“Topologische Fragen in der Theorie der Spannungsfunktionen,” Abh. Braunschweig. Wiss. Ges. **12** (1960), 4-65.

Topological questions in the theory of stress functions

By Georg Rieder

Presented by **H. Schaefer**

(Received on 20 June 1960)

Translated by D. H. Delphenich

Summary: *Displacement functions or stress functions may be used to solve elastic boundary-value problems. In the first case, the fulfillment of the compatibility conditions is secured beforehand by deriving the strain field from a displacement, and the displacement function is adapted afterwards to the equilibrium condition. In the second case, the derivation of the stress field from stress functions guarantees beforehand the fulfillment of the equilibrium condition, while the compatibility condition is to be satisfied by subsequent adaptation. The topological properties of displacement functions have been repeatedly studied from a geometric viewpoint, especially in connection with dislocation theory; the present paper contains a corresponding study of stress functions from a static viewpoint. It is shown that in order for the elastic field to be representable by stress functions in a spatial domain that is devoid of external forces and sources of external stress (briefly: an "unperturbed domain") the number of bounding surfaces plays the same role that the connectivity does for the representation of strain by displacement functions. In particular, a representation by stress functions is impossible in a multiply-bounded domain if the external forces are not in equilibrium on any individual surface; this also applies, for example, to the isolated single force, which is to be regarded as the limiting case of an assembly of forces on an infinitely-small hole. The stress functions that are given for this case in the plane and in space prove to be the stress functions of a state of internal stress whose singular stress sources were omitted when deriving the stress field; the intentional introduction of such "fictitious supplementary stresses" makes it possible to construct more solutions of this kind. A relation that is known for the plane annular domain between the boundary conditions for zero stress functions and the conditions for vanishing Volterra states of distortion is extended into space for multiply-connected domains. As examples, stress functions are set up for the isolated force and a double force with a moment in the "pierced" full space, and Schaefer's stress functions for the problems of Boussinesq and Cerutti on the half space are derived by* a *different method.*

1. Introduction. Basic topological concepts.

a) *Problem statement and notations.* – The goal of this paper is to compare and contrast the topological properties of the solution of elastic problems by stress functions and displacement functions, as one gets in a generally-valid form immediately from the geometry and statics of the general elastic continuum. We consider the solution in a connected spatial region of the continuum in whose interior there are neither external forces that act there nor any sort of sources of proper stresses [**2**, **30**, **31**] that lie there. In addition, the elastic properties of that region might be arbitrarily anisotropic and vary from position to position. It will only be assumed that the quadratic form of the elastic energy in it is positive-definite. Such a region and the elastic stress and dilatation fields in its interior shall be called “unperturbed” (*). Special solutions in special coordinate systems will enter in the background in this context and will be treated only as examples. For that reason, the coordinate-free symbolic notation [**9**] will mostly be employed here for the vector and tensor equations, since it seems to be best suited to such investigations (**).

In detail, vectors will be denoted by German symbols or by ones that have an arrow overhead. One also includes the tensors of rank four $\underline{c}$ and $\underline{s}$ of the elasticity constants (elasticity coefficients, resp.) and the identity tensor I of rank two. In rectangular Cartesian coordinates with the summation convention implied:

$\mathfrak{a} \cdot \mathfrak{b} \equiv a_i\, b_i$	the scalar product of the vectors $\mathfrak{a} \equiv (a_i)$ and $\mathfrak{b} \equiv (b_i)$,
$\mathfrak{a} \times \mathfrak{b} \equiv (\varepsilon_{ijk}\, a_j\, b_k)$	the vectorial product,
$\mathfrak{a}\ \mathfrak{b} \equiv (a_i\, b_j)$	the dyadic product
$\tilde{\underline{\beta}} \equiv (\beta_{ji})$	the tensor transpose of $\underline{\beta} \equiv (\beta_{ij})$
$\underline{\mu} \cdot\cdot\, \underline{\nu} \equiv \mu_{ij}\, \nu_{ji}$	the double scalar product (1.1)

The use of I as an index will characterize the first scalar of a tensor. The location of the symbolic ∇-vector in the formula will be determined by the rules of vector multiplication. Therefore, all terms with the same scalar, vectorial, or dyadic products that are included in subordinate brackets will be differentiated with no regard for the sequence. By contrast, the effect of differentiation is interrupted by a super-ordinate bracket or the sign of the double scalar multiplication, as long as no deviation from that rule is made known by being coupled by an arrow. The differentiation shall also remain under the other factors when an individual factor to be differentiated is denoted with a vertical arrow.

In regard to the analytical nature of the functions that appear, it shall be assumed only that they are single-valued and differentiable sufficiently often, up to isolated singularities, and that the singularities are arranged such that one can approximate them by differentiable functions arbitrarily well. That is, one must also allow, in particular, isolated jumps, simple and double covering, etc., whose exact analytical treatment **Schwartz** [**28**] has laid the foundations of in the theory of distributions. In what follows, we will not distinguish between the differentiation of “good” functions and the corresponding operations on distributions. It is obvious that the admissibility of those operations in each individual case must still be tested in particular (***). Corresponding assumptions shall also be true for the bounding surfaces of the spatial region [**36**].

(*) This terminology has nothing to do with the moving perturbations [**33**, **34**] that are defined in other places and in other contexts.

(**) Solutions for the isotropic medium in general coordinates can be found in **Brdicka** [**29**]. For general stress functions, cf., the survey paper of **Truesdell** [**27**].

(***) Another method of treating topological questions emerges from the properties of special function classes (**Slobodianskii** [**32**]). That path leads to far-reaching statements for special types of solutions, but it is less general and does not allow the relationships to statics and geometry to emerge as clearly.

Nothing will be assumed about the physical origin of the stresses in the spatial region in question beyond the restriction to static problems. The boundary of the unperturbed region can be identical with the outer surface of an elastic body. However, one can just as well treat an unperturbed slice of a larger, or even infinite, elastic body. The stresses in the region in question can arise as load stresses from external forces or also as proper stresses of proper stress sources in other parts of the body.

b) *Stress functions and four-potential.* – The mutually-independent work of **Gwyther**, **Finzi** [**27**], and **Krutkov** [**29**] that derived the stress field $\underline{\sigma}(\mathfrak{r})$ from a symmetric stress function tensor $\underline{\chi}(\mathfrak{r})$ ($\mathfrak{r}$ is the position vector):

$$\underline{\sigma} = \nabla \times \underline{\chi} \times \nabla \equiv \mathrm{Ink}\ \underline{\chi} \tag{1.2}$$

ensured that the equilibrium condition would be fulfilled:

$$\nabla \cdot \underline{\sigma} \equiv \mathbf{Div}\ \underline{\sigma} = 0 \tag{1.3}$$

in the absence of external forces in the entire domain of validity of (1.2) (*). Now, in the event that the outer surface of that spatial domain is not already identical with the outer surface of the total elastic body, one thinks of the domain as being excised from it, and the new outer surface will carry the forces that are required to establish equilibrium. One will then have that the force $d\vec{P}$ on a infinitesimal surface patch that is represented by the vector $d\,\mathfrak{f}$, which is perpendicular outward, will be:

$$d\vec{P} = d\mathfrak{f} \cdot \underline{\sigma} = d\mathfrak{f} \cdot \mathrm{Ink}\ \underline{\chi} \equiv d\mathfrak{f} \cdot (\nabla \times \underline{\chi} \times \nabla), \tag{1.4}$$

and from **Stokes**, the force $\vec{P}$ on a finite, simply-bounded, and simply-connected piece of the outer surface will be:

$$\vec{P} = \int d\mathfrak{f} \cdot (\nabla \times \underline{\chi} \times \nabla) = \oint d\mathfrak{r} \cdot \underline{\chi} \times \nabla. \tag{1.5}$$

Now, from a known argument in vector analysis, that will imply the vanishing of the total force on a simply-connected outer surface when one can contract the integration contour to a regular point of the outer surface at which the stress functions are also differentiable. However, from our assumptions, such a point must always exist. For multiply-connected outer surfaces, one employs the possibility of converting the entire outer surface into a simply-bounded, simply-connected surface with the aid of the canonical decomposition [**4**]. If one now defines the contour integral around that boundary then the cut curves will run in opposite directions to each other (cf., e.g., Fig. 8a, pp. 24); the contour integral must then vanish, under very general assumptions about differentiability along the cut curves. The resultant of all forces on a boundary surface is zero then, from the validity of (1.2), and independently of the connection number of the bounding surface.

(*) For an introduction to the differential operators Ink, Def, Div, and Rot, cf., **Kröner** [**2**].

From **Peretti** [**22**] or **Günther** [**13**], the moment of the force that acts upon a simply-bounded and simply-connected patch of the outer surface is:

$$\vec{M} = -\int d\mathfrak{f} \cdot (\nabla \times \underline{\chi} \times \nabla) \times (\mathfrak{r} - \mathfrak{r}_0) = -\oint d\mathfrak{r} \cdot \{ \underline{\chi} + (\underline{\chi} \times \nabla) \times (\mathfrak{r} - \mathfrak{r}_0), \tag{1.6}$$

in which $\mathfrak{r}_0$ is the position vector to the arbitrarily-chosen fixed reference point for the moment. The vanishing of the resultant moment will then follow from the same argument as above, so in total, one will have the vanishing of the resultant dyname on each closed outer surface of a spatial domain in which (1.2) is true. In other words: A stress field that can be represented by (1.2) can carry no resultant force and no result moment between two separated individually-closed outer surfaces of the same elastic body.

The analogue in the theory of vector fields defines the source-free current vector $\mathfrak{v}$ ($\mathfrak{r}$) of an incompressible fluid, which is known [**35**] to always be representable as the rotor of a vector potential $\mathfrak{B}$:

$$\mathfrak{v} = \nabla \times \mathfrak{B}. \tag{1.7}$$

One then has:

$$\oint d\mathfrak{f} \cdot \mathfrak{v} = 0 \tag{1.8}$$

there for any closed outer surface, independently of its connection number. That is, the net effect of all sources that are enclosed by a closed outer surface must vanish, and it can be exchanged with a flux field that can be represented by (1.7), indeed, with a fluid between different outer surfaces of the same spatial domain, but no additional fluid masses being added or taken away.

The definitive topological criterion for (1.2) and (1.7) is then the number of bounding outer surfaces of a spatial domain, independently of the connection number.

c) *Displacement functions and potential.* – It is known [**1**, **9**, **26**] that the derivation of the (elastic) extension $\underline{\varepsilon}$ from the displacement vector $\mathfrak{u}$ ($\mathfrak{r}$) by:

$$\underline{\varepsilon} = \tfrac{1}{2}(\nabla\mathfrak{u} + \mathfrak{u}\,\nabla) \equiv \text{Def}\,\mathfrak{u}, \tag{1.9}$$

will guarantee the fulfillment of the compatibility conditions (*):

$$\underline{\eta} = \text{Ink}\,\underline{\varepsilon} = \nabla\times\underline{\varepsilon}\times\nabla = 0. \tag{1.10}$$

Similar to the way that in the theory of vector fields, the Ansatz:

$$\mathfrak{v} = \nabla V \tag{1.11}$$

(*) The distinction between elastic, total, and supplementary strain [**5**, **6**, **7**] breaks down in a flux-free region (up to some assumptions that are yet to be discussed). The special way of characterizing elastic strain can be temporarily preserved.

ensures the absence of vortices [**9**, **15**, **35**]:

$$\nabla \times \mathfrak{v} = 0 \tag{1.12}$$

Now, conversely, the fulfillment of (1.12) guarantees the fulfillment of (1.11) only in a simply-connected domain, while in multiply-connected domain, the potential *V* will be multiple-valued in some cases; i.e., it cannot be defined in a physically-sensible way. The number of bounding outer surfaces plays no role in the topologically-decisive step in the proof, namely, stretching a simply-connected surface that is completely internal over the contour integral in question. For simple-connectivity, one can (cf., Fig 2) always go around all bounding surfaces and avoid the external region, while that is not possible for any closed curve for multiple-connectivity (Sec. 1).

