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On the differential geometry of ruled surfaces

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Summary. – *(Given in the introduction)*

The normal curvature and the geodesic torsion of the line of striction of a skew ruled surface in three-dimensional Euclidian space are metric invariants of the surface. Furthermore, these two quantities are functions of the variable whose values correspond to the points of the line of striction. They are also coupled by a relation, in general, and the skew ruled surfaces on which those quantities are coupled by relations of the same form will have some common properties, the study of which can lead to some results in regard to the metric differential geometry of ruled surfaces, which is not devoid of interest.

In the present article, which is dedicated to the study in question, I shall establish certain theorems in regard to the skew ruled surfaces on each of which the normal curvature and the geodesic torsion of its line of striction verify a first or second-degree algebraic relation with constant coefficients. In addition, after showing that when the geodesic curvature and the geodesic torsion of *one* orthogonal trajectory to the generators of a skew ruled surface verify an algebraic relation with constant coefficients, the geodesic curvature and geodesic torsion of any other orthogonal trajectory to the generators of the surface will necessarily be coupled by an algebraic relation with constant coefficients, with the aid of the theorems thus-established, I will arrive at the determination of certain classes of ruled surfaces on which the geodesic curvature and geodesic torsion of each orthogonal trajectory to their generators verifies an algebraic relation of degree at most two with coefficients that are constant on it, but which will generally vary from one of those curves to another. I will then show that the geodesic curvature and geodesic torsion of each orthogonal trajectory to the surface that is generated by the principal normals to a BERTRAND curve or a MANNHEIM curve or a cylindrical helix, as well as a skew surface with constant distribution parameter or the surface that is generated by the binormals to a skew curve, will verify an algebraic relation of degree at most two with coefficients that are constant on it. These coefficients will vary from of those curves to another, if one ignores the case of surfaces with constant distribution parameter.

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I.

1. – Consider a portion R of a real skew ruled surface *ds* in three-dimensional Euclidian space on which the distribution parameter is non-zero on all of the generators.

Let:

$$\bar{\rho} = \bar{\rho}(u) \tag{1.1}$$

be the vectorial equation with respect to the coordinate system that is chosen in space for the line of striction C of R, which is the locus of central points of the generators of that surface when the parameter u is the arc length of C.

If $\bar{e}(u)$, $\bar{n}(u)$, $\bar{z}(u) = \bar{e} \wedge \bar{n}$ are the unit vectors that determine the positive sense on the generator e of R that issues from a running point K (u) of C of the *central normal n* to R at K and the *central tangent z* to R at that same point (i.e., of the tangent to R at K that is perpendicular to the generator e) then – as one knows ([**4**], pp. 145) – one will have the following formulas for the derivatives of $\bar{e}$, $\bar{n}$, $\bar{z}$ with respect to the variable u ([1]):

$$\dot{\bar{e}} = \kappa\bar{n}, \qquad \dot{\bar{n}} = -\kappa\bar{e} + \sigma\bar{z}, \qquad \dot{\bar{z}} = -\sigma\bar{n}, \tag{1.2}$$

in which:

$$\kappa = \varepsilon\,|\dot{\bar{e}}|, \qquad \sigma = \frac{(\bar{e} \wedge \dot{\bar{e}}) \times \ddot{\bar{e}}}{|\dot{\bar{e}}|^2}, \tag{1.3}$$

and $\varepsilon = +1$ or -1 according to whether the mixed product $(\dot{\bar{\rho}} \wedge \bar{e}) \times \dot{\bar{e}}$, which is, from the hypothesis that was made for the distribution parameter of R, non-zero for all values of u that correspond to the points of C, is greater than or less than zero, resp.

The lines e, n, z issue from the point K (u) of the curve C along the directions whose positive senses are determined by the unit vectors $\bar{e}$, $\bar{n}$, $\bar{z}$, respectively, which are the axes of an oriented tri-rectangular trihedron [K; $\bar{e}, \bar{n}, \bar{z}$]; that trihedron will be called the *central trihedron that is associated with the surface* in what follows.

Moreover, the derivative $\dot{\bar{\rho}}$ at the point K (u) of the curve C is a unit vector, since u is the arc length of C. Since that vector is parallel to the tangent to C at K, it is parallel to the tangent plane [K; $\bar{e}, \bar{z}$] to R at that point. One can then put it into the form:

$$\dot{\bar{\rho}} \equiv \bar{t} = \bar{e}\cos\varphi + \bar{z}\sin\varphi, \tag{1.4}$$

in which φ is the oriented angle $(\bar{\varepsilon}, \bar{t})$ on the plane [K; $\bar{e}, \bar{z}$].

The three quantities κ; σ; φ are functions of the arc length u of the line of striction C of R that are defined, thanks to the hypothesis that was made for the distribution parameter of R, in the interval of values of u that correspond to the points of the curve C or, what amounts to the same thing, they are generators of R. Those three functions $\kappa(u)$, $\sigma(u)$, $\varphi(u)$, whose values on each generator of R are called the *curvature*, *torsion*, *and striction of the surface* on that generator by E. KRUPPA ([**3**], pp. 63), are metric

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([1]) Dots denote derivatives with respect to the variable u. One supposes that the derivation operations that are performed in what follows are legitimate within the intervals considered.

invariants of R, along with the arc length u of C, and they are precisely the metric invariants of the surface that was chosen to be fundamental to the metric differential geometry of ruled surfaces that was founded by G. SANNIA [**5**] and E. KRUPPA [**4**].

Two other metric invariants of the surface are the normal curvature κ_n and the geodesic torsion σ_g of its line of striction. Those two quantities are functions of the three fundamental invariants of the surface.

Indeed, from known formulas ([**3**], pp. 73), one will have:

$$\kappa_n = \kappa \cos \varphi - \sigma \sin \varphi, \qquad \sigma_g = \kappa \sin \varphi + \sigma \cos \varphi. \tag{1.5}$$

REMARK. – As one will see immediately, the surface R cuts the surface R_n that is generated by the central normals to R all along its line of striction C, which is obviously an orthogonal trajectory to the generators of R_n .

It results from this that the normal curvature κ_n and the geodesic torsion σ_g of the curve C, which is considered to be a curve that is traced on the surface R, are equal in absolute value at each point of C to the geodesic curvature κ'_g in the former case and the geodesic torsion σ'_g of C in the latter case, when it is considered to be a curve that is traced on the surface R_n .

One will then have:

$$\kappa'_g = \varepsilon' \kappa_n , \qquad \sigma'_g = \sigma_g$$

at each point of C, in which $\varepsilon' = +1$ or -1, and one chooses the positive sense along the direction of the normal to R_n in such a manner that one will have:

$$\kappa'_g = \kappa_n , \qquad \sigma'_g = \sigma_g . \tag{1.6}$$

2. – Thanks to (1.4), equation (1.1) of the curve C can be written:

$$\bar{\rho} = \int (\bar{e} \cos\varphi + \bar{z} \sin\varphi)\, du , \tag{2.1}$$

and the (vectorial) equation of the surface R, if one chooses its line of striction C to be the *director curve* in it, can be put into the form:

$$\bar{r} = \int (\bar{e} \cos\varphi + \bar{z} \sin\varphi)\, du + v\,\bar{e}(u) . \tag{2.2}$$

Having said that, let R' be a skew ruled surface that cuts R at a right angle along an orthogonal trajectory C' of the generators of R.

