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Proper stresses and proper stress sources.

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The term “proper stresses” (also “self-stresses”) will refer to those stresses that appear in a body that is free of outer surface stresses and in the absence of volume forces.

Ansätze and assumptions. *Kirchhoff’s* theorem on the uniqueness of the solution of the equations of elasticity does not contradict the possible existence of proper stresses, since it is based upon the assumption that the existence of proper stresses is excluded.

In fact, a transformation of a space integral into a sum of an outer surface integral and a space integral appears in this proof, and this transformation can only be performed when one assumes that the components of the deformation can be defined linearly from the derivatives of the displacement of the vector.

Now, proper stresses arise precisely when:

1. Either the deformation tensor cannot be derived from a continuous and single-valued displacement vector, or:

2. The displacements are large enough that this derivation is indeed possible, but nonlinear ([1]), in which case, the more precise kinematical relations:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right],$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z},$$

etc., with cyclic permutations, would be imperative.

Only case 1 shall be considered in what follows. In regard to case 2, it might be mentioned ([2]) only that it means a tilt from an unstressed position to a stressed one, as, e.g., for the overturning of a thin spherical shell.

([1]) Case 2 also involves a multi-valuedness in the case of a buckling process.
([2]) See the general reference on proper stresses by *Nemenyi* in this issue, pp. 58, as well.

Proper stresses of the only kind that will be considered in the sequel are then present in a body when the deformation tensor of dilatations and shears does not satisfy the consistency (i.e., compatibility) conditions.

The appearance of proper stresses in a body can be a transient phenomenon when local variations in density are provoked by variations in temperature, and indeed thermal (proper) stresses, although they will appear only to the extent that no permanent deformation comes about. However, permanent internal stresses can also be produced in a body – e.g., when the stresses that arise from temperature inhomogeneity lie above the elastic limit, or when local variations of density appear due to chemical reactions or re-crystallizations, as are produced in the so-called tempering process for steel, or finally, when local deformations are left behind that are due to loading forces.

While the transient, temperature stresses below the elastic limit that remain can, as is known ([3]), be calculated from dilatations alone, in the general case of permanent, internal stresses, one will no longer be able to associate them with only dilatations as their sources.

The permanent deformations that arise as a result excessive temperature stresses or load stresses will be, at the very least, variations of density and, at the very most, non-spherically-symmetric stretching and shearing.

In general, the calculation will also be possible to perform in this general case only under a certain assumption – which is generally satisfied precisely very often – namely:

I. *The body is once more isotropic and homogeneous (with overbarred elasticity constants E, G, and m) after the permanent deformation has set in.*

This assumption should be understood to mean that, as in any other elastic stress state, any anisotropy and inhomogeneity that might result from the stress are absent.

With that assumption, one will be able to calculate in must engineering applications, although one must exclude certain cases that were emphasized by *Love*, in which the material remains plastic without setting up or becomes noticeably inhomogeneous and anisotropic after setting up. The former case shows up at relatively high temperatures – e.g., for ice, as well – while the latter case must be assumed for the immense gravitational stresses in the Earth's interior.

Notations.

$\sigma_x, \sigma_y, \sigma_z$	The normal stresses, which will be abbreviated by σ
$\tau_{yz}, \tau_{zx}, \tau_{xy}$	The tangential stresses, which will be abbreviated by τ, and which will represent the proper stress state in a body, in particular.

$\sigma_x + \sigma_y + \sigma_z = s$

([3]) See, e.g., A. Föppl, *Vorlesungen über Mechanik*, Bd. V, pp. 293-308. – H. Lorenz, *Elastizität*, Oldenbourg, 1913, pp. 583-591. – A. and L. Föppl, *Zwanf und Drang*, Oldenbourg, 1920, Bd. II, pp. 266-314. – H. Winkel, *Festigkeitslehre*, Springer, 1927, pp. 482-494. – M. v. Laue, Zeitschrift für technische Physik, 1930, pp. 385-394.