Corresponding statement are true for stretching and displacement [**1**, **2**, **3**, **6**, **8**, **21**, **30**, **31**, **43**], along with other things. It is only for a simply-connected domain that the equation (1.10) guarantees the existence of a realizable single-valued displacement, and thus one that is physically sensible and requires no cut operations. The definitive topological criterion for (1.9) and (1.11) is then the connection number of the spatial domain, independently of the number of bounding outer surfaces.

d) *Basic topological concepts and basic operations.* – The topological characteristics of a bounded spatial domain, or also one that extends to infinity, that are essential for the present problem are, from the above, the connection number and the bounding number. In a spatial region, simply-connected and simply-bounded shall mean that one can contract any closed curve in the external region to a point without intersecting the internal region, and at the same time, bring two arbitrary points in the external region to a single point without piercing the internal region in so doing.

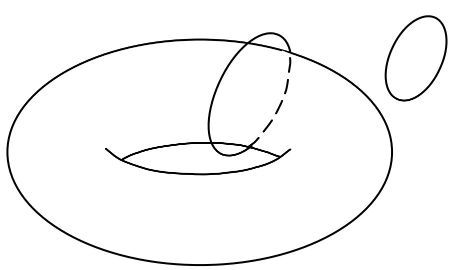


Figure 1. Doubly-connected region with two closed curves in external space

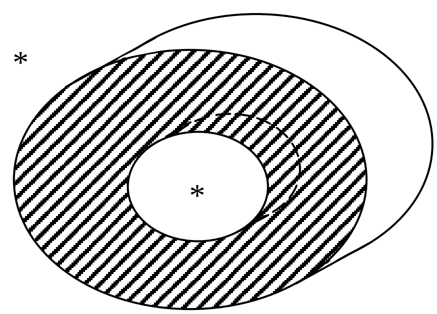


Figure 2. Doubly-bounded region with two points in external space (cross-section)

We define the connection number of an arbitrary spatial domain to be one plus the number of closed curves (“circuits”) in the external region that can be made to overlap another closed curve or divided up into simpler circuits or contracted to a point (i.e., “irreducible circuits”) without cutting through the internal region either. The full torus is drawn as an example of a doubly-connected domain in Fig. 1. As one can learn from a comparison with the topology of closed surfaces [**4**, **37**], a simply-bounded, (n – 1)-connected spatial region will be bounded by a surface of genus n and connection number

$2n + 1$ (i.e., a "pretzel"): The connection number of the internal region is equal to that of the exterior region, so in the example of Fig. 1, the connection number of the "unbounded" infinite medium is equal to that of the hollow torus.

We define the bounding number of an arbitrary spatial region to be the number of points in the external domain that cannot be brought together into a single point without crossing the internal region in so doing. An example of a doubly-bounded region is illustrated in Fig. 2. From the above definition, it is obviously simply-connected, since all of the closed curves in the external region and also in the internal region can be contracted to a point. Obviously, the bounding number is also equal to the number of bounding closed outer surfaces. However, an exchange of the external and internal region is not possible in this case, since for any reasonable application, the internal region must be connected in the sense that two arbitrary points of the internal region must be capable of being transferred to each other along a path that lies completely within the internal region (*).

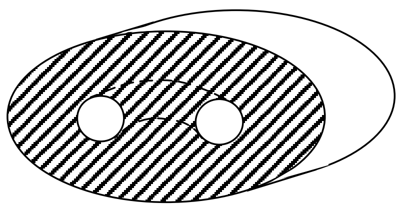


Figure 3a. Doubly-bounded and doubly-connected domain (cross-section)

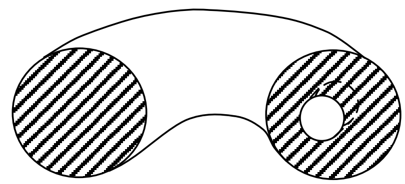


Figure 3b. Doubly-connected and doubly-bounded domain (cross-section)

Multiple connectivity and multiple boundedness can appear on the same spatial region at the same time, as the examples in Figs. 3 will show. In addition, Fig. 3a gives a topological picture of an infinite medium with a hollow torus that is better suited to the applications, since the infinite elastic medium mostly enters into it as a limiting case of the internal region of a very large, simply-connected outer surface (**). The connection numbers of the individual bounding surfaces in the same region are obviously independent of each other. In order to determine the bounding number, one must simply count the closed bounding surfaces, and in order to determine the connection number of the entire region, the genera of the individual bounding surfaces must be added, while one is added to the result.

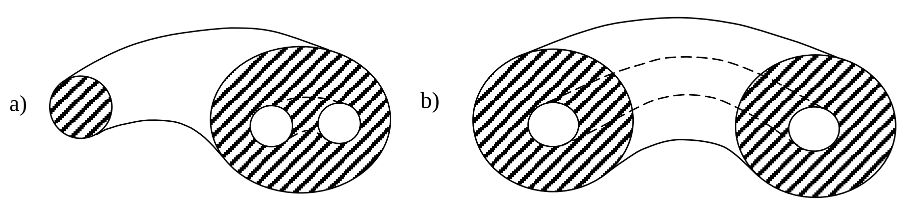
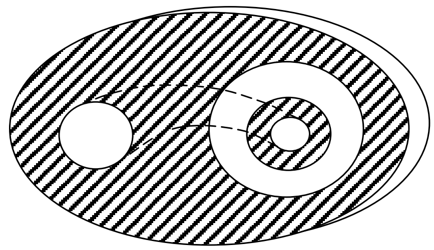


Figure 4. Further examples of multiple boundedness and multiple connectivity for the same region.
a), b) Doubly-bounded, triply-connected.

(*) The distinction between multiply-connected ("cyclic") and multiply-bounded ("periphrastic") domains is found already in **Maxwell** ([**44**], pp. 18).

(**) That corresponds to the topological structure of metric space ([**4**], pp. 262).



c) Triply-bounded, triply-connected.

In order to reduce the bounding number, one must drill a hole in the region and draw a "connecting hose" from one bounding surface to another. For the reduction to simple boundedness, obviously $p - 1$ connecting hoses will be required for a p-fold bounded region, independently of the connectivity of the individual outer surfaces. If one does not introduce any more than these absolutely necessarily connecting hoses then the connection number will not be changed in that way. The old irreducible circuits then continue to exist, and a new one can be introduced only by way of an extra connecting hose that couples bounding surfaces that are already connected to each other one more time. The geometric form of the connecting hose is arbitrary, within very broad restrictions; it can be contracted over a singular surface or also a singular line, as in Fig. 5.

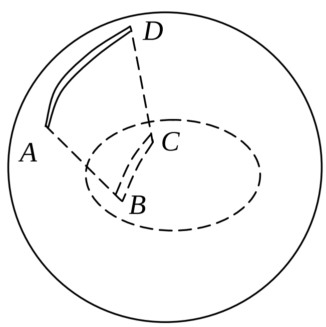


Figure 5. Doubly-bounded region with a flattened connection hose.

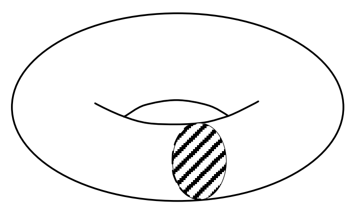


Figure 6. Doubly-connected region with a separating surface.

If one has reduced the bounding number of the spatial region by one by way of a connecting hose then its boundary will correspond topologically to a pretzel of genus n and connection $2n + 1$, where n is the sum of the genera of the original bounding surfaces; from our definition, the spatial domain will then be $(n + 1)$-connected. It can be reduced to simple connectivity by means of n simply-connected separating surfaces that span the irreducible circuits in the external region, as is illustrated in Fig. 6 with the example of the torus, and as is also known from the literature that was cited in c). The reduction to simple connectivity and simply bounding by this prescription is also easy to perform in the examples of Figs. 3 and 4. An extension to arbitrarily-high connectivity and bounding numbers then seems possible with not further assumptions.

The meaning of these basic topological operations is illuminated by b) and c). For a vector field that is free of sources and vortices in a given spatial region, multiple-connected vortex lines in external space can make the derivation from a potential impossible, and for multiple-boundedness, the non-vanishing total effect of the sources in the individual subsets of the external space can make the derivation from a vector potential impossible. It is only by reducing to simple connectedness and simple bounding that the simultaneous derivation from a potential and from a vector potential will become possible, at least formally. Corresponding statements will be true for an unperturbed stress field and an unperturbed extension field in a given spatial region. For multiple connection, dislocation lines of type 1 and 2 in external space can make the derivation from displacement functions impossible, and for multiple-boundedness, the derivation of force systems with non-vanishing dynames from stress functions in the individual subsets of the external region can become impossible. It is only by reducing to simple connection and simple bounding that the simultaneous derivation from displacement functions and stress functions will be possible, at least formally.

The following section will address the explanation, proof, and application of those laws.

2. Displacement functions and stress functions in simply-connected and simply-bounded domains.

The assumed existence of an unperturbed stress and strain field will imply their single-valuedness in relation to boundary-value problems and the essential existence of single-valued stress and displacement functions. The result will be compared with known theorems from the theory of vector fields.

a) *Displacement functions and potentials.* – The compatibility condition is fulfilled with the Ansatz (1.9) for the elastic extension. In order to fulfill the equilibrium condition (1.3) with the stress that is calculated from the strain by way of **Hooke**'s law:

$$\underline{\sigma} = \underline{c} \cdot\cdot \underline{\varepsilon}, \tag{2.1}$$

the displacement $\mathfrak{u}$ must satisfy the differential equation:

$$\mathrm{Div}\,(\underline{c} \cdot\cdot\ \mathrm{Def}\,\mathfrak{u}) = 0. \tag{2.2}$$

We shall prove the single-valuedness of $\underline{\varepsilon} = \mathrm{Def}\,\mathfrak{u}$ directly with the inclusion of the case in which the outer surface of the unperturbed region is not the outer surface of the elastic body, but the forces on the unperturbed region will be converted into proper stress sources or external forces on other perturbed subsets of the total body. If we exclude unstable cases then the linear functional relationship:

$$\mathfrak{P}_0 - \oint \underline{C}(\mathfrak{r},\mathfrak{r}') \cdot \mathfrak{u}(\mathfrak{r}')\, df' - \mathfrak{n} \cdot (\underline{c} \cdot\cdot \mathrm{Def}\,\mathfrak{u}) = 0 \tag{2.2a}$$

will be true as the equilibrium condition on the outer surface of the region with the outward-point unit normal vector $\mathfrak{n}$, in which the inequality:

$$\oint\oint \mathfrak{u}(\mathfrak{r})\cdot \underline{C}(\mathfrak{r},\mathfrak{r}')\cdot \mathfrak{u}(\mathfrak{r}')\,df\,df' \geq 0 \tag{2.2b}$$

is assumed for the functional. Equality shall apply only when either the second-rank tensor $\underline{C}(\mathfrak{r},\mathfrak{r}')$ or the displacement vector $\mathfrak{u}$ vanishes at each location on the integration surface, or in the event that the rest of the body is not fixed at any location in space, when the displacement $\mathfrak{u}$ ($\mathfrak{r}$) corresponds to a rigid motion on the outer surface.

