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On the equations of electromagnetic induction

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When an electromagnetic induction is given on the space-time manifold, that will imply the existence of an associated metric and an associated electromagnetic field.

1. Consider a domain D in space-time V_4 that is referred to local coordinates x^α and endowed with the world-metric ([1]):

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\alpha. \tag{1.1}$$

An electromagnetic induction is defined on the domain D when there exist two fields of antisymmetric tensors $H_{\alpha\beta}$, $G_{\alpha\beta}$ on D and two matrices $(\varepsilon^\alpha_\beta)$, (μ^α_β), which are called *induction matrices*, such that:

$$G_{\alpha\beta}\, e^\alpha = \varepsilon^\rho_\beta\, H_{\alpha\rho}\, e^\alpha, \qquad \mu^\rho_\beta\, \dot{G}_{\alpha\rho}\, e^\alpha = \dot{H}_{\alpha\beta}\, e^\alpha, \tag{1.2}$$

in which e^α denotes a proper vector that is oriented in time and *common to the two induction matrices*, and $\dot{G}_{\alpha\rho}$, $\dot{H}_{\alpha\beta}$ are the adjoint tensors to $H_{\alpha\beta}$, $G_{\alpha\beta}$ in the sense of the Riemannian metric (1.1). One supposes that the inductions thus-defined satisfy the Maxwell equations:

$$\nabla_\alpha\, \dot{H}^{\alpha\beta} = 0, \qquad \nabla_\alpha\, G^{\alpha\beta} = J^\beta. \tag{1.3}$$

Here, we shall confine oourselves to the case in which the medium that occupies D is isotropic; i.e., the one in which the induction matrices represent two homotheties. We will then have $\varepsilon^\alpha_\beta = \varepsilon\,\delta^\alpha_\beta$, $\mu^\alpha_\beta = \mu\,\delta^\alpha_\beta$, in which ε and μ are scalars. The relations (1.2) will become $G_{\alpha\beta}\, e^\alpha = \varepsilon\, H_{\alpha\beta}\, e^\alpha$, $\mu\dot{G}_{\alpha\beta} e^\alpha = \dot{H}_{\alpha\beta}\, e^\alpha$, and the matrices $(G_{\alpha\beta} - \varepsilon\, H_{\alpha\beta})$, $(\mu\dot{G}_{\alpha\beta} - \dot{H}_{\alpha\beta})$ will have rank 2. It that were true then the integration by parts of Maxwell's equations would yield the electrodynamical tensor:

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([1]) Greek indices vary from 0 to 3, and Latin indices vary from 1 to 3. $\partial_\lambda = \partial / \partial x^\lambda$.

(1.4) $$\tau_{\alpha\beta} = \tfrac{1}{4} g_{\alpha\beta}\,(G^{\rho\sigma} H_{\rho\sigma}) - G_{\rho\alpha} H^{\rho}{}_{\beta}\,.$$

That tensor is not symmetric. Upon adding an interaction term to it, one will get the symmetric tensor:

(1.5) $$t_{\alpha\beta} = \tau_{\alpha\beta} - (1 - \varepsilon\mu)\ \tau_{\alpha\rho}\, e^{\rho}\, e_{\beta}\,.$$

2. An impulse-energy tensor defines a matter-induction matrix in the form of:

(2.1) $$T_{\alpha\beta} = \rho\, u_{\alpha}\, u_{\beta} + t_{\alpha\beta}\,,$$

in which u_α is the unit velocity vector that is associated with the matter, and **u** is defined differently from **e** ([2]). The relativistic equations of induction are then composed of Maxwell's equations (1.3) and the Einstein equations $S_{\alpha\beta} = \chi\, T_{\alpha\beta}$. If ε, μ, e^α are assumed to be given then the field variables will consist of the $g_{\alpha\beta}$, $H_{\alpha\beta}$, J_α, u^α, ρ. By means of a supplementary hypothesis on the current vector J^α (for example, $J^\alpha = \delta\, u^\alpha$), one can prove that under convenient differentiability hypotheses, the system of Maxwell-Einstein equations will admit a well-defined solution if the hypersurface S that carries the Cauchy data ($g_{\alpha\beta}$, $\partial_\lambda\, g_{\alpha\beta}$, $H_{\alpha\beta}$) is not exceptional. That study exhibits the existence of two types of characteristic manifolds that are coupled with Einstein's equations and the Maxwell equations, respectively:

(2.2) $$\Delta_1 f \equiv g^{\alpha\beta}\, \partial_\alpha f\, \partial_\beta f = 0,$$

(2.3) $$\bar{\Delta}_1 f \equiv \bar{g}^{\alpha\beta}\, \partial_\alpha f\, \partial_\beta f = 0, \qquad \text{in which} \qquad \bar{g}^{\alpha\beta} = g^{\alpha\beta} - (1 - \varepsilon\mu)\, e^{\alpha} e^{\beta}\,.$$

Two cones will then be found to be defined at each point x: The characteristic cone C_x of the Einstein equations, which coincides with the elementary cone in space-time, and the characteristic cone $\bar{C}_x$ of Maxwell's equations, which is generally distinct from the first one. If $\varepsilon\mu > 1$ then $\bar{C}_x$ will be interior to C_x. Those two cones will coincide for $\varepsilon\mu = 1$. One will note that, in the language of the theory of wave propagation, $(\varepsilon\mu)^{-1/2}$ represents the speed of propagation of wave fronts (2.3) of the electromagnetic induction.

The second cone $\bar{C}_x$ defines the associated metric:

(2.4) $$d\bar{s}^2 = \bar{g}_{\alpha\beta}\, dx^\alpha\, dx^\alpha \equiv \left[g_{\alpha\beta} - \left(1 - \frac{1}{\varepsilon\mu}\right) e_\alpha\, e_\beta \right]\ dx^\alpha\, dx^\alpha,$$

in which $\bar{g}_{\alpha\beta}$ is the tensor that is conjugate to the tensor $\bar{g}^{\alpha\beta}$. One lets $\bar{V}_4$ denote the Riemannian manifold that is defined by the differentiable manifold that carries space-time V_4 and is endowed with the associated metric. In what follows, the overbarred quantities will be defined in terms of the associated metric, while the unbarred quantities will be defined in terms of the world-metric.

([2]) It was assumed that **e** agreed with **u** in a prior study [C. R. Acad. Sci. Paris **245** (1957), pp. 1782].

If $\bar{V}_4$ is referred to the same local coordinates x^α as V_4 the one can prove that if $\varepsilon\mu$ is *constant* in D then the Maxwell equations that relate to the induction can be expressed directly in terms of the associated metric in the form:

$$\bar{\nabla}_\alpha \overset{\bullet}{\bar{H}}{}^{\alpha\beta} = 0, \qquad \bar{\nabla}_\alpha \bar{G}^{\alpha\beta} = \bar{J}^\beta, \tag{2.5}$$

in which $\bar{H}_{\alpha\beta} = H_{\alpha\beta}$, $\bar{G}^{\alpha\beta} = (1/\mu)\bar{H}^{\alpha\beta} = G^{\alpha\beta}$, $\bar{J}^\beta = J^\beta$. If $\varepsilon\mu$ is variable then one can generalize those equations. An integration by parts of equations (2.5) will yield the symmetric tensor:

$$\bar{\tau}_{\alpha\beta} = \tfrac{1}{4}\bar{g}_{\alpha\beta}(\bar{G}^{\rho\sigma}\bar{H}_{\rho\sigma}) - \bar{G}_{\rho\alpha}\bar{H}^{\rho}{}_{\beta}, \tag{2.6}$$

which is such that in contravariant components:

$$\bar{\tau}^{\alpha\beta} = \tau^{\alpha\beta} - (1 - \varepsilon\mu)\ \tau^{\alpha\rho}\, e_\rho\, e^\beta = t^{\alpha\beta}, \tag{2.7}$$

which justifies the introduction of the symmetric tensor (1.5).