ε_x, ε_y, ε_z	The stretching components, which will be abbreviated by ε
γ_{yz}, γ_{zx}, γ_{xy}	The shear components, which will be abbreviated by γ, and which will give a deformation state that is derivable (compatible) from a displacement vector.
${}^0\varepsilon_x$, ${}^0\varepsilon_y$, ${}^0\varepsilon_z$ ${}^0\gamma_{yz}$, ${}^0\gamma_{zx}$, ${}^0\gamma_{xy}$	The components of a permanent deformation that originates in the proper stresses, which is not derivable from a displacement vector, and which will be abbreviated by ${}^0\varepsilon$, ${}^0\gamma$.
$\Theta = \varepsilon_x + \varepsilon_y + \varepsilon_z$ ${}^0\Theta = {}^0\varepsilon_x + {}^0\varepsilon_y + {}^0\varepsilon_z$	
u, v, w	A displacement vector
E	The modulus of elasticity
G	The shear modulus
$1 / m$	The sectional contraction ratio (i.e., the *Poisson* number)
${}^+\sigma$, ${}^+\tau$	The "fictitious" stress state that is derived from the deformation state ε, γ on the basis of *Hooke's* stress-deformation relationship.

Under assumption (I), one can write the six stress-deformation relations of *Hooke's* law in the following extended form:

$$\left.\begin{aligned} \varepsilon_x &= {}^0\varepsilon_x + \frac{1}{E}\left[\sigma_x - \frac{1}{m}(\sigma_y + \sigma_z)\right], \\ \gamma_{yz} &= {}^0\gamma_{yz} + \frac{1}{G}\tau_{yz}, \end{aligned}\right\} \tag{1}$$

with cyclic permutations of x, y, and z.

The deformation components ε and γ then refer to the deformation that is calculated from a stress-free initial state.

The inverse of eq. (1) reads: When one denotes the dilatation by $\Theta = \varepsilon_x + \varepsilon_y + \varepsilon_z$, as usual, one will have:

$$\left.\begin{aligned} \sigma_x &= \frac{Em}{(m+1(m-2)}(\Theta - {}^0\Theta) + \frac{Em}{m+1}(\varepsilon_x - {}^0\varepsilon_x), \\ \tau_{yz} &= G(\gamma_{yz} - {}^0\gamma_{yz}), \end{aligned}\right\} \tag{2}$$

with cyclic permutations of x, y, and z.

Under the assumption that was made initially, it is now obviously sufficient to next ignore all volume forces and outer surface forces, since their effects will simply be superposed when only the final total stresses remain below the elastic limit and the deformations that are calculated by establishing an ideal initial position will preserve their character as small magnitudes.

As is known, the equilibrium conditions will then read:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \tag{3}$$

etc., with cyclic permutations of *x*, *y*, and *z*.

The condition of vanishing outer surface stresses will serve as the associated boundary condition.

If one now substitutes eq. (2) into these equilibrium conditions then they will take on a form that represents the deformation components ε, γ that enter into a connected initial state as something that results from the initial deformations ${}^0\varepsilon$, ${}^0\gamma$.

In fact, that will yield:

$$\Delta u + \frac{m}{m-2}\frac{\partial \Theta}{\partial x} = \frac{2}{m-2}\frac{\partial {}^0\Theta}{\partial x} + 2\frac{\partial {}^0\varepsilon_x}{\partial x} + \frac{\partial {}^0\gamma_{xy}}{\partial y} + \frac{\partial {}^0\gamma_{xz}}{\partial z}, \tag{3a}$$

with cyclic permutations of *x*, *y*, and *z* as the equations of elasticity.

The condition of vanishing outer stresses is formulated similarly from eq. (2).

Here, ${}^0\varepsilon_x$, ${}^0\gamma_{yz}$, etc., are to be regarded as functions that are entirely independent of position and each other, and if one so desires, also functions of time whose values must not exceed certain absolute values, on the basis of the assumption I, but can be otherwise arbitrarily discontinuous. As long as the continuum hypothesis is admissible, one can then start from a stress-free initial state and impose arbitrary permanent deformations in an arbitrarily small volume element, independently of the neighboring elements, with the help of chemical, thermal, or mechanical experimental arrangements.

These Ansätze obviously imply the following:

II. *A proper stress state can always be regarded as originating in the collective effect of the permanent initial deformations* ${}^0\varepsilon$, ${}^0\gamma$ *and elastic deformations in a connected, originally stress-free body.*

Two types of problems in engineering physics now arise, namely:

a) The initial deformations ${}^0\varepsilon$, ${}^0\gamma$ are given, and one seeks the stresses that they produce.

b) The proper stresses are known by the deformation of mutually-isolated volume elements, with the help of the well-known method of sections ([4]), to a degree of precision that increases with the smallness of the volume element, and one asks what the sources of the proper stresses were – i.e., the initial deformations ${}^0\varepsilon$, ${}^0\gamma$.

