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Some questions concerning the theory of maxima and minima of multiple integrals

By **Johann Radon** in Göttingen

Translated by D. H. Delphenich

The problem in the following considerations is that of extending the method of parametric representation that was introduced into the calculus of variations by **Weierstrass** and was constructed completely for the integral $\int F(x, y, x', y')\,dt$ to the case of multiple integrals that contain first derivatives of the unknown functions. We shall address integrals over n-dimensional hypersurfaces in $(n + 1)$-dimensional spaces, and we will have to derive for them, on the one hand, the formal results in regard to the theory of the first and second variation and on the other hand, the main theorems of the more recent calculus of variations, namely, **HILBERT**'s independence theorem and the **Weierstrass** transformation of the total variation and **Kneser**'s transversality theorem that it implies. No weight will be placed upon achieving an arithmetically-rigorous basis. With a sufficient restriction of the concepts of "hypersurface" and "boundary of the hypersurface," that would present no difficulties, but it would nonetheless require considerable space.

The methods of the parametric representation were already applied to double integrals by **Kobb** (Acta mathematica, v. 16, 17), and that theory is presented thoroughly in **Kneser**'s textbook. Now, it is remarkable that the quite complicated calculations of those authors can be simplified considerably by a simple device and can then be easily adapted to the general case of n-fold integrals. By just that device, one will succeed in expressing the transversality condition, the independence theorem, etc. very simply and clearly.

I.

Let a hypersurface be given in the $(n + 1)$-dimensional space with the coordinates $x_0, x_1, \ldots, x_n$ by the parametric representation:

$$x_i = x_i\,(u_1, u_2, \ldots, u_n) \qquad i = 0, 1, \ldots, n, \tag{1}$$

and at the same, the determinant of the matrix:

$$\left\| \frac{\partial x_i}{\partial u_k} \right\|$$

may not vanish anywhere. We denote that determinant as follows:

$$p_h = (-1)^h \begin{vmatrix} \dfrac{\partial x_0}{\partial u_1} & \cdots & \dfrac{\partial x_{h-1}}{\partial u_1} & \dfrac{\partial x_{h+1}}{\partial u_1} & \cdots & \dfrac{\partial x_n}{\partial u_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} . \tag{2}$$

Now let Φ be a function of $x_0, \ldots, x_n$ and the derivatives $\partial x_i / \partial u_k$. We then consider the n-fold integral:

$$J = \int \Phi\left(x_i, \frac{\partial x_i}{\partial u_k} \right) du_1 \cdots du_n$$

when it is extended over the hypersurface. We would now like to consider only those functions Φ for which that integral is independent of the choice of the parameter u_i. In order to fix them more precisely, we would like to assume that our hypersurface is referred to a well-defined system of parameters. A positive normal direction will then be established by the associated p_h, and in that way a positive and negative side of the hypersurface will be distinguished at every point. We shall now consider only those parameter transformations that do not switch the sides of the surface, i.e., ones whose determinant:

$$\frac{\partial (u_1, \ldots, u_n)}{\partial (u'_1, \ldots, u'_n)} = \Delta \tag{3}$$

is positive. In that way, the integral will go to:

$$J' = \int \Phi\left(x_i, \frac{\partial x_i}{\partial u'_k} \right) du'_1 \cdots du'_n = \int \Phi\left(x_i, \frac{\partial x_i}{\partial u'_k} \right) \frac{1}{\Delta} du_1 \cdots du_n .$$

One easily concludes from this that if one is to have $J = J'$ for all hypersurfaces and all possible parametric representation then the identity:

$$\Phi\left(x_i, \frac{\partial x_i}{\partial u'_k} \right) = \Delta \, \Phi\left(x_i, \frac{\partial x_i}{\partial u_k} \right)$$

must exist. Now, instead of deriving relations between the derivatives of Φ with respect to $\partial x_i / \partial u_k$ by performing various differentiations, as **Kobb** and **Kneser** did, we shall make the easily-verified remark that the identity above will be fulfilled if and only if Φ has the form:

$$\Phi\left(x_i, \frac{\partial x_i}{\partial u_k}\right) = F\,(x_i, p_i)\,,$$

in which F is a positive-homogeneous function of the p_i such that:

$$F\,(x_i, k\,p_i) = k\,F\,(x_i, p_i) \qquad \text{when} \quad k > 0\,. \tag{4}$$

