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Electromagnetic inductions in general relativity and Fermat's principle

PHAM MAU QUAN

Communicated by A. LICHNEROWICZ

Translated by D. H. Delphenich

Summary

Page

Introduction

This paper is dedicated to the study of electromagnetic inductions in general relativity. In it, one will find a proof of FERMAT's principle that is based upon the properties of the characteristics of MAXWELL's equations.

The electric and magnetic inductions are introduced as GORDON, WEYL, and LICHNEROWICZ did with the aid of two antisymmetric tensors of order 2: viz., the electric field-magnetic induction tensor $H_{\alpha\beta}$ and the electric induction-magnetic field tensor $G_{\alpha\beta}$. Some constraint equations express the linear relations between inductions and fields. It is the two tensors $H_{\alpha\beta}$, $G_{\alpha\beta}$ collectively that constitute the electromagnetic field. The energetic context in which they are introduced is that of a "charged, conducting fluid," which was studied previously (*) and in which one recalls the essential results of the integration of the field equations.

In the presence of inductions, the characteristic manifolds of MAXWELL's equations are not identical to the characteristic manifolds of EINSTEIN's equations. One knows that the latter are tangent to the elementary cones $\mathfrak{C}_x$ at each of their points. The characteristic cones $\bar{\mathfrak{C}}_x$ for the MAXWELL equations are interior to the elementary cones. The characteristic manifolds that are tangent to the elementary cones $\bar{\mathfrak{C}}_x$ are time-like, and the bicharacteristics are the null-length geodesics of the associated metric (**):

$$d\bar{s}^2 = \left[g_{\alpha\beta} - \left(1 - \frac{1}{\varepsilon\mu}\right) u_\alpha u_\beta \right] dx^\alpha \, dx^\beta ,$$

in which u^α denotes the unitary world-velocity vector, and the scalars ε, μ denote the dielectric strength and magnetic permeability of each point of the medium considered.

Now, the characteristic manifolds of MAXWELL's equations play the role of electromagnetic wave surfaces, and the bicharacteristics play that of the corresponding electromagnetic rays. One is naturally led to introduce the Riemannian manifold $\bar{\mathfrak{V}}_4$ that is defined by the differentiable manifold that carries the space-time and is endowed with the associated metric $d\bar{s}^2$.

MAXWELL's equations can be expressed in that manifold in which they take on a simple, symmetric form as they do in LORENTZ's theory of electrodynamics. The electromagnetic rays are null-length geodesics of $\bar{\mathfrak{V}}_4$. The geometric study of electromagnetic rays in space yields the statement of FERMAT's principle.

In order to carry out that study, we begin by generalizing to the case of a charged, conducting, perfect fluid the notion of permanent motions that are linked to the existence of a connected, one-parameter group of global isometries whose trajectories are time-like and leave no point of $\mathfrak{V}_4$ invariant (***). If the motion of the fluid considered is permanent then a group of isometries that are induced by the group of the spacetime are induced in $\bar{\mathfrak{V}}_4$. We will then be in a situation in which "LICHNEROWICZ's method of descent" applies (****). Upon projecting the null-length geodesics of $\bar{\mathfrak{V}}_4$ onto the quotient manifold of $\bar{\mathfrak{V}}_4$ by the equivalence relation that is defined by its group of

(*) "Etude électromagnetique et thermodynamique d'un fluide relativiste chargé," J. Rat. Mech. Anal. **5** (1956), 473-583.

(**) GORDON found this metric by taking an algebraic route.

(***) A. LICHNEROWICZ, *Théories relativistes de la gravitation et de l'électromagnétisme*, chap. IV, III, pp. 83-90. Maison, 1955.

(****) *ibid.*, Book II, chap. 1.

isometries, we will obtain a theorem that generalizes FERMAT's principle in relativity. That theorem will be valid for a medium in an arbitrary state of permanent motion. In particular, in the case of a MINKOWSKI spacetime without gravitation, it will yield a proof of the relativistic formula for the composition of velocities as a consequence.

Notations employed

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha}, \qquad \partial_{\alpha\beta} = \frac{\partial^2}{\partial x^\alpha \, \partial x^\beta} : \qquad \text{partial derivatives,}$$

$$\nabla_\alpha \qquad : \qquad \text{covariant derivative,}$$

$\alpha, \beta, \ldots$ (all Greek indices) $= 0, 1, 2, 3$,
$i, j, \ldots$ (all Latin indices) $= 1, 2, 3$.

I. Electromagnetic inductions. Integration of MAXWELL's equations.

1. MAXWELL's equations in the spacetime $\mathfrak{V}_4$.

Let a domain $\mathcal{D}$ in the spacetime $\mathfrak{V}_4$ of general relativity, which is endowed with the world-metric:

(1.1) $$ds^2 = g_{\alpha\beta} \, dx^\alpha \, dx^\beta \qquad (\alpha, \beta = 0, 1, 2, 3),$$

be occupied with a material distribution that is schematized in the form of a fluid-electromagnetic field. Let **u** denote the unitary velocity vector at each point x of $\mathcal{D}$. One calls a frame at x a *proper frame* when it is an orthonormal frame whose first vector $\mathbf{V}^{(0)}$ coincides with **u** and whose other three vectors $\mathbf{V}^{(i)}$ are space-like and normalized by the condition that:

$$g_{\alpha\beta} \, V^{(i)\alpha} \, V^{(i)\beta} = -1.$$

Electromagnetic phenomena are characterized by two antisymmetric tensor fields of order 2: viz., the electric field-magnetic induction tensor $H_{\alpha\beta}$ and the electric induction-magnetic field tensor $G_{\alpha\beta}$, whose components relate to a proper frame at the point x considered will have the values:

$$(H_{\alpha\beta}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \qquad (G_{\alpha\beta}) = \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & H_3 & -H_2 \\ -D_2 & -H_3 & 0 & H_1 \\ -D_3 & H_2 & -H_1 & 0 \end{pmatrix},$$

and verify the relations:

(1.2) $$G_{0i} = \varepsilon \, H_{0i} , \qquad H_{ij} = \mu \, G_{ij} \qquad (i, j = 1, 2, 3),$$

in which the scalars ε and μ represent the dielectric strength and the magnetic permeability, respectively, of the medium considered.

We introduce the adjoint tensors:

$$\overset{*}{H}{}^{\alpha\beta} = \tfrac{1}{2}\eta^{\alpha\beta\gamma\delta} H_{\gamma\delta}, \qquad \overset{*}{G}{}^{\alpha\beta} = \tfrac{1}{2}\eta^{\alpha\beta\gamma\delta} G_{\gamma\delta}, \tag{1.3}$$

in which $\eta^{\alpha\beta\gamma\delta}$ is the completely-antisymmetric tensor that is attached to the volume element form of $\mathfrak{B}_4$. The relations (1.2) can then be written in the invariant form:

$$G_{\alpha\beta}\, u^{\alpha} = \varepsilon\, H_{\alpha\beta}\, u^{\alpha}, \tag{1.4}$$

$$\mu\, \overset{*}{G}{}^{\alpha\beta}\, u^{\alpha} = \overset{*}{H}{}^{\alpha\beta}\, u^{\alpha}, \tag{1.5}$$

which is valid in an arbitrary local coordinate system. These relations are called the *constraint equations*.

The tensor fields $H_{\alpha\beta}$ and $G_{\alpha\beta}$ must satisfy MAXWELL's equations, which are written:

$$\nabla_{\alpha}\, \overset{*}{H}{}^{\alpha\beta} = 0, \tag{1.6}$$

$$\nabla_{\alpha}\, G^{\alpha\beta} = J^{\beta}, \tag{1.7}$$

in which J^{β} is the electric current vector. The first group of MAXWELL equations can be further written:

$$\tfrac{1}{2}\eta^{\alpha\beta\gamma\delta}\, \nabla_{\alpha} H_{\beta\gamma} = 0.$$

This expresses the idea that there exists a local vector field φ_{α} such that $H_{\alpha\beta}$ is its rotation; i.e:

$$H_{\alpha\beta} = \partial_{\alpha}\, \varphi_{\beta} - \partial_{\beta}\, \varphi_{\alpha}.$$

The evolution of the electromagnetic field is well-determined when one knows J^{β} – i.e., the distribution of the electricity. For a non-conducting medium, one can assume that **J** is collinear to the velocity vector:

$$J^{\alpha} = \delta\, u^{\alpha}.$$

δ is called the *proper density* of the electric charge, and the electric current is a *convection current*. More generally, one is led to make the hypothesis:

$$J^{\alpha} = \delta\, u^{\alpha} + \sigma\, u_{\rho}\, H^{\rho\alpha},$$

in which δ is again the proper density of electric charges and σ is a scalar that characterizes the electric conductivity of the medium. **J** then possesses a component that is collinear to **u** and a component $\Gamma^{\alpha} = \sigma\, u_{\rho}\, H^{\rho\alpha}$ that is orthogonal to **u**. The first one represents the convection current, and the second one represents the conduction current, which satisfies OHM's hypothesis.

Equations (1.4), (1.5), (1.6), (1.7) constitute the equations of electromagnetism in the presence of inductions in matter. *In vacuo*, one has the equality:

$$\varepsilon\,\mu = 1, \tag{1.8}$$

and MAXWELL's equations become:

$$\nabla_\alpha \overset{*}{H}{}^{\alpha\beta} = 0, \tag{1.9}$$

$$\nabla_\alpha G^{\alpha\beta} = 0, \tag{1.10}$$

while the constraint equations reduce to:

$$G_{\alpha\beta} = \frac{1}{\mu} H_{\alpha\beta} = \varepsilon\, H_{\alpha\beta}\,. \tag{1.11}$$

In what follows, we will let $\mathcal{E}$ and $\mathcal{D}$ denote the vectors that figure in the left-hand side of MAXWELL's equations, namely:

$$\mathcal{E}^\beta \equiv \nabla_\alpha \overset{*}{H}{}^{\alpha\beta}, \tag{1.12}$$

$$\mathcal{D}^\beta \equiv \nabla_\alpha G^{\alpha\beta}. \tag{1.13}$$

One proves that their divergences are zero:

$$\nabla_\alpha \mathcal{E}^\alpha = 0, \tag{1.14}$$

$$\nabla_\alpha \mathcal{D}^\alpha = 0. \tag{1.15}$$

These two equations are called the *conservation conditions* that relate to MAXWELL's equations; they express the conservation of electricity. Therefore, one infers from (1.7) and (1.15) that:

$$\nabla_\alpha J^\alpha = 0.$$

2. Expressing the $G_{\alpha\beta}$ as functions of the $H_{\alpha\beta}$.

The constraint equations (1.4) and (1.5) express the linear character of the relations between the inductions and the fields. They show that the two tensor fields $H_{\alpha\beta}$ and $G_{\alpha\beta}$ are not independent of each other. One can express the $G_{\alpha\beta}$ as functions of the $H_{\alpha\beta}$.