If:

$$\bar{\rho}' = \bar{\rho}'(u) = \bar{\rho}(u) + v'(u)\,\bar{e}(u)$$

is the equation of the curve C' then the normal to R at the running point $K'(u)$ of that curve will be parallel to the vector:

(2.3) $$\bar{n}'(u) = \frac{\dot{\bar{\rho}}}{|\dot{\bar{\rho}}|} \wedge \bar{e} \equiv \bar{t}' \wedge \bar{e}\,,$$

which is a unit vector, since the unit vectors $\dot{\bar{\rho}}/|\dot{\bar{\rho}}| = \bar{t}'$, $\bar{e}$ are orthogonal, from the hypothesis that was made on the curve C'.

The generator e' of the surface R' that issues from the point $K'(u)$ of C' is situated on the plane $[K'; \bar{t}', \bar{n}']$ that is perpendicular at K' to the generator of R that issues from that point, and consequently, it is necessarily parallel to a unit vector of the form:

(2.4) $$\bar{e}' = \bar{t}' \cos\theta + \bar{n}' \sin\theta.$$

In order for the curve C' to be the line of striction of the surface C', in addition, it is necessary and sufficient, as one knows, that one should have:

(2.5) $$\bar{t}' \times \dot{\bar{e}}' = 0$$

at each point of C'.

However, from (2.4), one has:

$$\dot{\bar{e}}' = \dot{\bar{t}}' \cos\theta + \dot{\bar{n}}' \sin\theta + \dot{\theta}\,(-\bar{t}' \sin\theta + \bar{n}' \cos\theta)\,,$$

and if one takes into account the fact that one has $\bar{t}'^2 \equiv 1$, $\bar{t}' \times \dot{\bar{t}}' \equiv 0$, then thanks to the hypotheses that were made, $\bar{t}' \times \bar{n}' = 0$, $\sin\theta \neq 0$, the condition (2.5) will become:

(2.6) $$\dot{\theta} - \bar{t}' \times \dot{\bar{t}}' = 0.$$

For each function $\theta(u)$ that satisfies the differential equation (2.6), the vector $\bar{e}'(u)$ (2.4) will determine ∞^1 lines that issue from the points of the curve C' and determine a skew ruled surface R' that cuts R along C' at a right angle and admits that curve as its line of striction.

However, the curve C is an orthogonal trajectory of the generators of R that is chosen at random. Furthermore, the preceding considerations demand only that the ruled surface R should be skew.

One can then state the:

THEOREM I.

Each orthogonal trajectory of the generator of a ruled surface is the line of striction of the ∞^1 skew ruled surfaces that admit the generators of the surface as central normals.

Moreover, the vectorial equation of a skew ruled surface R_1 that is represented on the skew surface R that is defined by the equation (2.2) in such a manner that the homologous points of the two surfaces admit the same curvilinear coordinates u, v will – as one knows ([**5**], pp. 45) – necessarily have the form:

$$\bar r_1 = \int (\bar e \cos\varphi + \bar z \sin\varphi)\, du + \beta \bar n + v_1(v)(\bar e \cos\omega + \bar z \sin\omega)\,, \tag{2.7}$$

in which:

$$\beta = \text{const.}, \qquad \omega = \text{const.} \qquad (-\pi < \omega \leq \pi), \tag{2.8}$$

where the function v_1 (v) is arbitrary, when the generators of the two surfaces, as well as their lines of striction, in that representation correspond, and the two surfaces have normals that coincide at each pair of homologous point of their lines of striction, in addition.

For each system of values of the parameters β, ω that enter into y, equation (2.7) will determine a skew ruled surface. When $\beta \neq 0$ if $\omega = 0$ or π, that surface will not coincide with the surface R, and when the two surfaces are represented on each other in the indicated manner, they will admit central normals that coincide at the homologous of their lines of striction.

In regard to the fact that R is a skew ruled surface that is chosen at random, it will result that *to each skew ruled surface, one can associate* ∞^2 *skew ruled surfaces whose central normals coincide with the central normals of that surface.*

The set of surfaces that are represented on a skew ruled surface in the indicated manner will be called the *set* (N) *that is associated with that surface* in what follows.

The set (N) that is associated with the surface considered R is defined by equation (2.7). The surfaces of that set, including the surface R, correspond to the systems of values of the parameters β, ω that enter into that equation and belong to the open interval $(-\infty, +\infty)$ and the upper-closed interval $(-\pi, +\pi)$, respectively.

3. – The generator of a surface R_1 of the set (N) that is associated with the surface R, which is the homologue of the generator of R that issues from the running point K (u) of its line of striction C, is necessarily parallel to the tangent plane to R and K, and from (2.8) it is invariably coupled with the central trihedron that is associated with R.

That being the case, one can choose the positive sense along the direction of that line in such a manner that it will coincide with the positive sense on the common direction of the coincident normals to the surfaces R, R_1 at each pair of homologous points their lines of striction.

Upon letting $\bar e_1$, $\bar n_1$, $\bar z_1$ denote the unit vectors that determine the positive sense along the directions of the axes of the central trihedron that is associated with R_1 at the point K_1 of its line of striction C_1 that is homologous to the running point K (u) of the line of striction C of R, one will then have the relations:

$$\bar e_1 = \bar e \cos\omega + \bar z \sin\omega, \qquad \bar n_1 = \bar n\,, \qquad \bar z_1 = \bar e_1 \wedge \bar n_1 = -\bar e \sin\omega + \bar z \cos\omega. \tag{3.1}$$

Moreover, from (2.7), the line of striction C_1 of R_1 is defined by the equation:

$$\bar\rho_1 = \bar\rho_1(u) = \int (\bar e \cos\varphi + \bar z \sin\varphi)\, du + \beta\, \bar n(u)\,, \tag{3.2}$$

in which b is a constant, from (2.8).

Upon differentiating (3.2) with respect to u and making use of (1.2), one will get:

$$\frac{d\bar{\rho}_1}{du} = \bar{e}\,(\cos\omega - \beta\kappa) + \bar{z}\,(\sin\omega + \beta\sigma). \tag{3.3}$$

Upon letting du_1 denote the elementary arc length of the curve C_1 at its point K_1 that is the homologue of the running point K (u) of the curve C_1, one will then have:

$$\left(\frac{du_1}{du}\right)^2 = \left(\frac{d\bar{\rho}_1}{du}\right)^2 = (\cos\omega - \beta\kappa)^2 + (\sin\omega + \beta\sigma)^2.$$

With the aid of (1.5), that relation will affect the form:

$$\left(\frac{du_1}{du}\right)^2 = 1 - 2\,\beta\,\kappa_n + \beta^2\,(\kappa_n^2 + \sigma_g^2) = \frac{1}{\lambda^2}, \tag{3.4}$$

which shows that the ratio of the elementary arc lengths du_1, du of the curves C_1, C, resp., at their homologous points K_1, K, resp., is an algebraic function of the normal curvature κ_n and the geodesic torsion σ_g of the curve C, where the coefficient β that figures in it is, from (2.8), a constant whose absolute value is equal to the constant distance between homologous points of the two curves.