([4]) See, e.g., A. Föppl, *loc. cit.* – M. v. Laue, *loc. cit.*

One cannot say, in general, how far one has to go in the second problem with the reduction in size of the body in question in order to reach a certain required accuracy, and whether one even approaches a system of limit functions, at all.

Only a certain vague knowledge of the continuity of the causes of the proper stresses can give one a clue to that.

Moreover, the resolution of a body into independent volume elements by sectioning is also coupled in experimental engineering with the limits in such a way that the forces that are necessary for the sectioning can produce no permanent deformation that would negate the result, and furthermore that it is difficult to attach sufficiently many measuring marks to the outer surface of the body for the sectioning.

Determination of the proper stresses from the stress sources. Eq. (3) is definitive for problem *a*) . One can give it a somewhat more intuitive form when one introduces certain "fictitious stresses," namely, stresses that would correspond to the deformation ε, γ – i.e., the displacement *u*, *v*, *w* – of the stress-free initial state in the absence of initial deformations ${}^0\varepsilon$, ${}^0\gamma$.

If one denotes these fictitious stresses by ${}^+\sigma$ and ${}^+\tau$ then eq. (3) can obviously be, in turn, written in the following form:

$$\frac{\partial\,{}^+\sigma_x}{\partial x}+\frac{\partial\,{}^+\tau_{xy}}{\partial y}+\frac{\partial\,{}^+\tau_{xz}}{\partial z} = \frac{Em}{m+1}\left[\frac{1}{m-2}\frac{\partial\,{}^0\Theta}{\partial x}+\frac{\partial\,{}^0\varepsilon_x}{\partial x}+\frac{1}{2}\frac{\partial\,{}^0\gamma_{xy}}{\partial y}+\frac{1}{2}\frac{\partial\,{}^0\gamma_{xz}}{\partial z}\right], \tag{4}$$

etc., with cyclic permutation of *x*, *y*, and *z*, as the equilibrium conditions.

Now, since these fictitious stresses ${}^+\sigma$, ${}^+\tau$ are coupled with the desired proper stresses σ and τ by way of the initial stresses or source stresses ${}^0\sigma$, ${}^0\tau$ that correspond to the imposed initial deformations or stress sources ${}^0\varepsilon$, ${}^0\gamma$ by *Hooke's* law by the relations:

III. $$\sigma = {}^0\sigma - {}^+\sigma, \qquad \tau = {}^0\tau - {}^+\tau,$$

the fictitious stresses will be known immediately, as well as the actual ones.

The fictitious stresses that are associated with the proper stresses as a result of conditions II and III are equal to the stresses that would appear in a body that was initially free of deformations as result of the following fictitious volume force:

$$X = -\frac{Em}{m+1}\left[\frac{1}{m-2}\frac{\partial\,{}^0\Theta}{\partial x}+\frac{\partial\,{}^0\varepsilon_x}{\partial x}+\frac{1}{2}\frac{\partial\,{}^0\gamma_{xy}}{\partial y}+\frac{1}{2}\frac{\partial\,{}^0\gamma_{xz}}{\partial z}\right], \tag{4a}$$

etc., with cyclic permutations of x, y, and z.

With the help of this theorem, one can, *inter alia*, immediately answer the question of when a proper stress state can be generated by density variations alone. Namely, this corresponds to a result that is already known in a somewhat different form ([5]):

Only fictitious stress states that are derivable from a force potential function can be generated by pure density variations (e.g., by pure temperature variations). The potential function of the fictitious volume force is then the density variation ${}^0\Theta$ *itself, up to the factor* $Em / 3\,(n-2)$.

On the basis of the considerations above, one can also give the following meaning to a *minimal principle* that was expressed by *A. Föppl*, and is suitable for the direct methods of the calculus of variations.

The proper stresses that are installed in a body that is free of volume and outer surface forces for given initial deformations ${}^0\varepsilon$, ${}^0\gamma$ make the deformation work (6) that is defined by $\varepsilon - {}^0\varepsilon$ and $\gamma - {}^0\gamma$ (which coincides with the potential energy in this case; i.e., in the absence of external forces) take a minimum. Therefore, the deformation quantities ε and γ will be derivable from a displacement vector field and will be defined by the relations (1).