Indeed, by starting with (1.4), we can form the equality:

$$(G_{\alpha\beta}\, u_\gamma + G_{\beta\gamma}\, u_\alpha)\, u^\beta = \varepsilon\,(H_{\alpha\beta}\, u_\gamma + H_{\beta\gamma}\, u_\alpha)\, u^\beta. \tag{2.1}$$

On the other hand, (1.5) can be written in the following form:

$$G_{\alpha\beta}\,u_\gamma + G_{\beta\gamma}\,u_\alpha + G_{\gamma\alpha}\,u_\beta = \frac{1}{\mu}\,(H_{\alpha\beta}\,u_\gamma + H_{\beta\gamma}\,u_\alpha + H_{\gamma\alpha}\,u_\beta),$$

which is true for any group of given values for α, β, γ. By contracted multiplication of this relation with u^β and then subtraction of (2.1) from the equality thus obtained, we will get:

$$G_{\gamma\alpha} = \frac{1}{\mu}\,(H_{\alpha\beta}\,u_\gamma + H_{\beta\gamma}\,u_\alpha + H_{\gamma\alpha}\,u_\beta)\,u^\beta - \varepsilon\,(H_{\alpha\beta}\,u_\gamma + H_{\beta\gamma}\,u_\alpha)\,u^\beta.$$

We then deduce that:

(2.2) $$G_{\alpha\beta} = \frac{1}{\mu}\,H_{\alpha\beta} + \frac{1-\varepsilon\mu}{\mu}\,(H_{\sigma\alpha}\,u^\sigma\,u_\beta - H_{\sigma\beta}\,u^\sigma\,u_\alpha).$$

This is the desired relation. In contravariant components, one will have:

(2.3) $$G^{\alpha\beta} = \frac{1}{\mu}\,H^{\alpha\beta} + \frac{1-\varepsilon\mu}{\mu}\,(H^{\sigma\alpha}\,u_\sigma\,u^\beta - H^{\sigma\beta}\,u_\sigma\,u^\alpha).$$

3. The integration of the MAXWELL-EINSTEIN equations

The electromagnetic fields ($H_{\alpha\beta}$, $G_{\alpha\beta}$) and the metric $ds^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta$ are linked by the MAXWELL-EINSTEIN equations. If the medium is in motion, and the domain considered $\mathcal{D}_4$ is schematized in the form of a charged, conducting, perfect fluid then the EINSTEIN equations will be (*):

(3.1) $$S_{\alpha\beta} \equiv R_{\alpha\beta} - \tfrac{1}{2}R\,g_{\alpha\beta} = \chi\,T_{\alpha\beta},$$

$$T_{\alpha\beta} \equiv (\rho + p)\,u_\alpha\,u_\beta - p\,g_{\alpha\beta} - (u_\alpha\,q_\beta + u_\beta\,q_\alpha) + \tau_{\alpha\beta} - (1 - \varepsilon\mu)\;\tau_{\alpha\rho}\,u^\rho\,u_\beta\,,$$

$$\tau_{\alpha\beta} \equiv \tfrac{1}{4}\;g_{\alpha\beta}\,(G_{\rho\sigma}\,H^{\rho\sigma}) - G_{\rho\alpha}\,H^\rho{}_\sigma,$$

(3.2) $$q^\alpha = -\,\kappa\,\partial_\rho\vartheta\,(\,g^\rho_\alpha - u^\rho\,u_\alpha),$$

in which p is the pressure and ϑ is the temperature at each point of the fluid. MAXWELL's equations are:

(*) Cf., "Étude électromagnetique et thermodynamique d'un fluide relativiste chargé," J. Rat. Mech. Anal. **5** (1956), 473-583.

(3.3) $$\mathcal{E}^{\delta} = \tfrac{1}{2}\,\eta^{\alpha\beta\gamma\delta}\,\nabla_\alpha H_{\beta\gamma} = 0,$$

(3.4) $$\mathcal{D}_\delta = g^{\alpha\beta}\,\nabla_\alpha G_{\rho\beta} = \delta u_\beta + \sigma u^\alpha H_{\alpha\beta}\,.$$

One can add the following conservation equations to these equations:

(3.5) $$\nabla_\alpha T^{\alpha\beta} = 0,$$

(3.6) $$\nabla_\alpha q^\alpha = c\,\rho\,u^\alpha\,\partial_\alpha\vartheta - \frac{1}{\rho}\,u^\alpha\,\partial_\alpha\vartheta,$$

(3.7) $$\nabla_\alpha(\delta u^\alpha + \sigma u_\rho H^{\alpha\beta}) = 0.$$

The scalars κ, c, l, ε, μ, σ are assumed to be given: They characterize the fluid envisioned, which admits the following equation of state, moreover:

(3.8) $$\rho = \varphi(p,\ \vartheta).$$

The *field variables* are composed of the set:

$$\mathfrak{H}\,(g_{\alpha\beta},\ H_{\alpha\beta},\ \vartheta,\ u^\alpha,\ p,\ \delta),$$

where the vector u^α is normalized:

(3.9) $$g_{\alpha\beta}\,u^\alpha\,u^\beta = +\,1.$$

The problem that one then poses is that of integrating the field equations. One can study it by analyzing the Cauchy problem. In order to do that, one is given a hypersurface $\mathfrak{S}$ that is space-like and represented locally by:

$$x^0 = 0,$$

and the values of the quantities:

$$\mathcal{C}\,(g_{\alpha\beta},\ \partial_0\, g_{\alpha\beta};\ H_{\alpha\beta},\ \vartheta,\ \partial_0\,\vartheta)$$

on that surface, and one then proposes to determine the various fields $\mathcal{H}$ ($g_{\alpha\beta}$, $H_{\alpha\beta}$, ϑ, u^α, p, δ) outside of $\mathfrak{S}$ in their domains of existence. It suffices to study the possibility of calculating the values of the various quantities that were introduced and their successive derivatives on $\mathfrak{S}$.

If the spacetime $\mathfrak{V}_4$ is a differentiable manifold of class (C^2, piece-wise C^4) then one can suppose that $g_{\alpha\beta}$ has class (C^1, piece-wise C^3), $H_{\alpha\beta}$ has class (C^0, piece-wise C^2), and ϑ has class (C^2, piece-wise C^4).

For $g^{00} \neq 0$, the EINSTEIN equations are equivalent to the following set of two systems:

(3.10) $$R_{ij} \equiv -\tfrac{1}{2} g^{00}\, \partial_{00}\, g_{ij} + F_{ij} = \chi\, (T_{ij} - \tfrac{1}{2} T\, g_{ij}),$$

(3.11) $$S^0_\alpha = \chi\, [(\rho + p)\, u^0\, u_\alpha - p\, g^0_\alpha - (u^0\, q_\alpha + u_\alpha\, q^0) + \tau^0{}_\alpha - (1 - \varepsilon\mu)\, \tau^0{}_\rho\, u^\rho\, u_\alpha],$$

in which the F_{ij} and S^0_α have known values on $\mathfrak{S}$. Equations (3.11), when combined with the unitary character of u^α and the equation of state, yield the quantities p, u^α. (3.10) will then determine $\partial_{00}\, g_{ij}$ when $g^{00} \neq 0$.

MAXWELL's equations are equivalent to the following system:

(3.12) $$\mathcal{D}_i \equiv \frac{1}{\mu}\, [g^{00} - (1 - \varepsilon\mu)\, (u^0)^2]\, \partial_0\, H_{0i} + \frac{1}{\mu}\, [g^{0j} - (1 - \varepsilon\mu)\, u^0\, u^j]\, \partial_0\, H_{ji} + \Phi_i\,,$$

$$= \delta\, u_i + \sigma\, u^\rho\, H_{\rho\, i}\,,$$

(3.13) $$\mathcal{E}^{\,i} \equiv \tfrac{1}{2}\, \eta^{0\, jki}\, \partial_0\, H_{jk} + \Psi^i = 0,$$

and the two identities:

(3.14) $$\mathcal{D}^0 = \delta\, u^0 + \sigma\, u_\alpha\, H^{\alpha 0},$$

(3.15) $$\mathcal{E}^{\,0} = \tfrac{1}{2}\, \eta^{jki\, 0}\, \partial_i\, H_{jk} = 0,$$

in which the Φ_i and Ψ^i do not depend upon the $\partial_0 H_{\alpha\beta}$, but they do depend upon the $\partial_0 u^\alpha$, while the quantity $\mathcal{D}^0$ does not depend upon either $\partial_0 H_{\alpha\beta}$ or $\partial_0\, u^\alpha$. (3.15) expresses the idea that there exists a local vector potential for H_{ij} on $\mathfrak{S}$. Equation (3.14) will determine δ if $u^0 \neq 0$. In order to get $\partial_0\, H_{\alpha\beta}$, one must first seek to determine the derivatives of u^α, namely, $\partial_0\, u^\alpha$. That determination can be accomplished simultaneously with that of $\partial_0 p$, $\partial_0 \vartheta$, by means of the conservation equations that relate to the EINSTEIN equations (3.5), to which one adds the unitary character of u^α, the equations of thermal conduction (3.6), and the equation of state. The derivative $\partial_0 \delta$ is then calculated by means of the equation of conservation of electrical current, which can be written:

$$u^0\, \partial_0 \delta = \Omega,$$

in which Ω depends upon $\partial_0\, u^\alpha$, but not upon the $\partial_0 H_{\alpha\beta}$.

Once the $\partial_0 u^\alpha$ have been calculated, one substitutes them into (3.12) and (3.13), which will finally yield the $\partial_0 H_{\alpha\beta}$ if:

$$g^{00} - (1 - \varepsilon\mu)\, (u_0)^2 \neq 0.$$

If the hypersurface $\mathfrak{S}$ that carries the Cauchy data $\mathcal{C}$ is not exceptional then it will result from equations (3.10), (3.12), (3.13), (3.5), (3.6), (3.7) that the quantities $\partial_{00}\, g_{ij}$,

$\partial_0 H_{\alpha\beta}$, $\partial_{00}\vartheta$, $\partial_0\ u^\alpha$, $\partial_0 p$, $\partial_0\delta$ are well-determined and necessarily continuous upon traversing the hypersurface $\mathfrak{S}$. The same conclusions can be extended to higher-order derivatives of these quantities if one supposes that the givens are differentiable to a higher order than was assumed by hypothesis.