Similarly, upon differentiating the first and third relation in (3.1) with respect to u and making use of (1.2), one will get:

$$\dot{\bar{e}}_1 = (\kappa\cos\omega - \sigma\sin\omega)\,\bar{n}, \qquad \dot{\bar{z}}_1 = -\,(\kappa\sin\omega + \sigma\cos\omega)\,\bar{n}. \tag{3.5}$$

If one now lets κ_1, σ_1, φ_1 denote the curvature, curvature, and striction, resp., of the surface R_1 along its generator that is homologous to the running generator of R, and one takes into account that, from some known formulas ([**4**], pp. 145), one will have:

$$\frac{d\bar{e}_1}{du_1} = \kappa_1\,\bar{n}_1, \qquad \frac{d\bar{z}_1}{du_1} = -\sigma_1\,\bar{n}_1,$$

then from these formulas, with the aid of (3.4) and (3.5), one will arrive at the relations:

$$\frac{d\bar{e}_1}{du_1} = \kappa_1\,\bar{n}_1 = (\kappa\cos\omega - \sigma\sin\omega)\frac{du}{du_1}\bar{n} = \lambda\,(\kappa\cos\omega - \sigma\sin\omega)\,\bar{n},$$

$$\frac{d\bar{z}_1}{du_1} = -\sigma_1\,\bar{n}_1 = -\,(\kappa\sin\omega + \sigma\cos\omega)\frac{du}{du_1}\bar{n} = -\,\lambda\,(\kappa\sin\omega + \sigma\cos\omega)\,\bar{n}.$$

However, from the second relation (3.1), one will have $\bar{n} = \bar{n}_1$; one will then have:

$$(3.6) \qquad \kappa_1 = \lambda\,(\kappa \cos\,\omega - \sigma \sin\,\omega), \qquad \sigma_1 = \lambda\,(\kappa \sin\,\omega + \sigma \cos\,\omega)\,.$$

In addition, one will have:

$$\cos\,\varphi_1 = \frac{d\bar{\rho}_1}{du_1}\times\bar{e}_1 = \frac{du}{du_1}\frac{d\bar{\rho}_1}{du}\times\bar{e}_1 = \lambda\frac{d\bar{\rho}_1}{du}\times\bar{e}_1\,,$$

$$\sin\,\varphi_1 = \frac{d\bar{\rho}_1}{du_1}\times\bar{z}_1 = \frac{du}{du_1}\frac{d\bar{\rho}_1}{du}\times\bar{z}_1 = \lambda\frac{d\bar{\rho}_1}{du}\times\bar{z}_1\,,$$

since the tangent to the curve C_1 at its point K_1 is obviously situated on the tangent plane $[K_1;\ \bar{e}_1,\ \bar{z}_1]$ to R_1 at that same point.

If one replaces $\bar{e}_1,\ \bar{z}_1,\ \dfrac{d\bar{\rho}_1}{du}$ in these relations with their values in (3.1) and (3.3) then they will take the form:

$$(3.7) \qquad \begin{cases} \cos\varphi_1 = \lambda\{\quad \cos\omega(\cos\varphi - \beta\kappa) + \sin\omega(\sin\varphi + \beta\sigma)\}, \\ \sin\varphi_1 = \lambda\{-\sin\omega(\cos\varphi - \beta\kappa) + \cos\omega(\sin\varphi + \beta\sigma)\}. \end{cases}$$

Finally, if one takes into account the fact that the normal curvature κ_{1n} and the geodesic torsion σ_{1g} of the line of striction C_1 of R_1 are coupled with the fundamental invariants κ_1, σ_1, φ_1 of the surface by the relations:

$$\kappa_{1n} = \kappa_1 \cos\,\varphi_1 - \sigma_1 \sin\,\varphi_1\,, \qquad \sigma_{1g} = \kappa_1 \sin\,\varphi_1 + \sigma_1 \cos\,\varphi_1\,,$$

from some known formulas ([**3**], pp. 73), then upon replacing κ_1, σ_1, and $\cos\,\varphi_1$, $\sin\,\varphi_1$ with their values (3.6) and (3.7) and making use of (1.5), one will arrive at the relations:

$$\kappa_{1n} = \lambda^2\,\{\kappa_n - \beta(\kappa_n^2 + \sigma_g^2)\,\}, \qquad \sigma_{1g} = \lambda^2\,\sigma_g.$$

and finally, thanks to (3.4), at the formulas:

$$(3.8) \qquad \kappa_{1n} = \frac{\kappa_n - \beta(\kappa_n^2 + \sigma_g^2)}{1 - 2\beta\kappa_n + \beta^2(\kappa_n^2 + \sigma_g^2)}\,, \qquad \sigma_{1g} = \frac{\sigma_g}{1 - 2\beta\kappa_n + \beta^2(\kappa_n^2 + \sigma_g^2)}\,.$$

However, from the final remark of paragraph 1, the normal curvature and the geodesic torsion of the line striction of the surface R, as well as any other surface R_1 of the set (N) that is associated with R, are coupled by the relations (1.6) with the geodesic curvature and geodesic torsion of that curve, when it is considered to be a curve that is traced on the surface R_n that is generated by the central normals to R.

That being the case, if one lets κ'_g, κ'_{1g}; σ'_g, σ'_{1g} denote the geodesic curvatures and geodesic torsions of the lines of striction C, C_1 of the surfaces R, R_1, resp., when considered to be curves that are traced on the surface R_n, one will get the following relations from the two formulas (3.8):

$$\kappa'_{1g} = \frac{\kappa'_g - \beta(\kappa'^2_g + \sigma'^2_g)}{1 - 2\beta\kappa'_g + \beta^2(\kappa'^2_g + \sigma^2_g)}, \quad \sigma'_{1g} = \frac{\sigma'_g}{1 - 2\beta\kappa'_g + \beta^2(\kappa'^2_g + \sigma'^2_g)}, \tag{3.9}$$

where, from (2.8), the coefficient β that appears in them is equal in absolute value to the constant distance between the two points of the curves C, C_1 that are situated on the same generator of the surface R.

The line of striction C_1 of the surface R_1 is obviously an orthogonal trajectory of the generators of the surface R_n that is chosen at random, and the preceding considerations in regard to the geodesic curvature and geodesic torsion of that curve that is traced on the surface R_n, when combined with Theorem I, will permit one to state:

THEOREM II. – *If one chooses an orthogonal trajectory to the generators of a skew ruled surface to be a director curve then one can express the geodesic curvature and geodesic torsion of any other orthogonal trajectory of the generators of the surface as rational functions of the form* (3.9) *of the geodesic curvature and geodesic torsion of the director curve, and the single coefficient β that enters into those functions will be a constant whose absolute value is equal to the constant distance between two points of that trajectory and the director curve that is situated on the same generator of the surface.*

A remarkable consequence of that is the following proposition:

If the geodesic curvature and geodesic torsion of an orthogonal trajectory of the generators of a skew ruled surface verify an algebraic relation with constant coefficients then the geodesic curvature and geodesic torsion of any other orthogonal trajectory of the generators of the surface likewise verify an algebraic relation with constant coefficients.

