"Sui gruppi continui di movimenti rigidi negli iperspazii," Rend. Reale Accad. dei Lincei, classe sci. fis., mat. e nat. (5) **14** (1905), pp. 487-491.

On the continuous groups of rigid motions in hyperspaces

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In a note that was published last year in the Atti del R. Istituto Veneto, I extended the concepts of *principal directions and invariants* that I had already established for three-dimensional spaces to an arbitrary manifold. It is now important to observe, and it is easy to see, that the expressions $\gamma_{hi,kl}$ that relate to the principle n-tuple and are coupled to each other by the relations (7) in the cited note give the complete system of algebraic invariants that are common to fundamental quadratic form and to the quadrilinear one that has the Riemann symbols for its coefficients, and thus provide everything that is necessary for the expression of the second-order elements and invariant properties of the space considered.

While any $\gamma_{hk,hk}$ represents the curvature of a geodetic surface whose position is determined in V_n by the directions of the lines [h] and [k], the geometric significance of the invariants $\gamma_{hi,kj}$, at least three of which are distinct, appears to be more esoteric. Although I shall not address that geometric interpretation for now in the present writing, I propose to extend to an arbitrary V_n some of the fundamental results that were coupled to V_3 in my paper "Sui gruppi continui di movimenti, etc.," which was published in v. XII of the third series of the *Memorie della Società italiana delle Scienze.*

As one sees, the equations of the problem of the groups of rigid motions in an arbitrary V_n will take on a noteworthy simplicity for those V_n that admit a principal n-tuple that is endowed with the property that all of the $\gamma_{hi,kj}$ will be identically zero for it when they have at least three distinct indices. The second-order differential invariants will reduce to the single curvature $\gamma_{hk,hk}$ for such a manifold (which one calls *regular*). One is warned that, as would result from the note that will be cited many times, the regular manifolds will be the hypersurfaces, and more generally, all of the V_n for which the Riemann symbols can be put into the form of second-order minors of a symmetric determinant of order n.

1. Let:

$$\varphi = \sum_{r=1}^{n} a_{rs}\, dx_r\, dx_s$$

be the quadratic differential form that defines the metric of a V_n and let:

$$X(f) = \sum_{r=1}^{n} \xi^{(r)} \frac{\partial f}{\partial x_r}$$

be an infinitesimal transformation that corresponds to a rigid motion in that manifold. Killing's fundamental equations that refer to the fundamental form φ are written as follows in the notations of the absolute differential calculus:

$$(\alpha) \qquad \xi_{rs} + \xi_{sr} = 0.$$

Assume that one has a fundamental n-tuple [1], [2], …, [n] whose congruences have the covariant coordinate systems $\lambda_{h/r}$, and that γ_{ihk} denote its rotation coefficients; i.e., one poses:

$$(1) \qquad \lambda_{i/rs} = \sum_{h,k=1}^{n} \gamma_{ihk}\, \lambda_{h/r}\, \lambda_{k/s} ;$$

furthermore, set:

$$(2) \qquad \xi_r = \sum_{i=1}^{n} \eta_i\, \lambda_{i/r} .$$

If one differentiates:

$$\eta_i = \sum_{i=1}^{n} \xi^{(r)}\, \lambda_{i/r} ,$$

which is equivalent to (2), and takes (1) and (α) into account then one will easily recognize that (α) is equivalent to:

$$(\alpha_1) \qquad \frac{\partial \eta_i}{\partial s_j} = \sum_{i=1}^{n} \gamma_i\, l_j\, \eta_l + \delta_{ij} ,$$

in which the δ_{ij} represent indeterminates that are coupled by the relations:

$$\delta_{ij} + \delta_{ji} = 0.$$

In the general case, it has the significance that the rotations have in the case of $n = 3$.

2. – One infers from (α_1) by derivation that:

$$\frac{\partial}{\partial s_k}\frac{\partial \eta_i}{\partial s_h} - \frac{\partial}{\partial s_h}\frac{\partial \eta_i}{\partial s_k} = \sum_{l=1}^{n} \gamma_{il,hk}\, \eta_l + \sum_{g,l=1}^{n} (\gamma_{ghk} - \gamma_{gkh})\, \gamma_{lig}\, \eta_l + \frac{\partial \delta_{ih}}{\partial s_k} - \frac{\partial \delta_{ik}}{\partial s_h}$$

$$+ \sum_{g=1}^{n} (\gamma_{ghk} - \gamma_{gkh})\, \gamma_{lig}\, \eta_l + \sum_{g=1}^{n} (\gamma_{ghk}\, \delta_{gk} - \gamma_{igk}\, \delta_{gh}) ;$$

comparing this with known formulas for the inversion of the intrinsic derivation will give:

$$\frac{\partial \delta_{ih}}{\partial s_h} - \frac{\partial \delta_{ik}}{\partial s_k} = \sum_{l=1}^{n} \gamma_{il,hk}\, \eta_l + \sum_{g,l=1}^{n} (\gamma_{ghk} - \gamma_{gkh})\, \delta_{il} + \sum_{g,l=1}^{n} (\gamma_{lih}\, \delta_{ik} - \gamma_{lik}\, \delta_{ih}),$$

or the equivalent ones:

$$(\beta) \qquad \frac{\partial \delta_{ih}}{\partial s_k} = \sum_{l=1}^{n} \gamma_{ik,hi}\, \eta_l + \sum_{l=1}^{n} (\gamma_{lik}\, \delta_{hi} - \gamma_{lhk}\, \delta_{il}).$$