That is what the device that was mentioned in the introduction consists of, and its immediate consequence consists of the convenience of having a function F with only $2n + 2$ variables, while Φ includes $(n + 1)^2$. It further follows from (4) that:

$$F\,(x_i, p_i) = \sum_{h=0}^{n} p_h\,F_{p_h}\,, \tag{5}$$

$$0 = \sum_{h=0}^{n} p_h\,F_{p_h\,p_k}\,, \tag{6}$$

when the indices p_h mean derivatives with respect to the respective variables. Since the general solution of the equation:

$$\sum_{h=0}^{n} p_h\,\lambda_h = 0$$

is given by:

$$\lambda_h = \sum_{i=1}^{n} \xi_i \frac{\partial x_h}{\partial u_i}\,,$$

it will follow from (6) that the $(n + 2)(n + 1) / 2$ quantities $F_{p_h\,p_k}$ can be expressed in terms of $n\,(n + 1) / 2$ new quantities $\Phi_{rs} = \Phi_{sr}$ in the form:

$$F_{p_h\,p_k} = \sum_{r,s=1}^{n} \Phi_{rs} \frac{\partial x_i}{\partial u_r} \frac{\partial x_k}{\partial u_s}\,. \tag{7}$$

It should be pointed out that F, F_{p_i}, $F_{p_i\,p_k}$ are invariant under parameter transformations, and ones of index 1, 0, – 1, resp., while the Φ_{rs} are again dependent upon the choice of the system of parameters. However, the following important relation exists for the transition to the new parameters $u'_1, \ldots, u'_n$:

$$\Phi'_{rs} = \Delta^{-1} \sum_{\rho,\sigma=1}^{n} \Phi_{\rho\sigma} \frac{\partial u'_r}{\partial u_\rho} \frac{\partial u'_s}{\partial u_\sigma}\,, \tag{8}$$

in which Δ is defined by (3). As long as the determinant of the Φ is non-zero, one can now regard the Φ_{rs} as subdeterminants of a system of quantities φ_{rs} and obtain the result that the quadratic differential form:

$$\sum_{r,s=1}^{n} \varphi_{rs}\, du_r\, du_s$$

will transform as follows under the transition to new parameters:

$$\sum_{r,s=1}^{n} \varphi'_{rs}\, du'_r\, du'_s = \Delta^{-\frac{3}{n-1}} \sum_{r,s=1}^{n} \varphi_{rs}\, du_r\, du_s \,. \tag{9}$$

In order to express the Φ_{ik} in terms of the $F_{p_i p_k}$, we proceed as follows:

We denote the subdeterminant of $\partial x_i / \partial u_k$ in p_h by $P_{i,k}^{(h)}$ and set $P_{i,k}^{(i)} = 0$. From the elementary rules for determinants, the following relations will then exist:

$$\left.\begin{array}{c} P_{i,k}^{(h)} + P_{h,k}^{(i)} = 0\,, \\ \displaystyle\sum_{\lambda=0}^{n} P_{\lambda,\sigma}^{(h)} \frac{\partial x_\lambda}{\partial u_\rho} = \delta_{\rho\sigma}\, p_\kappa\,, \quad \sum_{\lambda=0}^{n} P_{\rho,\sigma}^{(\lambda)} \frac{\partial x_\lambda}{\partial u_\tau} = -\delta_{\tau\sigma}\, p_\rho\,, \\ \displaystyle\sum_{\lambda=1}^{n} P_{\rho,\lambda}^{(\kappa)} \frac{\partial x_\sigma}{\partial u_\lambda} = \delta_{\sigma\rho}\, p_\kappa - \delta_{\sigma\kappa}\, p_\rho\,, \\ \displaystyle\sum_{k=1}^{n} \frac{\partial P_{i,k}^{(s)}}{\partial u_k} = 0\,. \end{array}\right\} \tag{10}$$

In those expressions, $\delta_{ik} = 0$ when $i \neq k$, $\delta_{ii} = 1$.

