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ON THE PROLONGATION OF FIBER BUNDLES AND INFINITESIMAL STRUCTURES ([1])

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Translated by D. H. Delphenich

Introduction.

This paper has as its point of departure the fundamental paper of D. C. Spencer: "Deformation of structures defined by transitive continuous pseudogroups." D. C. Spencer has developed a new method for differential geometry that he calls "cohomological." This method, which was illustrated in the important treatise of I. M. Singer and S. Sternberg: "On the infinite groups of Lie and Cartan," has, moreover, proved to an elegant and essential formalism for the study of differential operators or linear differential systems in general (see the profound thesis of D. G. Quillen at Harvard 1964), which unfortunately appeared while this paper was being conceived). Our goal is to make more precise the general framework in which the method of D. C. Spencer is applied and to apply it to the study of infinitesimal structures in particular. The various chapters of this paper are each preceded by an explanatory note that distills the essential results that are obtained; we thus refer the reader to them for a more detailed introduction.

Professor B. Malgrange gave me the essence of the ideas that are contained in paragraph 4 of chapter II. In particular, I owe the elegant form of theorem II-4-b in that paragraph to him. It would profit me to present him with my complete acknowledgement here.

Permit me to express my profound gratitude to Professor A. Lichnerowicz, whose benevolent advice and counsel have always guided my research.

([1]) The subject of this paper was a part of the thesis presented on 16 June 1966 to the Faculté des Sciences de Paris for obtaining the degree of Doctor of Sciences. The other part of this article, which developed the ideas of D. C. Spencer on the theory of deformation will be presented in a later publication.

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CHAPTER ONE

LIE GROUPOIDS AND ASSOCIATED FIBER BUNDLES

In this paragraph, we recall the notion of Lie groupoid and associated fiber bundle. It seems necessary to us in what follows to introduce the notion of Lie groupoid, which is meanwhile equivalent to the well-known notion of principal fiber bundle with structure group.

1. Lie groupoid.

DEFINITION I.1.a – *A groupoid Φ on the set V (or, more precisely, with V as its set of units) is a set endowed with a map:*

$$(a, b): \Phi \to V \times V, \qquad z \mapsto (a(z), b(z))$$

and a law of internal composition that is associative and partial, and verifies the following axioms:

1) *If z and z′ are two elements of Φ then the composition $z \cdot z'$ is defined if and only if $a(z) = b(z')$, and one has:*

$$b(z \cdot z') = b(z) \qquad \text{and} \qquad a(z \cdot z') = a(z').$$

2) $\forall\, x,\ x \in V,\ \exists\ l_x$, *which is an element of Φ such that:*

$$a(l_x) = b(l_x) = x,$$

and:

$$\text{if } z \cdot l_x \text{ is defined then } z \cdot l_x = z,$$
$$\text{if } l_x \cdot z \text{ is defined then } l_x \cdot z = z.$$

3) $\forall\, z,\ z \in \Phi,\ \exists\ z^{-1}$, *which is an element of Φ such that:*

$$z \cdot z^{-1} = l_y, \quad \text{where} \quad y = b(z),$$
$$z^{-1} \cdot z = l_x, \quad \text{where} \quad x = a(z).$$

The maps *a* and *b* are called the *source* and *target* maps of Φ, respectively. From axiom 2, the element l_x that is associated with any element *x* of *V* is unique; it is called the *unit* of Φ at *x*.

One verifies that the set of elements of Φ whose source and target coincide with the same element of *V* form a group G_x that is called the *isotropy group* of Φ at *x*, and if z_0 is an element of Φ with source *x* and target *y* then:

$$z_0 : G_x \to G_y, \qquad z \mapsto z_0 \cdot z \cdot z_0^{-1}$$

is an isomorphism of the group G_x with G_y .

In the case where the map (a, b) is surjective, we say that the groupoid Φ is *transitive.*

DEFINITION I.1.b. – *Φ being a groupoid on V, Φ is a differentiable groupoid if there exists the structure of a differentiable manifold on Φ (of infinite class and paracompact) such that:*

1) *The map (a, b) is (indefinitely) differentiable.*

2) *The map:*

$$\Phi \to \Phi, \qquad z \mapsto z^{-1},$$

is differentiable.

3) *For any differentiable manifold W that is endowed with two differentiable maps f and g of W into Φ that verify:*

$$a \circ f = b \circ g,$$

the map:

$$f \cdot g\colon\ W \to \Phi,\ z \mapsto f(z) \cdot g(z)$$

that is thus defined is differentiable.

Following Matsushima, we say that a differentiable groupoid is a *Lie groupoid* if the map (a, b) is a submersion (i.e., surjective and everywhere of maximal rank). A Lie groupoid is thus transitive.

PROPOSITION I. 1. – *If Φ is a Lie groupoid on the manifold V then one has:*

1) *The isotropy groups of Φ are isomorphic Lie groups.*

2) *Upon setting:*

$$\Phi_x = \{z,\ z \in \Phi, \text{ such that } a(z) = x\},$$

Φ_x is a differentiable principal fiber bundle on V with the target projection b, whose structure group is the isotropy Lie group G_x .

Indeed:

a) Since the map (a, b) is a submersion, the isotropy groups of Φ are closed submanifolds of Φ (Thom lemma), and conditions 2 and 3 of definition I.1.b entail that their algebraic structure is compatible with their differentiable structure. They are therefore Lie groups that are isomorphic, since Φ, being a Lie groupoid, is transitive.

b) Likewise, since the map a is also a submersion, Φ_x is also a closed differentiable submanifold of Φ, and the map b is a submersion of Φ onto V: Φ_x is thus a differentiable fiber bundle over V. On the other hand, condition 3 of definition I.1.b entails that G_x is a

Lie group that operates on Φ_x on the right in a simply transitive fashion on the fibers. Then, by virtue of the theorem above, which generalizes a theorem of Gleason, Φ_x is a differentiable fiber bundle over V, with the Lie group G_x for its structure group (the principal fiber bundle with structure group being defined in sense of Steenrod, *Topology of Fiber Bundles*).

THEOREM. – *Let E be a differentiable fiber bundle on a differentiable manifold V. If G is a Lie group that operates in the differentiable manifold E in a simply transitive fashion on the fiber then E is a differentiable principal fiber bundle on V with structure group G.*

Proof:

LEMMA. – *Let E be a differentiable fiber bundle over V. Any point of V admits a neighborhood U that is endowed with a differentiable map s of U into E such that* $p \circ s = \mathrm{Id}$, *p being the projection of E onto V.*

In other words, this lemma assures the existence of local differentiable sections in the neighborhood of any point of V. This lemma is only an immediate consequence of proposition 2, page 80, of *Theory of Lie Groups* by C. Chevalley, which remains valid in the differentiable case.

Therefore, let U be an open subset of V that is endowed with a differentiable section s. Consider the map:

$$\varphi_s : U \times G \to E_U , \qquad (= p^{-1}(U)) \qquad (x, g) \mapsto s(x) \cdot g,$$

where the dot on the right-hand side denotes the action of the element g of G in an element of E.

The map φ_s is differentiable (like the map of the differentiable product manifold $U \times G$ into the differentiable manifold E_U). It is also bijective, and one may easily see that is tangent map is a bijection at every point. It is therefore a diffeomorphism: φ_s^{-1} exists and is differentiable.

It remains to see that the *coordinate change functions* (see Steenrod) are differentiable with values in G. This amounts to seeing that if s' denotes another differentiable section that is defined on U then the following map:

$$g: U \to G, \qquad x \mapsto g(x)$$

is such, that $s'(x) = s(x) \cdot g(x)$ is differentiable. Now, the map g is nothing but the composed map:

$$f \circ \varphi_s^{-1} \circ s',$$

where f is the canonical projection of $U \times G$ onto G. Since each of the maps f, φ_s^{-1}, s' is differentiable, the map g is therefore differentiable.

Q. E. D.

From proposition I-1, we easily see the corollary.

COLROLLARY I.1. – *The Lie groupoid* Φ *is locally isomorphic to the trivial groupoid* $\mathbb{R}^n \times G \times \mathbb{R}^n$, *where n is the dimension of V and G is a Lie group that is isomorphic to the isotropy group of* Φ.

The trivial Lie groupoid $\mathbb{R}^n \times G \times \mathbb{R}^n$ admits $\mathbb{R}^n$ for its space of units and the following composition law:

$$(z, g', y) \cdot (y, g, x) = (z, g' \cdot g, x).$$

In a more precise fashion, the corollary confirms that at every point of V there exists an open neighborhood U and a diffeomorphism:

$$\varphi : (a, b)^{-1}(U \times U) \to \mathbb{R}^n \times G \times \mathbb{R}^n, \qquad z \mapsto (y(z), g(z), x(z)),$$

such that if $a(z') = b(z)$ then one has:

$$x(z') = y(z)$$

and

$$\varphi(z' \cdot z) = \varphi(z') \cdot \varphi(z).$$

Examples.

1) V being a differentiable manifold, let $\Pi^k(V)$ denote the set of invertible jets of order k of V into V (see C. Ehresmann – *a* –). $\Pi^k(V)$ is a Lie groupoid on V with an isotropy group that is isomorphic to the group L_n^k.

2) If E is a (locally trivial) differentiable fiber bundle on V then the set $\Pi(E)$ of linear isomorphisms of the fibers of E onto other fibers of E is a Lie groupoid on V.

3) If Φ and Φ' are two Lie groupoids on V then let $\Phi \times \Phi'$ be the Whitney product of Φ and Φ', or the set of pairs (z, z') of elements of Φ and Φ' that verify $a(z) = a(z')$ and $b(z) = b(z')$. The natural law of composition:

$$(z_1, z_1') \cdot (z, z') = (z_1 \cdot z,\ z_1' \cdot z')$$

determines a Lie groupoid structure on V in $\Phi \times \Phi'$.

2. Associated fiber bundle.

DEFINITION I.2. – *Let* Φ *be a Lie groupoid on V, and let E be a differentiable manifold that is fibered over V; i.e., endowed with a submersion p onto V. We say that E* *is a fiber bundle that is associated to* Φ *if and only if the following conditions are verified:*

1) $\forall z, z \in \Phi$, with $a(z) = x$ and $b(z) = y$, *z determines a diffeomorphism of the fiber* E_x ($= p^{-1}(x)$) *onto the fiber* E_y :

$$\tilde{z} : E_x \to E_y, \quad e \mapsto \tilde{z}(e)\text{, which is denoted } z \cdot e,$$

and one has:

$$\widetilde{z \cdot z'} = \tilde{z} \circ \tilde{z}'.$$

2) *For any differentiable manifold W that is endowed with two differentiable maps f and g with values in* Φ *and E, respectively, such that:*

$$a \circ f = p \circ g,$$

the map $f \cdot g$, *which is then defined:*

$$f \cdot g : W \to E, \; x \mapsto f(x) \cdot g(x),$$

is differentiable.

PROPOSITION I.2. – *If E is a fiber bundle that is associated with a Lie groupoid* Φ *then E is a locally trivial differentiable fiber bundle with fiber F and structure group G, where F is a differentiable manifold that is diffeomorphic to any fiber of E, and G is a Lie group that is isomorphic to the isotropy group of* Φ.

Indeed, let F denote the fiber E_x of E. It is easy to see that E is the fiber bundle that is obtained by modeling F on the principal fiber Φ_x, where the structure group G_x operates on F, according to the definition I-2.

We remark that if E is a differentiable fiber bundle that is obtained by modeling the manifold F on the principal bundle Φ_x then E is an associated fiber bundle, in the sense of the definition I-2 of the Lie groupoid Φ.

Furthermore, when there exists an algebraic structure (group, vector space, algebra, etc.) on each fiber of E that is compatible with its differentiable structure and is such that z is an algebraic isomorphism for any element z of Φ, then E is a differentiable fiber bundle with algebraic structure (fibered into groups, vector spaces, algebras, etc., resp.).

1) *Canonical group fibration associated with a Lie groupoid.* Let Φ be a Lie groupoid on V. Let $G(\Phi)$ denote the set $(a, b)^{-1}(\Delta)$, where Δ is the closed diagonal manifold in $V \times V$. Since the map (a, b) is a submersion, from the Thom lemma, $G(\Phi)$ is a closed differentiable submanifold of $\Phi \cdot G(\Phi)$ is obviously a differentiable fiber bundle on V under the map a or b, and is such that each of its fibers is a Lie group that is the isotropy group of $\Phi \cdot G(\Phi)$ is, on the other hand, canonically associated with Φ, in a manner that is compatible with the algebraic structure of its fibers: It is therefore a differentiable bundle on V that is fibered by groups.

2) *A space of infinitesimal prolongation of order k* of a differentiable manifold *V* is, by definition, a fiber bundle that is associated with the Lie groupoid $\Pi^k(V)$.

3) If *E* and *E′* are two fiber bundles over *V* that are associated with the Lie groupoids Φ and Φ', respectively, then their Whitney product on *V* is again a fiber bundle over *V* that is associated with the groupoid $\Phi \times \Phi'$ that is the Whitney product of Φ and Φ' (see ex. 3, I-1).

3. Lie subgroupoid and groupoid extension.

Let Φ and Φ' be two groupoids on *V*. A *functor* of Φ' to Φ is a map *f*:

$$f: \Phi' \to \Phi,$$

such that:

$$a \circ f = a', \qquad b \circ f = b'$$

and

$$f(z \cdot z') = f(z) \cdot f(z').$$

When Φ and Φ' are differentiable groupoids, a functor is always assumed to be differentiable.

Lie subgroupoid.

We say that Φ' is a Lie subgroupoid of the Lie groupoid Φ if there exists an injective functor of Φ into Φ'.

As in the case of Lie groups, an injective functor is necessarily regular; i.e., everywhere of maximal rank: Φ' is realized by a differentiable submanifold of Φ.

DEFINITION I.3. – *Let E be a fiber bundle on V that is associated with the Lie groupoid* Φ. *A global (differentiable) section of V in E is called regular if and only if for every pair of elements x and y in V there exists a z that is an element of* Φ *with source and target at x and y, respectively, and is such that:*

$$z \cdot s(x) = s(y).$$

PROPOSITION I.3.a. – *Any regular section s of a fiber bundle E that is associated with a Lie groupoid* Φ *canonically defines a Lie subgroupoid* Φ' *of* Φ.

Proof.

Indeed, consider:

$$\Phi' = \{z, z \in \Phi, \text{ such that } z \cdot s(a(z)) = s(b(z))\}.$$

Φ' is a subgroupoid of Φ that is transitive on *V* – viz., the subgroupoid that leaves the section invariant.

Φ' is a Lie subgroupoid if Φ is a differentiable submanifold of Φ. Now, in fact, it will suffice for us to consider things locally – i.e., to suppose that Φ is a trivial Lie groupoid $\mathbb{R}^n \times G \times \mathbb{R}^n$ (see corollary I-1) and that E is a trivial fiber bundle $\mathbb{R}^n \times F$, where F is a differentiable manifold on which the Lie group G operates – that is associated to F in the following manner: If $z \in \Phi$, $z = (x, g, y)$, and $e \in E$, with $e = (y, f)$ then

$$z \cdot e = (x, g \cdot f).$$

Moreover, let there be the regular section s:

$$s : \mathbb{R}^n \to \mathbb{R}^n \times F, \qquad x \mapsto (x, s(x)).$$

Consider the differentiable map:

$$S: \mathbb{R}^n \times G \times \mathbb{R}^n \to F \times F, \qquad (x, g, y) \mapsto (s(x), g \cdot s(y)).$$

It is clear that Φ' is the set $S^{-1}(\Delta)$, where Δ is the diagonal submanifold of $F \times F$. However, since the section s is regular, one may suppose that G operates transitively on F, because otherwise one may take the orbit submanifold $G \cdot s(x)$ in place of F. Furthermore, the map S is transversal to Δ, so from the Thom transversality theorem, $S^{-1}(\Delta)$ is a closed differentiable submanifold of Φ.

Q. E. D.

Examples.

An infinitesimal structure of order k on the differentiable manifold V is the given of a (differentiable) section of a fiber bundle that is associated with the Lie groupoid $\Pi^k(V)$. The infinitesimal structure is regular if that section is regular. It thus determines a Lie subgroupoid of $\Pi^k(V)$, and that subgroupoid is what one calls a *G-structure on V* (G, a subgroup of L_n^k that is isomorphic to the isotropy subgroup of a subgroupoid).

1) Let T^* denote the cotangent bundle of V, so T^* is associated with the Lie groupoid $\Pi^1(V)$. The symmetric product $S^2(T^*)$, in the Whitney sense, of T^* with itself is again associated with $\Pi^1(V)$. Having said this, a *pseudo-Riemannian* structure on V is the given of a non-zero regular section of $S^2(T^*)$.