The proof of this follows from the fact that the equilibrium conditions (3) for the proper stresses σ and τ lead to the following known expressions for the principle of virtual work:

$$\int dV\,(\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_{yz}\,\delta\gamma_{yz} + \tau_{zx}\,\delta\gamma_{zx} + \tau_{xy}\,\delta\gamma_{xy}) = 0, \tag{5}$$

under the assumption that the $\delta\varepsilon$ and $\delta\gamma$ must be derivable from a displacement vector in agreement with eq. (2). The expression (5) will then be represented as the variation of the following integral of the deformation work:

$$\int dV\left[(\varepsilon_x - {}^0\varepsilon_x)^2 + (\varepsilon_y - {}^0\varepsilon_y)^2 + (\varepsilon_z - {}^0\varepsilon_z)^2 + \frac{(\Theta - {}^0\Theta)^2}{m-2}\right.$$
$$\left. + \tfrac{1}{2}\{(\gamma_{yz} - {}^0\gamma_{yz})^2 + (\gamma_{zx} - {}^0\gamma_{zx})^2 + (\gamma_{xy} - {}^0\gamma_{xy})^2\right]. \tag{6}$$

A remarkable special case of proper stresses is defined by the system of stresses that was first treated by *Weingarten*, and then by *Timpe*, *Volterra*, and *Cesaro*, and are called *Volterra distortions* ([6]).

In fact, whereas the proper stresses cannot be derived from the displacement vector *u*, *v*, *w* without stress sources ${}^0\varepsilon$, ${}^0\gamma$, there is the apparent exception by which stresses without external forces are possible for a multi-valued displacement vector in a multiply-connected body. The displacement vector will then exhibit a jump for any circuit that cannot be shrunk to a point without leaving the body.

([5]) Love, *Elasticity*, 1927, pp. 168.
([6]) See the survey that is presented in Love, *Elasticity*, Cambridge, 1927, pp. 221-228.

The physical meaning of this is that one must think of constraint layers (*Zwangsschichten*) as being pressed into surfaces that are positioned arbitrarily and by which the body will be converted into a simply-connected one, and that these constraint layers push the two bounding surfaces apart, and thus make the displacement of the original body discontinuous.

Weingarten has presented and proved the interesting theorem that only for a rigid displacement of the two bounding surfaces with respect to each other will a deformation state be possible that is derivable from a displacement vector, and thus satisfy the compatibility conditions.

The state of stress and deformation, but not the state of displacement, is therefore continuous and single-valued, and the positions of the constraint surfaces will not be derivable from it.

In light of our manner of thinking, this *Weingarten-Volterra* displacement state also seems to be single-valued, and the stress state will be generated by a distribution of initial deformations ${}^0\varepsilon$, ${}^0\gamma$ over one or more wedge-like domains inside of the body that become infinitely thin in such a way that the extensions ${}^0\varepsilon$ will become infinite in the direction of the common surface normal, and the shears ${}^0\gamma$, in a direction that is perpendicular to the surface normal, in such a way that ${}^0\varepsilon_n\, dn$ and ${}^0\gamma_t\, dn$ remain finite.

Weingarten's theorem shows that the special case of the distortions only occurs when the ${}^0\varepsilon_n$ and ${}^0\gamma_t$ are compelled by a rigid displacement of the boundary surfaces.

Calculation of the stress sources from the stresses. For the evaluation of the fabrication process of a work piece, it can be important to calculate the cause of the proper stresses – viz., the tensor of stress sources ${}^0\varepsilon$ and ${}^0\gamma$ – from the proper stresses that are mediated by the gradual sectioning of the body in question.

Although this problem that was denoted by *b*) above means only an inversion of the first problem, and one has to add the six eqs. (1) of the extended *Hooke* law and the three eqs. (3) for the six unknown stress components ${}^0\varepsilon$, ${}^0\gamma$ and the three components u, v, w of the displacement vector, nonetheless, the direct employment of (1) and (3) seems to break down for the solution of the ${}^0\varepsilon$, ${}^0\gamma$.