II.

4. – First suppose that the surface considered R is a BERTRAND surface.

In that case, the points of the line of striction C of R will correspond to the points of the line of striction C_1 of another skew ruled surface R_1, in such a manner that the normals to the surfaces at each pair of homologous points of their lines of striction will coincide, and in addition, the two surfaces will have equal strictions along their generators that issue from each pair of homologous points of those curves ([**5**], pp. 45).

The surface R_1, which is obviously also a BERTRAND surface, necessarily belongs to the set (N) that is associated with R, because two skew ruled surfaces whose central normals coincide will each belong to the set (N) that is associated with the other one, as one will see immediately.

Now, if β_1, ω_1 are the values of the parameters β, ω, resp., that figure in equation (2.7) of the set (N) that is associated with R, which corresponds to the surface R_1 of that set, which constitutes a pair of BERTRAND surfaces, along with R, then β_1 will necessarily be non-zero, and from formulas (5.7), the invariants κ, σ, φ of R will be functions of the arc length of its line of striction C' that necessarily verify the relation:

$$\frac{\cos(\varphi-\omega_1)-\beta_1(\kappa\cos\omega_1-\kappa\sin\omega_1)}{\sin(\varphi-\omega_1)+\beta_1(\kappa\sin\omega_1+\kappa\cos\omega_1)}=\frac{\cos\varphi}{\sin\varphi} \tag{4.1}$$

for all values of u that correspond to points of C.

The condition (4.1), which is, in addition, sufficient for the surface R to be a BERTRAND surface – as would result from formulas (3.7) – will take on the form:

$$\kappa_n \sin\omega_1 + \sigma_g \cos\omega_1 = \frac{\sin\omega_1}{\beta_1} \tag{4.2}$$

with the aid of (1.5).

The relation (4.2) will be verified in the case where the line of striction C of R is a line of curvature if one sets $\omega_1 = 0$ or π, where β_1 is an arbitrary non-zero constant, because in that case, one will have $\sigma_g = 0$.

The surface R_n that is generated by the central normals of R will then be a developable surface, and all of the surfaces of the set (N) that is associated with R whose generators that are homologous to each generator of R are parallel to that generator will then admit the same striction as R along those generators. Hence, *a skew ruled surface whose line of striction is a line of curvature can be considered to be a BERTRAND surface.*

If the invariants κ, σ, φ of the surface R whose line of striction is not a line of curvature verify a relation of the form (4.1), in which β_1, $\sin\omega_1$, $\cos\omega_1$ are constants, and the first two are non-zero, then the set (N) that is associated with R will contain only one surface that constitutes a pair of BERTRAND surfaces with R: namely, the surface R_1 in that set that corresponds to the values β_1, ω_1 of the parameters β, ω, resp., that are included in equation (2.7).

The surface R_1 is likewise a BERTRAND surface. Consequently, the normal curvature κ_{1n} and the geodesic torsion σ_{1g} of its line of striction C_1 must verify a relation of the form (4.2):

$$\kappa_{1n} \sin\omega' + \sigma_{1g} \cos\omega' = \frac{\sin\omega'}{\beta'},$$

in which β', ω' are the values of the parameters β, ω, resp. that figure in the equation of the form (2.7) for the set (N) that is associated with the surface R_1 that corresponds to the surface R, when it is regarded as a surface that belongs to that set. However, as one will see immediately, one will have $\beta' = -\beta_1$, $\omega' = -\omega_1$. Hence κ_{1n}, σ_{1g} must verify the relation:

$$\kappa_{1n} \sin\omega_1 - \sigma_{1g} \cos\omega_1 = -\frac{\sin\omega_1}{\beta_1}. \tag{4.3}$$

From (4.2), in order for the surface considered R to be a BERTRAND surface, it is necessary that normal curvature κ_n and the geodesic torsion σ_g of its line of striction C must verify a relation of the form:

$$A\ \kappa_n + B\ \sigma_g = C \tag{4.4}$$

whose coefficients A, B, C are constants, and the first and third one are either both non-zero or both zero, and in that case, the second one will be non-zero.

That condition is, in addition, sufficient for R to be a BERTRAND surface.

Indeed, if $A = C = 0$, $B \neq 0$ then the line of striction of R will be a line of curvature. Consequently, R can be considered to be a BERTRAND surface.

If $A \neq 0$, $C \neq 0$ then upon setting:

$$\frac{A}{\sqrt{A^2+B^2}} = \sin\,\omega_1\,, \qquad \frac{B}{\sqrt{A^2+B^2}} = \cos\,\omega_1\,, \qquad \frac{A}{C} = \beta_1\,,$$

one can give the relation (4.4) the form (4.2), and since that is equivalent to the relation (4.1), it will be sufficient for R to be a BERTRAND surface.

One can then state:

THEOREM III

In order for a skew ruled surface to be a BERTRAND surface, it is necessary and sufficient that the normal curvature and geodesic torsion of its line of striction verify a linear relation with constant coefficients, the first and third of which are either both non-zero or both zero, and in that case, the second one will non-zero.

It should be noted that in the case where the line of striction C of a BERTRAND surface R is a not a line of curvature, one can deduce from the relation (4.2) that the normal curvature κ_n and the geodesic torsion σ_g of the curve C must satisfy, which can be written:

$$\sin\,\omega_1\,(1 - \beta\,\kappa_n) = \beta_1 \cos\,\omega_1\,\sigma_g\,,$$

in which $\beta_1 \neq 0$, $\sin\,\omega_1 \neq 0$, that one has:

$$\sin\,\omega_1 = \frac{\beta_1^2\,\sigma_g^2}{(1-\beta_1\,\kappa_n)^2+\beta_1^2\,\sigma_g^2}\,. \tag{4.5}$$

Moreover, from the second formula (3.8), the geodesic torsion of the line of striction C_1 of the surface R_1 that constitutes a BERTRAND pair, along with R, will be coupled with κ_n, σ_g by the relation:

$$\sigma_{1g} = \frac{\sigma_g}{(1-\beta_1\,\kappa_n)^2+\beta_1^2\,\sigma_g^2}\,. \tag{4.6}$$

One immediately deduces from the two relations (4.5) and (4.6) that *the geodesic torsions* σ_g , σ_{1g} *of the lines of striction of the surfaces* R, R_1, *resp., at the points that are situated on the same generator of the surface that is generated by the common central normal to the two surfaces will verify the relation:*

$$\sigma_g\,\sigma_{1g} = \frac{\sin^2\omega_1}{\beta_1^2}\,. \tag{4.7}$$

REMARK. – From a known theorem ([**6**], pp. 143), the lines of striction of two skew ruled surfaces R, R_1 that constitute a BERTRAND pair are geodesics of those surfaces in the particular case where those curves are also isogonal trajectories of the generators of the two surfaces, or (what amounts to the same thing) in the case where the two surfaces are surfaces of constant striction. The common central normal to two surfaces at each pair of homologous points of their lines of striction will necessarily be the common principal normal to those curves at the same points then. Hence, in that case, *the lines of striction of two surfaces will constitute a pair of BERTRAND curves, and the relation* (4.7) *will reduce to the relation that (as is known* [**6**], pp. 35) *is verified by the torsion of the curves of such a pair at their points that are situated along the same generator of the surface that is generated by the common principal normals of those curves.*

5. – If the lines of striction C, C_1 of a pair of BERTRAND surfaces R, R_1, resp., are not lines of curvature of those surfaces then, as we saw in the preceding paragraph, their normal curvatures κ_n , κ_{1n}, resp., and geodesic torsions σ_g , σ_{1g}, resp., must satisfy two relations of the form (4.2) and (4.3), in which the constants β_1, ω_1 that enter into them will be the values of the parameters β, ω, resp., that are included in equation (2.7) for the set (N) associated to R, which corresponds to the surface R_1 .