The determination of the preceding quantities does not involve equations (3.11), (3.14), (3.15). Now, they do not contain any mixed derivative of the Cauchy givens, so they are constrained to verify the three equations (3.11), (3.14), (3.15) on the manifold $\mathfrak{S}$ or their equivalents:

$$\text{(I)}\qquad \begin{aligned} Q_\alpha^0 &\equiv S_\alpha^0 - \chi T_\alpha^0 = 0, \\ P^0 &\equiv \mathcal{D}^0 - (\delta u^0 + \sigma u_\alpha H^{\alpha 0}) = 0, \\ \mathcal{E}^0 &= 0, \end{aligned}$$

in which one has set:

$$Q_{\alpha\beta} = S_{\alpha\beta} - \chi\, T_{\alpha\beta}, \qquad\qquad P_\alpha = \mathcal{D}_\alpha - (\delta u_\alpha + \sigma u^\rho H_{\rho\alpha})\ .$$

Now consider a set $\mathcal{H}$ ($g_{\alpha\beta}$, $H_{\alpha\beta}$, ϑ, u^α, p, δ) that is a solution to equations (3.10), (3.12), (3.13), (3.5), (3.6), (3.7) that corresponds to Cauchy data $\mathcal{C}$ that satisfy equations (I) on $\mathfrak{S}$. By virtue of the conservative character of the left-hand sides of the MAXWELL-EINSTEIN equations and the conservation equations (3.5), (3.7), one will have:

$$\nabla_\alpha Q_\beta^\alpha = 0, \qquad \nabla_\alpha P^\alpha = 0, \qquad \nabla_\alpha \mathcal{E}^\alpha = 0.$$

By virtue of (3.11), (3.12), (3.13), these identities will reduce to the equations:

$$\begin{aligned} g^{00}\partial_0 Q_\alpha^0 &= A_\alpha^{i\beta}\partial_i Q_\beta^0 + B_\alpha^\beta Q_\beta^0, \\ \partial_0 P^0 &= -C^i\, \partial_i P^0 - (\partial_i C^i + \Gamma_{\alpha\beta}^\alpha C^\beta)\, P^0, \\ \partial_\alpha \mathcal{E}^\alpha &= -\Gamma_{\alpha 0}^\alpha\ \mathcal{E}^0, \end{aligned}$$

in which the $A_\alpha^{i\beta}$, B_α^β, C^α are continuous functions. These equations are linear and homogeneous with respect to the unknowns Q_α^0, P^0, $\mathcal{E}^0$. Since $Q_\alpha^0 = P^0 = \mathcal{E}^0 = 0$ on $\mathfrak{S}$, they will admit no other solution besides the identically zero solution. It then results that if equations (I) are verified by the Cauchy data on $\mathfrak{S}$ then they will likewise be verified in all of the domain of spacetime considered by the solution $\mathcal{H}$ ($g_{\alpha\beta}$, $H_{\alpha\beta}$, ϑ, u^α, p, δ) of the field equations.

The problem of integrating the field equations finally consists of the choice of Cauchy data that will render equations (3.11), (3.14), (3.15) compatible, which will permit one to calculate u^α, p, δ, and then the integration of the system of equations (3.10), (3.12), (3.13), (3.5), (3.6), (3.7), which will permit one to study the evolution of the fields $\mathcal{H}$ ($g_{\alpha\beta}$, $H_{\alpha\beta}$, ϑ, u^α, p, δ). If the givens of the problem are real-analytic then, with the aid of the CAUCHY-KOWALEWSKI existence theorem for partial differential equations, one can establish that the problem will admit one and only one real-analytic solution whose development in powers of x^0 we know, up to a coordinate change that preserves the hypersurface $\mathfrak{S}$ point-by-point, along with the Cauchy data on $\mathfrak{S}$. The method of FOURES will permit one to establish the existence and uniqueness of the solution under the hypothesis of simple differentiability.

4. The characteristic manifolds $\mathfrak{V}_3^M$

In regard to equations (3.12), one sees that if the hypersurface $\mathfrak{S}$ that carries the Cauchy data is such that one has:

$$g^{00} - (1 - \varepsilon\mu)\,(u^0)^2 = 0$$

on $\mathfrak{S}$ then the derivatives $\partial_0\, H_0$ of the electromagnetic field can be discontinuous upon traversing $\mathfrak{S}$. An infinitude of distinct solutions to MAXWELL's equations that correspond to the same CAUCHY data can exist. The manifold $\mathfrak{S}$ is a *characteristic manifold* for MAXWELL's equations; such a manifold will be denoted by $\mathfrak{V}_3^M$.

In an arbitrary local coordinate system, the characteristic manifolds $\mathfrak{V}_3^M$ that are defined by $f(x^\alpha) = 0$ will be the manifolds that satisfy the equation:

(4.1) $$[g^{\alpha\beta} - (1 - \varepsilon\mu)\, u^\alpha u^\beta]\, \partial_\alpha f\, \partial_\beta f = 0.$$

Discontinuities of the electromagnetic field can be produced when one traverses these manifolds that constitute the relativistic extension of the classical electromagnetic wave fronts. We suppose that the wave fronts are time-like, or, more rigorously, they are tangent to the elementary cone $ds^2 = 0$ on $\mathfrak{V}_4$; we confirm that this hypothesis is indeed in accord with the demands of relativistic physics. If that is true then:

$$\Delta_1 f \equiv g^{\alpha\beta}\, \partial_\alpha f\, \partial_\beta f = (1 - \varepsilon\mu)\,(u^\alpha\, \partial_\alpha f) \leq 0.$$

One then deduces that:

(4.2) $$\varepsilon\mu \geq 1.$$

Having said that, the generalization of HUGONIOT's hypothesis will permit one to evaluate what one can regard as the velocity of propagation of the electromagnetic wave in question here. In order to do that, consider two neighboring wave surfaces $(\mathfrak{V}_3^M)_0$ and $(\mathfrak{V}_3^M)_\vartheta$ that are defined by the equations:

$$f(x^\alpha) = 0, \qquad f(x^\alpha) = \vartheta,$$

and take ϑ to be infinitely small.

The streamline that issues from the point x of $(\mathfrak{B}_3^M)_0$ will cut $(\mathfrak{B}_3^M)_\vartheta$ at a point that is defined, up to higher-order infinitesimals, by $x + \eta\,\mathbf{u}$, where η is given by the relation:

(4.3) $$\eta\, u^\alpha\, \partial_\alpha f = \vartheta.$$

Let $\mathbf{n}$ be the normal vector ($\mathbf{n}^2 = 1$) that is normal to the wave surface $(\mathfrak{B}_3^M)_0$ at x. Its covariant components at x are:

(4.4) $$n_\lambda = \frac{\partial_\lambda f}{\sqrt{-g^{\alpha\beta}\partial_\alpha f \partial_\beta f}}.$$

The orthogonal trajectory to $\mathfrak{B}_3^M$ that issues from x will cut $(\mathfrak{B}_3^M)_\vartheta$ at a point that is written, up to higher-order infinitesimals, as $x + \eta_1\,\mathbf{n}$, where η_1 is determined by the relation:

$$\eta_1\, n^\lambda\, \partial_\lambda f = \vartheta.$$

One deduces from this that:

(4.5) $$\eta_1 = \frac{\vartheta}{n^\lambda \partial_\lambda f} = \frac{\vartheta\sqrt{-g^{\alpha\beta}\partial_\alpha f \partial_\beta f}}{g^{\alpha\beta}\partial_\alpha f \partial_\beta f} = \frac{-\vartheta}{\sqrt{-g^{\alpha\beta}\partial_\alpha f \partial_\beta f}}.$$

Introduce the vector $\mathbf{t} = \eta\,\mathbf{u} - \eta_1\,\mathbf{n}$. By virtue of (4.3) and (4.4), one will have:

$$\eta(\mathbf{u}\cdot\mathbf{n}) = -\,\eta_1$$

and

$$\mathbf{t}\cdot\mathbf{n} = (\eta\,\mathbf{u} - \eta_1\,\mathbf{n})\cdot\mathbf{n} = \eta\,(\mathbf{u}\cdot\mathbf{n}) + \eta_1 = 0.$$

The vector $\mathbf{t}$ will then be tangent to the wave surface. It is time-like since its square:

$$\eta_0^2 = (\mathbf{t})^2 = \eta^2 - \eta_1^2 - 2\eta\,\eta_1\,(\mathbf{u}\cdot\mathbf{n}) = \eta^2 + \eta_1^2$$

is positive.

The vector $\eta\,\mathbf{u}$ then appears to be the sum of two vectors, one of which is orthogonal to the wave surface and space-like, while the other one is tangent to that surface and time-like. The velocity of propagation V of the wave is found to be defined as the limit of the ratio of the moduli of these two vectors, namely:

$$V = \lim_{\vartheta\to 0}\left|\frac{\eta_1}{\eta_0}\right|.$$

One will then have:

$$V^2 = \lim_{\vartheta\to 0}\frac{\eta_1^2}{\eta_0^2},$$

so upon replacing η_1 and η_0 by their values, one will get:

$$V^2 = \frac{1}{\varepsilon\mu}.$$

The velocity of propagation of the electromagnetic waves is then $1\big/\sqrt{\varepsilon\mu}$. That value suggests two remarks: First, it generalizes the value that is obtained in classical electromagnetism. Moreover, by our hypothesis ($\varepsilon\mu \geq 1$), the velocity of propagation V is less than a limiting value $c = 1$; that limiting value will coincide with the value of the velocity of propagation of the electromagnetic wave *in vacuo* ($\varepsilon\mu = 1$).

II. Study of the characteristics

5. Definition of an associated metric

The integration of MAXWELL's equations involves the intervention of the symmetric contravariant tensor field:

(5.1) $$\bar{g}^{\alpha\beta} = g^{\alpha\beta} - (1 - \varepsilon\mu)\, u^\alpha u^\beta,$$

whose associated quadratic form represents the characteristic form of MAXWELL's equations. The study of its characteristic manifolds $\mathfrak{V}_3^M$ will become more suggestive if one introduces the Riemannian metric:

(5.2) $$d\bar{s}^2 = \bar{g}_{\alpha\beta}\, dx^\alpha dx^\beta,$$

in which the matrix of coefficients $(\bar{g}_{\alpha\beta})$ is the matrix inverse to the matrix $(\bar{g}^{\alpha\beta})$. One will easily obtain:

(5.3) $$\bar{g}_{\alpha\beta} = g_{\alpha\beta} - \left(1 - \frac{1}{\varepsilon\mu}\right) u_\alpha u_\beta$$

upon performing the calculations in the proper frame. Moreover, if g and $\bar{g}$ represent the determinant of the matrix ($g_{\alpha\beta}$) and that of the matrix $(\bar{g}_{\alpha\beta})$, respectively, then one will have the relation:

(5.4) $$g = \varepsilon\,\mu\,\bar{g}.$$

The metric $d\bar{s}^2$ will be called the *associated metric*. It plays a fundamental role in the study of the characteristic manifolds of MAXWELL's equations. The world metric $ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ is of the hyperbolic normal type. When referred to a proper frame, it will take the canonical form:

$$ds^2 = (\omega^0)^2 - (\omega^1)^2 - (\omega^2)^2 - (\omega^3)^2,$$

in which the (ω^α) are linearly-independent, local Pfaff forms. The associated metric itself will then take the form:

$$d\bar{s}^2 = \left[\delta_{\alpha\beta} - \left(1 - \frac{1}{\varepsilon\mu}\right)u_\alpha u_\beta\right]\omega^\alpha\,\omega^\beta,$$

in which $\delta_{\alpha\beta}$ if $\alpha \neq \beta$, $\delta_{00} = +1$, and $\delta_{ii} = -1$, and $u_0 = 1$, $u_i = 0$ in the proper frame. One then deduces that:

$$d\bar{s}^2 = \left(\frac{\omega^0}{\sqrt{\varepsilon\mu}}\right)^2 - (\omega^1)^2 - (\omega^2)^2 - (\omega^3)^2,$$

which shows that the associated metric is likewise of the hyperbolic normal type.