3. Let $(\mathbf{e}_\alpha)$ be an orthonormal frame in V_4 whose $\mathbf{e}_0$ vector is oriented in time and coincides with $\mathbf{e}$. It defines a system of linearly-independent local Pfaff forms ω^α such that:

$$ds^2 = (\omega^0)^2 - (\omega^1)^2 - (\omega^2)^2 - (\omega^3)^2, \qquad d\bar{s}^2 = \left(\frac{\omega^0}{\sqrt{\varepsilon\mu}}\right)^2 - (\omega^1)^2 - (\omega^2)^2 - (\omega^3)^2. \tag{3.1}$$

Upon setting $\omega^0/\sqrt{\varepsilon\mu} = \bar{\omega}^0$, $\omega^i = \bar{\omega}^i$, one will introduce the orthonormal frame $(\bar{\mathbf{e}}_\alpha)$ that is canonically associated with the frame $(\mathbf{e}_\alpha)$, and with respect to which, one will have:

$$d\bar{s}^2 = (\bar{\omega}^0)^2 - (\bar{\omega}^1)^2 - (\bar{\omega}^2)^2 - (\bar{\omega}^3)^2. \tag{3.2}$$

One should remark that the automorphism of the tangent space to V_4 at x that transforms to $(\mathbf{e}_\alpha)$ corresponds to a choice of local coordinates x'^α for the associated manifold $\bar{V}_4$, and thus, to a choice of units such that $(\varepsilon\mu)^{-1/2}$ will have the value unity.

Upon agreeing that the primed quantities must be taken with their values referred to the frame $(\bar{\mathbf{e}}_\alpha)$, one will have:

$$\bar{H}'_{0i} = \sqrt{\mu}\left(\sqrt{\varepsilon}\,E_i\right),\ \overset{*}{\bar{H}}{}'_{0i} = \sqrt{\mu}\left(\sqrt{\mu}\,H_i\right),\ \bar{G}'_{0i} = \frac{1}{\sqrt{\mu}}\left(\sqrt{\varepsilon}\,E_i\right),\ \overset{*}{\bar{G}}{}'_{0i} = \frac{1}{\sqrt{\mu}}\left(\sqrt{\mu}\,H_i\right), \tag{3.3}$$

in which E_i, B_i, D_i, H_i are the vectors that are induced by the tensors $H_{\alpha\beta}$, $G_{\alpha\beta}$ on the tri-plane $(\mathbf{e}_i)$ that is associated with $\mathbf{e}_0$ ($D_i = \varepsilon E_i$, $B_i = \mu H_i$). Let $\bar{F}_{\alpha\beta}$ be the tensor whose vectors that are induced on the tri-plane $(\bar{\mathbf{e}}_i)$ that is associated with $\bar{\mathbf{e}}_0$ are $\bar{E}'_i = \sqrt{\varepsilon} E_i$ and $\bar{H}'_i = \sqrt{\mu} H_i$. By definition, it will be called the *electromagnetic field that is associated with* the given induction. Conversely, one verifies that an electromagnetic field that is defined in the associated frame corresponds in space-time to an induction that is deduced from the field by a change of frame. The associated electromagnetic field satisfies the equations:

$$\bar{\nabla}_\alpha \overset{*}{\bar{F}}{}^{\alpha\beta} = 0, \quad \bar{\nabla}_\alpha \bar{F}^{\alpha\beta} = \bar{J}^\beta . \tag{3.4}$$

It corresponds to the symmetric tensor:

$$\underset{(a)}{\bar{\tau}}{}_{\alpha\beta} = \tfrac{1}{4} \bar{g}_{\alpha\beta} (\bar{F}^{\rho\sigma} \bar{F}_{\rho\sigma}) - \bar{F}_{\rho\alpha} \bar{F}^{\rho}{}_{\beta} \tag{3.5}$$

for which the values of the components in the associated frame will coincide with those of $\bar{\tau}_{\alpha\beta}$.

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