With the help of the second equation in (10), one will get from (7) that:

$$p_\lambda\, p_\mu\, \Phi_{rs} = \sum_{i,k=0}^{n} F_{p_i p_k} P_{ir}^{(\lambda)} P_{ks}^{(\mu)} \,. \tag{11}$$

Therefore, when $p_h \neq 0$, those formulas will allow one to calculate Φ_{rs} when one sets $\lambda = \mu = h$.

With those preparations, we shall go on to the transformation of the first variation. We first have:

$$\delta J = \int \delta F\, du_1 \cdots du_n \,.$$

Now, we have:

$$\delta F = \sum_{h=0}^{n} F_{x_h}\, \delta x_h + F_{p_h}\, \delta p_h$$

$$= \sum_{h=0}^{n} F_{x_h}\,\delta x_h + \sum_{i,h=0}^{n}\sum_{k=1}^{n} P_{i,k}^{(h)} \frac{\partial\,\delta x_i}{\partial u_k} F_{p_h}$$

$$= \sum_{h=0}^{n} F_{x_h}\,\delta x_h - \sum_{i,h=0}^{n}\sum_{k=1}^{n} \delta x_i \frac{\partial P_{i,k}^{(h)}}{\partial u_k} - \sum_{i,h=0}^{n}\sum_{k=1}^{n} P_{i,k}^{(h)}\,\delta x_i \frac{\partial F_{p_h}}{\partial u_k} + \sum_{i,h=0}^{n}\sum_{k=1}^{n} \frac{\partial}{\partial u_k}(F_{p_h} P_{i,k}^{(h)}\,\delta x_i)\ .$$

The second vanishes from the last equation in (10). If we then set:

$$\frac{\partial F_{p_h}}{\partial u_k} = \sum_{\rho}\left(F_{x_\rho\, p_h} \frac{\partial x_\rho}{\partial u_k} + F_{p_\rho\, p_h} \frac{\partial p_\rho}{\partial u_k} \right)$$

and express $\frac{\partial p_\rho}{\partial u_k}$ in terms of $\sum_{r,s} P_{rs}^{(\rho)} \frac{\partial^2 x_r}{\partial u_k\,\partial u_s}$ then when we recall the relation (10), that will imply the simple final result that:

$$\delta F = \sum_{i,h=0}^{n}\sum_{k=1}^{n} \frac{\partial}{\partial u_k}(F_{p_h} P_{i,k}^{(h)} \delta x_i) + T \cdot \sum_{h=0}^{n} p_h\,\delta x_h\ , \tag{12}$$

in which:

$$T = \sum_{i=0}^{n} F_{x_i\, p_i} - \sum_{r,s=0}^{n} \Phi_{rs}\, d_{rs}\ ,$$

$$d_{rs} = \sum_{r,s} p_h \frac{\partial^2 x_r}{\partial u_r\,\partial u_s} = -\sum_{h=0}^{n} \frac{\partial x_h}{\partial u_s}\frac{\partial p_h}{\partial u_r} = -\sum_{h=0}^{n} \frac{\partial x_h}{\partial u_r}\frac{\partial p_h}{\partial u_s}\ . \tag{13}$$

The quadratic differential form:

$$\sum_{r,s=1}^{n} d_{rs}\, du_r\, du_s = -\sum_{h=0}^{n} dp_h\, dx_h$$

transforms as follows under the transition to new parameters:

$$\sum_{r,s=1}^{n} d'_{rs}\, du'_r\, du'_s = \Delta \sum_{r,s=1}^{n} d_{rs}\, du_r\, du_s\ . \tag{14}$$