We shall now calculate, from **Gauss**'s theorem:

$$\begin{aligned}&\tfrac{1}{2}\int \mathfrak{u}\cdot \mathrm{Div}(\underline{c}\cdot\cdot\mathrm{Def}\ \mathfrak{u})\,d\tau + \tfrac{1}{2}\oint df\ \mathfrak{u}\cdot\{\mathfrak{P}_0 - \oint \underline{C}(\mathfrak{r},\mathfrak{r}')\cdot \mathfrak{u}(\mathfrak{r}')\,df' - \mathfrak{u}\cdot(\underline{c}\cdot\cdot\mathrm{Def}\ \mathfrak{u})\}\\ &= \tfrac{1}{2}\oint df\ \mathfrak{P}_0\cdot\mathfrak{u}\cdot - \tfrac{1}{2}\oint\oint \mathfrak{u}(\mathfrak{r})\cdot\underline{C}(\mathfrak{r},\mathfrak{r}')\cdot\mathfrak{u}(\mathfrak{r}')\,df\,df' - \tfrac{1}{2}\int \mathrm{Def}\ \mathfrak{u}\cdot\cdot\underline{c}\cdot\cdot\mathrm{Def}\ \mathfrak{u}\,d\tau\,.\end{aligned} \tag{2.3}$$

The left-hand side of this vanishes from (2.2) and (2.2a). On the right-hand side, the double surface integral cannot be negative, from (2.2b), and the volume integral is finite and positive wherever Def $\mathfrak{u}$ does not vanish.

If one now replaces u in (2.3) with the difference between two solutions with the same outer surface force density $\mathfrak{P}_0$ then the first integral on the right-hand side will vanish from (2.2a). Hence, Def $\mathfrak{u}^-$ must also vanish in the interior of the region, and at least one of the conditions under which the equality sign can appear in (2.2b) must be true for $\mathfrak{u}^-$ on the outer surface. Q. E. D. The usual boundary-value problem with "forces given on the outer surface" is included in that proof for $\underline{C}(\mathfrak{r},\mathfrak{r}') = 0$. The extension to the case of displacements that are given on the outer surface is trivial, and ultimately the proof can be extended with no difficulty to the case in which either the displacements or a boundary condition of type (2.2a) is true in finitely many sub-surfaces of the total surface. (cf., [**1**], pp. 170)

However, if the difference Def $\mathfrak{u}^-$ vanishes then the difference vector $\mathfrak{u}^-$ must also itself vanish, up to a rigid motion, since from **Cesàro** ([**1**], pp. 222):

$\mathfrak{u}^-$ (r) =

$$\mathfrak{u}^-(\mathfrak{r}_0) - \tfrac{1}{2}(\mathfrak{r}-\mathfrak{r}_0)\times\left[\nabla\times\mathfrak{u}^-\right]_{\mathfrak{r}=\mathfrak{r}_0} + \int_{\mathfrak{r}_0}^{\mathfrak{r}} d\mathfrak{r}'\cdot\{\mathrm{Def}\ \mathfrak{u}^-(\mathfrak{r}') + [\mathrm{Def}\ \mathfrak{u}^-(\mathfrak{r}')\times\nabla']\times(\mathfrak{r}'-\mathfrak{r})\}\,. \tag{2.4}$$

(∇' means differentiation with respect to the integration variable $\mathfrak{r}'$.) The integrand vanishes here, and therefore the integral, as well, and all that will remain will be the terms in front of the integral, which represent a rigid motion that is meaningless for elastic strain. The single-valuedness of the solution u of (2.2) in relation to the boundary-value problem is thus proved, up to a rigid motion.

Spatial single-valuedness follows from the vanishing of the incompatibility $\underline{\eta}$ (2.2). To prove this, we insert the elastic strain $\underline{\varepsilon}$ into **Cesàro**'s formula [cf., (2.4)] and drop the inconsequential terms in front of the integral. We will then find that the displacement vector:

$$\mathfrak{u}\,(\mathfrak{r}) = \int_{\mathfrak{r}_0}^{\mathfrak{r}} d\mathfrak{r}' \cdot \{\underline{\varepsilon}\,(\mathfrak{r}') \;+[\underline{\varepsilon}\,(\mathfrak{r}')\times\nabla']\times(\mathfrak{r}'-\mathfrak{r})\} \tag{2.5}$$

exists and is spatially single-valued precisely when each contour integral (2.5) with $\mathfrak{r}_0 = \mathfrak{r}$ vanishes in an unperturbed region. However, each such contour can be converted into a surface integral using **Stokes**'s theorem; if $\mathfrak{r}$ lies on the contour then:

$$\begin{aligned} \mathfrak{b}\,(\mathfrak{r}) &= \oint d\mathfrak{r}' \cdot \{\underline{\varepsilon}(\mathfrak{r}') + (\underline{\varepsilon}(\mathfrak{r}')\times\nabla')\times(\mathfrak{r}'-\mathfrak{r}) \\ &= \int d\mathfrak{f}' \cdot \{\nabla'\times\underline{\varepsilon}(\mathfrak{r}') + (\nabla'\times\underline{\varepsilon}(\mathfrak{r}')\times\nabla')\times(\mathfrak{r}'-\mathfrak{r})\} \end{aligned} \tag{2.6}$$

and after a recalculation that is performed in an Appendix:

$$\mathfrak{b}\,(\mathfrak{r}) = \oint d\mathfrak{r}' \cdot \underline{\eta}(\mathfrak{r}')\times(\mathfrak{r}'-\mathfrak{r})\,. \tag{2.7}$$

However, in a simply-connected, connected, unperturbed region, every closed curve can be spanned by a simply-connected surface that lies entirely in the interior, and since $\underline{\eta}$ will vanish there, according to (1.10), $\mathfrak{b}\,(\mathfrak{r})$ must also vanish. The spatial single-valuedness of $\mathfrak{u}$ is thus ensured. Q. E. D.

Intuitively, the assumption (1.10) means that at most distributions of dislocations of the same type as stress-free **Nye** structual curvatures can enter into the interior of the unperturbed region, so for arrangements of singular dislocations, that would mean only small-angle grain boundaries of types 1 and 2 with crystallographic dislocations or quasi-dislocations [**2**, **31**] (*). In such cases, $\mathfrak{u}$ will no longer mean the total dislocation, but only the contribution of the dislocation field to the elastic strain.

The analogous case in the theory of vector fields – namely, the gradient vector – is generally known, such that we can skip a special presentation of it.

b) *Stress functions and vector potential.* – The equilibrium condition (1.3) is fulfilled with the Ansatz (1.2) for the (total-) stress. In order to fulfill the compatibility condition (1.10) by the strain that is calculated from the stress using **Hooke**'s law:

$$\underline{\varepsilon} = \underline{s}\cdot\cdot\underline{\sigma}\,, \tag{2.9}$$

(*) Therefore, such distributions of dislocations will not be regarded as perturbations here, in contrast to the far-reaching questions that are posed in the theory of **Cosserat** continua (the continuum theory of lattice fields, resp.) [**2**, **6**, **21**, **30**, **31**, **33**, **34**].

the stress function tensor $\underline{\chi}$ must satisfy the differential equation:

$$\operatorname{Ink}(\underline{s}\cdot\cdot\operatorname{Ink}\underline{\chi}) = 0. \tag{2.10}$$

Now, the single-valuedness of $\underline{\sigma} = \operatorname{Ink}\ \underline{\chi}$ already follows from the single-valuedness of $\underline{\varepsilon} = \operatorname{Def}\ \mathfrak{u}$ that was just proved. Nevertheless, we shall carry out the proof one more time especially for the case of given forces on the outer surface, since the formulas and concepts that are introduced in it will be necessary in other places.

Using a double application of **Gauss**'s law, we convert:

$$\begin{aligned}&\tfrac{1}{2}\int \underline{\chi}\cdot\cdot[\operatorname{Ink}(\underline{s}\cdot\cdot\operatorname{Ink}\underline{\chi})]\,d\tau \\ &= \tfrac{1}{2}\oint \underline{\chi}\cdot\cdot[d\mathfrak{f}\times(\underline{s}\cdot\cdot\operatorname{Ink}\underline{\chi})\times\nabla]-\tfrac{1}{2}\oint(\underline{\chi}\times\nabla)\cdot\cdot\,(\underline{s}\cdot\cdot\operatorname{Ink}\underline{\chi})\times d\mathfrak{f}]+\tfrac{1}{2}\int\operatorname{Ink}\underline{\chi}\cdot\cdot\ \underline{s}\cdot\cdot\operatorname{Ink}\underline{\chi}\,d\tau.\end{aligned} \tag{2.11}$$

The left-hand side vanishes, due to (2.10), while the integrand of the volume integral on the right-hand side is positive and finite whenever $\underline{\sigma}$ does not vanish completely.

We couple the examination of the outer surface integral to the static interpretation of the stress functions $\underline{\chi}$ and $\underline{\chi}\times\nabla$, which, according to **Schaefer** [**10**], are to be regarded as the sectional moment and sectional force on the section with the vectorial element $d\mathfrak{r}$ of a rigid "crustal shell" that replaces the outer surface to the region. That will also yield an intuitive interpretation for formulas (1.5) and (1.6) of **Peretti** and **Günther**, along with it. Now, let $\underline{\chi}^{-}$ be the difference between two solutions of the same boundary-value problem with given forces on the outer surface. From the form of (1.5) and the analogous construction of equations (1.6) and (2.6-7), we must now conclude that a necessary and sufficient condition for the vanishing of the difference of the forces on the outer surface must be that $\underline{\chi}^{-}$, along with its first normal derivatives, must behave like a deformator on the outer surface, as long as it is statically required (*), such that one can write:

(*) That means a restriction on the components and derivatives that appear in (1.5) and (1.6). Other pieces of the stress function tensor (besides the strain term $\underline{s}\cdot\cdot\operatorname{Ink}\underline{\chi}$) obviously do not enter into the outer surface integral of (2.11), etc. In detail, that means only that the values of:

$$\mathfrak{n}\times\underline{\chi} \tag{2.11a}$$

are determined on the outer surface by statics. One obtains the derivatives that are determined by statics from:

$$\mathfrak{n}\times\underline{\chi}\times\nabla. \tag{2.11b}$$

An essential simplification comes about for planar boundary surfaces by way of the identity transformation (only $\underline{\chi}$ is differentiated):

$$\nabla = \nabla\cdot\mathfrak{n}\mathfrak{n} - (\nabla\times\mathfrak{n})\times\mathfrak{n}.$$

All terms, up to the first one, already drop out here with the boundary conditions for (2.11a); thus, only the normal derivatives for a section:

$$\mathfrak{n}\cdot\nabla\mathfrak{n}\times\underline{\chi}\times\mathfrak{n} \tag{2.11b′}$$

$$\underline{\chi}^{-} = \tfrac{1}{2}(\nabla\, \mathfrak{A}^{-} + \mathfrak{A}^{-}\, \nabla), \tag{2.12}$$

in which $\mathfrak{A}^{-}$ is determined uniquely on the outer surface by:

$$\mathfrak{A}^{-}(\mathfrak{r}) = \int_{\mathfrak{r}_0}^{\mathfrak{r}} d\mathfrak{r}' \cdot \{\underline{\chi}^{-}(\mathfrak{r}') + (\underline{\chi}^{-}(\mathfrak{r}')\times\nabla')\times(\mathfrak{r}'-\mathfrak{r})\}\,. \tag{2.12a}$$

The normal derivatives come into play by way of the last term in (1.6). From (1.9), one can also set:

$$\underline{s}\cdot\cdot\,\mathrm{Ink}\,\underline{\chi}' = \tfrac{1}{2}(\nabla\, \mathfrak{u}^{-} + \mathfrak{u}^{-}\, \nabla) \tag{2.13}$$

on the outer surface. One now integrates the expressions under the outer surface integrals of (2.11) over a simply-connected piece of the outer surface, and after a conversion that is performed in an Appendix, one will find that:

$$\begin{aligned} &\tfrac{1}{2}\int \underline{\chi}^{-}\cdot\cdot[d\mathfrak{f}\times(\underline{s}\cdot\cdot\,\mathrm{Ink}\,\underline{\chi}^{-})\times\nabla] - \tfrac{1}{2}\int(\underline{\chi}^{-}\times\nabla)\cdot\cdot[(\underline{s}\cdot\cdot\,\mathrm{Ink}\,\underline{\chi}^{-})\times d\mathfrak{f}] \\ &= \tfrac{1}{8}\oint d\mathfrak{r}\cdot\{-2(\mathrm{Def}\ \mathfrak{A}^{-})\cdot(\nabla\times\mathfrak{u}^{-}) + 2(\mathrm{Def}\ \mathfrak{A}^{-})\cdot(\nabla\times\mathfrak{A}^{-}) + (\nabla\times\mathfrak{A}^{-})\times(\nabla\times\mathfrak{u}^{-})\}\,. \end{aligned} \tag{2.14}$$

Since one can always contract the integration contour on a simply-connected outer surface to a point, the vanishing of the outer surface integrals in (2.11) will follow from that, and therefore the single-valuedness of the stress field $\underline{\sigma} = \mathrm{Ink}\,\underline{\chi}$ for the given boundary-value problem.