2) The exterior product $\Lambda^2 T^*$, in the Whitney sense, of T^* is likewise associated with $\Pi^1(V)$. A section of $\Lambda^2 T^*$ that is everywhere of the same rank is a regular section. In the case where the rank of the section (viz., the 2-form) is everywhere equal to the dimension of V, which must then be even, from a theorem of Lepage, one has what one calls an *almost-symplectic* structure on V. An almost-symplectic structure is therefore regular.

Groupoid extension.

Let Φ and Φ' be two groupoids on V. Φ is called a *groupoid extension* of Φ' if there exists a surjective functor φ of Φ onto Φ'.

Just as in the case of Lie groups, a surjective functor of a Lie groupoid Φ onto another Lie groupoid Φ' is necessarily everywhere of maximal rank: i.e., Φ is fibered over Φ'.

Moreover, to any extension Φ of Φ', Φ and Φ' being two Lie groupoids, there corresponds the following exact sequence of group bundles on V:

$$1 \to N(\varphi) \to G(\Phi) \to G(\Phi') \to 1. \tag{1}$$

If we call any functor ρ of Φ' to Φ such that $\varphi \circ \rho = \mathrm{Id}$ a *reduction* of Φ to Φ' then to any reduction of Φ' to Φ there corresponds a splitting of the exact sequence of group bundles. Moreover, that splitting is regular, in the sense that for any pair of elements x and y of V there exists a z, which is an element of Φ with source and target at x and y, respectively, such that:

$$g \in G(\Phi'), \qquad z \cdot \rho(g) \cdot z^{-1} = \rho(\varphi(z) \cdot g \cdot \varphi(z)^{-1}).$$

Conversely, we have the proposition:

PROPOSITION I.3.b. – *If Φ is a Lie groupoid extension of the Lie groupoid Φ' then any splitting of the exact sequence of group bundles* (1) *determines a reduction of Φ' to Φ when it has the property that for any point x of V, $G_x(\Phi')$ operates on $N(\varphi)|_x$ by the adjoint operation with no other fixed point than the neutral element.*

Indeed, let ρ denote the lift of this splitting. One can immediately confirm that the set of elements z of Φ such that:

$$z \cdot \rho(g) \cdot z^{-1} = \rho(\varphi(z) \cdot g \cdot \varphi(z)^{-1})$$

for any g of $G_x(\Phi')$ with $x = a(z)$ is a Lie subgroupoid of Φ that is isomorphic to Φ' by the functor φ.

CHAPTER II

PROLONGATIONS OF FIBER BUNDLES AND DIFFERENTIAL OPERATORS

Suppose is given (E, p, V), i.e., a differentiable manifold E fibered over the differentiable manifold V by the submersion p. Let $J_k(E, p, V)$, or, when there is no risk of confusion, simply $J_k(E)$, the set of jets of order k of (differentiable) sections of E. It is again a differentiable manifold that is fibered over V by the source map that is called the *prolongation of order k* of the fiber bundle E, and if s is a differentiable section of E then the map:

$$j^k s : V \to J_k(E, p, V), \quad x \mapsto j_x^k s$$

is a differentiable section of V in $J_k(E, p, V)$.

If E is a vector bundle then we can show that the prolongation of the bundle is also a vector bundle and by defining the Spencer operator on the prolongation of the bundle we can make a contribution to the study of differential operators.

1. Prolongation of Lie groupoids.

If Φ is a Lie groupoid on V then consider the set:

$$\Phi^k \subset J_k(\Phi, a, V),$$

which is such that if X is a jet of order k of a section of (Φ, a, V) then $X \in \Phi^k$ if and only if $bX \in \Pi^k(V)$, bX denoting the composition of jets.

PROPOSITION II.1.a. – *The set Φ^k admits a canonical structure of a Lie groupoid on V.*

Indeed, consider the following maps to be the source and target maps of Φ^k onto V:

$$a_k : \Phi^k \to V, \quad X \mapsto \alpha(X),$$

where $\alpha(X)$ is the source of the jet X, and:

$$b_k : \Phi^k \to V, \quad X \mapsto b(\beta(X)),$$

where $\beta(X)$ the target of the jet X.

If X and X' are two elements of Φ^k that satisfy $a_k(X) = b_k(X')$ then one defines the composition:

$$X \cdot X' = (X\, bX') \cdot X',$$

where $X\, bX'$ is the composition of jets, and the dot in the right-hand side is composed in the following manner: If $Z = j_x^k f$ and $Z' = j_x^k g$, f and g being two differentiable maps of

W into Φ such that $a \circ f = b \circ g$ then $Z \cdot Z'$ is the jet $j_x^k(f \cdot g)$ of the map $f \cdot g$ (see definition I-1-b), a jet that depends upon only the jet of order k of f and g at the point x.

When endowed with this law of internal and partial composition, Φ^k is obviously a groupoid on V. To show that it is, in fact, a Lie groupoid on V it will suffice to consider things locally; i.e., to suppose that Φ is a trivial Lie groupoid $\mathbb{R}^n \times G \times \mathbb{R}^n$, and this is left to the reader.

Q. E. D.

PROPOSITION II.1.*b*. – *If E is a differentiable fiber bundle that is associated with a Lie groupoid* Φ *then the prolongation of the bundle* $J_k(E)$ *is canonically associated with the Lie groupoid* Φ^k .

Indeed, let $Z \in \Phi^k$, with $a_k(Z) = x$, and let $X \in J_k(E)$ with source x. Set:

$$Z \cdot X = (Z\,(b\,Z)^{-1}) \cdot (X\,(bZ)^{-1}),$$

where the elements between parentheses are the compositions of jets and the dot in the right-hand side is defined as follows: If $Y = j_x^k f$ and $Y' = j_x^k g$, f and g being two maps of W into Φ and E, respectively, such that $a \circ f = p \circ g$ then one has $Y \cdot Y' = j_x^k(f \cdot g)$, which is the jet of the map $f \cdot g$ of W into E (see definition II.2), a jet that depends upon only the jet of order k of f and g at x.

$Z \cdot X$ is then a jet of a section of E with source y ($= b_k(Z)$), and Φ^k thus operates on $J_k(E)$. For the condition of differentiability (axiom 2 of definition I.2), it is again obviously sufficient to regard matters locally; i.e., to suppose that Φ is a trivial Lie groupoid $\mathbb{R}^n \times G \times \mathbb{R}^n$, and that E is the trivial bundle $\mathbb{R}^n \times F$, G being a Lie groupoid that operates on the manifold F; this is left to the reader.

Q. E. D.

Φ^k will be called the *prolongation of order k* of the groupoid Φ. It is an extension groupoid of the product groupoid $\Phi \times \Pi^k(V)$ by the canonical functor:

$$\rho\colon \Phi^k \to \Phi \times \Pi^k(V), \quad Z \mapsto (\beta(Z), bZ).$$

Furthermore, let ρ_r denote the canonical map that associates any jet of order k with the jet of lower order r. When applied to Φ^k, it is a surjective functor of Φ^k onto Φ^r. When applied to $J_k(E)$, it is a V-morphism of surjective bundles of $J_k(E)$ onto $J_r(E)$, and one has:

$$Z \in \Phi^k, \qquad X \in J_k(E), \qquad \rho_r(Z \cdot X) = \rho_r(Z) \cdot \rho_r(X).$$

2. Prolongation of vector bundles.

In all of what follows, E will denote a (differentiable and locally trivial) vector bundle on a manifold V. Recall that $\Pi(E)$ – viz., the set of all linear isomorphisms of fibers of E to fibers of E – is a Lie groupoid to which the fiber E is associated.

PROPOSITION II.2.a. – *The bundle prolongation $J_k(E)$ is a vector bundle.*

Indeed, if $J_k(E)$ is a fiber bundle that is associated with the Lie groupoid $\Pi^k(E)$ then it suffices to show that each fiber of $J_k(E)$ is a vector bundle such that if Z is an element of $\Pi^k(E)$ with source x and target y then Z is a linear isomorphism of $J_k(E)|_x$ onto $J_k(E)|_y$.

Therefore, let $X = j_x^k s$ and $X' = j_x^k s'$ (s and s' being two sections of E); set:

$$X + X' = j_x^k (s + s'), \qquad \lambda \in \mathbb{R},\ \lambda X = j_x^k (\lambda s).$$

When endowed with this law of composition, $J_k(E)|_x$ is obviously a vector bundle, and it is easy to confirm that the elements of $\Pi^k(E)$ are linear isomorphisms of the fibers of $J_k(E)$ to other fibers.

Q. E. D.

PROPOSITION II.2.b. – *For any integer k, we have the following exact sequence of vector bundles over V:*

$$0 \to E \otimes S^k(T^*) \to J_k(E) \xrightarrow{\rho_{k-1}} J_{k-1}(E) \to 0,$$

where $E \otimes S^k(T^*)$ is the tensor product, in the Whitney sense, of E with $S^k(T^*)$, which is the symmetric product, in the Whitney sense, of k examples of the cotangent bundle T^* of the base manifold V, and ρ_{k-1} is the canonical morphism that associates any jet of a section of order k with the jet of lower order k – 1.

This proposition is an immediate consequence of the lemmas above.

LEMMA 1. – *At any point of V, we have the following exact sequence of vector spaces:*

$$0 \to E \otimes S^k(T^*)|_x \to J_k(E)|_x \xrightarrow{\rho_{k-1}} J_{k-1}(E)|_x \to 0.$$

Indeed, in order to prove the lemma, one may obviously suppose that E is the trivial bundle $\mathbb{R}^n \times F$, n being the dimension of V, and F, a vector space that is isomorphic to the fiber of E. Since $J_k(E)$ is the trivial bundle $\mathbb{R}^n \times T_n^k(F)$, denoting the set of jets of order k of source 0 in $\mathbb{R}^n$ into F by $T_n^k(F)$ the lemma is nothing but the exact sequence of vector spaces:

$$0 \to E \otimes S^k(\mathbb{R}^{n*}) \to T_n^k(F) \xrightarrow{\rho_{k-1}} T_n^{k-1}(F) \to 0,$$

an exact sequence that one establishes immediately by means of the polynomial representation of jets of $T_n^k(F)$ by starting with the given of a basis for F and the canonical basis for $\mathbb{R}^n$.

LEMMA 2. – *For any Z that is an element of $\Pi^k(E)$ with source x and target y, we have the following isomorphism of exact sequences:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E \otimes S^k(T^*)|_x & \longrightarrow & J_k(E)|_x & \longrightarrow & J_{k-1}(E)|_x & \longrightarrow & 0 \\ & & \rho_0(Z)\downarrow & & \downarrow Z & & \downarrow \rho_{k-1}(Z) & & \\ 0 & \longrightarrow & E \otimes S^k(T^*)|_y & \longrightarrow & J_k(E)|_y & \longrightarrow & J_{k-1}(E)|_y & \longrightarrow & 0 \end{array}$$

where ρ_0 is the canonical functor of $\Pi^k(E)$ onto $\Pi(E) \times \Pi^1(V)$.

In order to establish this isomorphism, it again suffices to make a local study; i.e., by trivializing E in the neighborhood of x and y, which is left to the reader.

Q. E. D.

Remark.

Recall that the vector bundles on V form an additive category. It is easy to establish that:

$$J_k : E \to J_k(E)$$

is an exact functor of that additive category into itself, and if h is a V-morphism of differentiable vector bundles:

$$h : E \to E'$$

then one has:

$$\rho_{k-1} \circ J_k(h) = J_{k-1}(h) \circ \rho_{k-1},$$

and, in particular, the restriction of $J_k(h)$ to the sub-bundle $E \otimes S^k(T^*)$ has its values in the sub-bundle $E' \otimes S^k(T^*)$, and that is nothing but the morphism $h \otimes \mathrm{I}d$.

3. *D* operator and the Spencer exact sequence.

From the proposition II.2.b, we thus have, in particular:

$$0 \to J_k(E) \otimes T^* \to J_1[J_k(E)] \xrightarrow{\rho} J_k(E) \to 0.$$

One confirms immediately that $J_{k+1}(E)$ is a differentiable sub-bundle of $J_1[J_k(E)]$, and that the restriction of the morphism ρ to the sub-bundle is nothing but the canonical morphism ρ_k.

Consider a differentiable section s of $J_{k+1}(E)$; s and $j^1(\rho_k \circ s)$ are two sections of $J_1[J_k(E)]$, which compose with the morphism ρ to give the same section of $J_k(E)$. From the preceding exact sequence, $j^1(\rho_k \circ s) - s$ is a section of $J_k(E) \otimes T^*$. Therefore, we define an operator D; i.e., an $\mathbb{R}$-linear V-morphism of the sheaf of differentiable sections of $J_{k+1}(E)$ with values in the sheaf of differentiable sections of $J_k(E) \otimes T^*$:

$$D: J_{k+1}(E) \to J_k(E) \otimes T^*, \qquad s \mapsto j^1(\rho_k \circ s) - s,$$

an operator that we call the *Spencer operator* (see Spencer ([2])).

LEMMA 1. – *If s is a differentiable section of* $J_{k+1}(E)$ *and f is a differentiable function on the base manifold V then one has:*

$$D(fs) = f\,D(s) + (\rho_k \circ s) \otimes df,$$

where df is the exterior differential of f.

This lemma is an immediate consequence of the following remark: If s is a differentiable section of the vector bundle E, and f is a differentiable function on the base manifold V then one has:

$$j^1(fs) = f\,j^1(s) + s \otimes df,$$

where $s \otimes df$ is a differentiable section of $E \otimes T^*$, which is a vector sub-bundle of $J_1(E)$, from proposition II.2.b

Consider a V-morphism h of differentiable vector bundles:

$$h: E \to E'.$$

From the remarks made at the end of paragraph II.2, it is immediate that we have:

LEMMA 2.

$$D \circ J_{k+1}(h) = (J_k(h) \otimes \mathrm{I}d) \circ D.$$

In particular, $J_{k+1}(E)$ *is a differentiable sub-bundle of* $J_{k-r}[J_{r+1}(E)]$, *and the restriction of* $J_{k-r}(\rho_r)$ *to that subspace is nothing but the canonical morphism* ρ_{k-1} .

([2]) Recall that the sheaf of (differentiable) sections of a vector bundle is a sheaf of $\mathcal{D}$-modules, so it is, in particular $\mathbb{R}$-linear, $\mathcal{D}$ being the sheaf of differentiable functions on the base manifold. We also point out that in this paper any vector bundle and its sheaf of sections will be represented by the same symbol, the context making it precise in each case which interpretation one must consider.

LEMMA 2′.

$$D \circ \rho_{k+1} = (\rho_r \otimes \mathrm{I}d) \circ D.$$

Finally, s being a differentiable section of $J_{r+1}(E)$, we say that section is *integrable* if and only if $s = j^{k+1}\sigma$, where σ is a section of E.

LEMMA 3. – *If s is a differentiable section of $J_{r+1}(E)$ then one has $D(s) = 0$ – viz., the zero section of $J_{r+1}(E) \otimes T^*$ – if and only if s is an integrable section.*

Indeed, it is clear from the definition itself of the operator D that if s is an integrable section then $D(s) = 0$. We prove that if $D(s) = 0$ then s is an integrable section. Now, the property is obvious when $k = 0$; it thus suffices to prove this by recurrence on the integer k. Consider the section $\rho_k \circ s$ of $J_k(E)$; from Lemma 2′, one has:

$$D(\rho_k \circ s) = (\rho_{k-1} \otimes \mathrm{I}d) \circ D(s) = 0.$$

Since $D(\rho_k \circ s)$ is zero, by the recurrence hypothesis, one has:

$$\rho_k \circ s = j^k\sigma,$$

σ being a section of E.

One then has:

$$D(s) = j^1(\rho_k \circ s) - s = 0,$$

$$s = j^1(\rho_k \circ s) = j^1(j^k\sigma) = j^{k+1}\sigma.$$

THEOREM II.3.a. – *If E is a differentiable vector bundle on V then there exists one and only one operator D of $J_{k+1}(E)$ into $J_k(E) \otimes T^*$ such that:*

1) *If s is a differentiable section of $J_{k+1}(E)$ then $D(s) = 0$ – i.e., the zero section of $J_k(E) \otimes T^*$ – if and only if s is an integrable section.*

2) *If f is a differentiable function on V then:*

$$D(fs) = f\,D(s) + (\rho_k \circ s) \otimes df$$

in which df is the exterior differential of f.