It follows from this and (9) that the expression T is absolutely invariant under parameter transformations. $T = 0$ is the differential equation of the extremal hypersurfaces. If we now substitute (12) in the expression for δJ then the first term can be transformed into an integral over the (n – 1-dimensional) boundary of the hypersurface by partial integration. If that is given by $x_k = x_k(v_1, \ldots, v_{n-1})$ then one will ultimately get:

$$\delta J = \pm\int \begin{vmatrix} F_{p_0} & F_{p_1} & \cdots & F_{p_n} \\ \delta x_0 & \delta x_1 & \cdots & \delta x_n \\ \dfrac{\partial x_0}{\partial v_1} & \dfrac{\partial x_1}{\partial v_1} & \cdots & \dfrac{\partial x_n}{\partial v_1} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial x_0}{\partial v_{n-1}} & \dfrac{\partial x_1}{\partial v_{n-1}} & \cdots & \dfrac{\partial x_n}{\partial v_{n-1}} \end{vmatrix} dv_1 \cdots dv_{n-1} + \int T \cdot \sum_{i=0}^{n} p_i\, \delta x_i\, du_1 \cdots du_n\,. \tag{15}$$

The sign on the first integral depends upon the choice of the parameters $v_1, \ldots, v_{n-1}$.

One sees from this that for a fixed boundary, if one is to have an extremum then it is necessary that $T = 0$, i.e., that the hypersurface must be an extremal.

By contrast, if all that is prescribed is that the boundary should lie on a given hypersurface K then the δx_i will not be zero on that boundary. However, one sees that the subdeterminants of the first row in the first term of (15) must then be proportional to the quantities $\pi_0, \pi_1, \ldots, \pi_n$ when we denote the direction parameters of the normal to K by π_k . Thus, a further necessary condition for an extremum with a variable boundary will be:

$$\sum_{h=0}^{n} F_{p_h}\, \pi_h = 0\,, \tag{16}$$

which we shall refer to as the *transversality condition.*

Here, we should connect with the theory of the second variation, but we shall only go so far as to remark that (for a fixed boundary) the second variation of an extremal hypersurface can be put into the form:

$$\delta^2 J = \int \Psi(w)\, w\, du_1 \cdots du_n\,,$$

in which $w = \sum p_i\, \delta x_i$, and Ψ is a linear self-adjoint second-order differential expression of the form:

$$\Psi(w) = A\, w - \sum_{h,k=1}^{n} \frac{\partial}{\partial u_h}\left(\Phi_{hk} \frac{\partial w}{\partial u_k}\right). \tag{17}$$

The **Jacobi** transformation of the second variation that is associated with it and the connected considerations are developed in precisely the same way as when the integral has the form $\int f\left(x_0, \ldots, x_n, \dfrac{\partial x_0}{\partial x_1} \cdots \dfrac{\partial x_0}{\partial x_n}\right) dx_1 \cdots dx_n$. We infer from that theory the theorem ([1]) that in order for a minimum (maximum, resp.) to occur, it is necessary that the quadratic form:

([1]) Which was proved rigorously for $n = 2$ by **Mason**.

$$\sum_{i,k=1}^{n} \Phi_{ik}\, y_i\, y_k \tag{17.a}$$

must not be negative (positive, resp.) on the extremal hypersurface for any real system of values $y_1, \ldots, y_n$.

If the determinant $| \Phi_{ik} |$ is non-zero then we can also discuss the form with the coefficients φ_{ik} [cf., (9)], instead of the latter form, and it will be positive or negative or definite (indefinite, resp.) at the same time as the latter form.

II.

