"Étude algébrique du tenseur électromagnétique en présence d'induction," C. R. Acad. Sci. **246** (1958), 3018-3020.

Algebraic study of the electromagnetic tensor in the presence of induction

Note by **LOUIS MARIOT** and **PHAM MAU QUAN**

Presented by Georges Darmois

Translated by D. H. Delphenich

One generalizes an algebraic study of the electromagnetic energy tensor in the case where there exists an induction. In particular, one defines singular induction.

1. If space-time is referred to local coordinates x^α and endowed with a world-metric:

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{1.1}$$

then consider a medium that occupies a domain D in which there exists an electromagnetic induction $(H_{\alpha\beta}, G_{\alpha\beta})$ [1] that satisfies the constraint equations:

$$G_{\alpha\beta}\, u^\alpha = \varepsilon\, H_{\alpha\rho}\, u^\alpha, \qquad \mu\, \overset{\bullet}{G}_{\alpha\rho} u^\alpha = \overset{\bullet}{H}_{\alpha\beta}\, u^\alpha, \tag{1.2}$$

in which u^α is the unit velocity vector of the medium. One sets:

$$G_{\rho\alpha}\, u^\rho = D_\alpha, \qquad \overset{\bullet}{G}_{\rho\alpha} u^\rho = H_\alpha, \qquad H_{\rho\alpha}\, u^\rho = E_\alpha, \qquad \overset{\bullet}{H}_{\rho\alpha} u^\rho = B_\alpha, \tag{1.3}$$

and let $t_{\alpha\beta}$ be the symmetric electromagnetic energy tensor that is associated with the induction considered:

$$t_{\alpha\beta} = \tau_{\alpha\beta} - (1 - \varepsilon\mu)\, \tau_{\alpha\beta}\, u^\rho\, u_\beta, \qquad \tau_{\alpha\beta} = \tfrac{1}{4}\, g_{\alpha\beta}\, (G_{\rho\sigma}\, H^{\rho\sigma}) - G_{\rho\alpha}\, H^\rho{}_\beta\,. \tag{1.4}$$

2. The introduction of the associated electromagnetic field [1] in the manifold $\overline{V}_4$ that is defined by the differentiable manifold that carries V_4 and is endowed with the associated metric:

---

[1] C. R. Acad. Sci. Paris **246** (1958), pp. 707.

(2.1) $$d\bar{s}^2 = \bar{g}_{\alpha\beta}\, dx^\alpha\, dx^\beta, \qquad \bar{g}_{\alpha\beta} = g_{\alpha\beta} - \left(1 - \frac{1}{\varepsilon\mu}\right) u_\alpha\, u_\beta$$

will lead us to consider the case in which that field is singular in the sense of A. Lichnerowicz [2]; i.e., such that:

(2.2) $$\bar{\Psi} = \tfrac{1}{2}\bar{F}_{\alpha\beta}\,\bar{F}^{\alpha\beta} = 0, \qquad \bar{\Phi} = \tfrac{1}{2}\bar{F}_{\alpha\beta}\,\overset{*}{\bar{F}}{}^{\alpha\beta} = 0.$$

It that were true then the electrodynamical tensor $\underset{(a)}{\bar{\tau}}{}_{\alpha\beta} = \tfrac{1}{4}\bar{g}_{\alpha\beta}\,\bar{F}_{\rho\sigma}\,\bar{F}^{\rho\sigma} - \bar{F}_{\rho\alpha}\,\bar{F}^{\rho}{}_{\beta}$ could be put into the tensorial form:

(2.3) $$\underset{(a)}{\bar{\tau}}{}_{\alpha\beta} = \bar{\xi}^2\,\bar{l}_\alpha\,\bar{l}_\beta$$

in which $\bar{\xi} = \left|\sqrt{\varepsilon}\,\mathbf{E}\right| = \left|\sqrt{\mu}\,\mathbf{H}\right|$, and in which $\bar{l}_\alpha$ is an isotropic vector in $\bar{V}_4$. Equations (2.2) imply that for the induction one will have:

(2.4) $$\tfrac{1}{2}G_{\alpha\beta}\,H^{\alpha\beta} = 0, \qquad \tfrac{1}{2}G^{\alpha\beta}\,\overset{*}{H}{}^{\alpha\beta} = 0.$$

The electric field and induction vectors **E**, **D** are orthogonal to the magnetic field and induction vectors **B**, **H**. Conversely, (2.2) is a consequence of (2.4). By definition, such an induction will be called *singular.*