Moreover, if R^* is a surface of that set that corresponds to the values β^*, ω^* of the parameters β, ω, resp., then, from (3.8), the normal curvature κ_n^* and the geodesic torsion σ_n^* of the line of striction C^* of R^* will be coupled with the normal curvature κ_n and geodesic torsion σ_g of the line of striction C of R by the relations:

$$\kappa_n^* = \frac{\kappa_n - \beta^*(\kappa_n^2 + \sigma_g^2)}{(1-\beta^*\kappa_n)^2 + \beta^{*2}\sigma_g^2}\,, \qquad \sigma_g^* = \frac{\sigma_g}{(1-\beta^*\kappa_n)^2 + \beta^{*2}\sigma_g^2}\,. \tag{5.1}$$

The elimination of κ_n , σ_g from the relations (5.1) and (4.2) leads to the relation:

$$\sin\,\omega_1\,\{(\beta^{*2} - \beta^*\beta_1)(\kappa_n^2 + \sigma_g^2) + (2\beta^* - \beta_1)\,\kappa_n^* + 1\} - \beta_1 \cos\,\sigma_g^* = 0, \tag{5.2}$$

which must be satisfied by the normal curvature and the geodesic torsion of the line of striction of R^* in the case envisioned. That relation will reduce to the relation (4.2) or (4.3) when one sets $\beta^* = 0$ or $\beta^* = \beta_1$, respectively.

On the other hand, if the normal curvature and the geodesic torsion of the line of striction C^* of a skew ruled surface R^* verify a relation of the form (5.2) and one has $\beta_1 \neq 0$, sin $\omega_1 \neq 0$, then it will result from that relation, which can be written:

$$\sin\,\omega_1\;\frac{\kappa_n^* + \beta^*\,(\kappa_n^{*2} + \sigma_g^{*2})}{(1-\beta^*\,\kappa_n^*)^2 + \beta^{*2}\,\sigma_g^{*2}} + \cos\,\omega_1\;\frac{\sigma_g^*}{(1-\beta^*\,\kappa_n^*)^2 + \beta^{*2}\,\sigma_g^{*2}} = \frac{\sin\omega_1}{\beta_1},$$

when one recalls (3.6), that the set (N) that is associated with the surface R^* will contain a BERTRAND surface, since the normal curvature and geodesic torsion of the line of striction verify a relation of the form (4.2), or (what amounts to the same thing) the central normals to R^* are the common central normals to a pair of BERTRAND surfaces.

On the hand, if the normal curvature and geodesic torsion of the line of striction of a skew ruled surface verify a relation of the form (5.3), where the second and fourth of the constant coefficients A, B¸ C, D that enter into it are non-zero, and if one considers the relations:

$$\text{(5.4)} \qquad \beta^{*2} - \beta^*\beta_1 = \frac{A}{D}, \qquad 2\beta^* - \beta_1 = \frac{B}{D}, \qquad \beta_1\frac{\cos\omega_1}{\sin\omega_1} = -\frac{C}{D},$$

then upon eliminating β_1 from the first two, one will arrive at the relation:

$$D\beta^{*2} - B\beta^* + A = 0.$$

Now, if $4AD - B^2 \leq 0$ then one can associate two real values of β_1, cot ω_1 with each (real) root of the polynomial $D\beta^{*2} - B\beta^* + A$ with the aid of the last two relations in (5.4), such that if one replaces β_1, cot ω_1 in it with those values and β^* with the root considered then the relation (5.2) will reduce to the relation (5.3). As we have seen already, that will prove that the central normals to R^* are the common central normals of a pair of BERTRAND surfaces.

One can then formulate:

THEOREM IV

In order for the central normals to a real skew ruled surface whose line of striction is not a line of curvature to be the common central normals to a pair of real BERTRAND surfaces, it is necessary and sufficient that the normal curvature and geodesic torsion of its line of striction should verify a second-degree algebraic relation of the form (5.3) *such that the coefficients A, B, C, D that enter into it are constants for which one has* $4AD - B^2 \leq 0$, *while B and D are non-zero.*

One should note that it would result from the preceding considerations that *the common central normals to a pair of BERTRAND surfaces are, at the same time, the common central normals to a second pair of surfaces of that type that will not coincide with the first one, in general*.

Furthermore, if one takes into account the fact that the line of striction C^* of the surface considered R^* of the set (N) that is associated with R is an orthogonal trajectory that is chosen at random from the generators of the surface R_n that is generated by the central normals to R, one will deduce from the relation (5.2) and the final remark in paragraph 1 that when R is a BERTRAND surface whose line of striction C is not a line of curvature and the fact that one chooses C to be a director curve on the surface R_n, that the geodesic curvature and the geodesic torsion κ'_g, σ'_g of that orthogonal trajectory of the generators of R_n will verify a relation of the form:

$$A\,\{(\beta^2 - \beta_1\,\beta)(\kappa'^2_g + \sigma'^2_g) + (2\beta - \beta_1)\,\kappa'_g + 1\} + B\beta_1\,\sigma'_g = 0, \tag{5.5}$$

and the coefficients A, B, β_1, β that figure in it are constants, the first three of which are the same for all of those curves, where the first and third ones are non-zero and the fourth one varies from one of those curves to another.

One will then have:

THEOREM V

The geodesic curvature and geodesic torsion of each orthogonal trajectory to the generators of the surface R_n that is generated by the common central normals to a pair of Bertrand surfaces whose lines of striction are not lines of curvature will verify a second-degree algebraic relation of the form:

$$A'(\kappa^2_g + \sigma'^2_g) + B'\kappa'_g + C'\sigma'_g + D' = 0 \tag{5.6}$$

with coefficients that are constant on it, but vary from one of the curves to another. That relation must be linear along the line of striction of each of the surfaces whose common central normals are the generators of the surface.