From lemmas 1) and 3), the previously-defined Spencer operator verifies the two properties of the theorem,. It remains for us to see that these properties are indeed characteristic, or that if D' is an operator of $J_{k+1}(E)$ into $J_k(E) \otimes T^*$ that verifies these two properties then one has $D' = D$, the Spencer operator. Now, these two operators are identical if they coincide on the local sections of $J_{k+1}(E)$. Therefore, let s be a local section of $J_{k+1}(E)$:

$$s = f^i\, j^{k+1}\sigma_i \qquad (1 < i < q),$$

in which f^i and σ_i are differentiable functions on V and differentiable sections of E, respectively, and one has, from properties 1) and 2):

$$D(s) = j^k\sigma_i \otimes df^i = D'(s). \qquad \text{Q. E. D.}$$

The property 2) permits us to prolong in a natural fashion:

$$D: J_{k+1}(E) \otimes \Lambda^p T^* \to J_k(E) \otimes \Lambda^{p+1} T^*, \qquad s \otimes \omega \mapsto D(s) \wedge \omega + (\rho_k \circ s) \otimes d\omega,$$

in which the notations have an obvious significance. Likewise, for the prolonged operator we have:

$$D \circ (\rho_k \otimes \mathrm{I}d) = (\rho_k \otimes \mathrm{I}d) \circ D. \tag{1}$$

Denoting the composed operator $D \circ D$ by D^2, we have:

$$D^2 : J_{k+1}(E) \otimes \Lambda^p T^* \xrightarrow{D} J_k(E) \otimes \Lambda^{p+1} T^* \xrightarrow{D} J_{k-1}(E) \otimes \Lambda^{p+2} T^*.$$

LEMMA 1. – $D^2 = 0$, *the zero operator, which associates any differentiable section of* $J_{k+1}(E) \otimes \Lambda^p T^*$ *with the zero section of* $J_{k-1}(E) \otimes \Lambda^{p+2} T^*$.

Indeed, since it is the composition of two operators, D^2 is again an operator. It suffices for us to verify that for any local section s of $J_{k+1}(E) \otimes \Lambda^p T^*$, one has $D^2(s) = 0$. Now, locally:

$$s = j^{k+1}\sigma_i \otimes \omega^i \qquad (1 \le i \le q),$$

where σ_i are differentiable sections of E, and ω^i are exterior p-forms on V. Thus, one has:

$$D(s) = j^k\sigma_i \otimes d\omega^i,$$
$$D^2(s) = j^{k-1}\sigma_i \otimes d^2\omega^i = 0, \qquad \text{because } d^2 = 0.$$

Q. E. D.

The restriction of the operator D to the sub-sheaf:

$$E \otimes S^{k+1}(T^*) \otimes \Lambda^p T^*$$

of the sheaf $J_{k+1}(E) \otimes \Lambda^p T^*$ is, in fact, $\mathcal{D}$-linear, and thus defines a V-morphism of differentiable vector bundles from:

$$E \otimes S^{k+r}(T^*) \otimes \Lambda^p T^*$$

into $J_k(E) \otimes \Lambda^{p+r}\mathrm{T}^*$ that is denoted by δ. Formula (1) above shows that this morphism takes its values in the vector sub-bundle $E \otimes S^k(T^*) \otimes \Lambda^{p+1} T^*$, and one sees that δ is nothing but the morphism:

$$\delta: E \otimes S^{k+r}(T^*) \otimes \Lambda^p T^* \to E \otimes S^k(T^*) \otimes \Lambda^{p+r} T^*,$$

$$e \otimes a^{k+r} \otimes \omega \mapsto -(k+1)\, e \otimes a^k \otimes (a \wedge \omega).$$

We then have the following lemma, which was established by Koszul (Séminaire de Cartan, theorem, exposé 20, 1949-1950).

LEMMA 2. – *The following sequence of vector bundles is exact:*

$$0 \to E \otimes S^{k+1}(T^*) \xrightarrow{\delta} E \otimes S^k(T^*) \otimes T^* \xrightarrow{\delta} E \otimes S^{k-1}(T^*) \otimes \Lambda^2 T^* \xrightarrow{\delta} \ldots$$

THEOREM II.3.b. – *For any integer k, we have the following exact sequence of* $\mathbb{R}$-*linear sheaves:*

$$0 \to E \xrightarrow{j^{k+1}} J_{k+1}(E) \xrightarrow{D} J_k(E) \otimes T^* \xrightarrow{D} J_{k-1}(E) \otimes \Lambda^2 T^* \xrightarrow{D} \ldots$$

in which j^{k+1} *is the canonical operator that associates any differentiable section of E with the differentiable section* $j^{k+1}s$ *of* $J_{k+1}(E)$.

(It is intended that in Lemma 2 and the theorem above that in everything we have adopted the following convention:

$S^0(T^*) = \mathbb{R} \times V$, the trivial fiber bundle over V,

$S^k(T^*) = 0$, if $k < 0$,

and:

$J_0(E) = E$, $J_k(E) = 0$ if $k < 0$).

Proof.

1) We indeed have the following exact sequence:

$$0 \to E \xrightarrow{j^{k+1}} J_{k+1}(E) \xrightarrow{D} J_{k+1}(E) \otimes T^*,$$

from the characteristic property 1) of the operator D and the fact that the operator j^{k+1} is obviously injective.

2) The sequence that we defined is cohomological because $D^2 = 0$ (Lemma 1). It remains for us to prove the exactness of the sequence. Now, it is clear that the following sequence is exact:

$$J_1(E) \otimes \Lambda^p T^* \xrightarrow{D} E \otimes \Lambda^{p+1} T^* \to 0,$$

since the restriction of the operator D to the sub-sheaf $E \otimes T^* \otimes \Lambda^{p+1} T^*$ is already surjective, from Lemma 2.

We then prove by recurrence on the integer k that we have the following exact sequence of $\mathbb{R}$-linear sheaves:

$$J_{k+1}(E) \otimes \Lambda^{p-1} T^* \xrightarrow{D} J_k(E) \otimes \Lambda^p T^* \xrightarrow{D} J_{k-1}(E) \otimes \Lambda^{p+1} T^*.$$

Therefore, let s_k^p be a differentiable section of $J_k(E) \otimes \Lambda^p T^*$ such that $D(s_k^p) = 0$. The section $\rho_{k-1} \circ s_k^p$ of $J_k(E) \otimes \Lambda^p T^*$ is also such that $D(\rho_{k-1} \circ s_k^p) = (\rho_{k-1} \otimes \mathrm{Id}) \circ D(s_k^p) = 0$. Then, by the recurrence hypothesis, there exists a section σ_k^{p-1} of $J_k(E) \otimes \Lambda^{p-1} T^*$ such that:

$$D(\sigma_k^{p-1}) = \rho_{k-1} \circ s_k^p .$$

Since the morphism ρ_k is surjective, we may find s_{k+1}^{p-1}, which is a section of $J_{k+1}(E) \otimes \Lambda^{p-1} T^*$ such that:

$$\rho_k \circ s_{k+1}^{p-1} = \sigma_k^{p-1} .$$

Consider the section $s_k^p - D(s_{k+1}^{p-1})$ of $J_k(E) \otimes \Lambda^p T^*$. It is, in fact, a section of the sub-bundle $E \otimes S^k(T^*) \otimes \Lambda^p T^*$, because:

$$\rho_{k-1} \circ (s_k^p - D(s_{k+1}^{p-1})) = \rho_{k-1} \circ s_k^p - D(\rho_k \circ s_{k+1}^{p-1}) = 0.$$

It is, moreover, annulled by the morphism δ:

$$\delta(s_k^p - D(s_{k+1}^{p-1})) = D(s_k^p - D(s_{k+1}^{p-1})) = D(s_k^p) = 0.$$

From lemma 2, there thus exists a section n_{k+1}^{p-1} of:

$$E \otimes s^{k-1}(T^*) \otimes \Lambda^{p-1} T^*,$$

such that:

$$\delta(n_{k+1}^{p-1}) = s_k^p - D(s_{k+1}^{p-1}) .$$

Hence:

$$s_k^p = D(n_{k+1}^{p-1} + s_{k+1}^{p-1}),$$

when $n_{k+1}^{p-1} + s_{k+1}^{p-1}$ is a section $J_{k+1}(E) \otimes \Lambda^{p-1} T^*$.

Q. E. D.

If we let ρ_k denote the canonical morphism of $J_1[J_k(E)]$ onto $J_k(E)$, and again let D be the Spencer operator of $J_1[J_k(E)]$ into $J_k(E) \otimes T^*$ then the restriction of this morphism and that operator to the differentiable sub-bundle $J_{k+1}(E)$ of $J_1[J_k(E)]$ are the maps that were considered above that are denoted by the same letters.

COROLLARY. – *In order for a differentiable section s of* $J_1[J_k(E)]$ *to be a section with values in the sub-bundle* $J_{k+1}(E)$*, it is necessary and sufficient that:*

1) $D(\rho_k \circ s) = \rho_{k-1} \circ D(s)$,
2) $D^2(s) = 0$, *the zero section of* $J_{k-1}(E) \otimes \Lambda^2 T^*$.

The conditions are obviously necessary. They are sufficient; indeed, let s be a section of $J_1[J_k(E)]$ such that conditions 1) and 2) are verified. $D(s)$ is then a section of $J_k(E) \otimes T^*$ such that:

$$D \circ D(s) = D^2(s) = 0,$$

so, from the last theorem, there exists a section σ of $J_{k+1}(E)$ such that:

$$D(\sigma) = D(s).$$

However, $s - \sigma$ is then a section of $J_1[J_k(E)]$ such that:

$$D(s - \sigma) = D(s) - D(\sigma) = 0.$$

From the first characteristic property of the operator D, $s - \sigma = j^1\chi$, with χ a differentiable section of $J_k(E)$, and one has:

$$D(\chi) = D(\rho_k \circ s - \rho_k \circ s) = (\rho_{k-1} \otimes \mathrm{Id})\, D(s - \sigma) = 0,$$
$$\chi = j^k\eta,$$

with η a section of E and:

$$\begin{aligned} s &= j^1\chi + \sigma = j^1(j^k\eta) + \sigma, \\ &= j^{k+1}\eta + \sigma, \end{aligned}$$

a section of $J_{k+1}(E)$.

Q. E. D.

We conclude this paragraph by making the remark that the Spencer operator D splits the sheaf $J_1(E)$ into a direct sum of two $\mathbb{R}$-linear sheaves E and $E \otimes T^*$, or more precisely, two sections s and s' of $J_k(E)$ are identical if and only if:

$$\rho_{k-1} \circ s = \rho_{k-1} \circ s',$$

and:

$$D(s) = D(s').$$

4. A differential operator and its prolongation.

Let E and F be two vector bundles over the same base V. Recall that we call any $\mathbb{R}$-linear V-morphism ∂ of the sheaf E into the sheaf F an *operator*.

DEFINITION II.4. – *An operator* ∂ *of E into F will be called a differential operator of order k if, s being a differentiable section of E,* $j_x^k s = 0$ *entails that the* $\partial(s)$ *in F is zero at the point x.*

Any V-morphism of differential fiber bundles is obviously equivalent to a differential operator of order 0, which is nothing but a $\mathcal{D}$-linear operator ($\mathcal{D}$, the sheaf of differentiable functions on the base manifold V).

The Spencer operator D is, from its second characteristic property, a differential operator of order 1 of $J_k(E)$ into $J_{k-1}(E) \otimes T^*$.

If ∂ is a differential operator of order k of E into F, and ∂' is a differential operator of order r of F into G then the composition $\partial' \circ \partial$ is obviously a differential operator of order $r + k$ of E into G.

Consider the operator:

$$j^k: E \to J_k(E).$$

It is a differential operator of order k, and one easily establishes the following theorem (see R. Palais, chap. IV, *Seminar on the Atiyah-Singer Index Theorem*).

THEOREM II.4. – *To any differential operator* ∂ *of order k of E into F there corresponds one and only one V-morphism of the vector bundle* $h(\partial)$ *of* $J_k(E)$ *into F such that:*

$$\partial = h(\partial) \circ j^k.$$

Let ∂ be a differential operator of order k of E into F. We call the operator $j^k \circ \partial$ that maps E into $J_r(F)$ the *prolongation of order k* of ∂. We have, in an obvious fashion:

$$h(j^k \circ \partial) = J_r(h(\partial)),$$

or more exactly, the restriction of the V-morphism $J_r(h(\partial))$ to the sub-bundle $J_{k+1}(E)$ of $J_r[J_k(E)]$. For any integer $r \geq k$, let:

$$\mathcal{S}_r = \ker(h(j^{r-k} \circ \partial))$$

be the sub-sheaf of ∂-modules of the sheaf $J_r(E)$ that is formed from the sections that are annulled by the operator $h(j^{r-k} \circ \partial)$, and agree that:

$$\mathcal{S}_r = J_r(E) \qquad \text{for } r < k,$$

and that:

$$\mathcal{S}_r^p = \mathcal{S}_r \otimes \Lambda^p T^*$$

is the tensor product, in the Whitney sense, of the two sheaves of $\mathcal{D}$-modules. From the definition of $h(j^{r-k} \circ \partial)$, we have:

1) $$\rho_r \otimes \mathrm{Id}: \mathcal{S}_{r+1}^p \to \mathcal{S}_r^p,$$

2) $$D : \mathcal{S}_{r+1}^{p} \to \mathcal{S}_{r}^{p+1}.$$

Hence, one has the Spencer cohomology sequence relative to the differential operator ∂:

$$0 \to \Theta \xrightarrow{j^r} \mathcal{S}_r \xrightarrow{D} \mathcal{S}_{r-1} \xrightarrow{D} \mathcal{S}_{r-2}^{2} \xrightarrow{D} \ldots,$$

where Θ denotes the sheaf of solutions of ∂ – i.e., of sections of E that are annulled by the operator ∂. For $r \geq k$, the start of the sequence:

$$0 \to \Theta \xrightarrow{j^r} \mathcal{S}_r \xrightarrow{D} \mathcal{S}_{r-1}$$

is obviously exact.

Upon denoting the sub-sheaf of $\mathcal{D}$-modules of $\mathcal{S}_{r+1}^{p}$, we have the cohomology sequence of sheaves of $\mathcal{D}$-modules:

$$\cdots \xrightarrow{\delta} \mathcal{N}_{r+q}^{p-q} \xrightarrow{\delta} \mathcal{N}_{r+q-1}^{p-q+1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{N}_{r}^{p} \xrightarrow{\delta} \cdots,$$

whose corresponding cohomology sheaves will be denoted by $\mathcal{H}_r^p$.

Suppose that the operator ρ_{r-1}: $\mathcal{S}_r \to \mathcal{S}_{r-1}$ is surjective, so likewise for any p, the operator $\rho_{r-1} \otimes \mathrm{Id}$: $\mathcal{S}_r^p \to \mathcal{S}_{r-1}^p$ is also surjective. Let λ denote an arbitrary lift – i.e. a V-morphism of sheaves:

$$\lambda: \mathcal{S}_{r-1}^{p} \to \mathcal{S}_{r}^{p}, \qquad (\rho_{r-1} \otimes \mathrm{Id}) \circ \lambda = \mathrm{Id}.$$

The morphism:

$$D \circ \lambda \circ D: \mathcal{S}_r \to \mathcal{S}_{r-1}^{2}$$

is such that:

$$(\rho_{r-1} \otimes \mathrm{Id}) \circ D \circ \lambda \circ D = D \circ (\rho_{r-1} \otimes \mathrm{Id}) \circ \lambda \circ D = D^2 = 0.$$

It thus maps $\mathcal{S}_r$ into $\mathcal{N}_{r-1}^{2}$ and takes values that are annulled by the operators D or δ; it thus passes to the quotient to define a morphism:

$$m = D \circ \lambda \circ D: \mathcal{S}_r \to \mathcal{H}_{r-1}^{p}.$$

One immediately verifies that this morphism is a $\mathcal{D}$-linear operator ($\mathcal{H}_{r-1}^{p}$ being obviously a sheaf of $\mathcal{D}$-modules that is defined independently of the choice of the lift λ).

THEOREM II.4.b – *If the operator:*

$$\rho_{r-1} : \mathcal{S}_r \to \mathcal{S}_{r-1}$$

is surjective then this canonically defines a $\mathcal{D}$-linear operator:

$$m : \mathcal{S}_r \to \mathcal{H}^2_{r-1},$$

such that the following sequence of sheaves of $\mathcal{D}$-modules is exact:

$$\mathcal{S}_{r+1} \xrightarrow{\rho_r} \mathcal{S}_r \xrightarrow{m} \mathcal{H}^2_{r-1}.$$

Proof.