One can see that $\underline{\chi}$ itself is determined uniquely, up to an irrelevant deformator, by repeating the same argument with the special tensor:

$$\underline{s}^{-} \equiv (s^{-}_{ijkl}) = (\delta_{ik}\ \delta_{jl}), \tag{2.15}$$

with which, (2.10) will go to the equation:

$$\mathrm{Ink\ Ink}\ \underline{\chi}^{-} = 0, \tag{2.16}$$

which is certainly valid for vanishing $\underline{\sigma}^{-}$. With the auxiliary condition:

$$\nabla\cdot\underline{\chi}^{-} = 0, \tag{2.17}$$

one will have:

$$\Delta\Delta\,\underline{\chi}^{-} = 0. \tag{2.18}$$

---

are determined by statics on the outer surface. We shall refer to the conditions for (2.11a-b) as the "static boundary conditions." The symmetry requirement can be added to that. The component $\mathfrak{n}\cdot\underline{\chi}\cdot\mathfrak{n}$ is not included for planar boundary surfaces at all.

The fulfillment of the auxiliary condition (2.17) be guaranteed, along with the boundary-conditions. Since $\underline{\sigma}^{-} = 0$, $\underline{\chi}^{-}$ can correspond to only a deformator on the outer surface whose vanishing, along with that of its normal derivatives, will ensure that (2.17) is fulfilled. However, the entire stress function $\underline{\chi}^{-}$, along with its normal derivatives on the outer surface, will then vanish, and what will remain in Cartesian coordinates is the homogeneous biharmonic boundary-value problem, whose single solution is known to be zero.

Before we go on to the existence proof for the stress function tensor, we first work through the analogous problem in the theory of vector fields, namely, the existence proof for the vector potential of a source-free vector field $\mathfrak{v}$, and in fact by constructing such a vector potential. Now, since the operation "rot" can in no way be represented as a gradient construction, one should not expect that the vector potential can be represented as a path-independent line integral from the outset. It is possible to give it the form of a line integral at most when the path of integration is also established uniquely through each point (cf., Prob. 15, pp. 109 in **Phillips** [**35**]). However, one is then dependent upon the form of the region, since the path must run completely in the interior. We then employ another path that was likewise chose by **Phillips**, namely, the construction of the vector potential using the **Biot-Savart** law. As is known, the vector potential for the vector field $\mathfrak{v}$ that is source-free in infinite space is:

$$\mathfrak{B} = \frac{1}{4\pi}\nabla\times\int\frac{\mathfrak{v}(\mathfrak{r}')}{|\,\mathfrak{r}-\mathfrak{r}'\,|}\,d\tau', \tag{2.19}$$

from which, one will get the **Biot-Savart** law:

$$\mathfrak{B} = \frac{1}{4\pi}\int\frac{\mathfrak{v}(\mathfrak{r}')\times(\mathfrak{r}-\mathfrak{r}')}{|\,\mathfrak{r}-\mathfrak{r}'\,|^3}\,d\tau' \tag{2.20}$$

by simply performing the differentiations. The integration must be extended over all space.

If a source-free vector field is defined only in a finite region, and its normal component does not vanish on the entire outer surface then one must continue it source-free into the external space or, intuitively speaking, close the flux tube over the external region. From a suggestion of **Phillips**, that can perhaps come about by a source-free and vortex-free continuation of the vector field to infinity ([**35**], pp. 196): However, it is more preferable for the present problem to introduce (cf., Appendix) a surface-singular closing flux on the outer surface, which is always possible. One will not need to consider the entire external space for bending and topological alterations.

One can solve the corresponding problem for stress field that satisfies (1.3), namely, the determination of the associated stress function tensor $\underline{\chi}$, by a double application of (2.20). One first decomposes $\underline{\sigma}$ into constant right-vectors and by calculating the vector potential of the left-vectors from (2.20), one will get the asymmetric first-order stress function:

$$\underline{\varphi} = \underline{\chi} \times \nabla, \tag{2.21}$$

and indeed in all of infinite space. If one has introduced closing fluxes for the left-vectors on the outer surface by this calculation then the first-order stress function tensor will arise there. One now decomposes it into constant left-vectors and calculates the vector potential for the new right-vectors, from which, one will obtain the (second-order) symmetric stress function tensor $\underline{\chi}$. Now, since no further integrations need to be performed, one can isolate the field of stress functions on the outer surface and set it equal to zero in the external space. Upon differentiating it, that discontinuity on the outer surface will lead to simple and double singular stress distributions that can be interpreted intuitively as the stress state in an infinitely-thin shell that envelops the whole region and maintains the stress state in the enclosed elastic body in place of the outer surface forces, which drop out. The load stress state of the elastic body that is enclosed by the outer surface of the region is then replaced with the proper stress state of the elastic body, which is then extended to the "crustal hull." The crustal hull is obviously the elastic analogue of the singular outer surface flux in the theory of vectors.

When it becomes necessary to calculate directly the stress state in the crustal hull or any statically-equivalent stress state in a shell, one can also perform the calculation of the stress function tensor in a single integration step. In order to do that, one sets:

$$\underline{\chi} = \mathrm{Ink}\ \underline{\psi}, \tag{2.22}$$

with the auxiliary condition:

$$\nabla \cdot \underline{\psi} = 0, \tag{2.23}$$

and then obtains the equation:

$$\Delta\Delta \underline{\psi} = \underline{\sigma}, \tag{2.24}$$

as long as $\underline{\sigma}$ is symmetric and (1.3) is fulfilled in all of space, which can be arranged by the addition of singular stress state on the crustal hull. One gets the solution:

$$\underline{\chi}(\mathfrak{r}) = -\frac{1}{8\pi}\mathrm{Ink}\int |\mathfrak{r} - \mathfrak{r}'|\, \underline{\sigma}(\mathfrak{r}')\, d\tau', \tag{2.25}$$

or after performing the differentiations and from a conversion that is given in the Appendix:

$$\underline{\chi}(\mathfrak{r}) = -\frac{1}{8\pi}\int \left\{ \frac{1}{|\mathfrak{r}-\mathfrak{r}'|}[\underline{\tilde{\sigma}}(\mathfrak{r}') - \sigma_{\mathrm{I}}(\mathfrak{r}')\mathbf{I}] - \frac{1}{|\mathfrak{r}-\mathfrak{r}'|^3}(\mathfrak{r}-\mathfrak{r}')\times\underline{\tilde{\sigma}}(\mathfrak{r}')\times(\mathfrak{r}-\mathfrak{r}') \right\} d\tau', \tag{2.26}$$

in which one naturally has:

$$\underline{\tilde{\sigma}} = \underline{\sigma} \tag{2.27}$$

due to the symmetry of the stress tensor.

One can regard the crustal hull as a kind of static counterpart to **Schaefer**'s crustal shell [**10**]. The forces and moments are equal and opposite on crustal shells and crustal hulls. However, whereas for the crustal shell, the asymmetric stresses and moment stresses of the general **Cosserat** continuum [**21**] are allowed, and even requisite, along with the ordinary stresses, since otherwise the crustal shell could not replace the actual elastic body, in general, for the crustal hull that is added later on, from the Ansatz (1.2), only stresses that can be realized by singular simple and double coverings with symmetric stresses are allowed. Moreover, one can assume the elastic properties of the crustal hull arbitrarily. From the remarks in conjunction with (2.21), there is no doubt regarding the existence of such singular stress states.

c) *Summary.* – An unperturbed stress and strain field in a simply-connected and simply-bounded spatial region can be derived from displacement functions, as well as stress functions, which are determined uniquely in their own right, up to inessential contributions (rigid rotation, deformator, resp.).

3. Displacement functions in multiply-connected and multiply-bounded domains.

In this section, we will study the extent to which the uniqueness and existence proofs of § **2** for displacement functions must be modified for general regions. In essence, things that are known will be repeated in it. However, the theorems shall be summarized once more for the sake of comparison with the properties of stress functions, and will also be partially interpreted from other viewpoints.

First of all, the uniqueness proof (2.4) for the displacement vector that relates to an elastic strain that is given as a deformator will remain unchanged, since the integrand will also vanish in that case. By contrast, the uniqueness proof that relates to the boundary-value problem is valid only for the contribution $\mathfrak{u}^K$ of the outer surface forces, which are either given directly or can be determined later from the given displacements on the outer surface [the linear relation (2.1a)]. That is because even if the elastic strain can be derived formally from a vector field $\mathfrak{u}$ using (1.9) in the entire unperturbed region, $\mathfrak{u}$ no longer needs to be the actual displacement, since its spatial uniqueness is no longer guaranteed for multiple connectivity. That follows from (2.7) and the existence of contours that are entirely inside the internal region and cannot be spanned by simply-connected surfaces without them extending into the external region. However, (1.10) does not have to be fulfilled then; hence, $\mathfrak{b}$ ($\mathfrak{r}$) must not vanish either. $\mathfrak{u}$ ($\mathfrak{r}$) can then be multiple-valued; a simple conversion of (2.7):

$$\mathfrak{b}\ (\mathfrak{r}) = \int d\mathfrak{f}' \cdot \underline{\eta}(\mathfrak{r}') \times \mathfrak{r}' + \mathfrak{r} \times \int [d\mathfrak{f}' \cdot \underline{\eta}(\mathfrak{r}')] \tag{3.1}$$

will give one information about that kind of multiple-valuedness. After a complete circuit, the first part of this yields the constant "displacement jump" of the **Volterra** distortion state of the first kind (i.e., ordinary dislocations), while the second will give the rigid rotation ("rotation jump") of the **Volterra** distortion state of the second kind, corresponding to a small-angle grain boundary in the unperturbed region [**2**, **3**, **30**, **31**,

43]. The displacement jump then represents the “moment,” while the rotation jump represents the sum of all incompatibility lines that pierce the surface. From the rules of kinematics, the splitting of the rigid motion jump into a displacement and a rotation can be altered by a new choice of coordinate origin or the introduction of $\mathfrak{r} - \mathfrak{r}_0$ in place of $\mathfrak{r}$.

Naturally, a multiple-valued displacement is not realizable, and thus physically meaningless. One solves the contrary case by introducing an supplementary strain $\underline{\varepsilon}^Q$, which extends the elastic strain $\underline{\varepsilon} = \underline{\varepsilon}^E$ to the total strain $\underline{\varepsilon}^G$ that is derived from a true, single-valued displacement $\mathfrak{u}^G$ [**2**, **5**, **6**, **7**]:

$$\underline{\varepsilon}^G = \text{Def } \mathfrak{u}^G = \underline{\varepsilon}^Q + \underline{\varepsilon}^E . \tag{3.2}$$

The physical origin and meaning of the supplementary strain can be given more precisely by means of a more precise knowledge of the history of the material (e.g., plastic deformation, uneven heating, magnetization, electrification) (*). However, since only stress and elastic strain are of interest at the moment, they can also be regarded formally and varied at will around a deformator that is derived from a single-valued vector field, as long as one only drops the displacement jump and rotation jump in comparison to the elastic strain, so for any circuit in the entire unperturbed region, one will have:

$$\int d\mathfrak{f}' \cdot (\nabla' \times \underline{\varepsilon}^Q(\mathfrak{r}') \times \nabla') \times (\mathfrak{r}' - \mathfrak{r}) + \int d\mathfrak{f}' \cdot \underline{\eta}(\mathfrak{r}') \times (\mathfrak{r}' - \mathfrak{r}) = 0, \tag{3.3}$$

in which the integral is extended over a simply-connected surface that spans the contour.