In what follows, we will let $\bar{\mathfrak{V}}_4$ denote the Riemannian manifold that is defined by the differentiable manifold that carries $\mathfrak{V}_4$ and is endowed with the associated metric $d\bar{s}^2$. Furthermore, we will distinguish the quantities that are defined relative to $\bar{\mathfrak{V}}_4$ by an overbar. We shall call the real cone $\bar{\mathfrak{C}}_x$ at a point x that consists of directions tangent to $\bar{\mathfrak{V}}_4$ that are defined by the equation $d\bar{s}^2 = 0$ the *associated elementary cone.*

6. Study of the bicharacteristics.

In the Riemannian space $\bar{\mathfrak{V}}_4$, the characteristic manifolds of MAXWELL's equations that are defined locally by the $f(x^\alpha) = 0$ are solutions to the first-order partial differential equation:

(6.1) $$\bar{\Delta}_1 f \equiv \bar{g}^{\alpha\beta}\partial_\alpha f\,\partial_\beta f = 0.$$

They are tangent to the associated elementary cone $\bar{\mathfrak{C}}_x$ at each point. The elementary cones $\bar{\mathfrak{C}}_x$ of $\bar{\mathfrak{V}}_4$ are thus characteristic cones for MAXWELL's equations, and they admit the manifolds that are tangent to those cones for characteristic manifolds. However, the characteristic cones of MAXWELL's equations in the spacetime $\mathfrak{V}_4$ are generally different from the elementary cones $\mathfrak{C}_4$ ($ds^2 = 0$). They will coincide with the latter only in regions that are devoid of matter.

A characteristic manifold $\mathfrak{V}_3^M$ – i.e., a solution to (6.1) – can be generated by means of the characteristic bands of (6.1). Such a solution can be generated by means of the bands of $\bar{\mathfrak{V}}_4$ that are each composed of set of a curve $\mathcal{L}_0$ and a one-parameter family of elementary 3-planes that are tangent to those curves. The curves $\mathcal{L}_0$ are called the *bicharacteristics* of MAXWELL's equations.

In order to determine them, set:

$$2H(x^\lambda, y_\mu) = \bar{g}^{\alpha\beta}\,y_\alpha\,y_\beta\,,$$

and consider the partial differential equation:

(6.2) $$\bar{\Delta}_1 f \equiv 2H(x^\lambda, \partial_\mu f) = C,$$

where C is an arbitrary constant. Relative to the variables x^α, f, y_β, the characteristic bands of MAXWELL's equations (3.1) and (3.2) are given by the solutions to the differential system:

$$\frac{dx^0}{\dfrac{\partial H}{\partial y_0}} = \dots = \frac{dx^3}{\dfrac{\partial H}{\partial y_3}} = \frac{df}{2H} = -\frac{dy_0}{\dfrac{\partial H}{\partial x^0}} = \dots = -\frac{dy_3}{\dfrac{\partial H}{\partial x^3}} = du$$

that satisfy the first integral:

$$2H(x^\lambda, \partial_\mu f) = C$$

for the value C of the constant. If one introduces the auxiliary variable u then the functions $x^\alpha(u)$, $y_\beta(u)$ will be given by the canonical system:

(6.3) $$\frac{dx^\alpha}{du} = \frac{\partial H}{\partial y_\alpha}, \qquad \frac{dy_\alpha}{du} = -\frac{\partial H}{\partial x^\alpha}$$

that relates to the Hamiltonian function $H(x^\lambda, y_\mu)$. The first group of equations (6.3) is written out explicitly as:

(6.4) $$\dot{x}^\alpha = \bar{g}^{\alpha\beta} y_\beta \qquad \left(\dot{x}^\alpha = \frac{dx^\alpha}{du}\right).$$

Inversely:

(6.5) $$y_\beta = \bar{g}_{\alpha\beta} \dot{x}^\beta .$$

Having said that, the solutions $x^\alpha(u)$ to (6.3) will be extremals of the Lagrangian function L that is defined by:

$$2L = \bar{g}_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta ,$$

since upon passing from the variables $(x^\alpha, \dot{x}^\alpha)$ to the canonical variables (x^β, y_β), which are coupled by (6.4) and (6.5), one will have the classical relation:

$$H = \dot{x}^\alpha \frac{\partial L}{\partial \dot{x}^\alpha} - L = L$$

between H and L. These solutions are the extremals that satisfy the first integral:

(5.6) $$2L = C$$

for the value C of the constant. Now, from the existence of that first integral, the extremals thus defined will be also extremals of:

$$\sqrt{2L} = \sqrt{\bar{g}_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}$$

that satisfy (6.6). It then results that the $x^\alpha(u)$ define the geodesics of $\bar{\mathfrak{B}}_4$. If $C = 0$ then the differential system of the characteristics of (6.1) will admit the first integral f = const., and the manifolds $\mathfrak{B}_3^M$ can be generated by the bands of $\bar{\mathfrak{B}}_4$ that are defined by the null-length geodesics $\bar{\mathfrak{L}}_0$, with the associated 3-plane being the plane tangent to the elementary cone $\bar{\mathfrak{C}}_x$ along the tangent to $\bar{\mathfrak{L}}_0$.

We have proved the theorem:

Theorem. *The bicharacteristics of MAXWELL's equations are the null-length geodesics of the Riemannian manifold* $\bar{\mathfrak{B}}_4$*, which is endowed with the metric:*

$$d\bar{s}^2 = \bar{g}_{\alpha\beta}\, dx^\alpha\, dx^\beta.$$

In the language of the theory of the propagation of waves, the characteristic manifolds $\mathfrak{B}_3^M$ play the role of electromagnetic wave surfaces. The bicharacteristics $\bar{\mathfrak{L}}_0$ are the associated electromagnetic rays. We can thus state the following result:

Theorem. *In an isotropic medium with constant dielectric and magnetic variables* ε*,* μ*, the electromagnetic rays can be considered to be null-length geodesics of the Riemannian space* $\bar{\mathfrak{B}}_4$*, which is endowed with the metric:*

$$d\bar{s}^2 = \bar{g}_{\alpha\beta}\, dx^\alpha\, dx^\beta = d\bar{s}^2 = \left[g_{\alpha\beta} - \left(1 - \frac{1}{\varepsilon\mu}\right) u_\alpha u_\beta \right] dx^\alpha\, dx^\beta.$$

in which $g_{\alpha\beta}$ *is the fundamental metric tensor, and* u_α *is the unitary world-velocity vector that is defined at each point of the medium.*

7. The equations of electromagnetism in the associated metric

Define the antisymmetric tensor field $\bar{H}_{\alpha\beta}$ on $\bar{\mathfrak{B}}_4$ such that at each point (x^α) one has:

$$\bar{H}_{\alpha\beta} = H_{\alpha\beta}.$$

Upon raising the indices of $\bar{H}_{\alpha\beta}$ with the aid of the metric tensor on $\bar{\mathfrak{B}}_4$, we will have:

$$\bar{H}^{\alpha\beta} = \bar{g}^{\rho\alpha}\,\bar{g}^{\sigma\beta}\,\bar{H}_{\rho\sigma} = [(g^{\rho\alpha} - (1 - \varepsilon\mu)\,u^{\rho}\,u^{\alpha}]\,[g^{\sigma\beta} - (1 - \varepsilon\mu)\,u^{\sigma}\,u^{\beta})\,H_{\rho\sigma}\,,$$

so

$$\bar{H}^{\alpha\beta} = g^{\rho\alpha}\,g^{\sigma\beta}\,H_{\rho\sigma} + (1 - \varepsilon\mu)\,(g^{\rho\alpha}\,H_{\sigma\rho}\,u^{\sigma}\,u^{\beta} - g^{\sigma\beta}\,H_{\rho\sigma}\,u^{\rho}\,u^{\alpha}).$$

Upon comparing this equality with the relation (2.3) that gives the expression for $G^{\alpha\beta}$ as a function of $H_{\alpha\beta}$, we see that at each point (x^{α}), one will have:

$$G^{\alpha\beta} = \frac{1}{\mu}\bar{H}^{\alpha\beta}.$$

The study of the CAUCHY problem that relates to MAXWELL's equations in $\mathfrak{V}_4$ suggests that we should write the following equations in $\bar{\mathfrak{V}}_4$:

(7.1) $$\bar{\nabla}_{\alpha}\,\overset{*}{\bar{H}}{}^{\alpha\beta} = 0,$$

(7.2) $$\bar{\nabla}_{\alpha}\bar{G}^{\alpha\beta} = \bar{J}^{\beta},$$

in which $\bar{G}_{\alpha\beta}$ is a tensor that is proportional to $\bar{H}_{\alpha\beta}$. The first group (7.1) can once more be written:

$$\bar{\nabla}_{\alpha}\,\overset{*}{\bar{H}}{}^{\alpha\beta} = \frac{\sqrt{|\bar{g}|}}{2}\,\varepsilon^{\alpha\beta\gamma\delta}\partial_{\alpha}\bar{H}_{\gamma\delta} = 0.$$

It expresses the idea that there exists a local vector field $\bar{\varphi}_{\alpha}$ whose rotation is $\bar{H}_{\alpha\beta}$. Since $g = \varepsilon\mu\,\bar{g} \neq 0$ and $\bar{H}_{\alpha\beta} = H_{\alpha\beta}$ at the point considered, one sees that equations (7.1) are equivalent to MAXWELL's equations of the first group (16) in $\mathfrak{V}_4$. We can then identify the two vector potentials $\bar{\varphi}_{\alpha}$ and φ_{α}.

As for equations (7.2), one can write them as:

$$\bar{\nabla}_{\alpha}\bar{G}^{\alpha\beta} = \frac{1}{\sqrt{|\bar{g}|}}\partial_{\alpha}\left(\sqrt{|\bar{g}|}\,\bar{G}^{\alpha\beta}\right) = \frac{\sqrt{\varepsilon\mu}}{\sqrt{|g|}}\partial_{\alpha}\left(\sqrt{|g|}\,\frac{\bar{G}^{\alpha\beta}}{\sqrt{\varepsilon\mu}}\right) = \bar{J}^{\beta}$$

or

$$\frac{1}{\sqrt{|g|}}\partial_{\alpha}\left(\sqrt{|g|}\,\frac{\bar{G}^{\alpha\beta}}{\sqrt{\varepsilon\mu}}\right) = \frac{\bar{J}^{\beta}}{\sqrt{\varepsilon\mu}}.$$

Upon comparing this equation with MAXWELL's equations of the second group (1.7) in $\mathfrak{V}_4$, when they are written in the form:

$$\frac{1}{\sqrt{|g|}}\partial_\alpha\left(\sqrt{|g|}\,G^{\alpha\beta}\right) = J^\beta,$$

one sees that equations (7.2) will be equivalent to equations (1.7) if one takes:

$$\bar{G}^{\alpha\beta} = \sqrt{\varepsilon\mu}\, G^{\alpha\beta} \qquad \text{and} \qquad \bar{J}^\beta = \sqrt{\varepsilon\mu}\, J^\beta .$$

One is led to set:

$$\bar{G}^{\alpha\beta} = \sqrt{\frac{\varepsilon}{\mu}}\,\bar{H}_{\alpha\beta}. \tag{7.3}$$

In particular, if one makes the hypothesis that:

$$J^\beta = \delta u^\beta + \sigma u_\alpha H^{\alpha\beta}$$

then one will verify that:

$$\bar{J}^\beta = \delta \bar{u}^\beta + \sigma \bar{u}_\alpha \bar{H}^{\alpha\beta},$$

in which $\bar{u}_\alpha = dx^\alpha / d\bar{s}$. One will observe that the scalars ε, μ, σ, δ are the same in $\mathfrak{V}_4$ and $\bar{\mathfrak{V}}_4$. It should be noted that in classical physics, MAXWELL's equations lead to the study of the second-order, hyperbolic, linear operator:

$$\frac{1}{V^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}.$$

That operator will remain invariant under the LORENTZ group – i.e., the group of transformations that leave invariant the quadratic form in the differentials:

$$V^2\, dt^2 - dx^2 - dy^2 - dz^2,$$

and that is only the translation into the proper frame of the associated metric form, which is written:

$$d\bar{s}^2 = \frac{c^2}{\varepsilon\mu} dt^2 - dx^2 - dy^2 - dz^2$$

here. One sees that:

$$V = \sqrt{\frac{c^2}{\varepsilon\mu}}.$$

The quantity $\sqrt{\varepsilon\mu}$ that presents itself in our study can be interpreted as the index of refraction of the medium considered. We set:

$$n = \sqrt{\varepsilon\mu},$$

so n will be a dimensionless positive number ≥ 1.