The path that arrives at determining equations for the stress sources from the so-called compatibility conditions is passable, however.

If the initial deformations or stress sources ${}^0\varepsilon$, ${}^0\gamma$ do not, in fact, satisfy the consistency conditions, nor the stresses σ, τ themselves, but the deformation components ε, γ of the ideal initial state, as well as the associated stresses ${}^+\sigma$, ${}^+\tau$ that were called fictitious above, then they must satisfy these six consistency conditions.

As is known, the consistency relations read ([7]):

([7]) See, e.g., *Handbuch der Physik* (Geiger-Scheel), Springer, 1929, Bd. VI, art. *E. Trefftz*, "Math. Elastizitätstheorie," pp. 64.

$$\left.\begin{aligned}\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}-\frac{\partial^2\gamma_{xy}}{\partial x\,\partial y}=0,\\ 2\frac{\partial^2\varepsilon_x}{\partial y\,\partial z}-\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)=0,\end{aligned}\right\}\tag{7}$$

etc., with cyclic permutation of the *x*, *y*, and *z*.

If one now substitutes the relations (1) (the ones ${}^{+}\sigma = \sigma + {}^{0}\sigma$, ${}^{+}\tau = \tau + {}^{0}\tau$, resp., which are equivalent to them) into (7) then when one denotes $\sigma_x + \sigma_y + \sigma_z$ by s and ${}^{0}\sigma_x + {}^{0}\sigma_y + {}^{0}\sigma_z$ by ${}^{0}s$, one will get:

$$\left.\begin{aligned}&\frac{\partial^2\sigma_x}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}-\frac{1}{m-1}\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}\right)-2\frac{\partial^2\tau_{xy}}{\partial x\,\partial y}\\ &\quad=-\left[\frac{\partial^2\,{}^{0}\sigma_x}{\partial y^2}+\frac{\partial^2\,{}^{0}\sigma_y}{\partial x^2}-\frac{1}{m-1}\left(\frac{\partial^2\,{}^{0}s}{\partial x^2}+\frac{\partial^2\,{}^{0}s}{\partial y^2}\right)-2\frac{\partial^2\,{}^{0}\tau_{xy}}{\partial x\,\partial y}\right],\\ &\frac{\partial^2\,{}^{0}\sigma_x}{\partial y\,\partial z}-\frac{1}{m-1}\frac{\partial^2\,{}^{0}s}{\partial y\,\partial z}-\frac{\partial}{\partial x}\left(\frac{\partial\,{}^{0}\tau_{yz}}{\partial x}+\frac{\partial\,{}^{0}\tau_{yz}}{\partial y}+\frac{\partial\,{}^{0}\tau_{xy}}{\partial z}\right)\\ &\quad=-\left[\frac{\partial^2\sigma_x}{\partial y\,\partial z}-\frac{1}{m-1}\frac{\partial^2 s}{\partial y\,\partial z}-\frac{\partial}{\partial x}\left(\frac{\partial\tau_{yz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{xy}}{\partial z}\right)\right],\end{aligned}\right\}\tag{8}$$

etc., with cyclic permutations.

In this, the ${}^{0}\sigma$ and ${}^{0}\tau$ depend immediately upon the ${}^{0}\varepsilon$ and ${}^{0}\gamma$ linearly by *Hooke's* law, and can, if one so desires, be expressed directly in terms of the latter.

Six mutually-independent second-order partial differential equations for the six unknown functions ${}^{0}\varepsilon$ and ${}^{0}\gamma$ arise in this way, in which the proper stresses, which are perhaps given experimentally, play the role of perturbing functions. However, the differential equations are not associated with any outer surface conditions.

This apparent lack is connected with the arbitrariness in the choice of the manifold of equivalent stress-source states, which will be established below.

The general solution of these equations (which is, by no means, practicable, due to the arbitrariness in the outer surface conditions) must perhaps be achieved as follows:

One first sees immediately that if ${}^{\sigma}\varepsilon$, ${}^{\sigma}\gamma$ are the deformations that follow immediately from the proper stresses by *Hooke's* law then a possible stress-source system for the ${}^{0}\varepsilon$ and ${}^{0}\gamma$ will be given by:

$${}^{0}\varepsilon_x = {}^{\sigma}\varepsilon_x, \qquad {}^{0}\tau_{yz} = {}^{\sigma}\tau_{yz}, \qquad \text{etc.;}$$

i.e., one obviously arrives at a possible initial state when one makes the deformation that corresponds to the proper stresses die off. However, one could not use this solution for the problem of the first part of this paper – viz., the determination of the stresses from given stress sources – so one cannot expect that the physically-given system of sources, which is equivalent to just one of infinitely many, will coincide precisely with the deformation tensor of proper stresses.