One solves the equation:

$$\text{Div } [\underline{c} \cdot (\text{Def } \mathfrak{u} - \underline{\varepsilon}^Q)] = 0, \tag{3.4}$$

instead of (2.2), with the outer surface condition [cf., (2.2a)]:

$$\mathfrak{n} \cdot [\underline{c} \cdot (\text{Def } \mathfrak{u} - \underline{\varepsilon}^Q)] = \mathfrak{P}_0 - \oint \underline{C}(\mathfrak{r}, \mathfrak{r}') \cdot \mathfrak{u}(\mathfrak{r}')\, d\mathfrak{f}' \tag{3.4a}$$

and then subtracts the supplementary strain $\underline{\varepsilon}^Q$ in order to determine $\underline{\varepsilon}^E$. This will happen most simply when one sets the supplementary strain equal to zero everywhere, up to a singular surface, as in Fig. 6, pp. 7, where it will become infinite, like a sort of **Dirac** delta function, and one must avoid that surface in the derivation of the total strain from the displacement field. The elastic strain will then be identical with the total strain, up to the singular surface.

If one addresses the part on the right-hand side of (3.4):

$$\mathfrak{q} = -\text{ Div } (\underline{c} \cdot\cdot \underline{\varepsilon}^Q) \tag{3.5}$$

or

$$\mathfrak{Q} = \mathfrak{n} \cdot (\underline{c} \cdot\cdot \underline{\varepsilon}^Q), \tag{3.5a}$$

(*) In general, the supplementary strain also makes a stress-free contribution to the total displacement [**2**, **30**, **31**], but it is not interesting in the present context.

resp., then it can be regarded as the "internal force density" [**6**] and the inhomogeneous differential equation:

$$\mathrm{Div}\,(\underline{c}\cdot\cdot\,\mathrm{Def}\ \mathfrak{u}) = -\,\mathfrak{q} \tag{3.6}$$

will enter in place of (2.2), with the outer surface condition:

$$\mathfrak{u}\cdot(\underline{c}\cdot\cdot\,\mathrm{Def}\ \mathfrak{u}) = \mathfrak{Q} + \mathfrak{P}_0 - \oint \underline{C}\,(\mathfrak{r},\mathfrak{r}')\cdot\mathfrak{u}\,(\mathfrak{r}')\,d\mathfrak{f}'\,. \tag{3.6a}$$

Since these relations are linear, the contribution of the outer surface forces to the stress and strain fields can be separated from the contribution of the **Volterra** distortion state. Likewise, the subdivision of the problem into a summation problem in an infinite medium (i.e., inhomogeneous differential equation) and a boundary-value problem (homogeneous differential equation with boundary condition) is always possible [**2**].

Duhamel and **Neumann** have already solved the heat-stress problem by employing the internal force density back in 1840 ([**2**], pp. 58). If one assembles the supplementary strain on a singular surface then the internal force density will degenerate into a double-layer; **Burgers** [**2**, **8**] has determined the stress field of a ring dislocation with the help of that picture.

The uniqueness proof that relates to the boundary-value problem for the displacement will not be affected by a multiple boundary, since the boundary condition (2.2a) are also valid on the outer surface of a cavity. The same thing is also true for the proof of spatial uniqueness, since in a simply-connected region, every simply-connected surface that is bounded by a circuit can also be shifted completely inside the internal region by a suitable deformation when the boundary is multiple.

The analogue in the theory of vector fields is obviously the potential in a multiply-connected, source-free region, in which the holes in the pretzel can be crossed by vortex filaments. The methods of magnetic or electric double-layers that correspond to the introduction of singular supplementary strain do not need to be dealt with any further here.

4. Stress functions in simply-connected, multiply-bounded domains.

Once one assumes the existence of a stress function tensor, the uniqueness proof in § **2** can be adopted with no changes. Equations (2.9) to (2.18) are valid, moreover, for the internal region and each individual outer surface, and the topologically-definitive step in the proof – namely, the contraction of the integration circuit to a point – will be true due to the simple-connectivity in each individual outer surface. By contrast, the existence proof will be affected by multiple connectivity. A stress function will exist only when the forces on each individual outer surface are in equilibrium. Nevertheless, a substitute solution can always be given, but it is generally employable only with certain restrictions. We shall next consider the analogous case in the theory of vector fields.

a) *Vector potential in multiply-bounded domains.* – From § **2**, the exhibition of a vector potential by (2.20), and therefore, due to the known uniqueness up to a gradient, the exhibition of a vector potential at all, is possible only when the flux tubes can be closed in the external space. However, if one finds, e.g., a point source in a cavity then that closure in the external space will no longer be possible; at the very least, a singular flux tube must penetrate the vortex-free and source-free region.

Examples are easy to give. In the plane, the vector potential in, say, polar coordinates ρ, φ [**15**]:

$$\mathfrak{B}_1 = \varphi\, t \tag{4.1}$$

leads to the flux field of the planar point-source:

$$\mathfrak{v}_1 = \text{rot}\,\mathfrak{B}_1 = \mathfrak{e}_\rho \tag{4.2}$$

in each arbitrary annular region around the origin, with the exception of a separation line along which the jump in φ, which is required to preserve the single-valuedness of $\mathfrak{B}_1$, is laid. In the flux picture, that separation line can be interpreted as a singular flux tube that includes the backflow of the fluid that passes through the annular region. In spatial cylindrical coordinates [**15**], the same field can be interpreted as the flux of a line-source along the z-axis with back-flux that is singular on a surface. Another vector potential for the same flux vector is:

$$\mathfrak{B}_1' = -\frac{z}{\rho}\,\mathfrak{e}_\rho\,. \tag{4.3}$$

Its singularity is a vortex filament that goes to infinity in both directions and is linearly-decreasing in magnitude at the origin. It obviously represents the influx from the infinitely-distance cylinder ends, which exhausts the supply from the line-source, except at the center at the origin. This example shows the topological multiple-valuedness of a planar structure when considered spatially. If one imagines that the cylinder ends are both closed then one will have double-connectedness. One can even present the spatial structure as a simply-connected and simply-bounded body by closing one end and leaving the other one open, but that way of looking at things is meaningless in practice, especially since one would be led to a divergent vector potential.

In spatial spherical coordinates r, φ, ϑ [**15**], the vector potential:

$$\mathfrak{B}_2 = \frac{\cot\vartheta}{r}\,\mathfrak{e}_\varphi = \frac{z}{r\rho}\,\mathfrak{e}_\varphi \tag{4.4}$$

will lead to the flux field of the spatial point-source:

$$\mathfrak{v}_2 = \frac{1}{r^2}\,\mathfrak{e}_r \tag{4.5}$$

in each hollow spherical region around the origin (cf., also [**35**], Prob. 13, pp. 109). The singularities to be avoided lie on the positive and negative z-axis, as one will see from the second form of (4.4), which is a mixture of the cylindrical and spherical coordinates. There are singular vortex tubes of different sign for the vector potential, and therefore singular flux tubes for the flux vector through which the point-source is fed. If one discards one of those flux tubes in favor of the other one by adding a new vortex filament along the entire z-axis then the singularity of the new vector potential for (4.5)

$$\mathfrak{B}_2' = \left(\frac{\cot\vartheta}{r} + \frac{1}{\rho}\right)\mathfrak{e}_\varphi \tag{4.6}$$

will lie along only the positive z-axis. In general, in order to represent a gradient in a multiply-connected source-free region as a vortex of a vector potential in the region, one must drill through it enough times that the various outer surfaces will unite into a single one. The flux tubes can then be closed by means of the connecting hoses that are so arranged, as long as that is not possible already in the individual surfaces.

b) *Stress functions in multiply-connected domains.* – The non-vanishing productivity of the sources in a cavity in the theory of vector fields corresponds to a result dyname in the theory of elasticity. A representation by stress functions is not possible when not every individual outer surface of the unperturbed region is found to be in equilibrium. In the other case, there will be a contradiction when one applies the train of thought in § **1**.b to the individual outer surfaces. The existence proof (2.21-27) will also break down, since the multiple boundary of the unperturbed region can no longer be surrounded with a connected crustal hull that does not penetrate the internal region. The presentation of a stress function tensor from which the actual stress would follow from (1.2) is possible if, and generally only if, one removes a sufficient number of connecting hoses from the unperturbed region as in § **1**.d (cf., Fig. 5, pp. 7). The uniqueness proof (2.11-18) is true for simple-connectivity independently of the bounding number, since the individual integration circuits can be contracted to a point on each outer surface; it will first be perturbed for multiple-connectivity (cf., § **5**). By contrast, the existence proof is independent of the connection number.

Now, in order to find a stress function tensor in an unperturbed region that has been drilled through, we make use of the possibility of replacing the given external forces with fictitious internal forces [**5**, **6**, **7**, **38**], and thus with the divergence vector of a fictitious supplementary stress field $\underline{\sigma}^{K*}$. These fictitious internal forces will then be equivalent to the actual external forces, not relative to the resultant total stress, but probably relative to the displacement that enters in. We then assume a fictitious equilibrium stress:

$$\underline{\sigma}^{G*} = \text{Ink}\,\underline{\chi}\,, \tag{4.7}$$

which is coupled with the total stress $\underline{\sigma}^{G}$ that actually occurs by the relation:

$$\underline{\sigma}^{G*} = \underline{\sigma}^{K*} + \underline{\sigma}^{G}\,. \tag{4.8}$$

$\underline{\sigma}^{G*}$ will then be calculated by the method that was developed for the proper stress problem (cf., §§ **7** and **8**), and $\underline{\sigma}^{K*}$ will once more be subsequently subtracted. The load-stress problem will then be replaced by a fictitious proper stress problem ([1]).

Now, there are infinitely many fictitious supplementary stress fields for a given field of external forces; i.e., the total stress field that actually appears, but with the opposite sign, will belong to them. However, the practical advantage of these auxiliary quantities is that due to the greater solubility of the fictitious proper stress problems, one also gets to deal with simpler tensor fields. Since we are interested in only the unperturbed region itself in the present case, when the fictitious internal forces possess the same resultant dyname as the actual external forces only on each individual outer surface, and thus, for each individual bounding surface, it will even suffice for one to have:

$$-\oint d\mathfrak{f}\cdot\underline{\sigma}^{K*} = \oint \mathfrak{P}\,df \tag{4.9}$$

and

$$\oint d\mathfrak{f}\cdot\underline{\sigma}^{K*}\times(\mathfrak{r}-\mathfrak{r}_0) = -\oint \mathfrak{P}\times(\mathfrak{r}-\mathfrak{r}_0)\,df. \tag{4.10}$$

Equilibrium will then be established by the fictitious internal forces on each outer surface, and the rest of the problem can then be resolved as a boundary-value problem by the method of stress functions. One will obtain the solution of (4.9-10) that is most convenient for applications by contracting the fictitious supplementary stresses to singular lines that one removes from the stress calculation by differentiation, just as one does with the singular backflow lines in (4.1-6). The fictitious supplementary stress will then be restricted to infinitely-thin connecting hoses between the individual bounding surfaces and will no longer be recognizable in the stress field when one replaces its values at the exceptional places with the boundary values in the approximation that prevails in the neighborhood, as one usually does. Of course, the fictitious supplementary stresses will once more appear immediately when one seeks to construct new solutions by continuous superpositions of such solutions. If one would then like to contract the fictitious supplementary stress once more then that would require the solution of an additional proper stress problem.