III. Permanent motion of a charged, perfect fluid

8. Spacetime that is stationary in a domain

Consider a well-defined, four-dimensional domain $\mathfrak{D}_4$ in $\mathfrak{V}_4$ and suppose that the Riemannian manifold that is defined by $\mathfrak{D}_4$ is endowed with the world metric:

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

admits a connected, one-parameter group of global isometries that leaves no point of $\mathfrak{D}_4$ invariant and whose trajectories z are time-like. We suppose, moreover, that:

a) The z are homeomorphic to the real line $\mathfrak{R}$.

b) One can find a three-dimensional differentiable manifold $\mathfrak{D}_3$ that satisfies the same differentiability hypotheses as $\mathfrak{V}_4$ and is such that there exists a differentiable homeomorphism of the same class from $\mathfrak{D}_4$ to the topological product $\mathfrak{D}_3 \times \mathfrak{R}$ that maps z to the line factor.

Under these conditions, we say that the Riemannian spacetime $\mathfrak{V}_4$ is *stationary* in $\mathfrak{D}_4$. The trajectories z are called *time lines*. The manifold $\mathfrak{D}_3$ that is the quotient of the $\mathfrak{D}_4$ by the equivalence relation that is defined by the group is called *space*.

Let $\boldsymbol{\xi}$ be the infinitesimal generator of the group of isometries. Since no point of $\mathfrak{D}_4$ is invariant, $\boldsymbol{\xi} \neq 0$ at every point of $\mathfrak{D}_4$. One knows that this vector satisfies the KILLING equations:

(8.1) $$X\, g_{\alpha\beta} \equiv \nabla_\alpha\, \xi_\beta + \nabla_\beta\, \xi_\alpha = 0,$$

in which X denotes the Lie derivative operator that relates to the vector $\boldsymbol{\xi}$.

One can define a local coordinate system (x^α) in $\mathfrak{D}_4$ (which are said to be *adapted* to the stationary character) in the following manner: The (x^i) are an arbitrary local coordinate system on $\mathfrak{D}_3$; being given the (x^i) will then determine a time line. In order to determine a point on that line, one gives the manifold x^0 = const. to which it belongs, and these manifolds will be the manifolds that are homeomorphic to $\mathfrak{D}_3$ and are defined by the homeomorphism *b*), and are such that the components of $\boldsymbol{\xi}$ are:

$$\xi^0 = 1, \qquad \xi^i = 0.$$

In these adapted coordinate systems, one will have:

$$\xi_\alpha = g_{\alpha 0} ,$$

and the KILLING equations will translate into:

$$X\, g_{\alpha\beta} = \partial_0\, g_{\alpha\beta} = 0.$$

Therefore, the $g_{\alpha\beta}$ are independent of the variable x^0.

In what follows, we shall introduce only adapted local coordinate systems. Upon performing the decomposition of the fundamental quadratic form:

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

into squares, starting with the director variable dx^0, we will get:

(8.2) $$ds^2 = (g_{\alpha 0}\, dx^\alpha)^2 + d\hat{s}^2 ,$$

in which:

(8.3) $$d\hat{s}^2 = \hat{g}_{ij}\; dx^i\, dx^j = \left(g_{ij} - \frac{g_{0i}\, g_{0j}}{g_{00}} \right) dx^i\, dx^j$$

is independent of x^0 and defines a negative-definite Riemannian metric on $\mathfrak{D}_3$.

9. Permanent motion of a charged, perfect fluid

Consider a charged, perfect, conducting fluid that moves in a domain $\mathfrak{D}_4$. The motion of that fluid will be called *permanent* if the associated Riemannian spacetime $\mathfrak{V}_4$ is stationary in the domain $\mathfrak{D}_4$, and the group of isometries leaves invariant the unitary velocity vector u^α, the pressure p, the temperature ϑ, the electric charge density δ, the heat current vector q_α, and the induced electromagnetic fields $H_{\alpha\beta}$, $G_{\alpha\beta}$:

$$X\, g_{\alpha\beta} = X\, H_{\alpha\beta} = X\, G_{\alpha\beta} = \partial_0\, G_{\alpha\beta} = \partial_0\, u^\alpha = \partial_0 \vartheta = \partial_0\, q_\alpha = \partial_0\, p = \partial_0\, \delta = 0;$$

hence, the $g_{\alpha\beta}$, $H_{\alpha\beta}$, $G_{\alpha\beta}$, u^α, ϑ, q_α, p, δ do not depend upon the variable x^0.

Consider a motion of the fluid envisioned such that:

a) The associated Riemannian spacetime $\mathfrak{V}_4$ is stationary in $\mathfrak{D}_4$.

b) The group of isometries leaves invariant $H_{\alpha\beta}$, ϑ, c, l, κ; ε, μ, σ.

We shall show that if this is true then the motion of the fluid will be permanent. Now, the hypotheses *a* and *b* translate into the following conditions in adapted coordinates ([1]):

$$\partial_0 g_{\alpha\beta} = \partial_0 H_{\alpha\beta} = \partial_0 \vartheta = \partial_0 \kappa = \partial_0 c = \partial_0 l = \partial_0 \varepsilon = \partial_0 \mu = \partial_0 \sigma = 0.$$

It suffices to show that $\partial_0 G_{\alpha\beta} = \partial_0 u^\alpha = \partial_0 q_\alpha = \partial_0 p = \partial_0 \delta = 0$.

Let x be an arbitrary point of $\mathfrak{D}_4$. Choose an adapted coordinate system (x^0, x^i) such that the point x belongs to the manifold $\mathfrak{S}$ whose equation is $x^0 = 0$. Suppose that the manifold $\mathfrak{S}$ is space-like and that it is not an exceptional manifold of the Cauchy problem that relates to the field equations that correspond to the fluid considered. We then know (*cf.*, § 3) that the system of field equations will admit a well-defined solution $\mathcal{H}$ $(g_{\alpha\beta}, H_{\alpha\beta}, \vartheta, u^\alpha, p, \delta)$ for a system of Cauchy data $\mathcal{C}$ $(g_{\alpha\beta}, \partial_0 g_{\alpha\beta}; H_{\alpha\beta}; \vartheta, \partial_0 \vartheta)$ that is carried by $\mathfrak{S}$ and satisfies:

$$\text{(I)} \qquad Q^0_\alpha = 0, \qquad P^0 = 0, \qquad \mathcal{E}^0 = 0.$$

Moreover, equations (I), which are verified on $\mathfrak{S}$, are likewise verified on the entire domain of the spacetime $\mathfrak{D}_4$ being considered. Now consider the manifold $\mathfrak{S}'$ whose equation is:

$$x^0 = h,$$

which corresponds, point-by-point, to manifold $\mathfrak{S}$ under the homeomorphism that maps $\mathfrak{D}_4$ onto the topological product $\mathfrak{D}_3 \times \mathfrak{R}$ in which the time line z maps to the line factor. By virtue of the hypotheses *a*) and *b*), the solution $\mathcal{H}$ $(g_{\alpha\beta}, H_{\alpha\beta}, \vartheta, u^\alpha, p, \delta)$ is such that the quantities $\mathcal{C}$ $(g_{\alpha\beta}, \partial_0 g_{\alpha\beta}; H_{\alpha\beta}; \vartheta, \partial_0 \vartheta)$ will have equal values at the points z of $\mathfrak{S}$ and $\mathfrak{S}'$ that have the same local coordinates (x^i). These quantities verify equations (I′), which are identical to (I), for the solution $\mathcal{H}$. Thus, if one poses the Cauchy problem with the preceding data $\mathcal{C}$ being carried by $\mathfrak{S}'$ then one must first calculate the quantities u^α, p, δ by starting with equations (I′) and then integrate the field equation. Since all of the equations are identical, and the givens are identical, one will obtain a solution $\mathcal{H}'$ that is identical to the solution $\mathcal{H}$. In other words, the quantities $(g_{\alpha\beta}, H_{\alpha\beta}, \vartheta, u^\alpha, p, \delta)$ will

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([1]) In a previous paper "Sur une théorie relativiste des fluides thermodynamiques" [Ann. di Math. pura ed applicata, ser. IV, **38** (1955)], we studied the permanent motions of a pure thermodynamic fluid by supposing that, in addition to the conditions $\partial_0 g_{\alpha\beta} = \partial_0 \vartheta = 0$, one also satisfied the hypothesis that:

$$\partial_0 q^0 = c\,\rho\, u^0\, \partial_0 \vartheta - \frac{l}{\rho} u^0 \rho\,;$$

indeed, the latter hypothesis is a consequence of $\partial_0 \vartheta = 0$. The conditions $X\, g_{\alpha\beta} = X\, \vartheta = 0$ suffice to insure that the motion of the thermodynamic fluid is permanent.

have the same values, along with their derivatives at the points with the same local coordinates (x^i); they are thus invariant along the time lines.

It then results that the hypotheses *a*) and *b*) imply that:

$$\partial_0 u^\alpha = \partial_0 p = \partial_0 \delta = 0,$$

and by virtue of the defining equations of $G_{\alpha\beta}$ and q_α :

$$\partial_0 G_{\alpha\beta} = \partial_0 q_\alpha = 0.$$

Consequently:

$$X g_{\alpha\beta} = X H_{\alpha\beta} = X G_{\alpha\beta} = X u_\alpha = X \vartheta = X q_\alpha = X p = X \delta = 0.$$

The motion of the fluid being considered is therefore permanent. One can state the theorem:

Theorem. *If one is given a charged, perfect, conducting fluid in a domain* $\mathfrak{D}_4$ *then in order for the motion of that fluid to be permanent it is necessary and sufficient that:*

a) *The associated Riemannian spacetime* $\mathfrak{V}_4$ *must be stationary in* $\mathfrak{D}_4$.

b) *The group of isometries must leave invariant the fields* ϑ *and* $H_{\alpha\beta}$, *and the coefficients c, l,* κ, ε, μ, σ.