3. We shall study the proper values and directions of $t_{\alpha\beta}$ relative to $g_{\alpha\beta}$. In order to do that, we consider the values of the components of $t_{\alpha\beta}$ in a proper frame; i.e., an orthonormal frame ($\mathbf{e}_\alpha$) whose $\mathbf{e}_0$ vector coincides with **u**. We set:

(3.1) $$X = \sqrt{\varepsilon}\,E_1, \quad Y = \sqrt{\varepsilon}\,E_2, \quad Z = \sqrt{\varepsilon}\,E_3, \quad L = \sqrt{\mu}\,H_1, \quad M = \sqrt{\mu}\,H_2, \quad N = \sqrt{\mu}\,H_3.$$

It is easy to choose the proper frame that is called *simple* such that $\mathbf{e}_1$, $\mathbf{e}_2$ are found in the 2-plane that is defined by the vectors **E**, **D**, **B**, **H** and whose directions will coincide with the proper directions of (t_{ab}) (a, b = 1, 2). One will then have:

(3.2) $$Z = N = 0, \qquad XY + LM = 0.$$

In that frame, $t_{\alpha\beta}$ will have the components:

([2]) A. LICHNEROWICZ, *Théories relativistes de la gravitation et de l'électromagnetisme*, MASSON, Paris, 1955; L. MARIOT, *Thèse*, Paris, 1957.

(3.3)
$$(t_{\alpha\beta}) = \begin{pmatrix} \varepsilon\mu\dfrac{\zeta^2+\eta^2}{2} & 0 & 0 & \sqrt{\varepsilon\mu}\,\zeta\,\eta \\ 0 & -\dfrac{\zeta^2-\eta^2}{2} & 0 & 0 \\ 0 & 0 & \dfrac{\zeta^2-\eta^2}{2} & 0 \\ \sqrt{\varepsilon\mu}\,\xi\,\eta & 0 & 0 & \dfrac{\zeta^2+\eta^2}{2} \end{pmatrix},$$

in which:

(3.4)
$$\xi^2 = X^2 + L^2, \qquad \eta^2 = Y^2 + M^2.$$

In the regular case ($\zeta^2 - \eta^2 \neq 0$), $t_{\alpha\beta}$ will admit the four proper values:

$$\frac{s_0}{s_2} = (\varepsilon\mu - 1)\,\frac{\xi^2+\eta^2}{4} \pm \sqrt{(\varepsilon\mu-1)^2\left(\frac{\xi^2+\eta^2}{2}\right)^2 + 4\varepsilon\mu\left(\frac{\xi^2-\eta^2}{2}\right)^2}\,, \quad s_1 = -\,s_2 = \tfrac{1}{2}(\xi^2 - \eta^2).$$

One recovers the well-known results for $\varepsilon\mu = 1$.

4. If $\xi^2 - \eta^2 = 0$ then the induction will be singular, and $\xi = |\sqrt{\xi}\,\mathbf{E}| = |\sqrt{\mu}\,\mathbf{H}|$. The four proper values will then be: $s_0 = (\varepsilon\mu - 1)\,\xi^2$, $s_1 = s_2 = s_3 = 0$.

If $\varepsilon\mu > 1$, which corresponds to the study of the electromagnetic field in matter, then s_0 will be positive. The proper vectors **V** that are associated with the triple proper value of zero are found in the space-like 3-plane whose equation is $\sqrt{\varepsilon\mu}\,V^0 + V^3 = 0$, while the proper vector **W** that is associated with the positive proper value will be oriented in time and have the well-defined direction:

$$\sqrt{\varepsilon\mu}\,W^3 + W^0 = 0, \quad W^1 = W^2 = 0,$$

i.e., the direction of the vector $\boldsymbol{l} = \sqrt{\varepsilon\mu}\,\mathbf{u} + \mathbf{e}_3$.

In that case, $t_{\alpha\beta}$ will translate into the tensorial equation:

$$t_{\alpha\beta} = \xi^2\, l_\alpha\, l_\beta\,,$$

or, upon normalizing:

$$t_{\alpha\beta} = (\varepsilon\mu - 1)\,\xi^2\,\lambda_\alpha\,\lambda_\beta\,.$$

We point out that the direction of $\boldsymbol{l}$ or $\boldsymbol{\lambda}$ belongs to the cone $\bar{C}_x$ ($d\bar{s}^2 = 0$), and that it is orthogonal to **E**, **D**, **B**, **H**.

(Facultés des Sciences, Besançon, Dijon)