We now turn to the second part of our considerations. Let a one-parameter family of extremal hypersurfaces be given that simply covers a subset of $(n + 1)$-dimensional space, which we will refer to as a *field*. $p_0, \ldots, p_n$, $F_{p_0}, \ldots, F_{p_n}$ will then be single-valued functions of position in it when we understand $p_0, \ldots p_n$ to mean the values that are calculated for the extremal hypersurface that goes through the point $x_0, \ldots, x_n$. One can immediately derive a differential relation that the F_{p_0}, $\ldots$, F_{p_n} must satisfy then. Namely, if we let $\partial F_{p_i} / \partial x_k$ denote a differentiation in the sense that F_{p_i} will be thought of as a single-valued function of $x_0, \ldots, x_n$ then we will have:

$$\begin{aligned}\sum_{k=0}^{n} \frac{\partial F_{p_k}}{\partial x_k} &= \sum_{k=0}^{n} \left(F_{p_k x_k} + \sum_{h=0}^{n} F_{p_k\, p_h} \frac{\partial p_h}{\partial x_k} \right) \\ &= \sum_{k=0}^{n} \left(F_{p_k x_k} + \sum_{h=0}^{n} \sum_{r,s=1}^{n} \Phi_{rs} \frac{\partial x_k}{\partial u_r} \frac{\partial x_h}{\partial u_s} \frac{\partial p_h}{\partial x_k} \right) \\ &= \sum_{k=0}^{n} F_{p_k x_k} - \sum_{r,s=1}^{n} \Phi_{rs}\, d_{rs} \ .\end{aligned}$$

However, that is nothing but the expression T, and it will vanish since the hypersurfaces are extremals.

We then have the relation:

$$\sum_{k=0}^{n} \frac{\partial F_{p_k}}{\partial x_k} = 0 \ , \tag{18}$$

but that says only that the integral:

$$J^* = \int \sum_{k=0}^{n} \bar{p}_k\, F_{p_k}\, d\bar{u}_1 \cdots d\bar{u}_n \tag{19}$$

has the same value over all hypersurfaces with the same boundary that lie completely in the field and whose parameters are $\bar{u}_1, \ldots, \bar{u}_n$, and have a normal direction $\bar{p}_0, \ldots, \bar{p}_n$. With that, we have arrived at the analogue of **Hilbert**'s independence theorem.

That further implies the following transformation of ΔJ : Let a subset of an extremal be surrounded by a field such that the field itself belongs to an extremal field. We can then take the integral $\int F\,du_1\cdots du_n = J$ over the extremal and replace it with the integral J^*, which is taken over an arbitrary hypersurface that lies completely in the field and has the same boundary as the extremal hypersurface. Let the integral $\int F\,d\bar{u}_1\cdots d\bar{u}_n$, when taken over the latter surface, be denoted by $\bar{J}$, so we will then have:

$$\Delta J = \bar{J} - J = \bar{J} - J^* = \int E(\bar{x}_0,\ldots,\bar{x}_n, p_0,\ldots,p_n, \bar{p}_0,\ldots,\bar{p}_n)\,d\bar{u}_0\cdots d\bar{u}_n\,, \tag{20}$$

when we set:

$$E(\bar{x}_0,\ldots,\bar{x}_n, p_0,\ldots,p_n, \bar{p}_0,\ldots,\bar{p}_n) = F(\bar{x},\bar{p}) - \sum_{\kappa=0}^{n} \bar{p}_\kappa\, F_{p_\kappa}(\bar{x}, p)\,, \tag{21}$$

in which the p are the normals directions of the extremals, when regarded as functions of the field.

We now consider the curves of the field that are given by the differential equations:

$$\frac{dx_0}{F_{p_0}} = \frac{dx_1}{F_{p_1}} = \ldots = \frac{dx_n}{F_{p_n}}\,.$$

Assuming that $F \neq 0$, they will simply cover the field and shall be referred to as *transversal curves*. If one bounds an arbitrary region on an extremal hypersurface of the field then the transversal curve through its boundary will define a hypersurface that will be referred to as a *transversal hypersurface.* That will bound a certain region on each extremal hypersurface of the field, and when the integral $\int F\,du_1\cdots du_n$ is extended over that region, it will have the same value for all field extremals, which would follow from (15). That is the analogue of **Kneser**'s transversal theorem. When the **Hilbert** integral J^* is extended over a transversal surface, it will have the value zero, since its integrand will vanish identically there.