Remark. – If there exists a global vector potential for the electromagnetic field – i.e., a vector field φ_α such that:

$$H_{\alpha\beta} = \partial_\alpha \varphi_\beta - \partial_\beta \varphi_\alpha$$

(which is the case, in particular, when the domain is simply-connected) then one can replace the hypothesis that concerns the field $H_{\alpha\beta}$ with the equivalent hypothesis that concerns the vector potential; one supposes that the φ_α are invariant under the group of isometries.

10. Isometries that are induced in $\bar{\mathfrak{V}}_4$

In what follows, we shall consider a simply-connected domain $\mathfrak{D}_4$ that is occupied by a charged, conducting fluid. We suppose, more generally, that the motion of that medium is such that the associated Riemannian spacetime $\mathfrak{V}_4$ is stationary in $\mathfrak{D}_4$ and the group of isometries leaves invariant the unitary velocity vector **u** and the index $n = \sqrt{\varepsilon\mu}$ of the medium:

(10.1) $$X g_{\alpha\beta} = X u_\alpha = X n = 0.$$

That will be true, in particular, when the medium considered is a charged, perfect, conducting fluid in a state of permanent motion.

From (10.1), the quantities:

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} - \left(1 - \frac{1}{n^2}\right) u_\alpha u_\beta$$

are invariant under the group of isometries of $\mathfrak{V}_4$. It then results that the contravariant vector field $\boldsymbol{\xi}$ that is the generator of the group of isometries of $\mathfrak{V}_4$ will determine a connected, global group of isometries in the Riemannian manifold that is defined by the differentiable manifold that carries $\mathfrak{D}_4$ and is endowed with the associated metric:

$$d\bar{s}^2 = \bar{g}_{\alpha\beta}\, dx^\alpha\, dx^\beta,$$

and that group will leave no point of the corresponding domain $\bar{\mathfrak{D}}_4$ invariant, and the coordinate system (x^0, x^i) will be an adapted, local coordinate system for that group. One can take the infinitesimal generator of that group to be the vector $\boldsymbol{\zeta}$ that has the contravariant components:

$$\zeta^0 = \xi^0 = 1, \quad \zeta^i = \xi^i = 0.$$

It is obvious that the square of the vector will have the value:

$$(\boldsymbol{\zeta})^2 = \bar{g}_{00} = g_{00} - \left(1 - \frac{1}{n^2}\right)(u_0)^2. \tag{10.2}$$

Now, introduce the spatial quantity that takes the form of the vector **u** that relates to the time direction $\boldsymbol{\xi}$. Let:

$$-w^2 = \hat{g}_{ij}\; u^i\, u^j\,.$$

By virtue of the unitary character of **u**, one will have:

$$g_{\alpha\beta}\, u^\alpha\, u^\beta \equiv \frac{1}{g_{00}}(g_{0\alpha}u^\alpha)^2 + \hat{g}_{ij}u^i u^j = 1.$$

One will then deduce that:

$$(u_0)^2 = g_{00}\,(1 + w^2).$$

Upon substituting this value into (10.2) and replacing $1 / n^2$ with V^2 in it, it will become:

$$(\boldsymbol{\zeta})^2 = \bar{g}_{00} = g_{00}\,(V^2\, w^2 + V^2 - w^2).$$

One sees that the sign of $(\boldsymbol{\zeta})^2$ can change. In $\bar{\mathfrak{V}}_4$, the vector $\boldsymbol{\zeta}$ can be time-like, space-like, or isotropic; the same thing will be true for the trajectories of the isometries of $\bar{\mathfrak{V}}_4$.

IV. Geometric study of electromagnetic rays in space

11. A problem in the calculus of variations

We propose to interpret the electromagnetic rays in three-dimensional space geometrically. To that effect, we begin by briefly recalling a problem in the calculus of variations.

If one is given a differentiable manifold $\mathfrak{V}_{n+1}$ then let $\mathfrak{W}_{2(n+1)}$ be the fiber bundle of tangent vectors to the various points of $\mathfrak{V}_{n+1}$. If one adopts local coordinates (x^α) on $\mathfrak{V}_{n+1}$ then each element of $\mathfrak{W}_{2(n+1)}$ will consist of the union of the coordinates (x^α) of the corresponding point x and the $n + 1$ components $(\dot{x}^\beta)$ of the vector $\dot{x}$ in the natural frame at x that is associated with (x^α). The structure of a Finslerian manifold on $\mathfrak{V}_{n+1}$ is defined by being given a function $\mathcal{L}(x, \dot{x})$ with scalar values in $\mathfrak{W}_{2(n+1)}$ such that for fixed x, one will have $\mathcal{L}(x, \lambda\dot{x}) = \lambda\, \mathcal{L}(x, \dot{x})$. Such a function is represented by $\mathcal{L}(x^\alpha, \dot{x}^\beta)$ in local coordinates, and is homogeneous of the first degree with respect to $\dot{x}^\beta$.

Consider a differentiable manifold $\mathfrak{V}_{n+1}$ that is endowed with the structure of a Finslerian manifold and suppose that it admits a connected, one-parameter group of global isometries that has $\boldsymbol{\xi}$ for an infinitesimal generator and leaves no point of $\mathfrak{V}_{n+1}$ invariant ($\boldsymbol{\xi} \neq 0$). Furthermore, suppose that the trajectories z of the group are homeomorphic to the real line $\mathfrak{R}$, and let $\mathfrak{V}_n$ be the manifold of $\mathfrak{V}_{n+1}$ modulo the equivalence relation that is defined by the group. One knows that there exists a local coordinate system (x^0, x^i) that is adapted to the group of isometries such that $\boldsymbol{\xi}$ will have the contravariant components:

$$\xi^0 = 1, \qquad \xi^i = 1$$

in the associated natural frame, and the (x^i) are an arbitrary local coordinate system on $\mathfrak{V}_n$. When the (x^i) are given, that will determine a trajectory. In order to determine a point on that trajectory, one gives the manifold x^0 = const. that it belongs to.

In an adapted coordinate system (x^0, x^i), the hypothesis of isometry translates into the fact that the function $\mathcal{L}$ is locally independent of the variable x^0:

$$\mathcal{L} = \mathcal{L}(x^i, \dot{x}^j, \dot{x}^0)\,.$$

We shall show that it is possible to give the quotient manifold $\mathfrak{V}_n$ the structure of a Finslerian manifold by means of functions $L(z, \dot{z})$ in such a fashion that the geodesics of $\mathfrak{V}_{n+1}$, which are extremals to the integral:

$$\int_{x_0}^{x_1} \mathfrak{L}(x, \dot{x})\, du\,, \qquad \dot{x} = \frac{dx}{du}, \tag{11.1}$$

correspond to the extremals of:

(11.2) $$\int_{z_0}^{z_1} L(z, \dot{z})\, du\,, \qquad \dot{z} = \frac{dz}{du}.$$

In what follows, any Greek index will equal 0, 1, 2, …, n, and any Latin index will equal 1, 2, …, n, and we will suppose that:

$$\partial_{\dot{0}\dot{0}}\mathfrak{L} \neq 0, \qquad \partial_{\dot{\alpha}} = \frac{\partial}{\partial \dot{x}^\alpha}.$$

We give an extremal of (11.1) by its parametric representation $x^\alpha(u)$, where u is an arbitrary parameter. The differential system of the extremals of (11.1):

(11.3) $$\frac{dx^\alpha}{du} = \dot{x}^\alpha,$$

where $\dot{x}^\alpha$ satisfies:

(11.4) $$\frac{d}{du}\frac{\partial\mathfrak{L}}{\partial\dot{x}^\alpha} - \frac{\partial\mathfrak{L}}{\partial x^\alpha} = 0,$$

is characterized by the fact that it admits the relative integral invariant:

(11.5) $$\omega = \sum_\alpha \frac{\partial\mathfrak{L}}{\partial\dot{x}^\alpha} dx^\alpha = \partial_{\dot{k}}\mathcal{L}\, dx^k + \partial_{\dot{0}}\mathcal{L}\, dx^0 .$$

By virtue of the hypothesis $\partial_0\, \mathcal{L} = 0$, one will have the first integral:

(11.6) $$\partial_{\dot{0}}\mathcal{L} = h.$$

Since $\partial_{\dot{0}\dot{0}}\mathfrak{L} \neq 0$, one can solve (11.6) with respect to $\dot{x}^0$ and one will get the equivalent equation:

(11.7) $$\dot{x}^0 = \varphi\,(x^k,\ \dot{x}^l\,,\ h),$$

in which φ is a homogeneous function of degree one in $\dot{x}^l$, and φ depends essentially upon h.

Consider the family of extremals $(\mathfrak{C}_h)$ that correspond to a well-defined value of the constant h. The last term in ω has the value $h\ dx^0$ for that family and defines a relative integral invariant. It will then result that this family of extremals admits the relative integral invariant:

(11.8) $$\partial_{\dot{k}}\mathcal{L}\ \ dx^k.$$

Now, from the homogeneity of $\mathcal{L}$, one has:

$$\dot{x}^k \partial_{\dot{x}} \mathcal{L} + \dot{x}^0 \partial_{\dot{0}} \mathcal{L} = \mathcal{L}.$$

As a result, for any solution (11.6) or (11.7), the quantity can be expressed by a function L of the variables x^k, $\dot{x}^l$, h:

(11.9) $$L(x^k, \dot{x}^l, h) = \mathcal{L}[x^k, \dot{x}^l, \varphi(x^k, \dot{x}^l, h)] - h\,\varphi(x^k, \dot{x}^l, h),$$

and one will have:

$$\partial_{\dot{k}} L = \partial_{\dot{k}} \mathcal{L} + \partial_{\dot{0}} \mathcal{L}\, \partial_{\dot{k}} \varphi - h\, \partial_{\dot{k}} \varphi = \partial_{\dot{k}} \mathcal{L}.$$

Thus, from (11.8), the projections of $(\mathfrak{C}_k)$ onto $\mathfrak{B}_n$ will be defined by a differential system that admits the relative integral invariant:

$$\pi = \partial_{\dot{k}} L\, dx^k.$$

In other words, they are extremals of the integral:

(11.10) $$\int_{z_0}^{z_1} L(x^k, \dot{x}^l, h)\, du,$$

in which h has the chosen value.

One calls the correspondence that makes the function $\mathcal{L}(x^k, \dot{x}^l, \dot{x}^0)$ correspond to the function $L(x^k, \dot{x}^l, h)$ a *descent*. The inverse problem admits a solution (*).

12. Projection of null-length geodesics onto the Riemannian manifold $\mathfrak{B}_4$

We suppose that the Riemannian manifold $\overline{\mathfrak{B}}_4$ satisfies the hypotheses of paragraph 10. The function $\mathcal{L}$ is defined by the relation:

(12.1) $$\mathcal{L}^2 = \bar{g}_{\alpha\beta}\, x^\alpha x^\beta,$$

where the left-hand side is a non-degenerate quadratic form, since $\bar{g} = \det(\bar{g}_{\alpha\beta}) \neq 0$. We shall first study the extremals that correspond to the values of $\dot{x}^\alpha$ for which the left-hand side is positive. One knows, moreover, that it suffices for a geodesic to make it positive at a point in order for it to be positive all along the geodesic.

First case: $\bar{g}_{00}$ *is not annulled in the domain studied.* – The descent procedure leads us to form the equation:

(*) See A. LICHNEROWICZ: *Théories relativistes de la gravitation et de l'électromagnétisme*, Book II, chap. I.