Indeed, if the operator m is defined as before then one confirms immediately that:

$$m \circ \rho_r = 0.$$

It remains for us to prove the exactness of the sequence. The reasoning consists of two steps, which we make precise in the form of a Lemma.

LEMMA 1. – *If s is a section of $\mathcal{S}_r$ then there exists a section σ of $J_1[J_r(E)]$ that verifies the following conditions:*

1) $J_1[h(j^{r-k} \circ \partial)](\sigma) = 0,$
2) $\rho(\sigma) = s,$
3) $(\rho_{r-1} \otimes \text{Id}) \circ D(\sigma) = D(s).$

Indeed, take a lift λ of $\mathcal{S}^1_{r-1}$ to $\mathcal{S}^1_r$. Let η denote the section $-\lambda \circ D(s)$, which we consider to be a section of $J_r(E) \otimes T^*$, which is a sub-bundle of $J_1[J_r(E)]$. The section η is therefore such that:

1) $J_1[h(j^{r-k} \circ \partial)](\eta) = -(h(j^{r-k} \circ \partial) \otimes \text{Id}) \circ \lambda \circ D(s) = 0,$
2) $\rho(\eta) = 0,$
3) $D(\eta) = \lambda \circ D(s).$

The section $j^1 s$ is a section of $J_1[J_r(E)]$ such that:

1) $J_1[h(j^{r-k} \circ \partial)](j^1 s) = j^1[h(j^{r-k} \circ \partial)(s) = 0,$
2) $\rho(j^1 s) = s,$
3) $D(j^1 s) = 0.$

The section $\sigma = j^1 s + \eta$ is therefore the section that responds to the conditions of the lemma.

LEMMA 2. – *If s is a section of $\mathcal{S}_r$ such that $m(s) = 0$ then there exists a section χ of $J_1[J_r(E)]$ that verifies the three conditions of lemma* 1, *and the following fourth condition:*

4) $D \circ D(\chi) = 0.$

Indeed, always letting λ be a lift of $\mathcal{S}_{r-1}$ into $\mathcal{S}_r$, we say that $m(s) = 0$, i.e., that:

$$D \circ \lambda \circ D = \delta(\eta),$$

with η, a section of $\mathcal{N}_r^1$. We may consider η to be a section of $J_r(E) \otimes T^*$, which is a sub-bundle of $J_1[J_r(E)]$ such that:

1) $J_1[h(j^{r-k} \circ \partial)](\eta) = 0,$
2) $\rho(\eta) = 0,$
3) $(\rho_{r-1} \otimes \mathrm{Id}) \circ D(\eta) = -(\rho_{r-1} \otimes \mathrm{Id})(\eta) = 0.$

If σ is a section as in the Lemma then the section:

$$\chi = \sigma + \eta$$

is a section of $J_1[J_r(E)]$ that verifies the conditions of Lemma 2.

Now, from the corollary to the theorem II.3.b, the section χ of Lemma 2, in fact, has its values in the sub-bundle $J_{r+2}(E)$ of $J_1[J_r(E)]$. It is therefore a section of $\mathcal{S}_{r+1}$ such that $\rho_r(\chi) = s$.

Q. E. D.

Remark.

1) One of the important problems of analysis is to know whether the Spencer cohomology sequence relative to a given differential operator ∂ is exact or not ([3]).

2) ∂ being a differential operator of order k, for $r \geq k$, the exactness of the sequence:

$$0 \to \Theta \xrightarrow{j^r} \mathcal{S}_r \xrightarrow{D} \mathcal{S}_{r-1},$$

signifies that the integrable sections $j^r s$ of $\mathcal{S}_r$, are nothing but the prolongations of the sections of E that are solutions of ∂, i.e.:

$$\partial(s) = 0.$$

We say that the differential operator of order k is *completely integrable* to order r ($\geq k$) if and only if $\mathcal{S}_r$ is locally generated by the integrable sections, i.e.:

If s is a section of $\mathcal{S}_r$ then one has locally, in the neighborhood of any point of V: $s = f^i j^r \sigma_i$, $1 \leq i \leq p$, where f^i are differentiable functions on the base manifold V, and $j^r \sigma_i$ are integrable sections of $\mathcal{S}_r$.

If the differential operator is completely integrable of order r then the morphism:

([3]) Indeed, D. G. Quillen has proved that showing the exactness of the Spencer sequence is equivalent to finding the necessary and sufficient conditions on the section f of F for there to exist a section s of E such that $\partial(s) = f$.

$$\rho_r : \mathcal{S}_{r+1} \to \mathcal{S}_r$$

is obviously surjective. The operator *m*:

$$m: \mathcal{S}_{r+1} \to \mathcal{H}^2_{r-1}$$

if it defined is then zero. The operator *m* thus presents us with an obstruction, in an obvious sense, to the complete integrability of the differential operator of order *r*.

3) Let E_r denote the set:

$$\{e,\ e \in J_r(E) \quad \text{such that} \qquad h(j^{r-k} \circ \partial)(e) = 0\}.$$

The set E_r is what one calls the *linear differential system* that is associated with the differential operator $j^{r-k} \circ \partial$ of order *r*. The sets E_r are not necessarily vector subspaces of the fibers of $J_r(E)$, but fibered by fibers over *V*, so each fiber of E_r is a vector subspace of the fiber of $J_r(E)$; we may thus form the tensor product, in the Whitney sense, of E_r with $\Lambda^p T^*$, and we obviously have:

$$\mathcal{S}^p_r = E_r \otimes \Lambda^p T^*,$$

in which the right-hand side denote the differentiable sections of $J_r(E) \otimes \Lambda^p T^*$ with values in the subset of $E_r \otimes \Lambda^p T^*$.

We say that the differential system E_k is homogeneously linear if E_k is a vector sub-bundle of $J_r(E)$. Any homogeneous, linear, differential system can be considered as the associated system to the differential operator of order *k*:

$$p \circ j^k : E \xrightarrow{j^k} J_k(E) \xrightarrow{p} J_k(E) / E_k,$$

where $J_k(E) / E_k$ denotes the quotient bundle of $J_k(E)$ by the sub-bundle E_k, and *p* is the canonical projection morphism.

4) Theorem II.4.b is a revised form of a known proposition of Quillen (Singer and Sternberg, "On the infinite groups of Lie and Cartan.") In the analytic case, it is equivalent (*E* and *F* being analytic vector bundles, and the differential operator ∂ of order *k* transporting any analytic section of *E* to an analytic section of *F*) to the celebrated Cartan-Kaehler theorem on involutive linear differential systems (C. Buttin, "Existence of local solutions for analytic systems of equations.").

CHAPTER III

CONNECTIONS OF HIGHER ORDER IN A VECTOR BUNDLE

In this chapter, we shall introduce the theory of connections, whose role one knows of from differential geometry. Our essential result is that if E is a vector bundle that is associated with a Lie groupoid Φ then a connection of order k, in the sense of C. Ehresmann, in the Lie groupoid Φ canonically determines a splitting of the exact sequence of vector bundles:

$$0 \to J_k^0(E) \to J_k(E) \xrightarrow{\rho} E \to 0.$$

Conversely, any splitting of this sequence is determined by a connection of order k in the Lie groupoid $\Pi(E)$, the groupoid of all linear isomorphisms from fibers of E to fibers.

1. Connection is a Lie groupoid.

Let Φ be a Lie groupoid on the differentiable base manifold V. From geometric considerations, which we shall not recall, C. Ehresmann was led to make the following definition:

DEFINITION III.1.a – *A connection element of order k in* Φ *is a jet of a section X:*

$$X \in J_k(\Phi, b, V),$$

such that:

1) $\beta(X) = l_{\alpha(X)}$, $\alpha(X)$ *and* $\beta(X)$ *are the source and target of the jet X, respectively.*
2) $aX = \widehat{j^k_{\alpha(X)}}$, *the jet of order k of the constant map that maps V to the point* $\alpha(X)$.

PROPOSITION III.1a. – *The space* $Q_k(\Phi)$ *of all connection elements of order k of* Φ *is a differentiable fiber bundle over V under the source projection a that is associated with the prolonged Lie groupoid* Φ^k.

Proof.

Indeed, let X be a connection element of source x, and let Z be an element of Φ^k with source x and target y. Let Y denote:

$$Y = bZ, \qquad Y \in \Pi^k(V) \qquad \text{(see chap. II-1).}$$

One has a new connection element X' of source y:

$$X' = (Z\, Y^{-1}) \cdot (X\, Y^{-1}) \cdot \beta(Z)^{-1},$$

where the elements between parentheses are the compositions of jets, $\beta(Z)^{-1}$ must be considered as the jet of order k of the constant map of V onto $\beta(Z)^{-1}$, and the dots have the same significance as they did in the proof of proposition II.1.

Set $X' = Z \cdot X$; it is then clear that Φ^k operates on $Q_k(\Phi)$. It remains for us to verify the differentiability conditions, which we may do locally.

Now, if Φ is trivial Lie groupoid:

$$\Phi = \mathbb{R}^n \times G \times \mathbb{R}^n$$

then one has $Q_k(\Phi) = T^k_{n,e}(G) \times \mathbb{R}^n$, where $T^k_{n,e}(G)$ is, we recall, the set of jets of order k of $\mathbb{R}^n$ into G with source 0 and the neutral element e of G for target. Recall that Φ^k is then a trivial groupoid $\Phi^k = \mathbb{R}^n \times G^k \times \mathbb{R}^n$, where G^k is the semi-direct product of $T^k_{n,e}(G)$ with the product group $G \times L^k_n$. This being the case, the previously-determined operator Φ^k on $Q_k(\Phi)$ is defined by the operation of G^k on $T^k_{n,e}(G)$ in the following manner:

$$Z \in G^k, \qquad Z = (g, Y) \cdot X, \qquad \text{with} \quad (g, Y) \in G \times L^k_n \text{ and } X \in T^k_{n,e}(G),$$
$$Z\colon T^k_{n,e}(G) \to T^k_{n,e}(G), \qquad X' \mapsto (g, Y) \cdot X \cdot X \cdot (g, Y)^{-1},$$

which are products of elements of G^k that belong to the subgroup $T^k_{n,e}(G)$.

This indeed proves that $Q_k(\Phi)$ is a differentiable fiber bundle that is associated with the Lie groupoid Φ^k.

Q. E. D.

DEFINITION III.1*b*. – *A connection of order k in Φ is the given of a differentiable section of the bundle $Q_k(\Phi)$.*

Since the fiber of $Q_k(\Phi)$ is isomorphic to $T^k_{n,e}(G)$, it is therefore contractible. We remark that since the differentiable manifold V is assumed to be paracompact, there always exists a connection of order k in the Lie groupoid Φ.

From the preceding, we remark that the isotropy group G^k_x of Φ^k operates transitively on the fiber $Q_k(\Phi)|_x$. Any section of $Q_k(\Phi)$ is therefore regular; it canonically determines a Lie subgroupoid of Φ^k, namely, the subgroup that leaves it invariant. It is easy to prove that this subgroupoid is isomorphic to the groupoid $\Phi \times \Pi^k(V)$ by the surjective functor of Φ^k onto $\Phi \times \Pi^k(V)$. We thus have the proposition:

PROPOSITION III.1b. – *Any connection of order k in Φ canonically defines a reduction of $\Phi \times \Pi^k(V)$ in Φ^k.*

Likewise, one immediately establishes that:

PROPOSITION III.1c. – *If φ is a reduction of $\Phi \times \Pi^k(V)$ in Φ^k such that at a certain point x of V the isotropy group G of $\varphi(\Phi \times \Pi^k(V))$ leaves invariant a connection element of order k at x of Φ then there exists a connection of order k of Φ that defines that reduction.*

2. Connections in a vector bundle.

Suppose one is given (E, p, V); i.e., a differentiable manifold that is fibered by a submersion p on V. Let $F_k(E, p, V)$ – or simply $F_k(E)$, when there is no risk of confusion – denote the set of jets of order k of V into E, such that: $X \in F_k(E)$, $pX = \widehat{j_x^k}$ is the jet of order k of the constant map of V onto the point $x = \alpha(X)$, which is the source of X.

$F_k(E)$ is obviously a differentiable manifold that is fibered by the source map α onto V. In a more precise fashion, we have the proposition:

PROPOSITION III.2. – *If E is a fiber bundle that is associated with the Lie groupoid Φ then $F_k(E)$ is a fiber bundle that is associated with the product Lie groupoid $\Phi \times \Pi^k(V)$.*

The proof is identical to that of proposition II.1*b*, if we remark that the groupoid $\Phi \times \Pi^k(V)$ operates on $F_k(E)$ in the following fashion:

If $(z, Y) \in \Phi \times \Pi^k(V)$, with source x and target y, $X \in F_k(E)$, with source x,

and:

$X = j_x^k g$, where g is a map of V into the fiber E_x,

$Y = j_x^k f$, where f is a local diffeomorphism of V,

then one has:

$$(z, Y) \cdot X = j_y^k (z \cdot (g \circ f^{-1})),$$

in which $z \cdot (g \circ f^{-1})$ is, by definition of the operation (denoted by a dot) of Φ on E, a differentiable map that is defined in a neighborhood of y from V into the fiber E_y.

THEOREM III.2a. – *If E is a fiber bundle that is associated with the Lie groupoid Φ then any connection of order k in Φ determines a V-isomorphism of differentiable bundles of $F_k(E)$ into $J_k(E)$.*

Indeed, let C_x be a connection element of order k at x of Φ: $C_x = j_x^k f$, where f is a differentiable map that is defined in the neighborhood of x from V into Φ such that:

$a \circ f = \hat{x}$, the constant map of V to x,

$b \circ f = \text{Id}$.

C_x determines a diffeomorphism of $F_k(E)|_x$ with $J_k(E)|_x$ in the following manner:

$$C_x : F_k(E)|_x \to J_k(E)|_x , \qquad X = j_x^k g \mapsto X' = j_x^k (f \cdot g),$$

where g is a map of V into the fiber E_x and $f \cdot g$ is a section of E that is defined in a neighborhood of x.

Therefore, a connection C of order k in Φ determines a bijective map of $F_k(E)$ into $J_k(E)$ that diffeomorphically transforms the fiber $F_k(E)|_x$ onto the fiber $J_k(E)|_x$ at any point x of V. In order for this map to be a differentiable isomorphism of fibers, it remains for us to prove that if s is a differentiable section of $F_k(E)$ then $C \circ s$ is a differentiable section of $J_k(E)$; one may do this by (locally) trivializing E.

Q. E. D.

Obviously, we have the following injective V-morphism:

$$i: E \to F_k(E), \qquad e \mapsto j_x^k \hat{e} ,$$

where $x = p(e)$, and $\hat{e}$ is the constant map of V to the point e. When it is composed with the morphism C that is defined by a connection, we have an injective V-morphism:

$$C \circ i: E \to J_k(E),$$

such that:

$$\rho \circ C \circ i : J_k(E) \to E$$

(where r is the canonical morphism):

$$\rho \circ C \circ i = \text{Id}.$$

Indeed, let e be in E such that $p(e) = x$, and let:

$$C_x = j_x^k f ,$$

so we have:

$$C \circ i(e) = j_x(f \cdot \hat{e}).$$

Thus:

$$\rho \circ C \circ i(e) = f(x) \cdot e = e,$$

because, from the definition of a connection element at x, one has:

$$f(x) = l_x, \qquad \text{the unit at } x \text{ of } \Phi.$$

In the case where E is a differentiable vector bundle, $F_k(E)$ is also a vector bundle and the previously-defined morphisms i and C are morphisms of vector bundles. We thus have:

COROLLARY – *If E is a vector bundle that is associated with a Lie groupoid* Φ *then any connection of order k in* Φ *canonically determines a splitting of the following exact sequence of differentiable vector bundles:*

$$0 \to J_k^0(E) \to J_k(E) \xrightarrow{\rho} E \to 0.$$

In the case of vector bundles, we also have the following theorem:

THEOREM III.2b. – *If E is a differentiable vector bundle over V then any splitting of the exact sequence:*

$$0 \to J_k^0(E) \to J_k(E) \xrightarrow{\rho} E \to 0$$

is determined by a connection of order k in the Lie groupoid Π(*E*) *of all linear isomorphisms of fibers of E onto fibers that is associated with the bundle E.*

Proof.