We would now like to develop the consequences of the transformation (20). If we develop $F(\bar{x},\bar{p})$ in a **Taylor** series relative to the variables $\bar{p}$ then when we assume the continuity of the derivatives $F_{p_i p_k}$, that will give:

$$E(\bar{x}, p, \bar{p}) = \tfrac{1}{2}\sum_{i,k=0}^{n} (\bar{p}_k - p_k)(\bar{p}_i - p_i)\,F_{p_i p_k}(\bar{x}, p^*)\,, \tag{22}$$

when we set:

$$p_i^* = p_i + \Theta(\bar{p}_i - p_i)\,, \qquad 0 < \Theta < 1\,.$$

That further implies that:

$$E(\bar{x}, k_1 p, k_2 \bar{p}) = k_2\,E(\bar{x}, p, \bar{p})\,, \qquad k_1, k_2 > 0\,.$$

We can always introduce the direction cosines of the normal in question in place of p, $\bar{p}$ in the following discussion of the sign of E then, i.e., we can replace p_k with $\xi_k = \dfrac{p_k}{\sqrt{\sum_h p_h^2}}$ and $\bar{p}_k$ with $= \dfrac{\bar{p}_k}{\sqrt{\sum_h \bar{p}_h^2}}$. We shall now make the following assumptions:

a) Let the quadratic form with the coefficients Φ_{ik} be positive-definite along the extremal hypersurface H that is surrounded by a field.

b) Let $E(x, p, \bar{p})$ [$E(x, \xi, \bar{\xi})$, resp.] be positive along the same extremal hypersurface for all systems of values (x, p) [(x, ξ), resp.] of the extremal H and every arbitrary system of values $\bar{p}$ [$\bar{\xi}$, resp.], and let it vanish for only $p_i = \bar{p}_i$ [$\xi_i = \bar{\xi}_i$, resp.] ($i = 0, 1, \ldots, n$).

α) It then follows from the second assumption that: If we are given an arbitrary number ε, $0 < \varepsilon < 1$ then we can fix a neighborhood (ρ) of H inside of the field such that the $\bar{\bar{E}}$-function is likewise positive for all systems of values $\bar{\xi}$ that satisfy the relation $\sum_i \xi_i \bar{\xi}_i < 1 - \varepsilon$, and in the neighborhood (ρ) of each point.

We now consider the expression (22) for E to be a quadratic form with the variables $\bar{p} - p$ and the coefficients $F_{p_i p_k}(\bar{x}, p^*)$. For $\bar{x} = x$, $\bar{p}_i = p_i$, those coefficients will go to $F_{p_i p_k}(x, p)$, and after introducing the Φ_{rs}, the quadratic form will take on the value:

$$\sum_{i,k=0}^{n} F_{p_i p_k}(x, p)\, y_i\, y_k = \sum_{r,s=1}^{n} \Phi_{rs} \sum_i y_i \frac{\partial x_i}{\partial u_r} \sum_k y_k \frac{\partial x_k}{\partial u_s} .$$

However, since assumption *a*) says that the Φ_{rs} are the coefficients of a definite form, the form $\sum F_{p_i p_k}(x, p)\, y_i\, y_k$ will vanish only for $y_i = \rho\, p_i$ and will be nowhere negative. It is then semi-definite. The condition for that is that no root λ of the equation:

$$\begin{vmatrix} F_{p_0 p_0} - \lambda & F_{p_0 p_1} & \cdots & F_{p_0 p_n} \\ F_{p_1 p_0} & F_{p_1 p_1} - \lambda & \cdots & F_{p_1 p_n} \\ \vdots & \vdots & \ddots & \vdots \\ F_{p_n p_0} & F_{p_n p_1} & \cdots & F_{p_n p_n} - \lambda \end{vmatrix} = 0$$