If one sets:

$$\gamma_{ij,hkl} = \frac{\partial \gamma_{ij,hk}}{\partial s_l} + \sum_{g=1}^{n} (\gamma_{gil}\, \gamma_{gj,hk} + \gamma_{gjl}\, \gamma_{ig,hk} + \gamma_{ghl}\, \gamma_{ij,gk} + \gamma_{gki}\, \gamma_{ij,hg})$$

then one will have:

$$\gamma_{ij,hkl} = \sum_{r,s,t,u,v=1}^{n} a_{rs,tuv}\, l_i^{(r)}\, l_j^{(s)}\, l_h^{(t)}\, l_k^{(u)}\, l_l^{(v)},$$

and therefore the identity relations:

$$a_{rs,tuv} + a_{rs,vtu} + a_{rs,uvt} = 0$$

that BIANCHI recognized can also be written in an invariant form as follows:

$$\gamma_{rs,tuv} + \gamma_{rs,vtu} + \gamma_{rs,uvt} = 0.$$

If one takes that into account and applies the process of derivation and elimination to (β) that was applied before to (α_1) then one will finally arrive at:

$$(\gamma) \qquad \sum_{l=1}^{n} \{\gamma_{ij,hkl}\, \eta_l + \gamma_{lh,ji}\, \delta_{lk} - \gamma_{lk,ji}\, \delta_{lh} + \gamma_{lj,hk}\, \delta_{li} - \gamma_{li,hk}\, \delta_{lj}\} = 0.$$

Equations (α), (β), and (γ) constitute the simultaneous algebraic-differential system that the unknown functions of the problem must satisfy – viz., the translations and rotations.

3. – It is useful to separate the equations (γ) into three groups A, B, C according to whether two, three, or all of the indices i, j, h, k are distinct, respectively, and to assume that the reference n-tuple is a principal n-tuple.

If $\rho_1, \rho_2, \ldots, \rho_n$ denote the principal invariants of V_n then when the equations of the group A are combined with each other in a suitable way, while those of B are combined analogously, then that will give:

$$(A_0) \qquad \sum_{l=1}^{n} \eta_l \frac{\partial \rho_i}{\partial s_l} = 0,$$

$$(B_0) \qquad (\rho_k - \rho_h)\left(\sum_{l=1}^{n} \gamma_{hkl}\, \eta_l + \delta_{hk}\right) = 0.$$

If one supposes that the principal invariants are all distinct then (B_0) will give the following expression for the δ_{hk} :

$$(3) \qquad \delta_{hk} = \sum_{l=1}^{n} \gamma_{khl}\, \eta_l ,$$

with which, (α), (β), and (γ) will assume the forms:

$$(\alpha') \qquad \frac{\partial \eta_i}{\partial s_j} = \sum_{l=1}^{n} \gamma_{khl}\, \eta_l ,$$

$$(\beta') \qquad \frac{\partial \delta_{ih}}{\partial s_k} = \sum_{l=1}^{n} \{\gamma_{ik,hi} + \sum_{g=1}^{n} (\gamma_{gik}\, \gamma_{ghl} - \gamma_{ghk}\, \gamma_{gil})\}\eta_l ,$$

$$(\gamma') \qquad \sum_{l=1}^{n} \eta_l \frac{\partial \gamma_{ij,hk}}{\partial s_l} = 0,$$

respectively.

If (β') is compared to the equation that one obtains by differentiating (3) and eliminating the derivatives of the η by means of (α') then one will arrive at:

$$(a) \qquad \sum_{l=1}^{n} \eta_l \frac{\partial \gamma_{hik}}{\partial s_l} = 0.$$

If one observes that when (a) are satisfied identically, (γ') will also be satisfied identically then one can conclude that:

Any V_n whose principal invariants are all distinct will admit a transitive group of rigid motions if the rotations that relate to the principal n-tuple are constants, and only in that case. If that condition is verified then V_n will admit an n-parameter group for which the initial values of the translations will prove to be arbitrary.

If one assumes that the reference n-tuple is the principal n-tuple of V_n , while the translations are determined by integrating the system (α') *then the rotations that one obtains from* (3) *will be functions of the translations.*

One then infers from (α') that:

If the principal congruences are normal then the components of the translation along a well-defined principal direction will vary along only that direction.

Finally, it will follow from (A_0) that:

1. *If a V_n admits a group G of rigid motions then the principal invariants of V_n will be invariant under the group.*

2. *If the group is transitive then its principal invariants will be constants.*

4. – If one assumes that the reference *n*-tuple is a principal *n*-tuple then the equations of the group *C* will be satisfied identically if the manifold V_n is regular, and only in that case. The equations of the groups *A* and *B* will then assume the forms:

$$(A_1) \qquad \sum_{l=1}^{n} \eta_l \frac{\partial \gamma_{ij,ij}}{\partial s_l} = 0,$$

$$(B_1) \qquad (\gamma_{ij,ij} - \gamma_{ik,ik}) \left(\sum_{l=1}^{n} \gamma_{jkl}\, \eta_l + \delta_{jk} \right) = 0,$$

resp., and will substitute for the group (γ) completely.

They are satisfied identically for the manifolds with constant curvature, and only for them, and one will then get the well-known theorem about the group of rigid motions that pertains to them.

(A_1) says that:

If a regular manifold V_n admits a group of rigid motions G then the second-order differential invariants of V_n will all be invariant under G.

If the group is transitive then those invariants will be constants.

It is easy to geometrically characterize the case in which (B_1) determine the δ_{jk} as functions of the η_l whose expressions are given by (3). In that case, (α'), (β'), and (*a*) will also be valid and one will obtain them as in the preceding paragraph, and therefore the conclusions that were reached above will also be valid in the case that is now considered.

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