It also corresponds to the fact that one can superpose the particular solution above with any solution of the homogeneous system of differential equations (8) for the ${}^0\varepsilon$, ${}^0\gamma$.

For example, in the general case:

$${}^0\varepsilon_x = {}^\sigma\varepsilon_x + A_x\, e^{ax+by+cz},$$

$${}^0\gamma_{yz} = {}^\sigma\gamma_{yz} + \left(A_y \frac{c}{b} + A_z \frac{b}{c} \right) e^{ax+by+cz},$$

etc., with cyclic permutation of x, y, z and a, b, c, where the A_x, A_y, A_z, a, b, c are arbitrary real or complex constants.

The superposed solution of the homogeneous equations can also be an arbitrary linear expression in x, y, and z or a polynomial with certain relations between the coefficients.

Each of these systems of sources will generate the same proper stresses.

Furthermore, for the axially-symmetric case of polar coordinates:

The compatibility equations then reduce to the form:

$$\left.\begin{aligned} \frac{\partial^2\varepsilon_r}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial r^2} - \frac{\partial^2\gamma_{rz}}{\partial z\,\partial r} &= 0, \\ \frac{\partial\varepsilon_\varphi r}{\partial r} - \varepsilon_r &= 0, \end{aligned}\right\} \tag{9}$$

which can be derived immediately from considering the following known relations:

$$\varepsilon_r = \frac{\partial u}{\partial r}, \qquad \varepsilon_\varphi = \frac{u}{r}, \qquad \varepsilon_z = \frac{\partial w}{\partial z}, \qquad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}.$$

A solution of the homogeneous eq. (9) is, e.g.:

$$\begin{aligned} \varepsilon_\varphi &= A\, r \cos az, \\ \varepsilon_r &= 2Ar \cos az, \\ \varepsilon_z &= A \cos \frac{r^2}{b} \cos az, \\ \gamma_{rz} &= A \left(\frac{1}{a} - 2a \right) \frac{r^2}{b} \sin az\,. \end{aligned}$$

This solution [and every other solution of the homogeneous equations (9)] can be added to the solution:

$${}^0\varepsilon = {}^\sigma\varepsilon, \qquad {}^0\gamma = {}^\sigma\gamma,$$

without changing the proper stresses.

The foregoing consideration then leads to the following theorem:

One of the possible systems of stress-sources for a given system of proper stresses is the one that is associated with the system of proper stresses by Hooke's law. One can then add any solution of the homogeneous compatibility equations (7) *for the deformation tensor to this system of sources without changing anything.*

By contrast, for a prescribed system of proper stress sources ${}^0\varepsilon$, ${}^0\gamma$ *in a free body, there is only one system of proper stresses, namely, the one that is determined from the equilibrium conditions* (3) [(3*a*), *resp.*] *for vanishing outer surface stresses.*

The aforementioned *Weingarten-Volterra* distortions can also obviously be generated by sources that are continuously-distributed space, instead of the superficially-distributed stress sources, as for the stated authors, when one chooses the deformations that are associated with the proper stress to be the sources. Both possibilities must be solutions of eqs. (7), (8), or (9).

One can also immediately give the superposed solution of the compatibility conditions (7) that takes the one system of sources to the other one. Namely, it is the compatible deformation state that is composed of the negatively-taken *Weingarten* source wedge and the spatially distributed sources above.

Obviously, it is especially simple to carry over the arguments above to frameworks.

In an *n*-fold statically-indeterminate framework, the same system of proper stresses can be generated in a known way by false (incompatible) sections (*Ablängung*) in an infinitude of ways according to the choice of the superfluous indicated terms, and one can vary this manner of generation even further by superposing section defects (*Ablängungsfehlern*) of all rods that will generate no proper stresses by themselves alone. The question of the origin of the system of proper stresses also has a multi-valued answer here.