If one would then like to examine the singularities of the fictitious supplementary stress field, or – what amounts to the same thing – the singularities of the fictitious

---

([1]) The fictitious supplementary stresses $\underline{\sigma}^{K*}$ cannot be confused with the supplementary stresses $\underline{\sigma}^{Q}$ that arise in reality, such as heat strain, magneto-striction, plastic deformation, etc., which are connected with the supplementary strain by $\underline{\sigma}^{Q} = -\underline{c}\cdot\cdot\underline{\varepsilon}^{Q}$, and yield the total stress $\underline{\sigma}^{G} = \underline{\sigma}^{Q} + \underline{\sigma}^{D}$ when combined with the strain-induced stress $\underline{\sigma}^{D}$ [**5**, **7**]. There exists no reason, *loc. cit.*, for the introduction of $\underline{\sigma}^{K*}$ and $\underline{\sigma}^{G*}$, since only proper stresses will be considered, while $\underline{\sigma}^{D}$ is identical to $\underline{\sigma}^{G}$ here, due the fact that $\underline{\sigma}^{Q}$ = 0 in the unperturbed region, and thus does not need to be introduced specially. In regard to that, one should also confer the example that was cited in another place [**38**] of a horizontal place with diffusive dilatation centers in a gravitational field.

The load-stress problem with volume forces can also be made tractable with the help of fictitious supplementary stresses. The known special case in which the volume forces are derived from a potential [**27**] is distinguished by the fact that the determination of a fictitious supplementary stress field is possible with no special integration.

equilibrium field, then one can no longer employ the smoothed-out stress field for that, but one must immediately return to the stress functions. One then proceeds similarly to what one does in the theory of vector fields, in which one determines the boundary value of the contour integral:

$$S = \oint d\mathfrak{r} \cdot \mathfrak{B} \tag{4.11}$$

by contracting the contour around the singular line, along with the strength of the vortex line of the vector potential, i.e., the singular flux line of the flux vectors. In the theory of elasticity, the boundary-values of integrals (1.5) and (1.6) in **Peretti** and **Günther** enter in place of (4.11), and their application in the interior of the elastic body can be based upon the following Gedankenexperiment:

We appeal to the known definition of the total stress, according to which:

$$d\vec{P} = d\mathfrak{f} \cdot \underline{\sigma}^G , \tag{4.12}$$

is the force that one must apply to every arbitrary, oriented surface element $d\mathfrak{f}$ of an edge to any arbitrary intersection surface in the interior of the stressed body in order to maintain the given stress state. If we now imagine that such a finite cut has actually been made completely in the interior and that the forces that are required to maintain the stress state have been applied to both edges of the cut (Fig. 7) then that will mean nothing besides a new closed outer surface in the interior whose surface loading possesses the resultant dyname. One can then also replace the force distribution by a closed crustal hull as in § **2** b. The fact that the enclosed cavity is infinitely-thin plays no role in this case. If we then replace the total stress with the fictitious equilibrium stress that is derivable from a stress function tensor using (4.7) then, according to **Schaefer** [**10**], all of the surrounding material can be replaced with a crustal shell, and the assumptions have been made for the application of the **Peretti-Günther** contour integral (1.5-6) for the calculation of the forces and moments on the enclosed section of the edge of the cut. If one contracts the integration circuit to a regular point of $\underline{\sigma}^{G*}$ then the integral will tend to zero. However, by contracting to the piercing point of a singular line, one can infer conclusions about the type of fictitious singular supplementary stresses from the boundary value of the resultant dyname.

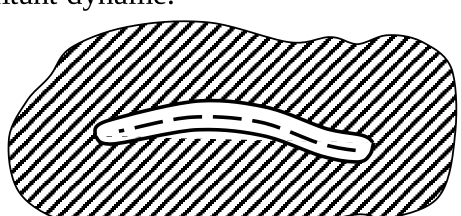


Figure 7. The interpretation of stress functions in the interior.
– – – – – cut surface ———— crustal shell (hull, resp.)

c) *Planar problems in a spatial context.* In regard to the peculiarities of planar stress (distortion, resp.) problems, we must especially consider the case in which the connecting hose does not contract to a singular line, but to a singular surface whose boundary can partially lie at infinity. We first consider Fig. 5, pp. 7. Since equation (2.10) will be valid everywhere in the unperturbed domain, with the exception of the singular surface that is bounded by the curved rectangle *ABCD*, one can calculate a fictitious elastic strain:

$$\underline{\varepsilon}^{E*} = \underline{s} \cdot \cdot \operatorname{Ink} \underline{\chi} \tag{4.13}$$

that is coupled to the fictitious equilibrium stress (4.7) by **Hooke**'s law, which must, however, be identical with the total strain $\underline{\varepsilon}^G$ that actually appears in the entire unperturbed region, when only the following assumptions are fulfilled:

1. The elastic strain in the outer surface of the flattened connection hose must be equal on both sides of the singular surface. By the continuity of the elastic constants, it will suffice for that to be true when Ink $\underline{\chi}$ has the same boundary value on both sides. If we next exclude the boundary of the curved rectangle *ABCD* then a **Volterra** state of distortion state can still remain; i.e., dislocation lines of type 1 and 2 traverse the boundary of the singular surface.

2. Obviously, the **Volterra** distortion state must also vanish, since otherwise the assumption of simple-connectivity of the unperturbed region would no longer be fulfilled. The two pieces of the boundary $\overline{AB}$ and $\overline{CD}$ will then penetrate the original simply-connected region and make it doubly-connected. However, by the continuity of the elastic constants, dislocation lines will make themselves known at those places by means of infinitely-high boundary values for the stress. If the curved rectangle lies completely at finite points then it will suffice in that case to also extend the continuity requirement for the boundary values of Ink $\underline{\chi}$ to the boundary.

3. However, that simple criterion will break down when the unperturbed region degenerates into an infinitely-long hose that is closed on both ends and the two bounding lines $\overline{AB}$ and $\overline{CD}$ go off to infinity in both directions. Their stress fields will then no longer exhibit discontinuities at finite points, and one will have only the possibility of calculating the displacement and rotation jumps that are present by (2.7) and (3.1); i.e., one must proceed precisely as in the case of double-connectedness and consider the infinitely-long hose as open at both ends. That is, one can now no longer distinguish between multiple-connectedness and multiple-boundedness.

That is true especially for planar problems, as was pointed out before in conjunction with (4.3). By introducing the **Airy** stress functions, which are independent of *z*, the planar state of distortion will appear to be the strain state of an infinitely-long cylinder, and the planar stress state can be interpreted in the same way when one alters the elastic constants without abandoning isotropy in such a way that the stress component σ_{zz} produces no further elastic strain. In fact, the integral conditions for the load-free boundary of a planar annulus were presented by **Michell** [**16**] on the basis of the single-valuedness requirement for the displacements. The fact that, from **Prager** [**17**], they appear as the natural boundary conditions for the variational problem, in addition, will also become comprehensible from a theorem of **Colonetti** [**20**] (cf., [**2**], pp. 62) that says that the energies of the proper stresses and the load stress can be added independently of each other. The exclusion of dislocations will then bring with it simply the vanishing of the proper stress component of the elastic energy.

The stress state of a plate can, in principle, be regarded as a section of the stress state in an infinitely-long cylinder, in general, due to outward-bending stresses that increase beyond all limits only under all kinds of imaginative assumptions about the loading and the *E*-modulus. What makes more sense is the introduction of the plate stress functions by **Schaefer** [**19**] as the jump in a null-stress function field (i.e., a deformator field). In that way, one will restrict the essential part of the stress function field to the singular jump surface. If one now carries the integration contour away to both sides of the singular surface ([**19**], Fig. 1) then one will decide in favor of the concept of double-connectivity for a planar annulus, and the same concept will also correspond to equation (25) in **Schaefer** [**18**] for a plate with an unloaded boundary. By contrast, the case of the disc with rigid inclusions, which is analogous according to [**18**], can be regarded as a case of double-connectedness (cf., § **5**), as well as double-boundedness. The second interpretation is expressed by saying that every path that is above or below the singular surface is forbidden, and a path in the surface through the annulus will be regarded as penetrating the interior region.

5. Stress functions in multiply-connected domains.

In the foregoing section, it was shown that it is not always possible to exhibit a stress function tensor in multiply-bounded unperturbed regions, while the proof of single-valuedness will not be affected by multiple-boundedness. Conversely, the existence proof will not be affected by multiple-connectedness, since a multiply-connected outer surface can be surrounded by a closed crustal hull [cf., (2.21), *et seq.*]. By contrast, the proof of single-valuedness will break down, since from (2.14), the decisive step in the proof, namely, the contraction of the integration contour to a point will not longer be possible for multiple-connectivity. A more detailed examination will show that, as could be expected, this is based upon the possibility of a **Volterra** distortion state appearing. An analogy that **Schaefer** examined many times [**18**, **19**, **39**] between null-stress functions and distortions can be extended to a spatial one in that way.

a) *The uniqueness proof for multiple-connectedness.* – A glimpse at (2.14) will show that the null-stress functions and the difference vector $\mathfrak{u}^-$ of the displacements will appear in precisely the same way in the contour integral of the generating vector $\mathfrak{A}^-$. We then combine the analogous equations for $\mathfrak{A}^-$ and $\mathfrak{u}^-$ once more and observe that due to multiple-connectivity, neither the spatial single-valuedness of $\mathfrak{A}^-$ nor that of $\mathfrak{u}^-$ can be assumed, so $\mathfrak{u}^-$ no longer must be an unconditionally physically-realizable, single-valued displacement. The axis of the rotational jump will go through the point with the position vector $\mathfrak{r}_0$, instead of the origin, as we have done up to now. The displacement field:

$$\mathfrak{u}^-(\mathrm{r}) = \int_{\mathfrak{r}_0}^{\mathfrak{r}} d\mathfrak{r}'\cdot\{\underline{\varepsilon}^-(\mathfrak{r}')+(\underline{\varepsilon}^-(\mathfrak{r}')\times\nabla')\times(\mathfrak{r}'-\mathfrak{r})\} \tag{5.1}$$

is associated with the displacement jump around a complete circuit of the edge 1 of the cut to edge 2:

$$\mathfrak{u}^-(\mathfrak{r}) = \mathfrak{u}_2^- - \mathfrak{u}_1^- = \mathfrak{b}_0^- + \mathfrak{d}^- \times (\mathfrak{r} - \mathfrak{r}_0), \tag{5.2}$$

with

$$\mathfrak{b}_0^- = \int d\mathfrak{f}'\cdot(\mathrm{Ink}\,\underline{\varepsilon}^-(\mathfrak{r}'))\times(\mathfrak{r}'-\mathfrak{r}_0) \tag{5.3}$$

and

$$\mathfrak{d}^- = -\int d\mathfrak{f}'\cdot\mathrm{Ink}\,\underline{\varepsilon}^-(\mathfrak{r}')\,. \tag{5.4}$$

If one lays the circuit on the outer surface of the region then due to the compatibility condition (1.10), the expressions (5.2-5.4) will vanish when the circuit encloses the interior region as in Fig. 8b. Only a circuit around the external region as in Fig. 8c can deliver a displacement and rotation jump.