(12.2) $$\tfrac{1}{2}\partial_{\dot{0}}\mathcal{L}^2 = \bar{g}_{00}\dot{x}^0 + \bar{g}_{0i}\dot{x}^i = h\,\mathcal{L}$$

and eliminate $\dot{x}^0$ from this equation and:

(12.3) $$L = \mathcal{L} - h\,\dot{x}^0 .$$

Upon decomposing $\mathcal{L}^2$ into squares by starting with the director variable $\dot{x}^0$, one will get:

$$\mathcal{L}^2 = \frac{1}{\bar{g}_{00}}\left(\frac{1}{2}\partial_{\dot{0}}\mathcal{L}^2\right) + \hat{\bar{g}}_{ij}\dot{x}^i\dot{x}^j ,$$

in which one sets:

$$\hat{\bar{g}}_{ij} = \bar{g}_{ij} - \frac{\bar{g}_{0i}\bar{g}_{0j}}{\bar{g}_{00}} ,$$

and one will see that $\hat{\bar{g}}_{ij}\dot{x}^i\dot{x}^j$ is negative if $\bar{g}_{00} > 0$ and positive if $\bar{g}_{00} < 0$. In the first case, one takes $h > \max \bar{g}_{00}$. Since $\tfrac{1}{2}\partial_{\dot{0}}\mathcal{L}^2 = h\,\mathcal{L}$, one will infer the equation:

(12.4) $$\mathcal{L} = \sqrt{\frac{\hat{\bar{g}}_{ij}\dot{x}^i\dot{x}^j}{1 - \dfrac{h^2}{\bar{g}_{00}}}} ,$$

which yields $\mathcal{L}$ as a function of the variables x^k, $\dot{x}^l$, h. One then infers from (12.2) that:

(12.5) $$\dot{x}^0 = \frac{h}{\bar{g}_{00}}\mathcal{L} - \frac{\bar{g}_{0i}\dot{x}^i}{\bar{g}_{00}} .$$

One deduces from (12.3), and by virtue of (12.4), that:

(12.6) $$L = \varepsilon\sqrt{\left(1 - \frac{h^2}{\bar{g}_{00}}\right)\hat{\bar{g}}_{ij}\,\dot{x}^i\dot{x}^j} + h\frac{\bar{g}_{0i}\dot{x}^i}{\bar{g}_{00}} ,$$

where ε is the sign of $\bar{g}_{00}$,

L is a function of x^k, $\dot{x}^l$, h that is homogeneous of degree one with respect to the $\dot{x}^l$. It defines a Finslerian manifold structure on the quotient manifold $\bar{\mathfrak{V}}_3$. Conversely, if one is given the preceding function $L(x^k, \dot{x}^l, h)$ locally in $\bar{\mathfrak{V}}_3$ then one will easily prove that there exists a function $\mathcal{L}(x^k, \dot{x}^l, \dot{x}^0)$ that is homogeneous and of degree one with respect to the $\dot{x}^\alpha$ and leads back to L by descent, and that this function is:

$$\mathcal{L} = \sqrt{\bar{g}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}} .$$

The corresponding extremal curves are thus geodesics of $\bar{\mathfrak{V}}_4$.

Thus, the geodesics of the Riemannian manifold $\bar{\mathfrak{V}}_4$ that correspond to the first integral:

$$\partial_{\dot{0}}\mathcal{L} = h$$

project onto the quotient manifold $\bar{\mathfrak{V}}_3$ along extremals of the integral:

$$\int_{z_0}^{z_1}\left[-\varepsilon\sqrt{\left(1-\frac{h^2}{\bar{g}_{00}}\right)\hat{\bar{g}}_{ij}\dot{x}^i\dot{x}^j} + h\frac{\bar{g}_{0i}\dot{x}^i}{\bar{g}_{00}}\right]du, \tag{12.7}$$

in which h has the same value. These extremals coincide with those of:

$$\int_{z_0}^{z_1}\left[\frac{\varepsilon}{h}\sqrt{\left(1-\frac{h^2}{\bar{g}_{00}}\right)\hat{\bar{g}}_{ij}\dot{x}^i\dot{x}^j} - \frac{\bar{g}_{0i}\dot{x}^i}{\bar{g}_{00}}\right]du . \tag{12.8}$$

From the expression for $\dot{x}^0$, one will have:

$$dx^0 = \frac{h}{\bar{g}_{00}}\sqrt{\frac{1}{1-\dfrac{h^2}{\bar{g}_{00}}}\hat{\bar{g}}_{ij}dx^i dx^j} - \frac{\bar{g}_{0i}dx^i}{\bar{g}_{00}} \tag{12.9}$$

along these extremals.

This being the case, one can define the null-length geodesics of $\bar{\mathfrak{V}}_4$ to be the limit curves that the time-like geodesics tend to as $\mathcal{L} \to 0$. It results from the relation:

$$h\,\mathcal{L} = \bar{g}_{0\alpha}\dot{x}^{\alpha}$$

that $h \to \infty$ when $\mathcal{L} \to 0$ and h has the same sign as $g_{0\alpha}\dot{x}^{\alpha}$. Now:

$$\mathcal{L}^2 \equiv \frac{1}{\bar{g}_{00}}(\bar{g}_{0\alpha}\dot{x}^{\alpha})^2 + \hat{\bar{g}}_{ij}\dot{x}^i\dot{x}^j = 0.$$

One then deduces that $\bar{g}_{0\alpha}\dot{x}^{\alpha}$ has a non-zero value and keeps a constant sign.

From (12.8), the projections of the null-length geodesics of $\bar{\mathfrak{V}}_4$ onto $\bar{\mathfrak{V}}_3$ are the extremals of the integral:

$$\int_{z_0}^{z_1}\left[\lim_{h\to\infty}\left(\frac{\varepsilon}{h}\sqrt{\left(1-\frac{h^2}{\bar g_{00}}\right)\hat{\bar g}_{ij}\dot x^i\dot x^j}-\frac{\bar g_{0i}\dot x^i}{\bar g_{00}}\right)\right]du\,.$$

Upon passing to the limit, one will deduce the following result:

Lemma 1. *The null-length geodesics of $\bar{\mathfrak{B}}_4$ project onto $\bar{\mathfrak{B}}_3$ along the extremals of the integral:*

(12.10) $$\int_{z_0}^{z_1}\left(\varepsilon\varepsilon'\sqrt{-\frac{1}{\bar g_{00}}\hat{\bar g}_{ij}\dot x^i\dot x^j}-\frac{\bar g_{0i}\dot x^i}{\bar g_{00}}\right)du\,,$$

in which ε is the sign of $\bar g_{00}$, and ε' is the sign of $\bar g_{0\alpha}\dot x^\alpha$.

From (12.9), *one will have:*

(12.11) $$dx^0=\varepsilon\varepsilon'\sqrt{-\frac{1}{\bar g_{00}}\hat{\bar g}_{ij}dx^i dx^j}-\frac{\bar g_{0i}dx^i}{\bar g_{00}}$$

along these extremals.

One remarks that $dx^0 = L\,du$.

Second case: $\bar g_{00} = 0$. – One will then have:

(12.12) $$\mathcal{L}^2 = 2\bar g_{0i}\dot x^0\dot x^i+\bar g_{ij}\dot x^i\dot x^j\,.$$

We suppose $\bar g_{0i}\dot x^i$ is finite and non-zero. The descent process leads us to eliminate $\mathcal{L}$ and $\dot x^0$ from (12.12), and we will get:

(12.13) $$\bar g_{0i}\dot x^i = h\,\mathcal{L},$$

(12.14) $$L = \mathcal{L} - h\,\dot x^0\,.$$

One infers from (12.13) that:

$$\mathcal{L}=\frac{\bar g_{0i}\dot x^i}{h}\,.$$

Upon substituting this into (12.12), one will get:

$$\frac{(\bar g_{0i}\dot x^i)^2}{h^2} = 2\bar g_{0i}\dot x^0\dot x^i+\bar g_{ij}\dot x^i\dot x^j\,.$$

One will then deduce that:

$$\dot{x}^0 = \frac{\bar{g}_{0i}\dot{x}^i}{2h} - \frac{\bar{g}_{ij}\dot{x}^i\dot{x}^j}{2\bar{g}_{0i}\dot{x}^i}.$$

Equation (12.14) will then determine L:

(12.15) $$L = \frac{\bar{g}_{0i}\dot{x}^i}{2h} + h\frac{\bar{g}_{ij}\dot{x}^i\dot{x}^j}{2\bar{g}_{0i}\dot{x}^i}.$$

Conversely, any function L of the preceding form will correspond to the function $\mathcal{L}$ that is defined by (12.12) by ascent. One notes that L presents itself with respect to the variables $\dot{x}^i$ as the quotient of a quadratic form with a linear form.

Therefore, in the case of $\bar{g}_{00} = 0$, the projections of the geodesics of $\bar{\mathfrak{B}}_4$ onto $\bar{\mathfrak{B}}_3$ will be the extremal curves of:

$$\int_{z_0}^{z_1} \left(\frac{\bar{g}_{0i}\dot{x}^i}{2h} + h\frac{\bar{g}_{ij}\dot{x}^i\dot{x}^j}{2\bar{g}_{0i}\dot{x}^i}\right) du$$

for the corresponding value of the constant h. These extremals coincide with those of:

$$\int_{z_0}^{z_1} \left(-\frac{\bar{g}_{0i}\dot{x}^i}{2h^2} - \frac{\bar{g}_{ij}\dot{x}^i\dot{x}^j}{2\bar{g}_{0i}\dot{x}^i}\right) du .$$

As before, the projections of the null-length geodesics will be defined by:

$$\int_{z_0}^{z_1} \lim_{h\to\infty}\left(-\frac{\bar{g}_{0i}\dot{x}^i}{2h^2} - \frac{\bar{g}_{ij}\dot{x}^i\dot{x}^j}{2\bar{g}_{0i}\dot{x}^i}\right) du .$$

Upon passing to the limit, one deduces the lemma:

Lemma 2. *In any domain of* $\bar{\mathfrak{B}}_4$ *where* $\bar{g}_{00} = 0$, *the null-length geodesics of* $\bar{\mathfrak{B}}_4$ *project onto* $\bar{\mathfrak{B}}_3$ *along extremals of the integral:*

(12.16) $$\int_{z_0}^{z_1} -\frac{\bar{g}_{ij}\dot{x}^i\dot{x}^j}{2\bar{g}_{0i}\dot{x}^i} du .$$

One will have:

$$dx^0 = -\frac{\bar{g}_{ij}\dot{x}^i\dot{x}^j}{2\bar{g}_{0i}\dot{x}^i} du$$

along these extremals.

