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Electromagnetic inductions in a relativistic anisotropic medium

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The anisotropy of a medium is translated with the aid three automorphisms ε, μ, σ of the tangent vector space at each point x of the space-time manifold V_4, which represent the dielectric strength, the magnetic permeability, and the electric condition of the medium, resp. The Maxwell equations admit a triple system of characteristic manifolds that are each tangent to a cone of order two.

1. Let D be a domain in space-time V_4 that is referred to a local coordinate system ([1]) (x^α). One lets T_x denote the tangent vector space to V_4 at x. Let $ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ be the world-metric, and let u^α be the unit velocity vector that is attached to each point x of the medium that is occupied by D. One recalls that a proper frame at the point x is an orthonormal frame whose basic vector $\mathbf{V}^{(0)}$ that is oriented in time and coincides with the vector $\mathbf{u}$¸ and the other three vectors $\mathbf{V}^{(i)}$, which are oriented in space define the tri-plane Π_x that is orthogonal to $\mathbf{u}$ and which one calls the *space associated* with the time direction $\mathbf{u}$.

The electromagnetic field is defined by the given on V_4 of two antisymmetric tensor fields of order 2, $H_{\alpha\beta}$ and $G_{\alpha\beta}$. The electric and magnetic induction and field vectors $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{B}$, $\mathbf{H}$, resp., are defined by ([2]):

$$D_\alpha = G_{\rho\alpha}\, u^\rho, \qquad E_\alpha = H_{\rho\alpha}\, u^\rho, \qquad B_\alpha = \overset{*}{H}_{\rho\alpha}\, u^\rho, \qquad H_\alpha = \overset{*}{G}_{\rho\alpha}\, u^\rho, \tag{1.1}$$

in which one sets:

$$\overset{*}{H}_{\alpha\beta} = \tfrac{1}{2}\, \eta_{\alpha\nu\gamma\delta}\, H^{\gamma\delta}, \qquad \overset{*}{G}_{\alpha\beta} = \tfrac{1}{2}\, \eta_{\alpha\nu\gamma\delta}\, G^{\gamma\delta},$$

in which $\eta_{\alpha\nu\gamma\delta}$ denotes the completely-antisymmetric tensor that is attached to the volume element form on V_4.

(*) Session on 21 October 1957.
([1]) Cf., J. rat. Mech. Anal. **5**, no. 3 (1956), 473-538.
([2]) Greek indices vary from 0 to 3, while Latin ones vary from 1 to 3.

In order to express the relations between inductions and fields, we given each point x of V_4 two automorphisms $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ of T_x; i.e., two linear maps of T_x to itself:

$$\mathbf{D} = \boldsymbol{\varepsilon}\,\mathbf{E}, \qquad \mathbf{B} = \boldsymbol{\mu}\,\mathbf{H}. \tag{1.2}$$

We denote the representative matrices of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ by the same letters, and their elements $(\varepsilon^\alpha_\beta)$ and (μ^α_β) define the components of two mixed tensors on V_4 that they are associated with.

2. Since $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are automorphisms, the matrices $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ will be invertible. A given field will define a well-defined induction, and conversely, if one knows an induction then it will correspond to one and only one field that is its induction. One lets $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ denote the inverse transformations to $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$, resp., or their representative matrices.

One infers from (1.1) that:

$$E_\alpha u^\alpha = D_\alpha u^\alpha = H_\alpha u^\alpha = B_\alpha u^\alpha = 0. \tag{2.1}$$

The fields and induction vectors are orthogonal to the unit velocity vector u^α: viz., they are vectors in the tri-plane Π_x. The automorphisms $\boldsymbol{\varepsilon}, \boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\tau}$ must make any vector in the tri-plane Π_x correspond to a vector in that tri-plane Π_x. We suppose, moreover, that $\mathbf{u}$ is a proper vector of the corresponding matrices. It will then result that $\boldsymbol{\varepsilon}, \boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\tau}$ leave $\mathbf{u}$ and Π_x invariant. In a proper frame, those two conditions are equivalent to $\varepsilon^0_i = \varepsilon^i_0 = 0$, $\mu^0_i = \mu^i_0 = 0$, $\lambda^0_i = \lambda^i_0 = 0$, $\tau^0_i = \tau^i_0 = 0$. One will note that if the matrices $\boldsymbol{\varepsilon}, \boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\tau}$ are symmetric then one of the preceding conditions will imply the other one.