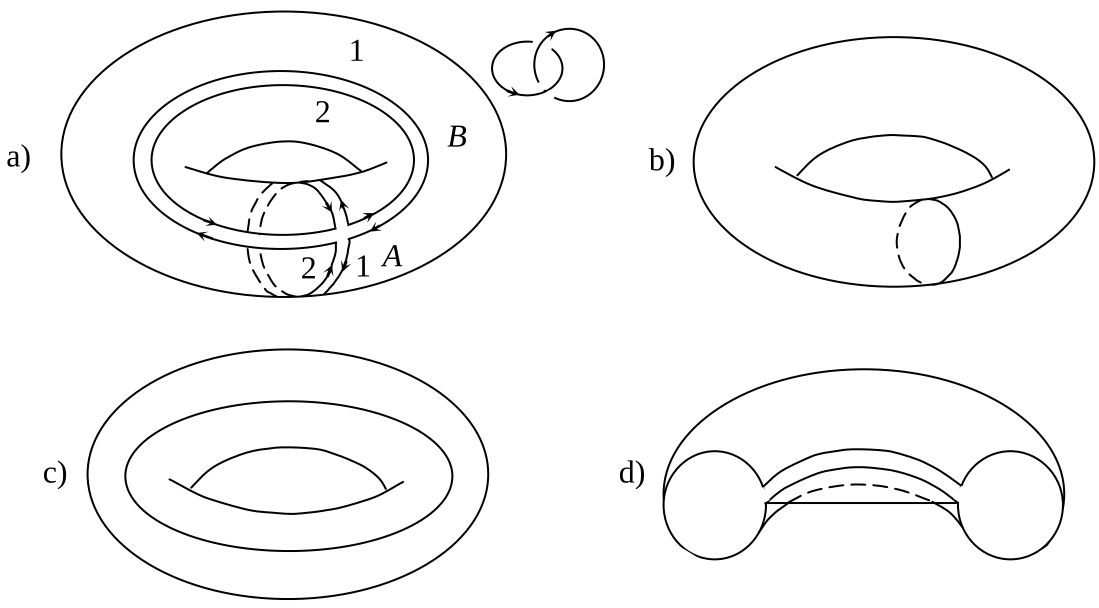


Figure 8. The crustal shell of the doubly-connected region.

a) Circuit around the outer surface, after canonical decomposition.
b) Circuit around the internal region.
c) Circuit around the external region.
d) Extension to simply-bounded shells by a surface that does not cut the interior (cross-section).

Correspondingly, the vector:

$$\mathfrak{A}^{-}(\mathfrak{r}) = \int_{\mathfrak{r}_0}^{\mathfrak{r}} d\mathfrak{r}' \cdot \{\underline{\chi}^{-}(\mathfrak{r}') + (\underline{\chi}^{-}(\mathfrak{r}') \times \nabla') \times (\mathfrak{r}' - \mathfrak{r})\} \tag{5.5}$$

can jump by the negative moment relative to $\mathfrak{r}$:

$$-\vec{M}^{-}(\mathfrak{r}) = \mathfrak{A}_2^{-} - \mathfrak{A}_1^{-} = -\vec{M}_0^{-} - \vec{P}^{-} \times (\mathfrak{r} - \mathfrak{r}_0) \tag{5.6}$$

after a complete circuit from the cut edge 1 to the cut edge 2, from which, according to **Peretti** [**22**] and **Günther** [**13**]:

$$\vec{M}_0^{-} = -\int d\mathfrak{f}' \cdot (\text{Ink}\, \underline{\chi}^{-}(\mathfrak{r}')) \times (\mathfrak{r}' - \mathfrak{r}_0) \tag{5.7}$$

is the moment relative to $\mathfrak{r}_0$, and:

$$\vec{P}^{-} = \int d\mathfrak{f}' \cdot \text{Ink}\, \underline{\chi}^{-}(\mathfrak{r}') \tag{5.8}$$

is the force on the surface patch that is bounded by the circuit. If the forces that act upon the outer surface are in equilibrium then (5.6-5.8) must vanish for a circuit that is defined by a canonical decomposition [**4**] (cf., Fig. 8a for the torus), when it yields the correct loading on the outer surface (*); above all, that will be true for the difference tensor $\underline{\chi}^{-}$.

However, these expressions also vanish for a circuit on the outer surface around the external region (Fig. 8c). One then cuts up the outer surface of the torus into such a circuit and spans a cut edge with a simply-connected, unloaded surface (Fig. 8d), which then extends the dissected toral outer surface torus to a simply-connected surface that is bounded by the other cut edge, so (5.6) will yield (up to sign) the resultant dyname of all forces on that surface on that cut edge, and therefore zero. $\mathfrak{A}^{-}$ can then jump only on a circuit around the internal region (Fig. 8b). From Fig. 6, pp. 7, (5.6) will then once more give (up to sign) the dyname that the cut-edge that is traversed in the positive sense experiences from the other cut-edge for a cut surface. For higher connectivity numbers, one previously makes the pretzel doubly-connected by a suitable decomposition and applies the same argument to the remaining hole.

In this analogy, one has the following correspondences:

$$\mathfrak{b}_0^{-} \leftrightarrow -\vec{M}_0^{-} \quad \text{and} \quad \mathfrak{d}^{-} \leftrightarrow -\vec{P}^{-}. \tag{5.9}$$

The quantities on the left-hand sides can be non-zero only for a circuit around the external region, while the quantities on the right-hand sides can be non-zero only for a circuit around the internal region.

The contour integral (2.14) will now be taken along the boundary to the simply-connected surface that one obtains by a canonical decomposition [**4**] of the multiply-connected outer surface to the region, and indeed, as usual, in the positive sense of

(*) In the case of a multiple boundary, one must, if necessary, have previously reduced the region to a simple boundary by means of connection hoses.

traversal, which makes the surface that is being circumscribed lie to the left (Fig. 8a). The entire circuit can then be composed of double circuits, one of which encloses one-half of the internal region (viz., double circuit *A*), while the other one encloses one-half of the external region (viz., double circuit *B*). The two sub-circuits of a double circuit will run in opposite directions. The numeration of the two sub-circuits will be chosen for each double circuit *A* such that circuit from 1 to 2 around the external region defines a right-handed screw with the circuit around the inner region, which leads from sub-circuit 1 of the double circuit *B* to sub-circuit 2 (Fig. 8a). For a higher connectivity number, the double circuits *A* and *B* will be associated with each other pair-wise in such a way that each double circuit *A* will penetrate the hole in the pretzel that enclosed the associated double circuit *B*.

The contribution of a double circuit will certainly vanish when the vectors $\mathfrak{A}^-$ and $\mathfrak{u}^-$ have the same values on both sub-circuits. It is only when $\vec{M}^-(\mathfrak{r})$ [$\mathfrak{b}^-(\mathfrak{r})$, resp.] are non-zero that it will contribute to (2.14). For the double circuit *A*, the sub-circuit 1 will be traversed in the positive sense, while sub-circuit 2 will be traversed in the negative sense. Since the vector $\mathfrak{A}^-$ cannot jump between these two edges, it will follow when one substitutes (5.2) in (2.14) that:

$$\begin{aligned} L^A &= \tfrac{1}{8}\oint d\mathfrak{r}\cdot\{-2(\mathrm{Def}\,\mathfrak{A}^-)\cdot[\nabla\times(\mathfrak{u}_1^- - \mathfrak{u}_2^-)] + 2[\mathrm{Def}\,(\mathfrak{u}_1^- - \mathfrak{u}_2^-)]\cdot(\nabla\times\mathfrak{A}^-) \\ &\qquad + (\nabla\times\mathfrak{A}^-)\cdot[\nabla\times(\mathfrak{u}_1^- - \mathfrak{u}_2^-)]\} \\ &= \tfrac{1}{8}\oint d\mathfrak{r}\cdot\{2(\mathrm{Def}\,\mathfrak{A}^-)\cdot(\nabla\times\mathfrak{b}^-(\mathfrak{r})) - 2(\mathrm{Def}\,\mathfrak{b}^-(\mathfrak{r}))\cdot(\nabla\times\mathfrak{A}^-) \\ &\qquad + (\nabla\times\mathfrak{A}^-)\times[\nabla\times\mathfrak{u}^-(\mathfrak{r})]\}. \end{aligned} \tag{5.10}$$

For the double circuit *B*, the sub-circuit 2 will be traversed in the positive sense, while for sub-circuit 2, it will be traversed in the negative sense. Since the vector $\mathfrak{u}^-$ cannot jump between these two edges, it will follow upon substituting (5.6) in (2.14) that:

$$\begin{aligned} L^B &= \tfrac{1}{8}\oint d\mathfrak{r}\cdot\{-2[\mathrm{Def}\,(\mathfrak{A}_2^- - \mathfrak{A}_1^-)]\cdot(\nabla\times\mathfrak{u}^-) + 2(\mathrm{Def}\,\mathfrak{u}^-)\cdot[\nabla\times(\mathfrak{A}_2^- - \mathfrak{A}_1^-)] \\ &\qquad + [\nabla\times(\mathfrak{A}_2^- - \mathfrak{A}_1^-)]\cdot(\nabla\times\mathfrak{u}^-)\} \\ &= \tfrac{1}{8}\oint d\mathfrak{r}\cdot\{2(\mathrm{Def}\,\vec{M}^-)\cdot(\nabla\times\mathfrak{u}^-) - 2(\mathrm{Def}\,\mathfrak{u}^-)\cdot(\nabla\times\vec{M}^-(\mathfrak{r})) \\ &\qquad - (\nabla\times\vec{M}^-(\mathfrak{r}))\times(\nabla\times\mathfrak{u}^-)\}. \end{aligned} \tag{5.11}$$

Now, for all circuits, the vector $d\mathfrak{r}$ certainly always lies tangential to each surface that spans a circuit, so, from (5.2), one will have:

$$d\mathfrak{r}\cdot\mathrm{Def}\,\mathfrak{b}^-(\mathfrak{r}) = 0 \tag{5.12}$$

for the double circuit *A*, and from (5.6), one will have:

$$d\mathfrak{r}\cdot\text{Def }\vec{M}^{-}(\mathfrak{r}) = 0, \tag{5.13}$$

and furthermore, for each surface that is spanned by the double circuit *A*, one will have:

$$\nabla\times\mathfrak{b}^{-}(\mathfrak{r}) = 2\;\mathfrak{d}^{-}, \tag{5.14}$$

and for each surface that is spanned by the double circuit *B*:

$$\nabla\times\vec{M}^{-}(\mathfrak{r}) = 2\vec{P}. \tag{5.15}$$

Substitution in (5.10) and (5.11) will yield:

$$\begin{aligned} L^A &= \tfrac{1}{8}\oint d\mathfrak{r}\cdot\{2(\text{Def }\mathfrak{A}^{-})\cdot 2\mathfrak{d}^{-} - (\nabla\times\mathfrak{A}^{-}\times 2\mathfrak{d}^{-}\} \\ &= \tfrac{1}{2}\oint d\mathfrak{r}\cdot\nabla\,\mathfrak{A}^{-}\cdot\mathfrak{d}^{-} = -\tfrac{1}{2}\vec{M}^{-}(\mathfrak{r})\cdot\mathfrak{d}^{-} \end{aligned} \tag{5.16}$$

for the double circuit *A* and:

$$\begin{aligned} L^B &= \tfrac{1}{8}\oint d\mathfrak{r}\cdot\{-2(\text{Def }\mathfrak{u}^{-})\cdot 2\vec{P}^{-} - 2\vec{P}^{-}\times(\nabla\times\mathfrak{u}^{-})\} \\ &= -\tfrac{1}{2}\oint d\mathfrak{r}\cdot\nabla\mathfrak{u}^{-}\cdot\vec{P}^{-} = -\tfrac{1}{2}\mathfrak{b}^{-}(\mathfrak{r})\cdot\vec{P}^{-}, \end{aligned} \tag{5.17}$$

in which $\mathfrak{r}$ is chosen to be the intersection point of the two double circuits. If one sets that equal to the moment reference point $\mathfrak{r}_0$ then from (5.6), (5.16) will imply:

$$L_0^A = -\tfrac{1}{2}\vec{M}_0^{-}\cdot\mathfrak{d}^{-}, \tag{5.18}$$

and from (5.2), (5.17) will imply:

$$L_0^B = -\tfrac{1}{2}\vec{P}^{-}\mathfrak{b}_0^{-}. \tag{5.19}$$

A simultaneous jump in $\mathfrak{A}^{-}$ and $\mathfrak{u}^{-}$ for the same circuit will not happen, as was shown already; however, it would make no contribution to the integral, anyway. From (5.12-15), one will then have:

$$L^{*} = -\tfrac{1}{2}\oint d\mathfrak{r}\cdot(\vec{P}^{-}\times\mathfrak{d}^{-}) = 0, \tag{5.20}$$

due to the constancy of the integrand. For more than double connectivity, one does the same thing for each hole in the pretzel. For a canonical decomposition [**4**], there will always exist an $\mathfrak{r}_0$ that is common to all double circuits.