One deduces from that theorem, combined with the fact that common principal normals of a pair of BERTRAND curves are (as one will easily see) the common central normals to ∞^1 pairs of BERTRAND surfaces, that *the geodesic curvature and geodesic torsion of each orthogonal trajectory to the generators of the surface that is generated by the common principal normals of a pair of BERTRAND surfaces will verify a second-degree algebraic relation of the form* (5.6) *with coefficients that are constants on them, but which will vary from one of those curves to another. That relation on each curve of the pair reduces to the linear relation that is verified by the curvature and torsion of that curve.*

6. – Now suppose that the line of striction C of the surface considered R is an asymptotic line of that surface, and consequently, that one will have:

$$\kappa_n = 0 \tag{6.1}$$

at each point of C.

In that case, the curve C will be a geodesic of the surface R_n that is generated by the central normals to R, since the surfaces R¸ R_n will cut at a right angle along that curve. The central normals to R will also be the binormals to the curve C.

In addition, in that case, formulas (3.8), which express the normal curvature κ_{1n} and the geodesic torsion σ_{1g} of the line of striction C_1 of a surface R_1 of the set (N) that is associated with R as functions of the normal curvature κ_n and the geodesic torsion σ_g of the line of striction C of R will become:

$$\kappa_{1n} = -\frac{\beta_1 \sigma_g^2}{1+\beta_1^2 \sigma_g^2}, \qquad \sigma_{1g} = \frac{\sigma_g}{1+\beta_1^2 \sigma_g^2}, \tag{6.2}$$

by virtue of (6.1).

Upon eliminating σ_g from these two relations, one will deduce that the normal curvature κ_{1n} and the geodesic torsion σ_{1g} of the line of striction C_1 of a surface R_1 of the surface (N) that is associated with the surface R when the central normals of R are the binormals of its line of striction C must verify a relation of the form:

$$\beta_1(\kappa_{1n}^2 + \sigma_{1g}^2) + \kappa_{1n} = 0, \tag{6.3}$$

and, from (2.8), the coefficient β_1 that enters into it will be equal to the absolute value of the constant distance between two points of the curves C, C_1 that are situated on the same central normal to R.

On the other hand, the central normals to a skew ruled surface R_1 are the binormals to a skew curve when the normal curvature and the geodesic torsion of its line of striction C_1 verify a relation of the form (6.3).

Indeed, it will result from this relation, with the aid of the first formula in (3.8), that the set (N) that is associated with the surface R_1 will contain a surface R whose line of striction is an asymptotic line. Consequently, the common central normals to the surface R, R_1 are the binormals to that curve.

One then has:

THEOREM VI

In order for the central normals of a skew ruled surface to be the binormals of a skew curve, it is necessary and sufficient that the normal curvature and geodesic torsion of its line of striction will verify a second-degree algebraic relation of the form (6.3)*, and the single coefficient that enters into it will be a constant.*

If one replaces κ_{1n}, σ_{1g} with the values as functions of the fundamental invariants of a skew ruled surface κ_1, σ_1, φ_1 in the condition (6.3), which is, from Theorem VI, necessary and sufficient for that surface to enjoy the indicated property, then that condition will assume the form:

$$\beta_1(\kappa_{1n}^2 + \sigma_{1g}^2) + \kappa_1 \cos\varphi_1 - \sigma_1 \sin\varphi_1 = 0,$$

and it was given in that form by E. KRUPPA ([**4**], pp. 165).

Furthermore, from what was presented in the final remark in paragraph 1, the normal curvature κ_{1n} and the geodesic torsion σ_{1g} of the line of striction C_1 of the surface R_1 are equal to the geodesic curvature and geodesic torsion κ'_{1g}, σ'_{1g}, respectively, of that curve, when it is considered to be a curve that is traced on the surface R_n that is generated by the common central normals of the surfaces R, R_1. From (6.3), κ'_{1g}, σ'_{1g} must then verify the relation:

$$\beta_1(\kappa'^2_{1n} + \sigma'^2_{1g}) + \kappa'_{1g} = 0. \tag{6.4}$$

However, R_1 is a surface of the set (N) that is associated with R that is chosen at random. Consequently, its line of striction C_1 will be an orthogonal trajectory to the generators of the surface R_n that is chosen at random. In addition, from Theorem I, each orthogonal trajectory of the generators of the surface that is generated by the binormals to a skew curve will be a line of striction of the ∞^1 skew ruled surfaces that admit the binormals to that curve as their central normals. One can then state:

THEOREM VII

The geodesic curvature and geodesic torsion of each orthogonal trajectory of the generators of the surface that is generated by the binormals of a skew curve C verify a second-degree algebraic relation of the form (6.4), *and the single coefficient that enters into it is a constant whose absolute value is equal to the constant distance between the two points of that trajectory and the curve C that is situated along the same generator of the surface.*

If one takes into account the characteristic property of MANNHEIM curves that the principal normals of a curve of that type C_1 are binormals of another skew curve C in that theorem and the fact that the curve C_1 is an asymptotic line of the surface that is generated by its principal normals then one will deduce that *the geodesic curvature and geodesic torsion of each orthogonal trajectory to the generators of the surface that is generated by the principal normals of a MANNHEIM curve C_1 verify a second-degree algebraic relation of the form* (6.4), *and the single coefficient that figures in it will be equal in absolute value to the constant distance between the two points of that trajectory and the curve whose binormals are the principal normals of the curve C_1 that is situated on the same generator of the surface. That relation will reduce on the curve C_1 to the relation that is verified by the curvature and the torsion of that curve.*

7. – Now consider the developable surface R_d that is tangent to the surface considered R all along its line of striction C.

The generators of the surface R_d are the tangents to the surface R at the points of the curve C that are conjugate to the tangents to R at those same points, and as is known ([**2**], pp. 189), the generator of that surface that issues from the running point K (u) of C is parallel to the vector:

$$\bar{d} = \sigma\bar{e} + \kappa\bar{z}, \tag{7.1}$$

in which κ; σ are the curvature and torsion of R on its generator that issues from the point K.

If one lets ψ denote the angle $(\bar{t}, \bar{d})$, in which $\bar{t}$ is the unit vector (1.4) that is parallel to the tangent to C at K then, by virtue of (1.4) and (7.1), one will have:

$$\cos\,\psi = \frac{\kappa \sin\varphi + \sigma\cos\varphi}{\sqrt{\kappa^2 + \sigma^2}}.$$

One will quickly deduce from that formula, which will take on the form:

$$\cos\,\psi = \frac{\sigma_g}{\sqrt{\kappa_n^2 + \sigma_g^2}} \tag{7.2}$$

with the aid of (1.5), that in order for the line of striction of a skew ruled surface R to be an isogonal trajectory of the generators of the developable surface R_d that is tangent to R all along the curve C, it is necessary and sufficient that the normal curvature and the geodesic torsion of that curve must verify a relation of the form:

$$A\ \kappa_n + B\ \sigma_g = 0, \tag{7.3}$$

and the coefficients A, B that enter into it will be constants, at least one of which is non-zero. One can then distinguish three cases according to whether one has $A = 0$, $B \neq 0$, or $A \neq 0$, $B = 0$, or $A \neq 0$, $B \neq 0$.

If $A = 0$, $B \neq 0$ then the line of striction C of R will be a line of curvature, since one has $\sigma_g = 0$ at each point of C. In this case, the surface R_n that is generated by the central normals to R is developable, and the orthogonal trajectories to the generators of R_n are lines of curvature of that surface. Consequently, their geodesic torsion will be everywhere zero.