Let λ be a lift of the splitting, and let $(e_1, e_2, \ldots, e_q)$ be a basis system for the vector space E_x, which is the fiber of E at x. Set:

$$\lambda(e_i) = j_x^k s_i ,$$

where one may obviously take differentiable sections s_i that are defined in the same neighborhood of x in V, and are such that in this neighborhood the s_i form a basis for the sheaf E of locally free $\mathcal{D}$-modules. On this same neighborhood of x, consider the differentiable section f of $(\Pi(E), b, V)$ such that:

1) $a \circ f = \hat{x}$, $\hat{x} : V \to x$,
2) $f(y) \cdot e_i = s_i(y)$.

One must also have:

$$f(x) = l_x , \qquad \text{the unit of } \Pi(E) \text{ at } x,$$

because for any i:

$$f(y) \cdot e_i = s_i(y) = e_i .$$

The jet $X = j_x^k f$ is then a connection element of order k at x of $\Pi(E)$, and this connection element is obviously defined in a manner that is independent of the choice of basis $(e_1, \ldots, e_q)$ of E, and the lift λ at the point x precisely.

To any point x of V, we thus associate a connection element or order k in $\Pi(E)$; in other words, we have a section that we denote by C of V in the connection space $Q_k[\Pi(E)]$; it is a differentiable section. Indeed, let Φ be the subgroupoid of $\Pi^k(E)$ leaves this section invariant. It is a transitive subgroupoid on V, because $\Pi^k(E)$ operates in a transitive fashion on the space $Q_k[\Pi(E)]$. The subgroupoid Φ is also the subgroupoid of $\Pi^k(E)$ that leaves the lift invariant, considered as a differentiable section of $J_k(E) \otimes E^*$

($J_k(E) \otimes E^*$ being associated in a canonical fashion with the Lie groupoid $\Pi^k(E)$). Since Φ is transitive on V, the section λ is a regular section. Φ is therefore a Lie subgroupoid of $\Pi^k(E)$, and the section C, which is left invariant by a Lie subgroupoid, is a differentiable section.

Q. E. D.

The corollary to theorem III.2a and the last theorem thus show the equivalence between the notion of connection in a vector bundle that is associated with a Lie groupoid, a notion that is due to C. Ehresmann, and that of "sur-connection" that was introduced by P. Libermann as a splitting of the exact sequence:

$$0 \to J_k^0(E) \to J_k(E) \xrightarrow{\rho} E \to 0.$$

3. Covariant derivation and connections of order 1.

Therefore, suppose we are given a connection of order k in the vector bundle E, which, from our studies, amounts to being given a splitting of the preceding exact sequence. Letting λ_k denote the lift of the splitting, we have the following differential operator:

$$\nabla_k : E \to J_{k-1}(E) \otimes T^*, \qquad s \mapsto \nabla_k s = D \circ \lambda_k(s),$$

in which D is the Spencer operator.

The differential operator ∇_k obviously verifies the following properties by setting $\lambda_{k-1} = \rho_{k-1} \circ \lambda_k : E \to J_{k-1}(E)$, which is the induced lift of the connection of order k – 1:

1) $(\rho_{k-2} \otimes \mathrm{Id}) \circ \nabla_k(s) = D \circ \lambda_{k-1}(s) = \nabla_{k-1}(s)$,
2) $D \circ \nabla_k(s) = 0$,
3) $\nabla_k(fs) = f\, \nabla_k(s) + \lambda_{k-1}(s) \otimes df$,

for any differentiable function f on V. In a more precise fashion, we have the proposition:

PROPOSITION III.3a. – *If one is given a lift λ_{k-1} of E in $J_{k-1}(E)$ and a differential operator:*

$$\nabla_k : E \to J_{k-1}(E) \otimes T^*$$

that verifies the three preceding properties then there exists one and only one lift λ_k of E into $J_k(E)$ such that if D is the Spencer operator then:

1) $\lambda_{k-1} = \rho_{k-1} \circ \lambda_k$
2) $\nabla_k = D \circ \lambda_k$.

Proof.

Therefore, let s be a differentiable section of E. Take s to be a section of $J_k(E)$ such that:

$$\rho_{k-1}(\sigma) = \lambda_{k-1}(s).$$

From property 1 of ∇_k, we have:

$$(\rho_{k-1} \otimes \mathrm{Id})(\nabla_k(s) - D(\sigma)) = (\rho_{k-1} \otimes \mathrm{Id}) \circ \nabla_k(s) - D(s) \circ \lambda_{k-1}(s) = 0.$$

It then results that $\nabla_k(s) - D(\sigma)$ is a section of the sub-bundle:

$$E \otimes S^{k-1}(T^*) \otimes T^* \qquad \text{of} \qquad J^{k-1}(E) \otimes T^*,$$

such that:

$$D(\nabla_k(s) - D(\sigma)) = 0;$$

from Lemma 2 of II.3.b:

$$\nabla_k(s) - D(\sigma) = D(\chi),$$

with χ a section of $E \otimes S^{k-1}(T^*)$. The section $\sigma+\chi$ of $J_k(E)$ is therefore such that:

1) $\rho_{k-1}(\sigma+\chi) = \rho_{k-1}(s)$,
2) $D(\sigma+\chi) = \nabla_k(s)$.

Such a section is obviously unique, and we thus define a morphism τ of sheaves:

$$\tau : E \to J_k(E),$$

which associates any section s of E with a section of $J_k(E)$ that verifies the conditions 1) and 2) above. If f is a differentiable function on V then one obviously has, by the third property of the operator ∇_k :

$$\tau(fs) = f\, \tau(s).$$

The morphism τ is therefore a $\mathcal{D}$-linear morphism: There exists a morphism λ_k of the vectorial bundle E into $J_k(E)$ such that:

$$s \in E, \qquad \lambda_k(s) = \tau(s).$$

The morphism λ_k is the lift that answers the question.

Q. E. D.

As in the particular case where $k = 1$, we have this corollary, which is a classical theorem of differential geometry:

COROLLARY. – *Suppose one is given a differential operator:*

$$\nabla: E \to E \otimes T^*,$$

such that if f is a differentiable function on V then:

$$s \in E, \qquad \nabla(fs) = f\,\nabla(s) + s \otimes df.$$

There then exists one and only one lift λ of E into $J_1(E)$, which is defined by a connection of order 1 in E such that:

$$s \in E, \qquad \nabla(s) = D \circ \lambda(s).$$

The differential operator ∇ that is associated with any connection of order 1 on E is what one calls the *covariant derivative* of the connection, and we shall recall some of its imprtant properties here. Since these properties are classical (see, for example, J. L. Koszul, *Lectures on Fiber Bundles and Differential Geometry*), we shall do so without giving the proofs.

Therefore, let a connection of order 1 in a Lie groupoid Φ be defined on V. For any vector bundle E over V that is associated with Φ, we let the same symbol ∇ denote the covariant derivative that is associated with that connection:

$$\nabla: E \to E \otimes T^*, \qquad s \mapsto \nabla(s),$$

and we let $\nabla_X(s)$ denote the value, in an obvious sense, of $\nabla(s)$ on the vector X on V (a vector field being a differentiable section of the tangent bundle T).

PROPOSITION III.3.b. –

a) *If E is a vector bundle that is associated with a Lie groupoid Φ then the same is true for the dual bundle E^*, and the covariant derivatives in E and E^* that are associated with the same connection of order* 1 *in Φ are coupled in the following way:*

$$s \in E,\ \omega \in E^*,\ X \in T, \qquad [\nabla_X(\omega)](s) = X \cdot \omega(s) - \omega[\nabla_X(s)],$$

where $X \cdot \omega(s)$ is the Lie derivative with respect to the vector field X of the function $\omega(s)$.

b) *If E and F are two vector bundles associated with the same Lie groupoid Φ then the same thing is true for the fiber bundle $E \otimes F$, and the covariant derivatives in E, F, and $E \otimes F$, respectively, that are associated with the same connection of order* 1 *in Φ are coupled by the following relation:*

$$s \in E,\ w \in F,\ X \in F, \qquad \nabla_X(s \otimes w) = \nabla_X(s) \otimes w + s \otimes \nabla_X(w).$$

Let $\mathcal{G}(\Phi)$ denote the bundle of Lie algebras that corresponds to the canonical Lie group bundle $G(\Phi)$ of the Lie groupoid Φ (see example 1 of I.2). For any connection of order 1 in Φ, one defines a differentiable section of $\mathcal{G}(\Phi) \otimes \Lambda^2 T^*$ that is called the *curvature tensor* of the connection, and which we denote by *R*. We also let ∇ denote the operator:

$$\nabla: E \otimes \Lambda^p T^* \to E \otimes \Lambda^{p+1} T^*,$$

$\nabla = D \circ (\lambda \otimes \mathrm{Id})$, λ being the lift of *E* in $J_1(E)$ that is defined by the connection.

PROPOSITION III.3.c. – *If one is given a connection of order* 1 *in a Lie groupoid* Φ, *where R denotes its curvature tensor:*

1) *In the vector bundle E that is associated with* Φ, *the operator:*

$$\nabla^2 : E \xrightarrow{\nabla} E \otimes T^* \xrightarrow{\nabla} E \otimes \Lambda^2 T^*,$$

is, in fact, $\mathcal{D}$*-linear, and it is equal to the linear representation* $\mathcal{R}(R)$ *of R, in an obvious sense,* $\mathcal{G}(\Phi)$ *being a Lie algebra bundle that operates on E.*

2) *If* $\mathcal{R}(R) = 0$, *or, what amounts to the same thing, if* $\nabla^2 = 0$ *then the connection in E will be called integrable. In a more precise fashion: At any point x of V there exist local sections* $s_1, \ldots, s_q$ *such that the sections* s_i *form a basis for the neighborhood of x of the sheaf E of locally free* $\mathcal{D}$*-modules and:*

$$\lambda(s_i) = j^1 s_i\,, \quad \forall i\,, \quad \text{or, in other words,} \quad \nabla(s_i) = 0.$$

CHAPTER IV

PROLONGATIONS OF THE TANGENT BUNDLE

We show that there exists a canonical structure of a sheaf of $\mathbb{R}$-Lie algebras in the sheaf $J_k(T)$, $J_k(T)$ being the prolongation bundle of order k of the tangent bundle T of a differentiable manifold V. A regular infinitesimal structure of order k on V is then equivalent to the given of a differentiable vector sub-bundle E_k of $J_k(T)$ such that the sheaf E_k is a sub-sheaf of $\mathbb{R}$-Lie algebras of the sheaf $J_k(T)$.

1. Structure of the sheaf of $\mathbb{R}$-Lie algebras.

T being the tangent bundle to a differentiable manifold V, recall that the Poisson bracket [,] of vector fields (i.e., differentiable sections of T) defines the structure of a sheaf of $\mathbb{R}$-Lie algebras in the sheaf of sections of T, or, in a more precise fashion, it verifies the following axioms:

If X, Y, Z are sections of T and α, β are real numbers then:

1) $[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Y]$,
2) $[X, Y] = -[Y, X]$,
3) $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$.

Moreover, if f is a differentiable function on V then we know that:

$$[X, fY] = f\,\{X, Y\} + (X \cdot f)\, Y,$$

where $X \cdot f$ is the function that is the Lie derivative of f with respect to the vector field X.

PROPOSITION IV.1 – *For any integer k, there exists one and only one structure of a sheaf of* $\mathbb{R}$*-Lie algebras in the sheaf* $J_k(T)$ *such that:*

1) *If* σ *and* η *are two integrable sections of* $J_k(T)$ *with* $\sigma = j^k X$ *and* $\eta = j^k Y$ *then one has:*

$$[\sigma, \eta] = j^k[X, Y],$$

2) *If* σ *and* η *are two differentiable sections of* $J_k(T)$ *and f is a differentiable function on V then one has:*

$$[\sigma, f\eta] = f[\sigma, \eta] + (\rho(\sigma) \cdot f)\, \eta,$$

$\rho(\sigma)$ *being the vector field that is the image of the section* σ *by the canonical morphism of* $J_k(T)$ *on T.*

Proof:

Suppose that such a structure exists, so we immediately have these two Lemmas:

LEMMA. – *If ρ_r is the canonical morphism of $J_k(T)$ onto $J_r(T)$, $r \leq k$, then:*

$$\rho_r([\sigma, \eta]) = [\rho_r(\sigma), \rho_r(\eta)].$$

Indeed, this lemma is obvious from the first property when σ and η are integrable sections. The same is true when σ and η are of the form $f\, j^k X$ and $g\, j^k Y$. The lemma is therefore exact by linearity, since any section of $J_k(T)$ is locally of the form $\sum_{i=1}^{m} f^i j^k X_i$.

LEMMA 2. – *If D is the Spencer operator then one has the relation:*

$$D([\sigma, \eta]) = [D(\sigma), \rho_{r-1}(\eta)] + [\rho_{r-1}(\sigma), D(\eta)] \\ + D(\sigma) \barwedge (\rho \otimes 2\mathrm{Id}) \circ D(\eta) - D(\eta) \barwedge (\rho \otimes 2\mathrm{Id}) \circ D(\sigma),$$

where the elements in the right-hand side are sections of $J_{k-1}(T) \otimes T^*$ whose values on a vector field X are given by the following formulas:

$$[D(\sigma), \rho_{k-1}(\eta)](X) = [D_X(\sigma), \rho_{k-1}(\eta)] = D_{[X, \rho(\eta)]}(\sigma),$$
$$[D(\sigma) \barwedge (\rho \otimes 2\mathrm{Id}) \circ D(\eta)](X) = (D(\sigma))(\rho \circ D_X(\eta)),$$

respectively.

It is, moreover, sufficient to verify this for σ and η of the form $f\, j^k Y$ and $g\, j^k Z$, which is immediate.

Having established these lemmas, the existence of such a structure is, on the one hand, known for the sheaf T, which is the case where $k = 0$, so it suffices for us to prove the proposition by recurrence on the integer k. Now, we let σ and η be two sections of $J_k(T)$ that one may assume to have the form:

$$\sigma = f^i j^k X_i \qquad \text{and} \qquad \eta = g^i j^k Y_i , \qquad 1 \leq i \leq m,$$

since this is true locally. Set:

$$[\sigma, \eta] = [f^i j^k X_i , g^i j^k Y_i],$$

where the right-hand side is a section χ of $J_k(T)$ that is perfectly determined, by reason of the two properties of the proposition. From Lemma 1 and Lemma 2, $\rho_{k-1}(\chi)$ and $D(\chi)$ depend upon only $\rho_{k-1}(\sigma)$,$\rho_{k-1}(\eta)$, $D(\sigma)$, and $D(\eta)$. From the remark at the end of II.3, that section χ is perfectly determined independently of the choice of integrable sections $j^k X_i$ and $j^k Y_i$ by which σ and η are expressed. By the proposition, the bracket $[\sigma, \eta]$, thus defined, indeed determines the structure of a sheaf of $\mathbb{R}$-Lie algebras in the sheaf $J_k(T)$, precisely. Q. E. D.

We remark that, from the second property, the bracket between sections of $J_k^0(T)$ is, in fact, $\mathcal{D}$-linear. $J_k^0(T)$ is therefore canonically a Lie-algebra bundle.

We also verify that a regular infinitesimal structure of order k on V is essentially the given of a morphism h of differentiable vector bundles:

$$h: J_k(T) \to F,$$

where F is a certain differentiable vector bundle on V such that the sub-sheaf of $\mathcal{D}$-modules:

$$\mathcal{S}_k = \ker(h)$$

is also a sub-sheaf of $\mathbb{R}$-Lie algebras on $J_k(T)$. It is useful for us to establish the following proposition:

PROPOSITION IV.1.b. – *If h is a V-morphism of vector bundles from $J_k(T)$ into a bundle F, such that:*

$$\mathcal{S}_k = \ker(h)$$

is a sub-sheaf of $\mathbb{R}$-Lie algebras of $J_k(T)$ then the sub-sheaf of prolongation:

$$\mathcal{S}_{k+r} = \ker J_r(h)$$

that is the kernel of the morphism $J_r(h)$ that maps $J_{r+r}(T)$ into $J_r(F)$ is also a sub-sheaf of $\mathbb{R}$-Lie algebras of $J_{r+r}(T)$ for any integer r.

