In a note that was published last year in the Atti del R. Istituto Veneto, I extended the concepts of principal directions and invariants that I had already established for three-dimensional spaces to an arbitrary manifold. It is now important to observe, and it is easy to see, that the expressions $\gamma_{hi,kl}$ that relate to the principle $n$-tuple and are coupled to each other by the relations (7) in the cited note give the complete system of algebraic invariants that are common to fundamental quadratic form and to the quadrilinear one that has the Riemann symbols for its coefficients, and thus provide everything that is necessary for the expression of the second-order elements and invariant properties of the space considered.

While any $\gamma_{hk,hk}$ represents the curvature of a geodetic surface whose position is determined in $V_n$ by the directions of the lines $[h]$ and $[k]$, the geometric significance of the invariants $\gamma_{hi,kj}$, at least three of which are distinct, appears to be more esoteric. Although I shall not address that geometric interpretation for now in the present writing, I propose to extend to an arbitrary $V_n$ some of the fundamental results that were coupled to $V_3$ in my paper “Sui gruppi continui di movimenti, etc.” which was published in v. XII of the third series of the Memorie della Società italiana delle Scienze.

As one sees, the equations of the problem of the groups of rigid motions in an arbitrary $V_n$ will take on a noteworthy simplicity for those $V_n$ that admit a principal $n$-tuple that is endowed with the property that all of the $\gamma_{hi,kj}$ will be identically zero for it when they have at least three distinct indices. The second-order differential invariants will reduce to the single curvature $\gamma_{hk,hk}$ for such a manifold (which one calls regular). One is warned that, as would result from the note that will be cited many times, the regular manifolds will be the hypersurfaces, and more generally, all of the $V_n$ for which the Riemann symbols can be put into the form of second-order minors of a symmetric determinant of order $n$.

1. Let:

$$\varphi = \sum_{r=1}^{n} a_{rs} \, dx_r \, dx_s$$

be the quadratic differential form that defines the metric of a $V_n$ and let:
be an infinitesimal transformation that corresponds to a rigid motion in that manifold. Killing’s fundamental equations that refer to the fundamental form $\varphi$ are written as follows in the notations of the absolute differential calculus:

$$\xi_{rs} + \xi_{sr} = 0.$$  (\(\alpha\))

Assume that one has a fundamental $n$-tuple $[1], [2], \ldots, [n]$ whose congruences have the covariant coordinate systems $\lambda_{h/r}$, and that $\gamma_{hk}$ denote its rotation coefficients; i.e., one poses:

$$(1) \quad \lambda_{i/rs} = \sum_{h,k=1}^{n} \gamma_{hk} \lambda_{h/ri} \lambda_{k/is};$$

furthermore, set:

$$(2) \quad \xi_r = \sum_{i=1}^{n} \eta_i \lambda_{i/r}.$$  

If one differentiates:

$$\eta_i = \sum_{r=1}^{n} \xi^{(r)} \lambda_{i/r},$$

which is equivalent to (2), and takes (1) and (\(\alpha\)) into account then one will easily recognize that (\(\alpha\)) is equivalent to:

$$\frac{\partial \eta_i}{\partial s_j} = \sum_{1 \leq l \leq n} \gamma_{i/lj} \eta_l + \delta_{ij},$$  (\(\alpha_i\))

in which the $\delta_{ij}$ represent indeterminates that are coupled by the relations:

$$\delta_{ij} + \delta_{ji} = 0.$$  

In the general case, it has the significance that the rotations have in the case of $n = 3$.

2. – One infers from (\(\alpha_i\)) by derivation that:

$$\frac{\partial}{\partial s_k} \frac{\partial \eta_i}{\partial s_h} - \frac{\partial}{\partial s_h} \frac{\partial \eta_i}{\partial s_k} = \sum_{l=1}^{n} \gamma_{ilhk} \eta_l + \sum_{g,l=1}^{n} (\gamma_{gkh} - \gamma_{ghk}) \gamma_{l/gk} \eta_l + \frac{\partial \delta_{ik}}{\partial s_h} - \frac{\partial \delta_{ih}}{\partial s_k} + \sum_{g=1}^{n} (\gamma_{gkh} - \gamma_{ghk}) \gamma_{l/gk} \eta_l + \sum_{g=1}^{n} (\gamma_{gkh} \delta_{gk} - \gamma_{l/gk} \delta_{gh});$$
comparing this with known formulas for the inversion of the intrinsic derivation will give:

$$\frac{\partial \delta_h}{\partial s_i} - \frac{\partial \delta_k}{\partial s_i} = \sum_{l=1}^{n} \gamma_{d, hk} \eta_l + \sum_{g, l=1}^{n} (\gamma_{ghk} - \gamma_{gkh}) \delta_l + \sum_{g, j=1}^{n} (\gamma_{lkh} \delta_{ik} - \gamma_{lik} \delta_{hj}),$$

or the equivalent ones:

$$(\beta) \quad \frac{\partial \delta_h}{\partial s_i} = \sum_{l=1}^{n} \gamma_{d, hk} \eta_l + \sum_{l=1}^{n} (\gamma_{lkh} \delta_{ik} - \gamma_{lik} \delta_{hj}).$$