One will be led to set:

$$\varepsilon^\alpha_\beta = \varepsilon\,\delta^\alpha_\beta + e^\alpha_\beta, \qquad \mu^\alpha_\beta = \mu\,\delta^\alpha_\beta + m^\alpha_\beta, \tag{2.2}$$

$$\lambda^\alpha_\beta = \lambda\,\delta^\alpha_\beta + l^\alpha_\beta, \qquad \tau^\alpha_\beta = \tau\,\delta^\alpha_\beta + t^\alpha_\beta, \tag{2.3}$$

in which $\varepsilon, \mu, \lambda, \tau$ are scalars ($\lambda = 1 / \varepsilon$, $\tau = 1 / \mu$) and e^α_β, l^α_β, m^α_β, t^α_β are such that:

$$e^\alpha_\beta u^\beta = l^\alpha_\beta u^\beta = m^\alpha_\beta u^\beta = t^\alpha_\beta u^\beta = 0.$$

The medium considered is called *isotropic* if the transformations $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are homotheties. One will then have ([1]):

$$\varepsilon^\alpha_\beta = \varepsilon\,\delta^\alpha_\beta, \qquad \mu^\alpha_\beta = \mu\,\delta^\alpha_\beta.$$

3. Starting from the constraint equations (1.2), which can be written explicitly as:

(3.1) $$G_{\alpha\beta}\, u^{\alpha} = \varepsilon^{\rho}_{\beta}\, H_{\alpha\rho}\, u^{\alpha}, \quad \overset{*}{H}_{\alpha\beta}\, u^{\alpha} = \mu^{\rho}_{\beta}\, \overset{*}{G}_{\alpha\beta}\, u^{\alpha}\,,$$

one can express the $G_{\alpha\beta}$ as functions of the $H_{\alpha\beta}$:

$$G_{\alpha\beta} = \tau H_{\alpha\beta} + (\tau - 1)(H_{\sigma\alpha}\, u^{\sigma}\, u_{\beta} - H_{\sigma\beta}\, u^{\sigma}\, u_{\alpha}) + (e^{\rho}_{\alpha} u_{\beta} - e^{\rho}_{\beta} u_{\alpha})\; u^{\sigma} H_{\rho\sigma}$$
$$+\ \varepsilon_{\alpha\beta\gamma\delta}\, u^{\gamma}\, \varepsilon^{\mu\nu\rho\sigma}\, u_{\nu}\, H_{\rho\sigma}\,,$$

in which $\varepsilon_{\alpha\beta\gamma\delta}$, $\varepsilon^{\mu\nu\rho\sigma}$ are the Kronecker symbols.

4. The electromagnetic field ($H_{\alpha\beta}$, $G_{\alpha\beta}$) satisfy Maxwell's equations:

(4.1) $$\mathcal{E}^{\delta} \equiv \tfrac{1}{2}\, \varepsilon^{\mu\nu\rho\sigma}\, \nabla_{\alpha} H_{\gamma\delta} = 0,$$

(4.2) $$\mathcal{D}_{\beta} \equiv g^{\alpha\rho}\, \nabla_{\alpha}\, G_{\rho\beta} = J_{\beta}\,,$$

in which the electric current vector J_{β} verifies the hypothesis that:

(4.3) $$J_{\beta} = \delta\, u_{\beta} + \sigma^{\alpha}_{\beta}\; H_{\rho\alpha}\, u^{\rho},$$

in which δ is a scalar that represents the proper density of the electric charge, and $(\sigma^{\alpha}_{\beta})$ is a new automorphism that introduces the electric conductivity of the medium, such that the conduction current $\Gamma_{\beta} = \sigma^{\alpha}_{\beta}\; H_{\rho\alpha}\, u^{\rho}$ will satisfy the generalized Ohm hypothesis $\mathbf{\Gamma} = \boldsymbol{\sigma}$ **E**. One remarks that equations (4.1) express the idea that there exists a local vector potential for $H_{\alpha\beta}$.

5. The $\varepsilon^{\alpha}_{\beta}$, μ^{α}_{β}, σ^{α}_{β} are given functions of x^{α}, the field variables are the $g_{\alpha\beta}$, $H_{\alpha\beta}$, which verify the Maxwell-Einstein equations that correspond to the schema considered in the domain D. The Einstein equations must determine the $g_{\alpha\beta}$ and u^{α}; in particular, consider the Maxwell equations, for which we study the Cauchy problem. We are given the values of ($H_{\alpha\beta}$) on the hypersurface S whose local equation is $x^0 = 0$, and we seek to determine the values of the oblique derivatives $\partial_0\, H_{\alpha\beta}$ on S. The Maxwell equations are equivalent to the set of two systems:

(5.1) $$\mathcal{E}^{k} \equiv \tfrac{1}{2}\, \eta^{0ijk}\, \partial_0\, H_{ij} + \Psi^{k} = 0,$$

(5.2) $$\mathcal{D}_{i} \equiv \frac{1}{\mu}\{(g^{00} - (1 - \varepsilon\mu)\, u^0\, u^0)\; \delta^{j}_{i} - \mu\, (e^{j}_{i} u^{0} - e^{0}_{i} u^{j})\, u^{0} +$$
$$+\ \frac{1}{\varepsilon}(g^{0\alpha} e^{0}_{\alpha}\, u^{j} - g^{j\alpha} e^{0}_{\alpha}\, u^{0})\, u_{i} + \frac{1}{2}\mu\, g^{0\lambda} \varepsilon_{\lambda i\gamma\delta} u^{\gamma} t^{\delta}_{\mu} \varepsilon^{\mu\nu 0 j} u_{\nu}\}\, \partial_0\, H_{0j} + \Phi_{i}$$

$$= \delta u_i + \sigma_i^{\alpha} H_{\rho\alpha} u^{\rho}$$

(in which Ψ^k and Φ_i are known quantities in S) and to the two identities that are verified on S :

(5.3) $$\mathcal{E}^0 \equiv \tfrac{1}{2}\, \eta^{ijk0}\, \partial_i H_{jk} = 0,$$

(5.4) $$\mathcal{D}^0 \equiv g^{0\beta}\, \mathcal{D}_\beta = \delta u^0 + g^{0\beta}\, \sigma_\beta^{\alpha} H_{\rho\alpha} u^{\rho},$$

in which $\mathcal{D}^0$ does not depends upon $\partial_0 H_{\alpha\beta}$. One notes that (5.3) expresses the idea that the tensor H_{ij} that is induced on S is locally derived from a potential vector.

If the hypersurface S is not exceptional then equation (5.4) will provide a value for δ, equations (5.1) will determine the values of $\partial_0 H_{ij}$, and equations (5.2) will determine those of $\partial_0 H_{0j}$ on S. The calculations can be performed by means of successive derivations.

The characteristic manifolds of Maxwell's equations are necessarily such that:

(5.5) $$\Omega \equiv \det\,(A_i^j) = 0,$$

in which the A_i^j represent the coefficients of $(1/\mu)\, \partial_0 H_{0j}$ in (5.2). An analysis of that equation will show that there generally exists a triple system of characteristic manifolds that are tangent to a second-order cone.

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