Further questions:

1. It was established by the foregoing that one can always exhibit infinitely many proper stress source fields that are associated with a given proper stress field – e.g., one that is mediated by sections. One now asks whether, and in what way, one can find a distinguished representation (give invariants of the source field, resp.).

One sees immediately that there are the special cases of pure dilatation fields, as well as pure form-changing fields, which differ fundamentally and cannot be converted into each other. The following examples of this might be given:

A pure dilatation source field is the field of a temperature variation or a variation of density that is produced by tempering, for which the elasticity limit of the corresponding proper stresses is not exceeded.

A pure form-changing source field is the field that is generated by the permanent torsion of the external layers of a twisted rod, or even simpler, the field that arises when one twists two thin-walled tubes with respect to each other and attaches the ends to each other.

From time to time, it is important in engineering physics to know whether a calculated proper stress state is produced by a pure dilatation source field or a pure form-changing source field or by an essentially mixed field.

The source field that is immediately given by applying *Hooke's* law will generally be ostensibly mixed, and it would be important to have a criterion for when a given source field can be traced back to a pure dilatation field and when it can be traced back to a pure form-changing field.

The question can thus be formulated as follows:

When is it possible to convert a stress source field ${}^0\varepsilon$, ${}^0\gamma$ *that is calculated in some way into either a pure dilatation field* $\varepsilon_x = \varepsilon_y = \varepsilon_y$ *or into a pure form-changing (dilatation-free) field* $\sum \varepsilon = 0$ *by superposing a deformation that is free of self-stresses* [*from eq.* (7)]?

2. There is often great interest in bringing a proper stress state to smaller stress values by afterglow (*Nachglühen*). The process consists of exposing the body to a uniform temperature change that rises very gradually and once more falls over a long period of time, and which is, on the one hand, not so large that it changes the form and crystalline structure of the body, but, on the other hand, intermittently reduces the plasticity limit in such a way that the maximal shear stress comes down everywhere to the value that corresponds to the annealing temperature.

The associated question then reads:

Into which proper stress state does a given proper stress state go when the plasticity limit is lowered by a certain amount?

The answer will, in any case, depend upon whether one introduces the greatest principal stress difference, following *Mohr*, or the quadratic mean value of the three principal stress differences, following *v. Mises.*

The problem can be solved with no difficulty for many spherically-symmetric stress states and many axially-symmetric ones, without going into the displacement state that leads from the original state to the desired one.

However, from a technical standpoint, it would also be desirable to give the solution for the general case.

3. The non-fulfillment of the compatibility conditions for a form-changing field was sometimes interpreted as a change of form in a body space with a non-Euclidian line element ([8]), in such a way that indeed, in place of the line element:

$$ds^2 = dx^2 + dy^2 + dz^2,$$

a *Riemannian* line element:

([8]) See *Trefftz* [*loc. cit.*, rem. ([7])].

$$ds^2 = g_{11}\, dx^2 + g_{22}\, dy^2 + g_{33}\, dz^2 + 2g_{12}\, dx\, dy + 2g_{23}\, dy\, dz + 2g_{31}\, dz\, dx$$

enters that is not transformable into the latter one.

The deformation components ε and γ can be expressed in terms of the variations $\delta\, ds$ and $\delta\, dx$, $\delta\, dy$, $\delta\, dz$ in the usual way for curvilinear, oblique coordinates by differentiating these variations with respect to the coordinates and, if the line element is not Euclidian then they will no longer need to fulfill the compatibility conditions in Euclidian space. On the contrary, the right-hand side of eq. (7), which is then non-vanishing, must be employed in order to determine the displacement vector δx, δy, δz, and certain relations between the coefficients g_{ik} of the line element.

As a closer consideration shows, the non-fulfillment of the compatibility conditions can then be interpreted as a curvature of the geometric field. Any visualization of this interpretation in space is, admittedly, not possible, although in many cases of superficial form-changing fields, one might perhaps visualize it as follows:

The deformation components of a planar membrane with self-stresses are ones that do not fulfill the compatibility conditions in the plane, but in certain cases one will be able to assemble the surface elements without stress when one places them in a suitably curved surface or allows the membrane to arch up from the plane.

Up to now, it has not been shown, in general, how the discovery of the associated coefficients g_{ik} of the non-Euclidian line element – and thus, the given of the spatial curvature – can benefit the physical problem.

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