is negative. Now, that equation has the root $\lambda = 0$, which is a simple root, since the coefficient of λ has the value $|\Phi_{ik}|\sum_i p_i^2$, where $|\Phi_{ik}|$ is the determinant of the Φ_{ik}. All of the remaining roots will then be positive. If we now go on from the $F_{p_i p_k}(x, p)$ to the $F_{p_i p_k}(\bar{x}, p^*)$ then the root will remain $\lambda = 0$, and the corresponding values of y_i that make the form $\sum y_i\, y_k\, F_{p_i p_k}(\bar{x}, p^*)$ vanish will be proportional to the p^*, as one easily sees. When one chooses the $\bar{x}$, p^* [$\bar{p}$, resp.] to be sufficiently close to x, p, the latter form will be likewise semi-definite (positive) then. When the E-function is written in terms of the ξ, from (22), it can therefore vanish only when:

$$\bar{\xi}_i - \xi_i = \rho\,\xi_i^* = \rho[\xi_i + \Theta(\bar{\xi}_i - \xi_i)] \;, \tag{23}$$

in which ρ means a positive constant, or when $\bar{\xi}_i = \xi_i$. However, it follows from (23) that since $\sum \bar{\xi}_i^{\,2} = \sum \xi_i^2 = 1$, one will have:

$$(1-\rho\Theta)(1-\sum \xi_i\,\bar{\xi}_i) = -\,\rho\,,$$
$$(1-\rho\Theta)(1-\sum \xi_i\,\bar{\xi}_i) = \;\;\rho\sum \xi_i\,\bar{\xi}_i\;.$$

However, those equations are incompatible, since ρ should be positive, but $\sum_i \xi_i\,\bar{\xi}_i$ will be positive when the direction $\bar{\xi}_i$ lies sufficiently close to ξ_i. It will then follow that:

β) One can determine a neighborhood (ρ') of H and a positive number $\varepsilon < 1$ such that $E(\bar{x}, \xi, \bar{\xi})$ does not vanish in the neighborhood (ρ') of H and for $1 > \sum_i \xi_i\,\bar{\xi}_i > 1 - \varepsilon$.

If we then apply the theorem that was proved in α) then it will follow that a neighborhood (ρ'') of H inside of the field can be given in which the E-function vanishes for only $p_i = \bar{p}_i$, and is otherwise everywhere positive.

The conditions *a*) and *b*) are then sufficient for the existence of a strong minimum.

One also succeeds in proving the necessity of the condition $E \geq 0$, but that proof will be reserved for a later work.

In conclusion, let it be remarked that the form in which have treated the problem will give rise to a tangible extension of the concept of a surface integral, which will be suggested for the case of $n = 2$. It corresponds completely to **Weierstrass**'s generalization of the concept of a curve integral in the treatment of the problem of the integral $\int F(x, y, x', y')\,dt$. We cover the surface over which the integral is to be extended with a net of triangles whose sides have lengths that do not exceed a positive number η and whose angle is larger than a positive number ε that is arbitrarily small, but fixed. Moreover, if the surface is one-sided, as we would like to assume, then a positive side will also be distinguished in each triangle, as long as that is true for the entire surface. If we now denote the projections of one such triangle onto the coordinate planes (their areas, resp.) by δ_1, δ_2, δ_3, in which, e.g., δ_1 is the projection onto the *yz*-plane, which is counted as positive or negative according to whether the positive side of the triangle points away from (towards, resp.) the *yz*-plane, and form the sum:

$$\sum F(x, y, z, \delta_1, \delta_2, \delta_3) \; , \tag{24}$$

which extends over all triangles, and in which x, y, z means an arbitrary point of the triangle in question, then we can show that when the surface has a continuously-rotating tangent plane [i.e., the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ are continuous, along with their first derivatives], that sum will converge to the integral $\int F(x, y, z, p_1, p_2, p_3)\, du\, dv$ for $\eta = 0$, but a fixed ε. It would then be an obvious generalization of the definition of the surface integral to consider it to be limiting value of (24), when it exists.

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