Proof:

It suffices to prove the proposition in the case where $r = 1$. Therefore, let σ and η be two sections of $\mathcal{S}_{k+1}$:

$$\rho_k \circ J_1(h)([\sigma, \eta]) = h \circ \rho_k([\sigma, \eta]) = h([\rho_k(\sigma), \rho_k(\eta)]),$$

so, since ρ_k maps $\mathcal{S}_{k+1}$ into $\mathcal{S}_k$, one thus has:

$$\rho_k \circ J_1(h)([\sigma, \eta]) = 0.$$

On the other hand:

$$D \circ J_1(h)([\sigma, \eta]) = (h \otimes \mathrm{Id}) \circ D([\sigma, \eta]);$$

however, from Lemma 2, it is immediate that:

$$D([\sigma, \eta]) \text{ is a section of } \mathcal{S}_k^1 \ (= \mathcal{S}_k \underset{\mathcal{D}}{\otimes} T^*),$$

since the Spencer operator D maps $\mathcal{S}_{k+1}$ into $\mathcal{S}_k^1$. Therefore:

$$D \circ J_1(h)([\sigma, \eta]) = 0.$$

From the remark at the end of II.3, the section $J_1(h)([\sigma, \eta])$ is such that:

1) $\rho_k \circ J_1(h)([\sigma, \eta]) = 0$,
2) $D \circ J_1(h)([\sigma, \eta]) = 0$

is a zero section; $[\sigma, \eta]$ is a section of $\mathcal{S}_{k+1}$.

Q. E. D.

2. Notions of torsion.

The structure of a sheaf of ℝ-Lie algebras on $J_k(T)$ amounts to the given of an operator – i.e., an ℝ-linear V-morphism of sheaves:

$$J_k(T) \underset{\mathbb{R}}{\wedge} J_k(T) \to J_k(T),$$

where $J_k(T) \underset{\mathbb{R}}{\wedge} J_k(T)$ is the exterior product, in the Whitney sense, of ℝ-linear sheaves. That operator is a differentiable operator of order 1 if one of the factors on the right-hand side is fixed; it thus gives (see II.4) a V-morphism of differentiable vector bundles:

$$\tau : J_1[J_k(T)] \wedge J_1[J_k(T)] \to J_k(T).$$

In particular, since one has $J_{k+1}(T) \subset J_1[J_k(T)]$:

$$\tau : J_{k+1}(T) \wedge J_{k+1}(T) \to J_k(T).$$

DEFINITION IV.2. – *Since a connection of order $k + 1$ in T is a splitting of the exact sequence:*

$$0 \to J_{k+1}^0(T) \to J_{k+1}(T) \xrightarrow{\rho} T \to 0,$$

where λ denotes the lift, we call the morphism $-\tau \circ \lambda$:

$$-\tau \circ \lambda : T \wedge T \to J_k(T)$$

the torsion tensor of the connection.

If X and Y are vector fields on V then one has:

$$- \tau \circ \lambda(X \wedge Y) = D_X(\lambda(Y)) - D_Y(\lambda(X)) - \rho_k([\lambda(X), \lambda(Y)]),$$

a formula that one establishes immediately by expressing the morphism τ as a function of the operator D and the bracket. In the particular case of the connection of order 1, one has:

$$- \tau \circ \lambda(X \wedge Y) = \nabla_X(Y) - \nabla_Y(X) - [X, Y],$$

and we recover the classical notion of the torsion of a covariant derivative $\nabla(= D \circ \lambda)$ into T.

We remark that the lift λ of T to $J_{k+1}(T)$ may be considered to be a differentiable section of $J_{k+1}(T) \otimes T^*$ such that:

$$(\rho \otimes \mathrm{Id})(\lambda) = \mathrm{Id}, \qquad \text{the “identity” of } T \otimes T^*.$$

Moreover, we have:

$$D: J_{k+1}(T) \otimes T^* \to J_k(T) \otimes \Lambda^2 T^*,$$

with:

$$(D(\lambda))(X \wedge Y) = D_X(\lambda(X)) - D_Y(\lambda(X)) - \rho_k \circ l([X, Y]).$$

The following proposition is therefore established immediately, since $\tau \circ \lambda$ is regarded as a differentiable section of $J_k(T) \otimes \Lambda^2 T^*$:

PROPOSITION IV.2.a. – *In the case of the connection of order* 1, *one has:*

$$- \tau \circ \lambda = D(\lambda).$$

The case of the connection of order 1 in T is well-known; it is, nevertheless, useful for us to recall the following two propositions:

Consider the morphism δ that was introduced in II.3:

$$\delta: T \otimes S^k(T^*) \otimes \Lambda^p T^* \to T \otimes S^{k-1}(T^*) \otimes \Lambda^{p+1} T^*,$$

$$e \otimes a^k \otimes w \mapsto - k\, e \otimes a^{k-1} \otimes (a \wedge w)$$

PROPOSITION IV.2.b. – *Since the curvature tensor R of a connection of order* 1 *in T is a section of $T \otimes T^* \otimes \Lambda^2 T^*$, one has:*

$$\nabla(- \tau \circ \lambda) = - \delta(R),$$

where $\nabla(- \tau \circ \lambda)$ is the covariant derivative of the torsion of the connection.

In effect:

$$\nabla(-\,\tau \circ \lambda) = \nabla[D(\lambda)] = \nabla^2(\mathrm{Id}),$$

in which Id denotes the "identity" section of $T \otimes T^*$. The equality thus posed, which is called the *Bianchi identity*, is then an immediate consequence of the assertion 1) of proposition III.3.c.

From theorem III.2.b, any connection of order 1 in T is defined by a connection of order 1 in the Lie groupoid $\Pi^1(V)$. Since $T \otimes S^k(T^*)$ is also canonically associated with the Lie groupoid $\Pi^1(V)$ for any integer k, a connection of order 1 in this groupoid thus also defines a covariant derivative operator:

$$\nabla\colon T \otimes S^k(T^*) \otimes \Lambda^p T^* \to T \otimes S^k(T^*) \otimes \Lambda^{p+1} T^*.$$

Having made this recollection, we have:

PROPOSITION IV.2.c. – *If the connection of order* 1 *in T is without torsion (i.e., the section $-\,\tau \circ \lambda$ is a zero section of $T \otimes \Lambda^2 T^*$) then one has the following anti-commutative diagram,* $\delta \circ \nabla = -\,\nabla \circ \delta$:

$$\begin{array}{ccc} T \otimes S^k(T^*) \otimes \Lambda^p T^* & \xrightarrow{\ \delta\ } & T \otimes S^{k-1}(T^*) \otimes \Lambda^{p+1} T^* \\ \downarrow \nabla & & \downarrow \nabla \\ T \otimes S^k(T^*) \otimes \Lambda^{p+1} T^* & \xrightarrow{\ \delta\ } & T \otimes S^{k-1}(T^*) \otimes \Lambda^{p+2} T^* \end{array}$$

This proposition is a direct consequence of the formula that was given in Koszul (*Fiber bundles and differential geometry*): If ω is a section of T^* then:

$$\delta \circ \nabla(\omega) = d\omega,$$

where ∇ is the covariant derivative in T^* that corresponds to a connection without torsion in T, δ is the anti-symmetrization morphism:

$$\delta\colon T^* \otimes T^* \to \Lambda^2 T^*, \quad a \otimes b \mapsto a \wedge b,$$

and $d\omega$ is the exterior derivative of ω.

3. Linear differential system associated with a regular infinitesimal structure.

Let X be a vector field on V. We know that X defines a local one-parameter group of transformations on V (R. Palais, *Lie Theory of Transformation Groups*, definition II, page 33) that is presently denoted by Exp $t\,X$, where t is the real parameter:

> For any x in V, there exists a positive real number ε such that for any $t \in (-\,\varepsilon, \varepsilon)$, Exp $t\,X$ is a diffeomorphism of a neighborhood of x into an open subset of V.

Consider the map that is defined for any x in V:

$$(-\varepsilon, \varepsilon) \to \Pi_x^k(V), \qquad t \mapsto j_x^k(\operatorname{Exp} t\,X).$$

This map is a germ of a differentiable curve in $\Pi_x^k(V)$. Therefore, when the parameter t = 0, its tangent vector $\varphi_x(X)$ belongs to $T_x[\Pi_x^k(V)]$, the tangent vector space at the point l_x of $\Pi_x^k(V)$. One proves that this tangent vector depends only upon the jet of order k at x of the section X of T.

LEMMA (P. LIBERMANN). – *For any integer k, there exists a canonical linear isomorphism* φ *of* $J_k(T)|_x$ *with* $T_x[\Pi_x^k(V)]$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} J_k(T)|_x & \xrightarrow{\varphi} & T_x[\Pi_x^k(V)] \\ \downarrow \rho_r & & \downarrow \rho_r \\ J_r(T)|_x & \xrightarrow{\varphi} & T_x[\Pi_x^k(V)] \end{array}$$

The isomorphism is the preceding map that associates any jet of a section $j_x^k(X)$ with the vector $\varphi(X)$. This lemma may be established by direct local calculation (see P. Libermann, "Pseudogroupes infinitésimaux," prop. 1, par. 2).

Recall that a regular infinitesimal structure of order k on V determines a Lie sub-groupoid Φ in $\Pi^k(V)$ (see I-3). If Φ is a Lie sub-groupoid then $T_x(\Phi_x)$ is a vector sub-bundle of $T_x[\Pi_x^k(V)]$. From the lemma, we thus have:

$$\cup\, T_x(\Phi_x) \approx E_k, \qquad \text{subset of } J_k(T).$$

The subset E_k of $J_k(T)$ is what one calls the *differential system* that is associated to the regular infinitesimal structure. We then state the following theorem, which we shall not prove, in order to not overextend our presentation.

THEOREM IV.3. – *The differential system* E_k *associated with a regular infinitesimal is a homogeneous, linear, differential system; i.e.,* E_k *is a differentiable vector sub-bundle of* $J_k(T)$. *It has no constant term; i.e., the restriction of the morphism* ρ *to* E_k *is surjective:*

$$E_k \xrightarrow{\rho} T \to 0,$$

and the sheaf of sections E_k *is a sub-sheaf of* $\mathbb{R}$*-Lie algebras of the sheaf of* $\mathbb{R}$*-Lie algebras* $J_k(t)$.

Conversely, any homogeneous, linear, differential system E_k *with no constant term, and such that the sheaf of sections* E_k *is a sub-sheaf of* $\mathbb{R}$*-Lie algebras of the* $\mathbb{R}$*-Lie* *algebra $J_k(T)$ may be considered to be the differential system that is associated with a regular infinitesimal structure of order k on V.*

Remarks.

1) An idea of the proof may be found in Rodriguès ("The first and second fundamental theorems of Lie for pseudogroups," Am. J. Math. 84). One shows that the sheaf of $\mathbb{R}$-Lie algebras $J_k(T)$ is canonically isomorphic to the sheaf of $\mathbb{R}$-Lie algebras of right-invariant vector fields on the principal bundle with structure group $\Pi_k(V)$. The sheaf E_k is then isomorphic to the sheaf of right-invariant vector fields on the principal bundle with structure group Φ_x . In particular, the Lie algebra bundle E_k^0 - i.e., the kernel of the morphism ρ of E_k onto T – is isomorphic to the Lie algebra bundle $\mathcal{G}(\Phi)$.

2) If F is a fiber bundle that is associated with the Lie groupoid $\Pi_k(V)$ then F is a prolongation space of $(V, \Lambda(V))$, $\Lambda(V)$ being the pseudogroup of local diffeomorphisms of V, in the sense that any local diffeomorphism f with source U and target U' can be lifted to a diffeomorphism $\tilde{f}$ of fibers:

$$\begin{array}{ccc} F|_U & \xrightarrow{\tilde{f}} & F|_{U'} \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & U' \end{array}$$

Since an infinitesimal structure is the given of a differentiable section S of F, recall that the pseudogroup of local automorphisms of the structure $\Gamma(S)$ is the set of local diffeomorphisms f of V such that:

$$\tilde{f} \circ S \circ f^{-1} = S,$$

or, in other words, it leaves the section S invariant.

The differential system E_k of the infinitesimal structure S is then a prolongation space of $(V, \Gamma(S))$.

We also let Θ denote the sheaf of solutions of E_k – i.e., the set of local sections X of T such that j^kX is a section of $J_k(T)$ with values in the sub-bundle E_k . The sheaf Θ is called the *sheaf of infinitesimal automorphisms* of the structure, in the sense that the local one-parameter group Exp tX that is defined by the sections X that are solutions of E_k is an element of $\Gamma(S)$ for fixed t. It is immediate from Theorem IV-3 that the sheaf of infinitesimal automorphisms of a regular infinitesimal structure is a sheaf of $\mathbb{R}$-Lie algebras in V.

3) In practice, in order to find, for example, the differential system E_1 that is associated with the infinitesimal structure of order 1 on V that is defined by the given of a section S of $T_q^p = \overset{p}{\otimes} T \overset{q}{\otimes} T^*$, one proceeds as follows:

Consider the operator *S:*

$$S: T \to T_q^p, \quad X \mapsto \mathcal{L}(X) \cdot S,$$

$\mathcal{L}(X) \cdot S$ being a local section of T_q^p that is the Lie derivative of the tensor S with respect to the local vector field X. The operator S is a differential operator of order 1; it thus defines a morphism of bundles:

$$h(S) : J_1(T) \to T_q^p;$$

the system E_1 is the kernel of the morphism:

$$E_1 = \{e \in J_1(T) \text{ such that } h(S)(e) = 0.\}$$

CHAPTER V

DIFFERENTIAL SYSTEM OF A *G*-STRUCTURE

Here, we recall the terminology that was established (D. Bernard, "Géo. dif. des *G*-structures") to refer to any regular infinitesimal structure of order 1 on a differentiable manifold *V* as a *G-structure*. We shall then apply the Spencer theory to the study of *G*-structures. In particular, the fundamental theorem II.4b permits us to define new tensors that are the obstructions to integrability (in the well-known sense, which we shall make precise, moreover) of the structure, the primary obstruction being the structure tensor of D. Bernard.

1. Type and degree of the structure.

Therefore, let there be a regular infinitesimal structure of order 1 given on *V* – i.e., a regular section *S* of an infinitesimal prolongation space of order 1 on *V*. Let Φ denote the Lie sub-groupoid of $\Pi^1(V)$ that leaves the section invariant, and let E_1 denote the homogeneous, linear, differential system that is associated with the structure (Theorem IV-3). Recall that E_1 is a differentiable vector sub-bundle of $J_1(T)$, and set N_1 in such a way that the following sequence of vector bundles is exact:

$$0 \to N_1 \to E_1 \xrightarrow{\rho} T \to 0.$$

Since E_1 is a sub-sheaf of $\mathbb{R}$-Lie algebras of $J_1(T)$, N_1 is a sub-bundle of Lie algebras of $J_1(T) = T \otimes T^* \cdot N_1$ is, moreover, a Lie algebra bundle that is associated with Φ that is currently called the *isotropy bundle of Lie algebras* of the structure, being identified with the bundle $\mathcal{G}(\Phi)$, which is the Lie algebra bundle of the isotropy groups of Φ.

E_1 being a differentiable vector sub-bundle of $J_1(T)$, we thus consider (see Remark 3 of II-4) the canonical morphism:

$$h: J_1(E) \to J_1(E) / E_1 = F,$$

$J_1(T) / E_1$ being the quotient vector bundle of $J_1(T)$ by E_1 . Furthermore, let $\mathcal{S}_k$ denote the subsheaf $\ker(J_{k-1}(h))$ in $J_k(T)$:

$$J_{k-1}(h) : J_k(T) \to J_{k-1}(F).$$

From Proposition IV.1b, the subsheaf of $\mathcal{D}$-modules $\mathcal{S}_k$ is also a subsheaf of $\mathbb{R}$-Lie algebras of $J_k(T)$. The sheaf $\mathcal{S}_k$ are also defined by the following recurrence relation: If $\mathcal{S}_1$ is the subsheaf E_1 of $J_1(T)$ then $\mathcal{S}_k$ is the subsheaf of $J_k(T)$ such that:

1) $\rho_{k-1} : \mathcal{S}_k \to \mathcal{S}_{k-1}$,

2) $D\colon \mathcal{S}_k \to \mathcal{S}^1_{k-1} \;\; (= \mathcal{S}_{k-1} \underset{\mathcal{D}}{\otimes} T^*)$.

Let $\mathcal{N}_k$ be the subsheaf $\ker(\rho_{k-1})$ in $\mathcal{S}_k$:

$$0 \to \mathcal{N}_k \to \mathcal{S}_k \xrightarrow{\rho_{k-1}} \mathcal{S}_{k-1}\,.$$

$\mathcal{N}_k$ is the subsheaf of sections of $T \otimes S^k(T^*)$ with values in the set N_k, the subset N_k being defined by recurrence as the subset of $T \otimes S^k(T^*)$ such that:

$$\delta\colon T \otimes S^k(T^*) \to T \otimes S^{k-1}(T^*) \otimes T^*, \qquad\qquad N_k = \delta^{-1}(N_{k-1} \otimes T^*).$$

Since N_1 is a vector sub-bundle of $T \otimes T^*$ that is associated with the Lie groupoid Φ, the subsets N_k are also differentiable vector sub-bundles $T \otimes S^k(T^*)$ that are associated with the Lie groupoid Φ; the sheaf of $\mathcal{D}$-modules $\mathcal{N}_k$ is therefore locally free:

$$\mathcal{N}_k = N_k\,.$$

DEFINITION V.1a. – *The regular infinitesimal structure S will be called of type p if there exists an integer p such that:*

$$N_k \neq 0 \qquad \text{and} \qquad N_{p+1} = 0 \qquad (\text{so } N_k = 0 \;\; \forall\, k > p).$$

In the contrary case, the structure will be called of infinite type.

