots \wedge \mathbf{f}_k = \det(A)\, \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_k$$

- Hence, one can associate Π_k with a *unique* line through the origin in $\Lambda_k(\mathbb{R}^n)$; i.e., a point in $P\Lambda_k$.

- This defines an embedding of $G(k, n)$ in $P\Lambda_k$.

    - It is not onto since not every k-form is decomposable.

- In the particular case of bivectors, the image of the embedding is a quadric hypersurface – called the *Klein hypersurface* – that is defined by the algebraic equation:

$$\mathbf{F} \wedge \mathbf{F} = 0 .$$

Real projective geometry and special relativity

- One first encounters the projection of homogeneous coordinates onto inhomogeneous ones when one wishes to look at the projection of a velocity four-vector v^μ that is expressed in arbitrary space-time coordinates (ct, x, y, z) onto three-dimensional proper-time coordinates:

$$(v_x, v_y, v_z) = \left(\frac{v^1}{v^0}, \frac{v^2}{v^0}, \frac{v^3}{v^0} \right)$$

in a rest space.

  - The homogeneous v^0 component that one must divide by is:

$$v^0 = \frac{dt}{d\tau} = \sqrt{1 - \frac{v^2}{c^2}}$$

    which also relates to changing the parameterization of the worldline that has **v** for its velocity vector from τ to t:

$$\frac{dx^i}{dt} = \frac{d\tau}{dt}\frac{dx^i}{d\tau}$$

- This leads one to observe some fundamental aspects of the role of projective geometry at this level:

    - It relates to the projection of four-dimensional objects into the three-dimensional rest spaces of measuring devices.

    - It relates to the fact that there is a subtle difference between a line as a set of points and a line as a *parameterized* set of points.

Complex projective geometry in SR

- One must notice that the fundamental transformations of special relativity come from the group $SL(2; \mathbb{C})$.

- This group can also be described as the group of projective transformations of $\mathbb{C}P^1$, which is diffeomorphic to S^2 as a real manifold.

    - One way of describing S^2 geometrically is by the one-point compactification of the complex plane by the addition of a point at infinity (i.e., the *Riemann sphere*).

- The 2-sphere in spacetime that one identifies $\mathbb{C}P^1$ with is the *light sphere* in a chosen rest space; it has a radius of 1 light sec.

- The projection of homogeneous coordinates (z^0, z^1) in $\mathbb{C}^2 - \{0\}$ to the inhomogeneous coordinate z^1/z^0 on $\mathbb{C}P^1$ associates a 2-spinor with each point on the light sphere (minus some point).

- The action of $SL(2; \mathbb{C})$ on $\mathbb{C}P^1$ is then defined by either its linear action on 2-spinors (homogeneous coordinates) or its nonlinear action on the light sphere (inhomogeneous coordinates) by *Möbius* transformations.

Projective geometry and electromagnetism

- Key to making the connection between projective geometry and electromagnetism is the application of the Plücker-Klein embedding to 2-forms on $\mathbb{R}^4$.

- A decomposable 2-form on $\mathbb{R}^4$ represents either a 2-plane in $\mathbb{R}^4$ or an *elementary* electromagnetic field.

  - "Elementary" means that its source is connected.

  - Examples are static electric and magnetic fields and electromagnetic waves that are due to single sources.

  - Such 2-forms can look like:

$$F = dt \wedge E,\ \tfrac{1}{2}\varepsilon_{ijk} B^i dx^j \wedge dx^k,\ k \wedge E$$

  - More general (i.e., rank four) 2-forms represent linear superpositions of elementary fields.

    - Although they do not define unique 2-planes in $\mathbb{R}^4$, they do define (non-unique) 4-frames.

Pre-metric electromagnetism

- The Lorentzian metric on $\mathbb{R}^4$ is not necessary for making the Maxwell equations well-defined, only the Hodge * star isomorphism *as it acts on 2-forms*.

    - However, * is equivalent to a conformal class of Lorentzian metrics.

- One can replace the * isomorphism with an isomorphism $\tilde{\kappa}$ of 2-forms with 2-forms that comes from the composition of two other isomorphisms:

    - A *linear electromagnetic constitutive law:*

    $$\kappa: \Lambda^2(\mathbb{R}^4) \to \Lambda_2(\mathbb{R}^4), \qquad F \mapsto \kappa(F) = \mathfrak{h}$$

    that takes electromagnetic field strengths (E, $\mathbf{B}$) to electromagnetic excitations ($\mathbf{D}$, H).

    - Poincaré duality:

    $$\#: \Lambda_2(\mathbb{R}^4) \to \Lambda^2(\mathbb{R}^4), \qquad \mathfrak{h} \mapsto \#\mathfrak{h} = i_{\mathfrak{h}}V$$

    in which $V = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ is the volume element on $\mathbb{R}^4$.

- The pre-metric form of the Maxwell equations is then:

$$dF = 0, \quad d\#\mathfrak{h} = \#\mathbf{J}, \qquad \mathfrak{h} = \kappa(F)$$

Reduction from transformations of 2-forms to Lorentz transformations

- So far, the only frames on $\Lambda^2(\mathbb{R}^4)$ that we can speak of are general 6-frames.
  - These are parameterized by the elements of $GL(6; \mathbb{R})$.
- Not all of these frames can be constructed from 4-frames on $\mathbb{R}^4$, which are parameterized by $GL(4; \mathbb{R})$.
  - E.g.: $b^i = dt \wedge dx^i$, $\quad b^{i+3} = \frac{1}{2}\varepsilon_{ijk}\, dx^j \wedge dx^k$, $i = 1, 2, 3$
- The key to making the reduction from linear 6-frames to Lorentzian 4-frames is the isomorphism of $SO(3; \mathbb{C})$ with $SO_0(3, 1)$.
- If one can introduce a complex structure on $\Lambda^2(\mathbb{R}^4)$ then this is straightforward.

$$GL(6; \mathbb{R}) \to GL(3; \mathbb{C}) \to SL(3; \mathbb{C}) \to SO(3; \mathbb{C})$$

- Since the Hodge * would define such a complex structure, one must assume that the isomorphism $\tilde{\kappa}$ behaves analogously: $\tilde{\kappa} = -\lambda^2 I$ for some function λ.
  - Caveat: not all physically meaningful constitutive laws have this property, but electromagnetic waves still propagate in those media.