If $A \neq 0$, $B = 0$ then the line of striction C of R will be an asymptotic line of that surface, and this case was discussed in the preceding paragraph.

Finally, if $A \neq 0$, $B \neq 0$ then upon eliminating κ_n, σ_g from the relation (7.3) and the two relations (3.8) that are verified by the normal curvatures κ_n, κ_{1n} and the geodesic torsions σ_g, σ_{1g} of the line of striction C of R and the line of striction C_1 of a surface R_1 of the set (N) that is associated with R, resp., then one will deduce that the normal curvature κ_{1n} and the geodesic torsion σ_{1g} of the curve C_1 must verify the relation:

$$\beta_1 A(\kappa_{1n}^2 + \sigma_{1g}^2) + A\ \kappa_{1n} + B\ \sigma_{1g} = 0, \tag{7.4}$$

in which β_1 is a constant that is, from (2.8), equal in absolute to the constant distance between the two points on the curves C, C_1 that are situated along the same common central normal to the surface R, R_1, resp.

The central normals to the surface R are, at the same time, the normals to the developable surface R_d that is tangent to R along its line of striction C. Furthermore, from Theorem I, an isogonal trajectory of the generators of a developable surface is the line of striction of ∞^1 skew ruled surfaces that admit the normals to the surface at the points of that curve as central normals.

That being the case, one will deduce from the relation (7.4) that the normal curvature κ_n and the geodesic torsion σ_g of the line of striction of a skew ruled surface must verify a relations of the form:

$$A_1(\kappa_n^2 + \sigma_g^2) + B_1\kappa_n + C_1\,\sigma_g = 0, \tag{7.5}$$

and the coefficients A_1 , B_1 , C_1 that enter into it are constants such that at least the last two are non-zero when the central normals to the surface are the normals to a developable surface at the points of an isogonal (but not orthogonal) trajectory to its generators.

On the other hand, the condition (7.5) that is verified by the normal curvature and the geodesic torsion of the line of striction of a skew ruled surface R is sufficient for the central normals to the surface to be the normals to a developable surface at the points of an isogonal trajectory of its generators, because, with the aid of formulas (3.8), one will deduce from that relation, which will take on the form:

$$A\{\beta_1(\kappa_n^2 + \sigma_g^2) + \kappa_n\} + B\,\sigma_g = 0$$

when one sets $B_1 = A$, $A_1 = B_1\,\beta_1$, $C_1 = B\,B_1$, that the normal curvature and geodesic torsion of the line of striction of a surface of the set (N) that is associated with R will verify a relation of the form (7.3). That surface will coincide with R if $A_1 = 0$.

One can then state:

THEOREM VIII

In order for the central normals to a skew ruled surface to be the normals to a developable surface at the points of an isogonal (but not orthogonal) trajectory to its generators, it is necessary and sufficient that the normal curvature and the geodesic torsion of its line of striction should verify a second-degree algebraic relation of the form (7.5), *and that the coefficients that enter into it must be constants, at least the last two of which are non-zero.*

In addition, from the final remark in paragraph 1, the geodesic curvature and the geodesic torsion κ'_{1g} , σ'_{1g} of the line of striction C_1 of a surface R_1 of the set (N) that is associated with R, when it is considered to be a curve that is traced on the surface R_n that is generated by the central normals to R, must verify the relation:

$$A\beta_1(\kappa_n'^2 + \sigma_g'^2) + A\kappa'_{1g} + B\sigma'_{1g} = 0 \tag{7.6}$$

in the case envisioned, to which one will arrive upon replacing κ_{1n}, σ_{1g} with κ'_{1g} , σ'_{1g} , respectively, in the relation (7.4). The coefficients A, B that enter into the relation (7.6) are the coefficients of the relation (7.3), while β_1 is a constant whose absolute value is, from (2.8), equal to the constant distance between two points of the curve C_1 and the line of striction C of R that is situated along the same generator of the surface R_n .

That fact, when combined with the fact that, from Theorem I, each orthogonal trajectory of the generators of the surface R_n that is generated by the normals to a developable surface at the points of an isogonal trajectory of its generators is the line of striction of ∞^1 skew ruled surfaces that admit the generators of R_n as central normals, will permit us to formulate:

THEOREM IX

The geodesic curvature and geodesic torsion of each orthogonal trajectory to the generators of the surface that is generated by the normals to a developable surface at the points of an isogonal (but not orthogonal) trajectory C to its generators verify a second-degree algebraic relation of the form (7.6) *with coefficients that are constant on it. The first two of the coefficients A, B,* β_1 *that enter into that relation, both of which are non-zero, are the same on all of those curves, while the third one will vary from one to another. That relation reduces along the curve C to the linear and homogeneous relation that is verified by the geodesic curvature and geodesic torsion of that curve.*

If one takes into account the fact that the principal normals to a cylindrical helix are the normals along that curve to the cylindrical surface whose helix is a geodesic then one can deduce from the theorem that *the geodesic curvature and geodesic torsion of each orthogonal trajectory to the generators of the surface that is generated by the principal normals to a cylindrical helix are coupled by a second-degree algebraic relation of the form* (7.6)*, and the coefficients A, B,* β *that enter into its are constant on it. The coefficient* β *varies from one of those curves to another, while the first two coefficients are invariable; the latter are the coefficients that enter into the linear and homogeneous relation that is verified by the curvature and torsion of the helix.*

8. – Let R_1 be a surface of the set (N) that is associated with the surface considered R that corresponds to the values β_1, ω_1 of the parameters β, ω, resp., that enter into equation (2.7) of that set.

The normal curvature κ_{1n} and the geodesic torsion σ_{1g} of the line of striction C_1 of R_1 are the functions (3.8) of the normal curvature κ_n and the geodesic torsion σ_g of the line of striction C of R.

Upon eliminating β_1 from the two relations (3.8), one will easily arrive at the relation:

$$\frac{\sigma_{1g}}{\kappa_{1n}^2+\sigma_{1g}^2}=\frac{\sigma_g}{\kappa_n^2+\sigma_g^2}, \tag{8.1}$$

which shows that the ratio $\dfrac{\kappa_n^2+\sigma_g^2}{\sigma_g}$ is the same for all surfaces of the set (N) that is associated with R.

The value of that ratio at the running point K (u) of the line of striction C of R is equal to the distribution parameter p_n of the surface R_n , along its generator that issues from the point K (u) of the curve C, which is:

$$p_n = \frac{(\dot{\bar{\rho}} \wedge \bar{n}) \times \dot{\bar{n}}}{\dot{\bar{n}}^2}.$$

If one replaces $\dot{\bar{\rho}}$, $\dot{\bar{n}}$ with their values from (1.4) and (1.2) in that formula then it will take on the form:

$$p_n = \frac{\kappa \sin\varphi + \sigma\cos\varphi}{\kappa^2 + \sigma^2},$$

or finally, with the aid of (1.5)

$$p_n = \frac{\sigma_g}{\kappa_n^2 + \sigma_g^2}. \tag{8.2}$$

One immediately deduces from that formula that in order for the surface R_n that is generated by the central normals of a ruled surface to be a surface with a constant distribution parameter, it is necessary and sufficient that one must have:

$$a\,(\kappa_n^2 + \sigma_g^2) - \sigma_g = 0, \tag{8.3}$$

in which a is a constant.