13. FERMAT's principle

We have established that electromagnetic rays are the null-length geodesics of the Riemannian manifold $\bar{\mathfrak{V}}_4$. We can interpret this geometrically in space if the medium considered is in a state of permanent motion. Indeed, Lemmas 1 and 2 provide an immediate proof of the following:

Theorem. *If the motion of the medium considered is permanent and such that:*

$$\bar{g}_{00} = g_{00}\,(V^2 w^2 + V^2 - w^2) \neq 0$$

then the electromagnetic rays in space will be the lines that realize an extremum for the integral:

$$\int_{z_0}^{z_1} \left(\varepsilon\varepsilon' \sqrt{-\frac{1}{\bar{g}_{00}} \hat{\bar{g}}_{ij} \dot{x}^i \dot{x}^j} - \frac{\bar{g}_{0i}\dot{x}^i}{\bar{g}_{00}} \right) du \tag{13.1}$$

for the variations with fixed extremities, where ε is the sign of $\bar{g}_{00}$ and ε' is the sign of $\bar{g}_{0\alpha}\dot{x}^\alpha$. The time that it takes for a ray to go from the point z_0 to the point z_1 is given by:

$$\int_{z_0}^{z_1} dx^0 = \int_{z_0}^{z_1} \left(\varepsilon\varepsilon' \sqrt{-\frac{1}{\bar{g}_{00}} \hat{\bar{g}}_{ij} \dot{x}^i \dot{x}^j} - \frac{\bar{g}_{0i}\dot{x}^i}{\bar{g}_{00}} \right) du\,. \tag{13.2}$$

This length of time is an extremum.

One will obtain an analogous statement in the case where $\bar{g}_{00} = 0$ by replacing (13.1) and (13.2) with:

$$\int_{z_0}^{z_1} -\frac{\bar{g}_{ij}\dot{x}^i\dot{x}^j}{2\bar{g}_{0i}\dot{x}^i}\,du \tag{13.3}$$

and

$$\int_{z_0}^{z_1} dx^0 = \int_{z_0}^{z_1} -\frac{\bar{g}_{ij}\dot{x}^i\dot{x}^j}{2\bar{g}_{0i}\dot{x}^i}\,du\,, \tag{13.4}$$

respectively.

One finds that the preceding theorem proves the equivalence of the geodesic principle and the principle of least time.

In particular, if the universe is *static* in the sense of LEVI-CIVITA – i.e., if the streamlines coincide with the time lines – then the spacetime $\mathfrak{V}_4$ will be orthogonal. Let:

$$ds^2 = U\,(dx^0)^2 + g_{ij}\,dx^i\,dx^j$$

be the world metric of $\mathfrak{V}_4$. Since the u^i are zero, one will then deduce the associated metric:

$$d\bar{s}^2 = \frac{U}{n^2}(dx^0)^2 + g_{ij}\,dx^i\,dx^j.$$

One can put (13.2) into the form:

$$\int_{z_0}^{z_1} dx^0 = \int_{z_0}^{z_1} \frac{n}{\sqrt{U}}\,d\sigma, \tag{13.5}$$

in which one sets:

$$ds^2 = -\,g_{ij}\,dx^i\,dx^j.$$

That integral recalls an integral that presents itself in the study of holonomic fluids. U is then the principal potential of gravitation. The gravitational field will act, one might say, as a type of pressure on electromagnetic rays.

If $U = 1$ then one can prove that the spacetime $\mathfrak{V}_4$ is Euclidian. The statement of the theorem becomes:

$$\delta\int_{z_0}^{z_1} dx^0 = \delta\int n\,d\sigma = 0.$$

We recover the exact statement of FERMAT's principle in optics. The theorem that we have established then constitutes its generalization to general relativity.

14. Interpretation of the sign of $\bar{g}_{0\alpha}\dot{x}^\alpha$

We have interpreted $\dot{x}^0$ as the time variable. The interpretation of the sign ε' of $\bar{g}_{0\alpha}\dot{x}^\alpha$ is simple. Indeed, the equation:

$$\mathcal{L}^2\,du^2 = \frac{1}{\bar{g}_{00}}(\bar{g}_{0\alpha}dx^\alpha)^2 + \hat{\bar{g}}_{ij}dx^i dx^j = 0 \tag{$\bar{C}_x$}$$

represents the characteristic cone $(\bar{\mathfrak{C}}_x)$ of MAXWELL's equations at the point x. The two sheets of that cone are symmetric with respect to the elementary hyperplane:

$$\bar{g}_{0\alpha}dx^\alpha = 0. \tag{π_x}$$

Let $M\,(x^\alpha)$ denote the vertex of the cone $(\bar{\mathfrak{C}}_x)$. Take a pair of neighboring points to M that have the spatial coordinates $(x^i + dx^i)$ and belong to the two sheets of the cone, respectively, and are symmetric with respect to π_x. Let:

$$M_1\,(x^i + dx^i,\ x^0 + dx^0), \qquad M_1'\,(x^i + dx^i,\ x^0 + d'x^0).$$

One can say that MM_1 represents, up to higher-order infinitesimals, the infinitesimal displacement that is associated with an electromagnetic ray that goes from the spatial point $A(x^i)$ to the spatial point $A'(x^i + dx^i)$ during the time interval dx^0. Likewise, MM'_1 can be considered to represent the infinitesimal displacement that is associated with an electromagnetic ray that goes from the point $A'(x^i + dx^i)$ to the point $A(x^i)$ during the time interval $d'x^0$.

The two points M_1 and M'_1 are symmetric with respect to the elementary hyperplane π_x, so one must have:

$$\bar{g}_{0\alpha} dx^\alpha = -\bar{g}_{0\alpha} d'x^\alpha .$$

One then deduces that:

$$d'x^0 = dx^0 + 2\,\frac{\bar{g}_{0i} dx^i}{\bar{g}_{00}}.$$

This relation shows that, except for the static case, the time that it takes for a ray to go from the spatial point $A(x^i)$ to the spatial point $A'(x^i + dx^i)$ will not be the same as the time that it takes for another ray to go from $A'(x^i + dx^i)$ to $A(x^i)$.

15. MINKOWSKI spacetime and the relativistic law of the composition of velocities

We now place ourselves in the case of a spacetime with no gravitation – namely, MINKOWSKI space – and refer it to a reduced Galilean coordinate system. We have the world metric:

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \tag{15.1}$$

In the present case, **u** will represent the unitary velocity vector whose components are classically determined by starting with the spatial velocity $\boldsymbol{\beta}$, if the limiting velocity c is taken to be unity.

An easy calculation gives the associated metric:

$$d\bar{s}^2 = \frac{V^2-\beta^2}{1-\beta^2}(dx^0)^2 + 2\frac{1-V^2}{1-\beta^2}\beta_i\, dx^0 dx^i - \sum_i (dx^i)^2 - \frac{1-V^2}{1-\beta^2}(\beta_i\, dx^i)^2 . \tag{15.2}$$

This metric is of the hyperbolic normal type, as is the world metric (15.1). Meanwhile, it is interesting to note that there is a change of order in the signature of that metric upon passing to $V^2 = \beta^2$. One easily exhibits this by choosing the x^1-axis to be parallel to the velocity $\boldsymbol{\beta}$ of the medium. One then gets the metric:

$$d\bar{s}^2 = \frac{V^2-\beta^2}{1-\beta^2}(dx^0)^2 + 2\frac{(1-V^2)\beta}{1-\beta^2} dx^0 dx^1 - \frac{1-V^2\beta^2}{1-\beta^2}(dx^1)^2 - (dx^2)^2 - (dx^3)^2 , \tag{15.3}$$

which one can put into the canonical form by a decomposition into squares. If $V^2 \neq \beta^2$ then one will get:

$$d\bar{s}^2 = \frac{1-\beta^2}{V^2-\beta^2}\left[\frac{V^2-\beta^2}{1-\beta^2}dx^0 + \frac{(1-V^2)\beta}{1-\beta^2}dx^1\right]^2 - \frac{(1-\beta^2)V^2}{1-\beta^2}(dx^1)^2 - (dx^2)^2 - (dx^3)^2,$$

and one will see that this metric will have the signature "+ – – –" for $V^2 > \beta^2$, and that it will have the signature "– + – –" for $V^2 < \beta^2$. For $V^2 = \beta^2$, one will get:

$$d\bar{s}^2 = 2V\,dx^0\,dx^1 - (1 + V^2)\,(dx^1)^2 - (dx^2)^2 - (dx^3)^2.$$

One likewise verifies that this metric has the signature "+ – – –" by putting it into the form:

$$d\bar{s}^2 = \frac{V^2}{(1-V^2)}(dx^0)^2 - \frac{1}{1+V^2}[(1+V^2)dx^1 + Vdx^0]^2 - (dx^2)^2 - (dx^3)^2.$$

Starting with the associated metric (15.2), we seek to express FERMAT's theorem by taking the arc length σ of the electromagnetic ray to be a parameter. We have to replace $\dot{x}^\alpha$ in (13.2) with:

$$\lambda^i = \frac{dx^i}{d\sigma},$$

where $d\sigma^2 = -\sum (dx^i)^2$. It will then become:

$$\int_{z_0}^{z_1} dx^0 = \int_{z_0}^{z_1}\left\{\varepsilon\varepsilon'\sqrt{\frac{(1-\beta^2)}{(V^2-\beta^2)}[V^2-\beta^2+(1-V^2)(\beta_i\lambda^i)^2]} - \frac{(1-V^2)(\beta_i\lambda^i)}{V^2-\beta^2}\right\}d\sigma, \tag{15.4}$$

and one can deduce from this that:

$$\frac{dx^0}{d\sigma} = \frac{1}{W} = \varepsilon\varepsilon'\sqrt{\frac{(1-\beta^2)}{(V^2-\beta^2)}[V^2-\beta^2+(1-V^2)(\beta_i\lambda^i)^2]} - \frac{(1-V^2)(\beta_i\lambda^i)}{V^2-\beta^2}.$$

If $V^2 - \beta^2 \neq 0$ then that relation will give:

$$1 - \beta^2 - (1-\beta^2)\,W^2 - (1 - V^2)\,(1 - W\beta_i\,\lambda^i)^2 = 0. \tag{15.5}$$

If one interprets **V** as the absolute velocity of the propagation of the electromagnetic wave considered and **W** as its relative velocity then one will obviously have:

$$\mathbf{V}^2 = \frac{1}{(1+\mathbf{W}\cdot\boldsymbol{\beta})^2}[\mathbf{W}^2 + \boldsymbol{\beta}^2 + 2\mathbf{W}\cdot\boldsymbol{\beta} + (\mathbf{W}\cdot\boldsymbol{\beta})^2 - \mathbf{W}^2\boldsymbol{\beta}^2]. \tag{15.6}$$

One verifies that this relation will remain valid in the case where $V^2 = \beta^2$ by direct calculation upon starting with (11.4). It is the relativistic formula for the composition of velocities. It is easy to verify that one can put it into the form (*):

$$\mathbf{V} = \frac{1}{1+\mathbf{W}\cdot\boldsymbol{\beta}}\left[\left(1+\frac{\mathbf{W}\cdot\boldsymbol{\beta}}{\beta^2}\right)\boldsymbol{\beta}+\sqrt{1-\beta^2}\left(\mathbf{W}-\frac{\mathbf{W}\cdot\boldsymbol{\beta}}{\beta^2}\boldsymbol{\beta}\right)\right].$$

We thus obtain a proof of the relativistic law of composition of velocities by starting with FERMAT's principle.

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(*) See A. LICHNEROWICZ, *Eléments de calcul tensoriel*, chap. VII, pp. 173-175.