We remark that the type of a regular infinitesimal structure S depends upon only on the linear algebraic structure of the fibers of N_1 . We thus have the following examples:

Examples.

1) A Lorentzian structure on V – i.e., S is a section of $S^2(T^*)$ that is everywhere of maximal rank is of type 1: $N_2 = 0$ (see V. Guillemin and S. Sternberg, "An Algebraic Model for Transitive Differential Geometry," par. 3, A).

2) An almost-complex or almost-symplectic structure on V is of infinite type (see, more generally, Y. Matsushima, Theorem 3, "Algèbre de Lie linéaire semi-involutive").

It is clear that if one is given:

$$\delta(N_k) \subset N_{k-1} \otimes T^*,$$

for any integer q then the restriction of the morphism δ to the vector sub-bundle $N_k \otimes \Lambda^q T^*$ of $T \otimes S^k(T^*) \otimes \Lambda^q T^*$ has its values in $N_{k-1} \otimes \Lambda^{q+1} T^*$. Therefore, consider the cohomological sequence:

$$N_{p+1} \otimes \Lambda^{q-1}T^* \xrightarrow{\delta} N_p \otimes \Lambda^q T^* \xrightarrow{\delta} N_{p-1} \otimes \Lambda^{q+1}T^*,$$

which we denote by $H_p^q(S)$, the corresponding cohomology space $H_p^q(S)$ is obviously a vector bundle over V that is associated with the Lie groupoid Φ; the sheaf $H_p^q(S)$ is nothing but the cohomology sheaf $\mathcal{H}_p^q$ of the sequence (see chap. II.4):

$$\mathcal{N}_{p+1}^{q-1} \xrightarrow{\delta} \mathcal{N}_p^q \xrightarrow{\delta} \mathcal{N}_{p-1}^{q+1}.$$

DEFINITION V.1b. – *The degree of the regular infinitesimal structure S is the smallest integer k such that if $p \geq k$ then one has, for any integer q:*

$$H_p^q(S) = 0.$$

This definition is meaningful by virtue of the following theorem:

THEOREM V.1 – *For any regular infinitesimal structure S, there exists an integer k such that if $p \geq k$:*

$$H_p^q(S) = 0 \qquad \textit{for any integer } q.$$

The algebraic equivalence of this theorem – i.e., when one regards only the linear algebraic structure of the fibers of N_1 – has been proved by I. Singer and S. Sternberg (section 6, chap. IV, "The infinite groups of Lie and Cartan"). We thus refer the reader to that paper for the proof of that theorem.

Example.

The degree of a 0-deformable structure is 1. (One then says that it is *involutive*.)

2. *S*-connection.

DEFINITION V.2. – *An S-connection is a connection of order* 1 *in the Lie groupoid* Φ.

In the case where the infinitesimal prolongation space F for the infinitesimal structure in question is defined as the given of a differentiable section S is a vector bundle, an S-connection is nothing but a connection of order 1 in $\Pi^1(V)$ that determines a covariant derivation ∇ in the vector bundle F (see chap. III-3) such that the section S has null covariant derivative:

$$\nabla(S) = 0.$$

Any S-connection obviously defines a splitting of the exact sequence:

$$0 \to T \otimes T^* \to J_1(T) \xrightarrow{\rho} T \to 0.$$

We also recall that when one is given a covariant derivative ∇ in T, A. Lichnerowicz introduced (*Géom. des groupes de transformation*, par. 19) a new covariant derivative $\bar{\bar{\nabla}}$ in T, that he calls "associated" to ∇ that has its torsion tensor opposite to that of ∇, or, in a more precise fashion, if X and Y are two vector fields on V:

$$[X, Y] = \bar{\bar{\nabla}}_X(Y) - \nabla_Y(X) = \nabla_X(Y) - \bar{\bar{\nabla}}_Y(X).$$

Having said this, we have:

PROPOSITION V.2 – *If ∇ is the covariant derivative in T that is defined by an S-connection then the associated covariant derivative $\bar{\bar{\nabla}}$ is such that the corresponding splitting of the exact sequence:*

$$0 \to T \otimes T^* \to J_1(T) \xrightarrow{\rho} T \to 0$$

has a lift λ that maps T into E_1, or, in other words, that splitting is, in fact, a splitting of the exact sequence:

$$0 \to N_1 \to E_1 \xrightarrow{\rho} T \to 0.$$

Conversely, any splitting of the latter sequence determines a covariant derivative $\bar{\bar{\nabla}}$ such that its associated ∇ is defined by an S-connection.

Proof:

We shall not give a general proof of the proposition, but only a direct verification in the case where, for example, the infinitesimal structure considered is the given of a section S of $T \otimes T^*$ (case of a 0-deformable structure).

Indeed, we first make it rigorous that in this case a section σ of $J_1(T)$ is, in fact, a section of E_1 if and only if when X is an arbitrary vector field on V, one has:

$$[\rho(\sigma), S(X)] - S([\rho(\sigma), X]) + D_{S(X)}(\sigma) - S(D_X(\sigma)) = 0.$$

Therefore, let a covariant derivative ∇ in T be given that is defined by an S-connection – i.e., for any pair of vector fields X and Y:

$$\nabla_X(S(Y)) - S(\nabla_X(Y)) = 0, \tag{1}$$

and let $\bar{\bar{\nabla}}$ be the associated covariant derivative:

$$\nabla_X(S(Y)) - \bar{\bar{\nabla}}_{S(Y)}(X) - [X, S(Y)] = 0, \tag{2}$$

$$\nabla_X(Y) - \bar{\bar{\nabla}}_Y(X) - [X, Y] = 0. \tag{3}$$

From these three relations, one easily deduces:

$$[X, S(Y)] - S(X, Y] + \bar{\bar{\nabla}}_{S(Y)}(X) - S(\bar{\bar{\nabla}}_Y(X)) = 0, \tag{4}$$

or again, λ denoting the lift of the splitting that corresponds to the covariant derivative $\bar{\bar{\nabla}}$:

$$[X, S(Y)] - S[X, Y] + D_{S(Y)}(\lambda(X)) - S(D_Y(\lambda(X)) = 0.$$

This proves precisely that for any vector field X, $\lambda(X)$ is a section of E_1.

Conversely, let λ be a lift of the splitting of the exact sequence:

$$0 \to N_1 \to E_1 \xrightarrow{\rho} T \to 0.$$

For any pair of vector fields X and Y, we get the latter relation, or again the relation (4), by letting ∇ denote the covariant derivative associated to $\bar{\bar{\nabla}}$, which therefore verifies the relations (2) and (3), so we immediately deduce that ∇ verifies the relation:

$$\nabla_X(S(Y)) - S(\nabla_X(Y)) = 0. \tag{1}$$

The covariant derivative ∇ is therefore indeed defined by an S-connection.

Q. E. D.

COROLLARY. – *Any S-connection without torsion canonically determines a splitting of the exact sequence:*

$$0 \to N_1 \to E_1 \xrightarrow{\rho} T \to 0.$$

More generally, we have the following theorem (for which we recall the notations of V-1, by agreeing that $N_0 = T$).

THEOREM V.2. – *Any S-connection without torsion canonically determines a lift:*

$$\lambda_k : N_k \to \mathcal{S}_{k+1},$$

i.e., a $\mathcal{D}$-linear morphism of the sheaf of $\mathcal{D}$-modules such that:

$$\rho_k \circ \lambda_k = \text{Identity}.$$

Proof.

From the preceding corollary, the theorem is true for $k = 0$. We shall therefore prove it by recurrence on the integer k.

Therefore, let there be given an S-connection without torsion, where shall let the same symbol ∇ denote the corresponding covariant derivative in the fiber bundle N_k, the latter being all vector bundles that are associated with the Lie groupoid Φ. Furthermore, suppose that by the recurrence hypothesis this connection canonically defines lifts:

$$\lambda_{k-p} : N_{k-p} \to \mathcal{S}_{k-p+1}$$

for any integer p with $k \geq p \geq 1$, and such that if η is a section of N_{k-1} then one has:

$$D \circ \lambda_{k-1}(\eta) = \nabla(\eta) + (\lambda_{k-1} \otimes \mathrm{Id}) \circ \delta(\eta).$$

If one takes a section η of N_k and considers the section χ of $\mathcal{S}_k^1$:

$$\chi = \nabla(\eta) + (\lambda_{k-1} \otimes \mathrm{Id}) \circ \delta(\eta)$$

then one obviously has:

$$(\rho_{k-1} \otimes \mathrm{Id})(\chi) = \delta(\eta)$$
$$D(\chi) = \delta \circ \nabla(\eta) + \nabla \circ \delta(\eta),$$

by the recurrence hypothesis, so:

$$D(\chi) = 0,$$

from proposition IV.2c.

One then shows that there exists a section σ of $J_{k+1}(T)$ such that:

1) $\rho_k(\sigma) = \eta$,
2) $D(\sigma) = \chi$.

Indeed, let σ' be a section of $J_{k+1}(T)$ such that $\rho_k(\sigma') = \eta$.
The section $D(\sigma') - \chi$ is therefore a section of $J_k(T) \otimes T^*$ that verifies:

$$(\rho_{k-1} \otimes \mathrm{Id})(D(\sigma') - \chi) = \delta(\eta) - \delta(\eta) = 0,$$
$$D(D(\sigma') - \chi) = D^2(\sigma') - D(\chi) = 0;$$

there thus exists a section of η' of $T \otimes S^{k+1}(T^*)$, such that:

$$D(\sigma') - \chi = \delta(\eta').$$

The section $\sigma = \sigma' - \eta'$ answers the question. Since this section verifies conditions 1 and 2, it is, in fact, a section of $\mathcal{S}_{k+1}$; it is obviously unique and depends $\mathcal{D}$-linearly upon the section η. We thus have a lift of N_k to $\mathcal{S}_{k-1}$.

Q. E. D.

3. Integrable structures.

DEFINITION V.3. – *The infinitesimal structure S will be called integrable if and only if for any point of V there exist infinitesimal automorphisms X_i – i.e., solutions of E_1*

that are defined on a neighborhood of x and are such that in this neighborhood the X_i define a basis for the sheaf T of locally free $\mathcal{D}$-modules and:

$$[X_i, X_j] = 0 \qquad \text{for any } i \text{ and } j.$$

In this neighborhood U of x, consider the lift:

$$\lambda : T|_U \to E_1|_U$$

such that for any i:

$$\lambda(X_i) = j^1 X_i\,.$$

This lift obviously defines an S-connection without curvature or torsion on this neighborhood, and we let the same symbol ∇ denote the covariant derivative on $T|_U$ and in the vector bundle $N_k|_U$.

Such a neighborhood U will be called a *flat* neighborhood. In the sequel, we agree that to simplify the notations the fiber bundles and sheaves considered will always be taken to be their restriction to a given flat neighborhood – for example, $T|_U$ will be denoted by simply T.

We thus have a canonical isomorphism of vector bundles for the lift considered: $E_1 \cong T \oplus N_1$ (direct sum, in the Whitney sense) and with that isomorphism, a covariant derivative without curvature:

$$\nabla : E_1 \to E_1 \otimes T^*, \qquad X + n \mapsto \nabla(X) + (\delta + \nabla)(n),$$

where X and n are sections of T and N, respectively. This covariant derivative is defined by a lift:

$$\lambda: E_1 \to J_1(E_1), \qquad \nabla = D \circ \lambda.$$

This lift is such that for any section $\sigma = X + n$ of E_1 one has:

1) $D^2 \circ \lambda(\sigma) = \nabla(\nabla(X)) + \delta(n)) + \delta \circ \nabla(n) = 0,$
2) $(\rho \otimes \mathrm{Id}) \circ D \circ \lambda(\sigma) = \nabla(X) + \delta(n) = D(\sigma).$

From the corollary to Theorem II.3b, it therefore has its values in $J_2(T)$. One immediately verifies that, more precisely, it has values in the sheaf $\mathcal{S}_2$:

$$\lambda: E_1 \to \mathcal{S}_2\,.$$

We thus have the exact sequence of sheaves of $\mathcal{D}$-modules:

$$0 \to N_2 \to \mathcal{S}_2 \xrightarrow{\rho_1} E_1 \to 0.$$

The sheaf $\mathcal{S}_2$ is locally free; the prolonged differential system E_2 (see remark 2 in II-4), which is, in this case:

$$E_2 = J_2(T) \cap J_1(E_1),$$

is then a differentiable vector sub-bundle of $J_2(T)$, with a canonical isomorphism of vector bundles:

$$E_2 \cong E_1 \oplus N_2 \cong T \oplus N_1 \oplus N_2 .$$

In a general fashion, by a simple recurrence argument, we have that for any integer k the prolonged differential system:

$$E_k = J_k(T) \cap J_{k-1}(E_1)$$

as a differentiable vector sub-bundle of $J_k(T)$, with a canonical isomorphism:

$$E_k \cong E_{k-1} \oplus N_k \cong T \oplus N_1 \oplus \ldots \oplus N_{k-1} \oplus N_k .$$

This isomorphism determines a covariant derivative ∇_{k-1} without curvature on E_{k-1} such that with the isomorphism considered, one has:

$$\nabla_{k-1} : E_{k-1} \to E_{k-1} \otimes T^*,$$

$$X + n_1 + \ldots + n_{k-1} \mapsto \nabla(X) + (\delta + \nabla)(n) + \ldots + (\delta + \nabla)(n_{k-1}) .$$

Since the covariant derivative ∇_{k-1} is without curvature, assertion 2 of Proposition III-3c permits us to conclude, in addition, that the differential system E_1 is completely integrable to order $k - 1$; i.e.:

$$\mathcal{S}_{k-1} = E_{k-1}$$

is locally generated by integrable section (see remark 2 of II-4), or again that any element of E_{k-1} is the jet of order $k - 1$ of a solution of E_1 .

From that local study, we therefore have the following global theorem:

THEOREM V.3. – *If the infinitesimal structure S is integrable then the differential system E_1 is completely integrable to all orders and for any integer k the prolonged differential system E_k is then a differentiable vector sub-bundle of $J_k(T)$ such that we have the following exact sequence of vector bundles:*

$$0 \to N_k \to E_k \xrightarrow{\rho_{k-1}} E_{k-1} \to 0.$$

Remark.

Let Θ denote the sheaf of infinitesimal automorphisms of the infinitesimal structure S; i.e., the sheaf of solutions of E_1 . The sheaf Θ will be called *of finite or infinite type* according to whether the space of its germs at an arbitrary point of the base manifold V is a vector space of finite or infinite dimension, respectively.

From the preceding theorem, it is immediate that if the infinitesimal structure S is integrable then the sheaf Θ has the same type as the structure (definition V-1-a).

4. Structure tensor of D. Bernard.

Consider the exact sequence of vector bundles:

$$0 \to N_1 \to E_1 \xrightarrow{\rho} T \to 0.$$

Any splitting of that sequence is equivalent to the given of a section λ of $E_1 \otimes T^*$ that is a lift of the splitting such that:

$$\rho \otimes \mathrm{Id} : E_1 \otimes T^* \to T \otimes T^*, \quad (\rho \otimes \mathrm{Id})(\lambda) = \mathrm{Id}., \qquad \text{the "identity" section.}$$

Any other splitting of the sequence is a lift λ' of the form:

$$\lambda' = \lambda + \eta,$$

where η is an arbitrary section of $N_1 \otimes T^*$, which is a sub-bundle of $E_1 \otimes T^*$. Hence, for the corresponding torsion tensor (see IV.2):

$$D(\lambda') = D(\lambda + \eta) = D(\lambda) + \delta(\eta).$$

This entails that we have a canonical section t_S of the quotient vector bundle $T \otimes \Lambda^2T^*/ \delta(N_1 \otimes T^*)$, which is defined independently of any lift λ:

$$t_S = D(\lambda) \bmod \delta(N_1 \otimes T^*).$$

We have the following obvious proposition, upon recalling that from Proposition V.2 any splitting of the exact sequence considered that defines a covariant derivative without torsion in T is determined by an S-connection.