One will then have:

THEOREM X

In order for the surface that is generated by the central normals of a skew ruled surface R to be a surface with a constant distribution parameter, it is necessary and sufficient that the normal curvature and the geodesic torsion of the line of striction of R should verify a second-degree algebraic relation of the form (8.3)*, and the single coefficient that enters into it should be equal to the constant distribution parameter of that surface.*

Furthermore, if one takes into account the final remark in paragraph 1 then one will deduce immediately from the two relations (8.1) and (8.3) that the geodesic curvature and geodesic torsion κ'_{1g}, σ'_{1g} of the line of striction C_1 of a surface R_1 of the set (N) that is associated with R, when considered to be a curve that is traced on the surface R_n that is generated by the central normals of R, will verify the relation:

$$a\,(\kappa'^2_{1n} + \sigma'^2_{1g}) - \sigma'_{1g} = 0, \tag{8.4}$$

in which a is a constant when R_n is a surface with a constant distribution parameter that is equal to a.

That realization, combined with the fact that, from Theorem I, the generators of a ruled surface with a constant distribution parameter are the common central normals to ∞^2 skew ruled surfaces, will permit one to formulate:

THEOREM XI

The geodesics curvature and the geodesic torsion of each orthogonal trajectory of the generators of a ruled surface with a constant distribution parameter verify a second-degree algebraic relation of the form (8.4), *and the single coefficient that enters into it is equal to the constant distribution parameter of the surface.*

9. – Finally, suppose that the line of striction C of the surface R is a curve on that surface whose normal curvature and geodesic torsion are both constants:

(9.1) $$k_n = c_1 , \qquad \sigma_g = c_2 ,$$

in which c_1 , c_2 are constants, and at least the second one is non-zero.

In that case, from what was said in paragraph 7, the curve C will be an isogonal trajectory of the generators of the developable surface R_d that is tangent to R along that curve.

In addition, from what was said in paragraph 6, the central normals to R are binormals of a skew curve C_1 : viz., the line of striction of the surface R_1 of the set (N) that is associated with R, which corresponds to the values $\beta_1 = \dfrac{c_2}{c_1^2 + c_2^2}$, ω_1 of the parameters β, ω, resp., that figure in equation (2.7) of that ensemble.

Moreover, in that case, thanks to (9.1), the surface R_n that is generated by the central normals to R will be a surface with a constant distribution parameter:

(9.2) $$p_n = \frac{c_2}{c_1^2 + c_2^2} \neq 0.$$

That being the case, from Theorem XI, the geodesic curvature and geodesic torsion κ'_{1g} , σ'_{1g} of the curve C_1 whose binormals are the central normals to R, when considered to be a curve on the surface R_n , must verify the relation:

(9.3) $$p_n\,(\kappa'^2_{1g} + \sigma'^2_{1g}) - \sigma'_{1g} = 0.$$

However, the curve C_1 is necessarily a geodesic on the surface R_n that is generated by its binormals. One will then have $\kappa'_{1g} = 0$, $\sigma'_{1g} = \sigma_1$, in which σ_1 is the torsion of that curve, and one will deduce from the relation (9.3) that one must have either $\sigma'_{1g} = \sigma_1 = 0$ or $\sigma'_{1g} = \sigma_1 = 1/p_n \neq 0$.

If $\sigma_1 = 0$ then the curve C_1 will be a plane curve, and the surface R_n that is generated by its binormals will be a cylindrical surface, which is excluded by the hypothesis that was made for the distribution parameter of R.

Hence, in the case envisioned, the central normals to the surface R are the binormals of a skew curve with constant torsion:

$$\sigma_1 = \frac{1}{p_n} = \frac{c_1^2 + c_2^2}{c_2}. \tag{9.4}$$

On the other hand, the surface R_b that is generated by the binormals of a skew curve C with constant torsion c is (as one knows, [**1**], pp. 104) a skew surface with a constant distribution parameter $p_n = 1 / c$.

Now, if one chooses the curve C on the surface R_b to be the director curve and one takes into account the fact that the geodesic curvature and geodesic torsion of C, when considered to be a curve that is traced on R_b , are:

$$\kappa'_{1g} = 0, \qquad \sigma'_{1g} = \sigma_1 = c, \tag{9.5}$$

since C is a geodesic of R_b , with the aid of (3.9) and (9.5), one will get the following expressions for the geodesic curvature and geodesic torsion κ'_g, σ'_g, resp., of an orthogonal trajectory of the generators of the surface R_b :

$$\kappa'_g = -\frac{\beta'\sigma^2}{1+\beta'^2c^2}, \qquad \sigma'_g = \frac{c}{1+\beta'^2c^2}, \tag{9.6}$$

which shows that κ'_g, σ'_g are constants on each orthogonal trajectory of the generators of the surface R_b , and the coefficient β' that enters into formulas (9.6) is constant on each of those curves.

The preceding considerations permit one to state:

THEOREM XII

The geodesic curvature and geodesic torsion of the orthogonal trajectories of the generators of a skew ruled surface are not constant on each of the curves, as in the case where the generators of the surface are binormals to a skew curve with constant torsion.

One should note that, from a theorem of X. ANTOMANI ([**4**], pp. 168), *if an orthogonal trajectory of the generators of the surface that is generated by the binormals to a skew curve C_1 is a curve on the surface with constant geodesic curvature then all of the orthogonal trajectories of the generators of the surface will necessarily be curves with constant geodesic curvature.*

In that case, the curve C_1 will necessarily be a curve with constant torsion.

Indeed, C_1 is an orthogonal trajectory of the generators of the surface R_b that is generated by its binormals, and at the same time, a geodesic of that surface. Upon letting κ'_{1g}, σ'_{1g} denote the geodesic curvature and geodesic torsion of C_1, one will have $\kappa'_{1g} = 0$, $\sigma'_{1g} = \sigma_1$, in which σ_1 is the torsion of that curve.

Now, if another orthogonal trajectory C' of the generators of R_b is a curve with constant geodesic curvature: viz., $\kappa'_g = c' \neq 0$, and one chooses the curve C_1 on the surface R_b to be the director curve then, from the first formula in (3.9), one will have the following expression for the geodesic curvature κ'_g of the curve C':

$$\kappa'_g = -\frac{\beta'\sigma_1^2}{1+\beta'^2\sigma_1^2} = \text{const.},$$

which proves that C_1 must be a curve with constant torsion in the case envisioned, since β' is a constant on the curve C'.

It results from that fact, when combined with Theorem XII, that if an orthogonal trajectory of the generators of the surface that is generated by the binormals of a skew curve is a curve on the surface that has constant geodesic curvature ($\neq 0$) *then the geodesic torsion and geodesic curvature of the orthogonal trajectories of the generators of the surface will both be constant along each of those curves.*

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