If one sets:

$$\gamma_{j, hkl} = \frac{\partial \gamma_{j, hk}}{\partial s_i} + \sum_{g=1}^{n} (\gamma_{gil} \gamma_{gji, hk} + \gamma_{gil} \gamma_{gij, hkk} + \gamma_{gkl} \gamma_{ij, gk} + \gamma_{gkl} \gamma_{gk} \gamma_{ij, gk})$$

then one will have:

$$\gamma_{j, hkl} = \sum_{r, s, t, u, v=1}^{n} a_{r, s, t, u, v} l^{(r)}(s) l^{(t)}(u) l^{(v)}(v),$$

and therefore the identity relations:

$$a_{rs, tuv} + a_{rs, vtu} + a_{rs, utv} = 0$$

that BIANCHI recognized can also be written in an invariant form as follows:

$$\gamma_{rs, tuv} + \gamma_{rs, vtu} + \gamma_{rs, utv} = 0.$$

If one takes that into account and applies the process of derivation and elimination to $$(\beta)$$ that was applied before to $$(\alpha)$$ then one will finally arrive at:

$$(\gamma) \quad \sum_{l=1}^{n} \{ \gamma_{j, hkl} \eta_l + \gamma_{lh, ji} \delta_h - \gamma_{li, ji} \delta_h + \gamma_{lj, hkl} \delta_i - \gamma_{lji, h} \delta_j \} = 0.$$

Equations $$(\alpha)$$, $$(\beta)$$, and $$(\gamma)$$ constitute the simultaneous algebraic-differential system that the unknown functions of the problem must satisfy – viz., the translations and rotations.

3. – It is useful to separate the equations $$(\gamma)$$ into three groups A, B, C according to whether two, three, or all of the indices i, j, h, k are distinct, respectively, and to assume that the reference n-tuple is a principal n-tuple.

If $\rho_1, \rho_2, \ldots, \rho_n$ denote the principal invariants of $V_n$ then when the equations of the group A are combined with each other in a suitable way, while those of B are combined analogously, then that will give:
\[ (A_0) \quad \sum_{t=1}^{n} \eta_i \frac{\partial \rho_j}{\partial S_i} = 0, \]

\[ (B_0) \quad (\rho_k - \rho_h) \left( \sum_{t=1}^{n} \gamma_{hkl} \eta_t + \delta_{hk} \right) = 0. \]

If one supposes that the principal invariants are all distinct then \((B_0)\) will give the following expression for the \(\delta_{hk}\):

\[ (3) \quad \delta_{hk} = \sum_{t=1}^{n} \gamma_{hkl} \eta_t, \]

with which, \((\alpha), (\beta),\) and \((\gamma)\) will assume the forms:

\[ (\alpha') \quad \frac{\partial \eta_t}{\partial s_j} = \sum_{t=1}^{n} \gamma_{bkl} \eta_t, \]

\[ (\beta') \quad \frac{\partial \delta_{hi}}{\partial s_k} = \sum_{l=1}^{n} (\gamma_{ik} \eta_h + \sum_{g=1}^{n} (\gamma_{gik} \gamma_{glh} - \gamma_{gih} \gamma_{gkl}) \eta_l) \]

\[ (\gamma') \quad \sum_{t=1}^{n} \eta_t \frac{\partial \gamma_{j,hl}}{\partial S_t} = 0, \]

respectively.

If \((\beta')\) is compared to the equation that one obtains by differentiating \((3)\) and eliminating the derivatives of the \(\eta\) by means of \((\alpha')\) then one will arrive at:

\[ (a) \quad \sum_{t=1}^{n} \eta_t \frac{\partial \gamma_{ukl}}{\partial S_t} = 0. \]

If one observes that when \((a)\) are satisfied identically, \((\gamma')\) will also be satisfied identically then one can conclude that:

**Any \(V_n\) whose principal invariants are all distinct will admit a transitive group of rigid motions if the rotations that relate to the principal \(n\)-tuple are constants, and only in that case. If that condition is verified then \(V_n\) will admit an \(n\)-parameter group for which the initial values of the translations will prove to be arbitrary.**

If one assumes that the reference \(n\)-tuple is the principal \(n\)-tuple of \(V_n\), while the translations are determined by integrating the system \((\alpha')\) then the rotations that one obtains from \((3)\) will be functions of the translations.

One then infers from \((\alpha')\) that:
If the principal congruences are normal then the components of the translation along a well-defined principal direction will vary along only that direction.

Finally, it will follow from \((A_0)\) that:

1. If a \(V_n\) admits a group \(G\) of rigid motions then the principal invariants of \(V_n\) will be invariant under the group.

2. If the group is transitive then its principal invariants will be constants.

4. If one assumes that the reference \(n\)-tuple is a principal \(n\)-tuple then the equations of the group \(C\) will be satisfied identically if the manifold \(V_n\) is regular, and only in that case. The equations of the groups \(A\) and \(B\) will then assume the forms:

\[
(A_1) \quad \sum_{i=1}^{n} \eta_i \frac{\partial \gamma_{ij,ij}}{\partial s_i} = 0,
\]

\[
(B_1) \quad (\gamma_{ij,ij} - \gamma_{ik,ik}) \left( \sum_{i=1}^{n} \gamma_{jki} \eta_i + \delta_{jk} \right) = 0,
\]

resp., and will substitute for the group \((\gamma)\) completely.

They are satisfied identically for the manifolds with constant curvature, and only for them, and one will then get the well-known theorem about the group of rigid motions that pertains to them.

\((A_1)\) says that:

If a regular manifold \(V_n\) admits a group of rigid motions \(G\) then the second-order differential invariants of \(V_n\) will all be invariant under \(G\).

If the group is transitive then those invariants will be constants.

It is easy to geometrically characterize the case in which \((B_1)\) determine the \(\delta_{jk}\) as functions of the \(\eta\) whose expressions are given by (3). In that case, \((\alpha{'})\), \((\beta{'})\), and \((a)\) will also be valid and one will obtain them as in the preceding paragraph, and therefore the conclusions that were reached above will also be valid in the case that is now considered.