PROPOSITION V.4a. – *The nullity of the section t_S is a necessary and sufficient condition for there to exist an S-connection without torsion.*

Remark A.

1) The section t_S is therefore a given of the infinitesimal structure S that D. Bernard defined ("Géom. diff. des G-structures") that one calls the *structure tensor,* for that reason.

2) The section t_S is automatically null if:

$$\delta(N_1 \otimes T^*) = T \otimes \Lambda^2T^*.$$

This is the case of a Lorentzian structure; i.e., S is a regular section of $S^2(T^*)$ that is everywhere of maximal rank. In this case, $N_2 = 0$, so one has an isomorphism of vector bundles:

$$\delta: N_1 \otimes T^* \to T \otimes \Lambda^2 T^*.$$

In the case of a Lorentzian structure, there thus exists one and only S-connection without torsion.

Let G be a pesudogroup of differentiable transformations on V. Recall that a fiber bundle F – i.e., one endowed with a submersion p onto V – will be called a *prolongation bundle* of (V, Γ) if any element f of Γ with source U defines a diffeomorphism $\tilde{f}$ of $F|_U$ onto $F|_{f(U)}$ such that the following diagram is commutative:

$$\begin{array}{ccc} F|_U & \xrightarrow{\tilde{f}} & F|_{f(U)} \\ \downarrow p & & \downarrow p \\ U & \xrightarrow{f} & f(U) \end{array}$$

An element f of Γ with source x and target y thus defines a diffeomorphism:

$$\tilde{f}_x : F_x \to F_y \quad (F_x = p^{-1}(x)).$$

In a more precise fashion, the prolongation space F will be called the *prolongation of order k of* (V, Γ) if we have the following relation:

$$(\tilde{f}_x = \tilde{g}_x) \quad \Leftrightarrow \quad (j_x^k(f) = j_x^k(g)).$$

If s_U denotes the restriction to U of a differentiable section s of F on V then $\tilde{f} \circ s_U \circ f^{-1}$ is a differentiable section of F that is defined on $f(U)$. We say that the section s is an invariant of (V, Γ) if for any element f of Γ with source U one has:

$$\tilde{f} \circ s_U \circ f^{-1} = s_{f(U)} .$$

In the case where F is a prolongation space of order k of (V, Γ), such a section s will be called an *invariant of order k* of (V, Γ).

Having recalled these notions, it is immediate to show that the structure tensor t_S is an invariant of order 1 of (V, Γ), where Γ denotes the pseudogroup of local automorphisms of the infinitesimal structure S (remark 2 of IV.3). We say simply that t_S is an *invariant of order* 1 of the structure.

PROPOSITION V.4.b (D. Bernard). – *The structure tensor t_S is an invariant of order* 1 *of the structure that is an obstruction to its integrability; i.e.:*

$$t_S = 0, \qquad \textit{viz., the zero section of } T \otimes \Lambda^2 T^* / \delta(N_1 \otimes T^*),$$

is a necessary condition for the infinitesimal structure S to be integrable.

Indeed, on any flat open set U there exists an S-torsion without torsion; therefore the restriction of t_U to U must be zero. It then follows from this that if the infinitesimal structure S is integrable then t_S must be zero.

Recall that since the morphism ρ of E_1 onto T is surjective, a bundle morphism m is defined such that the following sequence of sheaves of $\mathcal{D}$-modules is exact:

$$0 \to N_2 \to \mathcal{S}_2 \xrightarrow{\rho_1} E_1 \xrightarrow{m} H_0^2(S),$$

where $H_0^2(S)$ is the vector bundle $T \otimes \Lambda^2 T^* / \delta(N_1 \otimes T^*)$. It is immediate to show that the morphism m considered, as a differentiable section of $H_0^2(S) \otimes E_1^*$, is an invariant of order 2 of the structure. We shall show that the morphism m, which is an obstruction to the complete integrability of E_1, is zero if the structure tensor t_S is zero. In a more precise fashion, we have the following theorem:

THEOREM V.4. – *If the structure tensor is zero then we have the exact sequence of vector bundles:*

$$0 \to N_2 \to E_2 \to E_1 \to 0,$$

where E_2 is the prolonged differential system of order 2 *of E_1* .

Proof.

It suffices to prove that the morphism:

$$\mathcal{S}_2 \xrightarrow{\rho_1} E_1$$

is surjective. Since the structure tensor t_S is null, there exists a lift λ of T to E_1 that determines a covariant derivative ∇ without torsion on T. From theorem V.2, we already know that $\rho_1(\mathcal{S}_2)$ contains the subsheaf N_1 of E_1 . It thus remains to show that the morphism:

$$m \circ \lambda : T \to E_1 \to H_0^2(S)$$

is zero, which is an immediate consequence of the following lemma.

LEMMA. – For any section X of T, $\nabla^2(X)$ is a section of $\delta(N_1 \otimes T^*)$.

Now, indeed, since the covariant derivative ∇ is without torsion, it is determined by an S-connection whose curvature tensor R is a section of $N_1 \otimes \Lambda^2 T^*$ ($N_1 = \mathcal{G}(\Phi)$), and we have, for any pair of vector fields Y and Z:

$$\nabla^2(X)(Y \wedge Z) = R(Y \wedge Z)(X).$$

Now, from the Bianchi identity (Proposition IV.2.b), when it is applied to the case where the torsion tensor is null, one has:

$$R(Y \wedge Z)(X) + R(Z \wedge X)(Y) + R(X \wedge Y)(Z) = 0,$$

or again:

$$R(Y \wedge Z(X) = R(X \wedge Z)(Y) - R(X \wedge Y)(Z),$$

which expresses the idea that for any fixed X, $\nabla^2(X)$ is a section of $\delta(N_1 \otimes T^*)$.

Q. E. D.

Remark B.

1) If S is an absolute parallelism structure then we have $N_1 = 0$, and:

$$0 \to E_1 \xrightarrow{\rho} T \to 0.$$

Identifying E_1 with T, the morphism m: $E_1 \to T \otimes \Lambda^2 T^* = H_0^2(S)$ is nothing but the curvature tensor of the canonical connection; the morphism m is therefore zero. The nullity of the structure tensor t_S in this case must say that the canonical S-connection, which is without curvature, is also without torsion. An absolute parallelism structure with zero structure tensor is therefore integrable.

2) In the case where the structure S is involutive – i.e., of degree 1 ($N_1 \neq 0$, which entails that S is of infinite type; see Matsushima, "Algébres de Lie semi-involutives") – we may recursively apply Theorem II.4.b to confirm that if the structure tensor t_S is zero, or if the morphism $m : E \to H_0^2(S)$ is zero, then for any integer k, we have the exact sequence of sheaves of $\mathcal{D}$-modules on V:

$$0 \to N_k \to \mathcal{S}_k \xrightarrow{\rho_{k-1}} \mathcal{S}_{k-1} \to 0,$$

or furthermore, of vector bundles:

$$0 \to N_k \to E_k \xrightarrow{\rho_{k-1}} E_{k-1} \to 0,$$

the prolongation systems E_k being differentiable vector sub-bundles of $J_k(T)$. However, in the same case, we cannot actually conclude the complete integrability of E_1, since the existence of formal solutions does not imply the existence of solutions of E. D. C. Spencer has conjectured that if the structure S is, moreover, elliptic – i.e., E_1 defines an elliptic differential operator on T with values on $J_1(T)/E_1$ (see D. C. Spencer) – then the infinitesimal structure S is completely integrable (indeed, we have the Newlander-Nirenberg theorem, which asserts that an almost-complex structure with zero structure tensor is integrable).

5. Obstructions of higher order. Case of structures of finite type.

In a general fashion, suppose that the infinitesimal structure S is such that for any $k \leq p$, we have the exact sequence of vector bundles:

$$0 \to N_k \to E_k \xrightarrow{\rho_{k-1}} E_{k-1} \to 0,$$

E_k being differentiable vector sub-bundles of $J_k(T)$, respectively. From Theorem II.4.b, a morphism:

$$m : E_p \to H^2_{p-1}(S)$$

is then defined such that the following sequence of sheaves of $\mathcal{D}$-modules is exact:

$$0 \to N_{p+1} \to \mathcal{S}_{p+1} \xrightarrow{\rho_p} E_p \xrightarrow{m} H^2_{p-1}(S).$$

This morphism defines a section of $H^2_{p-1}(S) \otimes E^*_p$, which we nevertheless denote by $O(p)$, which is obviously an invariant of order p of the structure. The nullity of that section entails that we have the exact sequence:

$$0 \to N_{p+1} \to \mathcal{S}_{p+1} \xrightarrow{\rho_p} E_p \to 0,$$

or furthermore the exact sequence of vector bundles, with N_{p+1} a differentiable vector bundle of $J_{p+1}(T)$:

$$0 \to N_{p+1} \to E_{p+1} \to E_p \to 0.$$

We have therefore defined, in an obvious sense, $O(p)$ as an obstruction of order p to the complete integrability of the structure.

In the case where the structure tensor t_S is zero, Theorem V.2 asserts that the restriction of the morphism m to the sub-bundle N_p of E_p is, in fact, zero; the morphism m then passes to the quotient to define a morphism m':

$$m' : E_{p-1} \to H^2_{p-1}(S),$$

or again, a section $O'(p)$ of $H^2_{p-1}(S) \otimes E^*_{p-1}$, which is an invariant of order $p - 1$ of the structure, and is, in this case, the true obstruction of order k to the complete integrability of the structure.

If the infinitesimal structure is of degree p then the nullity of these obstructions up to order p obviously again entails that for any integer k, E_k is a differentiable fiber sub-bundle of $J_k(T)$, and we have the exact sequence of vector bundles:

$$0 \to N_k \to E_k \xrightarrow{\rho_{k-1}} E_{k-1} \to 0.$$

As in the remark B.2. of V.4, we may not conclude that the structure is completely integrable, in general. In the particular case of a structure of finite type, we have the theorem:

THEOREM V.5.a – *If the infinitesimal structure S is of type* $p-1$ *– i.e.,* $N_p = 0$ *– and all of its obstructions up to order p are zero then the differential system* E_1 *is completely integrable to all orders.*

Proof:

Indeed, for any $k \geq p$ we have the exact sequence:

$$0 \to E_k \to E_{k-1} \to 0.$$

It suffices for us to prove that E_p is completely integrable. From the assertion 2 of the proposition III.3.c, this amounts to verifying that the canonical isomorphism:

$$\lambda : E_p \to E_{p-1}$$

defines a covariant derivative $\nabla = D \circ \lambda$ without curvature in E_p . Now, we obviously have:

$$(\lambda \otimes \mathrm{Id}) \circ D \circ \lambda = D \circ \lambda',$$

where λ' is the canonical isomorphism of E_p into E_{p+2} . Thus:

$$\nabla^2 = D \circ (\lambda \otimes \mathrm{Id}) \circ D \circ \lambda = D \circ D \circ \lambda' = 0.$$

Q. E. D.

COROLLARY (see remark in V.3). – *With the same hypotheses as in the preceding theorem, the sheaf* Θ *of infinitesimal automorphisms of the structure is of finite type.*

Proof:

Indeed, let $(j^{p+1}X_i)_{1 \leq i \leq q}$ be integrable sections that form a basis for E_{p+1}, the locally free sheaf of $\mathcal{D}$-modules in the neighborhood *U* of a point *x* of *V*. In a neighborhood of *x*, any section *Y* of Θ is of the form:

$$j^{p+1}Y = f^i \, j^{p+1}X_i \, ,$$

where f^i are differentiable functions of *V* that are defined in a neighborhood of *x*. We thus have:

$$D(j^{p+1}\,Y) = j^p \, X_i \otimes df^i = 0.$$

Now, the $j^p X_i$ also define a basis for E_p, which is a sheaf of locally free $\mathcal{D}$-modules; this relation entails that:

$$df^i = 0,$$
$$f^i = a^i = \text{constant}.$$

We thus have:

$$Y = a^i X_i .$$

Q. E. D.

Remark A.

Always with the same hypotheses as in Theorem V.5.a, let X denote a global section of Θ. It is immediate, from the proof of the corollary, that the set:

$$\{x, x \in V \text{ such that } j_x^p X = 0\}$$

is an open subset. On the other hand, that set is obviously closed, like the set of "zeroes" of a differentiable section. If V is connected then the map:

$$j_x^p : H^0(V, \Theta) \to E_p|_x , \qquad X \mapsto j_x^p X$$

is therefore injective; one concludes from this that $H^0(V, \Theta)$ is a finite-dimensional vector space, and a simple argument due to Palais shows that the group of global automophisms of the structure is then a Lie group whose Lie algebra is nothing but the Lie subalgebra of $H^0(V, \Theta)$ that is defined by the elements that generate a global one-parameter group (R. Palais, "Lie Theory of Transformation Groups"). We thus recover the theorem of S. Kobayashi that says that the group of global automorphisms of an absolute parallelism structure is a Lie group.

In a more precise fashion, it is immediate that we have the following theorem, which clarifies the preceding corollary:

THEOREM V.5.b. – *If the infinitesimal structure S is of type p –* 1, *and all of its obstructions are zero up to order p then the sheaf Θ of infinitesimal automorphisms of the structure is a locally constant sheaf of $\mathbb{R}$-Lie algebras.*

In particular, if V is simply connected then the sheaf Θ is a constant sheaf – i.e., isomorphic to $V \times \mathcal{G}$, where $\mathcal{G}$ is a Lie algebra that is isomorphic to Θ_x . If G is a simply-connected Lie group that has $\mathcal{G}$ for its Lie algebra then G is a local group of automorphisms on V, in the sense of Palais (definition II, page 33, *loc. cit.*).

Remark B.

We return to the general case. Recall that, from Proposition IV.1.b, the sheaf $\mathcal{S}_k$ is a sheaf of $\mathbb{R}$-Lie algebras. Therefore, in the case where the nullity of the obstructions to complete integrability up to order p, E_k is, for any $k \leq p + 1$, a differentiable vector sub-bundle of $J_k(T)$ without constant term – i.e., the morphism $\rho : E_k \to T$ is surjective, and the sheaf of sections $E_k \cong \mathcal{S}_k$ is a subsheaf of $\mathbb{R}$-Lie algebras of $J_k(T)$; , from Theorem IV.3, E_k thus defines a Lie sub-groupoid Φ_k of $\Pi^k(V)$ that is called the *holonomic prolongation groupoid* of Φ of order k. The bundle of isotropy Lie algebras $\mathcal{G}(\Phi_k)$ is nothing but the bundle E_k^0 that is defined by the exact sequence:

$$0 \to E_k^0 \xrightarrow{\rho} T \to 0,$$

and the isotropy Lie algebra structure on E_k^0 is what one calls the *derived structure* of the Lie algebra structure on the fibres of $N_1 \cong \mathcal{G}(\Phi)$ (see Y. Matsushima, "Algèbres de Lie linéaires semi-involutive," or V. Guillemin and S. Sternberg, "Transitive differential geometry"). We obviously have, for any $q \leq k$:

$$\Phi_k \xrightarrow{\rho_q} \Phi_q \to 1.$$

One may prove, moreover, that E_k is a vector bundle that is associated with the Lie groupoid Φ_{k+1} .

In IV.2, we defined a morphism:

$$\tau : J_k(T) \wedge J_k(T) \to J_{k-1}(T) .$$

One immediately confirms that we have:

$$\tau : E_k \wedge E_k \to E_{k-1} .$$

This morphism, when considered as a section of $E_{k-1} \otimes \Lambda^2 E_k^*$ is then left invariant by the holonomic prolongation group Φ_{k+1} , which, from the preceding, operates in an obvious fashion on the fiber bundle considered. This morphism τ is therefore an invariant of order $k + 1$ of the structure, and it is that invariant that V. Guillemin and S. Sternberg considered in the cited paper.

If the infinitesimal structure S is of type $p - 1$ then we have:

$$0 \to E_p \xrightarrow{\rho_{p-1}} E_{p-1} \to 0.$$

Upon identifying E_{p-1} with E_p, the morphism:

$$\tau : E_{p-1} \wedge E_{p-1} \rightarrow E_{p-1}$$

defines a Lie algebra structure on each of the fibers of E_{p-1} . The preceding considerations show that E_{p-1} , when endowed with that Lie algebra structure defined on each of its fibers, is a bundle of Lie algebras that is associated with the holonomic prolongation Lie groupoid Φ_p . One may prove that this Lie algebra structure is nothing but the Lie algebra structure of Θ_x , which is the space of germs at x of the sheaf of $\mathbb{R}$-Lie algebras Θ.

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