Now, the physical interpretation is simple: The sum of all $L_0^A + L_0^B$ means nothing more than the work that is done by constructing the **Volterra** distortion state, up to sign. If we substitute that result in (2.11) then it will follow that (*):

$$\begin{aligned} &\tfrac{1}{2}\int \underline{\chi}^{-} \cdot\cdot [\mathrm{Ink}(\underline{s} \cdot\cdot \mathrm{Ink}\,\underline{\chi}^{-})]\,d\tau \\ &= -\tfrac{1}{2}\sum_{\lambda} (\vec{P}_{\lambda}^{-} \cdot \mathfrak{b}_{0\lambda}^{-} + \vec{M}_{0\lambda}^{-} \cdot \mathfrak{d}_{\lambda}^{-}) + \tfrac{1}{2}\int \mathrm{Ink}\underline{\chi}^{-} \cdot\cdot \underline{s} \cdot\cdot \mathrm{Ink}\,\underline{\chi}^{-}\,d\tau\,, \end{aligned} \tag{5.21}$$

in which the sum extends over all holes in the pretzel. The equation says that proper stresses from a **Volterra** distortion state can remain even for vanishing boundary loading on each hole (**).

b) *The boundary-value for multiple-connectedness.* – From (5.21), a solution that fulfills all static boundary conditions can be included for every hole in the pretzel can include the proper stresses of a **Volterra** distortion state. They must thereupon be examined and, if necessary, reduced by the addition of a suitable proper stress field for the pure load stress state. Now, since it no longer happens, as in the proof of uniqueness, that the entire outer surface of the pretzel is circumnavigated, it will no longer be necessary for all circuits to go through one point (canonical decomposition [**4**]), as in the uniqueness proof, and one can now construct the two circuits of type *A* and *B* at the individual holes on the pretzel, independently of the other holes. In what follows, we will require only single circuits that define a right-hand screw for the same hole (cf., the accompanying sketch to Fig. 8a).

Now, let a solution of the boundary-value problem take the form of a stress function tensor $\underline{\chi}^1$ that correctly recovers the outer surface loading. One will then establish a distortion state that possibly exists when one substitutes the associated elastic strain $\underline{s} \cdot\cdot \mathrm{Ink}\,\underline{\chi}^1$ in **Cesaro**'s formula (5.1) and integrates for each hole in the pretzel over a circuit around the external region (type *B*), in which one then defines:

$$\mathfrak{b}^1(\mathfrak{r}) = \oint d\mathfrak{r}' \cdot \{\underline{s}(\mathfrak{r}') \cdot\cdot \mathrm{Ink}\,\underline{\chi}^1(\mathfrak{r}') + [\underline{s}(\mathfrak{r}') \cdot\cdot \mathrm{Ink}\,\underline{\chi}^1(\mathfrak{r}') \times \nabla'] \times (\mathfrak{r}' - \mathfrak{r})\}\,, \tag{5.22}$$

in which $\mathfrak{r}$ is a point on the circuit. If one places $\mathfrak{r}_0$ on the circuit then one can determine the quantities $\mathfrak{b}_0^1$ and $\mathfrak{d}^1$ by a comparison with (5.2-4) (***).

(*) The connecting hoses that are endowed with fictitious supplementary stresses in multiply-bounded regions (§ **4**) are left out of the volume integration.

(**) The physical content of (5.21) was known already from the ground-breaking paper of **Volterra** [**43**]. What is new is its connection with the theory of stress functions. Furthermore, the **Volterra** paper differs from the standpoint of modern proper stress theory only by the fact the compatibility is not expressly included in the definition of regular strain (*loc. cit.*, pp. 404), but is obviously assumed to be self-explanatory.

(***) There is a different $\mathfrak{r}_0$ for each individual hole in the pretzel then. However, for the sake of clarity, we shall avoid introducing a special index for the hole number (e.g., $\mathfrak{r}_{01}$) in what follows.

The problem of determining a proper stress state with equal and opposite displacement and rotation jumps can likewise lead one to a boundary-value problem with stress functions now. One must replace the boundary values with null-stress functions for the generating vector $\mathfrak{A}^{1-}$ whose jump properties are to be determined on a circuit around the internal region by $\mathfrak{r}_0$ (type *A*) on each hole such that:

$$\mathfrak{b}_0^{1-} = -\,\mathfrak{b}_0^{1} \qquad \text{and} \qquad \mathfrak{d}^{1-} = -\,\mathfrak{d}^{1}. \tag{5.23}$$

In the general case, that can happen in such a way that one solves the boundary-value problem for any vector field $\mathfrak{A}$ that is defined in the external region with undetermined jump vectors $\vec{M}_0$ and $\vec{P}$ along a circuit around the internal region through $\mathfrak{r}_0$. Due to the linearity of all equations, this general solution can be a linear combination of two times three linearly-independent components of the jump vectors. If one then applies (5.22) to that general solution and decomposes it according to (5.2) then one will get a displacement and rotation jump on the circuit *B* as a function of the jump vectors of $\mathfrak{A}$ to the associated circuit *A*, and thus, a linear relation:

$$\begin{aligned} \mathfrak{b}_0 &= \mathfrak{b}_0(\vec{M}_0, \vec{P}) \\ \mathfrak{d} &= \mathfrak{d}(\vec{M}_0, \vec{P})\,. \end{aligned} \tag{5.24}$$

If the coefficient determinant of that equilibrium system does not vanish, which can be assumed on all physically-sensible cases, then that relation can be inverted to:

$$\begin{aligned} \vec{M}_0 &= \vec{M}_0(\mathfrak{b}_0\,,\,\mathfrak{d}), \\ \vec{P} &= \vec{P}(\mathfrak{b}_0\,,\,\mathfrak{d}). \end{aligned} \tag{5.25}$$

Substitution of (5.23) will then yield the desired vector field $\mathfrak{A}^1$, and since the boundary-value problem for the components of $\vec{M}_0$ and $\vec{P}$ was solved already by linear superposition of the individual component solutions of the stress function vector field $\underline{\chi}$, such that:

$$\underline{\chi} = \underline{\chi}^1 + \underline{\chi}^{1-} \tag{5.26}$$

will be the desired solution of the boundary-value problem that is free of the proper stress of the **Volterra** distortion state. One can regard this process as a three-dimensional extension of the method of solution that **Prager** [**17**] gave in conjunction with his eqs. (14-16). The exchange of the external and internal regions and the physical meaning of $\mathfrak{A}$ and $\mathfrak{u}$ corresponds to the disc-plate analogy that **Schaefer** [**18**] formulated in space. Moreover, if one contracts the outer surface of a torus in Fig. 8 to a singular circle whose radius one subsequently extends to infinity then the relationship to the stress function of the rod that is subjected to a dyname that **Schaefer** [**39**] will become obvious.

c) *Null-stress functions for special Ansätze.* – Up to now, no further assumptions have been made about the generating vector $\mathfrak{A}$ of the null-stress functions, other than the given jump locations and the necessary differentiability properties. That is entirely permissible when equation (2.10) is solved directly in the internal region and (1.2) is employed to calculate the stress. By contrast, if one replaces (2.10) with any simpler differential expression that guarantees the fulfillment of (2.10) only by the simultaneous fulfillment of an auxiliary condition or a special Ansatz for $\underline{\chi}$ then, in general, those auxiliary conditions should also be observed for the null-stress functions. Obviously, one can later add an arbitrary deformator that is defined in the whole interior to $\underline{\chi}$, as long as one employs the original equation (1.2) for the calculation of the stresses. However, when the auxiliary conditions are given, Ink $\underline{\chi}$ can often be replaced with a simpler differential expression, which does not by any means need to vanish for an arbitrary deformator. A more important case is that of multiply-connected regions when the deformator, with a multiple-valued generating vector $\mathfrak{A}$, is initially defined only in the exterior, and the "null-stress function" that it is derived from actually leads to zero stresses only in the exterior, while in the interior the proper stress state describes **Volterra** distortions and can therefore no longer be a pure deformator. The deformator in external space must then be specialized in such a way that by determining the boundary conditions on the outer surface of the region, it will, at the same time, also guarantee the preservation of the auxiliary conditions (the special Ansätze, resp.) in the whole interior.

In isotropic, homogeneous media, we will mainly employ the bipotential equation as a replacement for (2.10). In particular, we cite:

1. The **Airy** stress function in the plane. The auxiliary condition here consists of saying that only the *zz*-component of the stress function tensor is non-vanishing and depends upon only x and y, in addition. As is known, the only associated null-stress functions are linear functions of x and y. There are no other deformators that fulfill the auxiliary conditions. For the behavior of the **Airy** stress function in a planar annulus, cf., **Prager** [**17**].

2. The stress functions of the plate according to **Schaefer** [**18**, **19**]. According to **Schaefer** [**19**], for an ordinary plate (i.e., one that is free of **Cosserat** moment stresses that are perpendicular to the plane of the plate), the stress functions Φ_1 and Φ_2 (and the **Airy** stress function Φ_3) can be regarded as the components of the generating vector of a null-stress function field, which is independent of z, but is defined only in the upper half-space. Its continuation into the lower half-space will be set to zero. For that reason, the incompatibilities will not vanish in the boundary plane $z = 0$, but will yield singular simple and double coverings that correspond to the plates and disc stress states. One can regard this derivation as the limiting case of Fig. 11, pp. 38, which gives a representation of a singular loading on the curve (C) by a stress function $\underline{\chi}^a$ that is defined in the external space of (B) when the surface (F) degenerates to the plane $z = 0$, and (B)

degenerates to the disc (plate, resp.). A singular loading outside of the curve (C) will be given when one makes the upper half-space doubly-connected by means of a singular curve (D) (a "bar," (*Stange*) [**19**]) and introduces the corresponding multiple-valuedness of $\mathfrak{A}$ [**19**]. The considerations of this section can be adapted to annuli, when one carries out the passage to the limit in Fig. 11 with multiply-connected (B) (cf., on this [**18**]). When there are no loads, Φ_1 and Φ_2 can be derived from a biharmonic function [**18**, **19**] with help of the disc-plate analogy for isotropic, homogeneous plate materials.

One will get information about the permissible null-stress function most rapidly when one introduces the distribution $\delta^{(0)}(z)$, $\delta^{(1)}(z)$, $\delta^{(2)}(z)$ (cf., Appendix).

One then writes:

$$\underline{\chi} = \operatorname{Def} \mathfrak{A}\,(x, y)\; \delta^{(0)}(z), \tag{5.27}$$

and one will then get by differentiation:

$$\begin{aligned} \underline{\sigma} &= \nabla \times \underline{\chi} \times \nabla \\ &= \delta^{(2)}(z)\; \mathfrak{k} \times \operatorname{Def} \mathfrak{A} \times \mathfrak{k} + \delta^{(1)}(z)\; (\mathfrak{k} \times \operatorname{Def} \mathfrak{A} \times \nabla + \nabla \times \operatorname{Def} \mathfrak{A} \times \mathfrak{k}) \\ &= \delta^{(1)}(z) \cdot \tfrac{1}{2} (\mathfrak{k} \times \nabla\mathfrak{A} \times \nabla + \mathfrak{A} \times \nabla\mathfrak{k} \times \nabla). \end{aligned} \tag{5.28}$$

The first term includes the bending moment, while the second one includes the transverse forces and disc stresses. Since both dyads in the second bracket are composed of the same vectors, that expression can vanish only for $\nabla \times \mathfrak{A}$; i.e., only a vector field $\mathfrak{A}$ that corresponds to a rigid rotation is allowed ([**18**], eq. (24d)).

3. The **Kröner-Marguerre** solution in space [**25**, **12**, **2**]. We employ **Kröner**'s formulation ([**2**], (II.18)), which uses a tensor:

$$\underline{\gamma} = \frac{1}{2G}\left(\underline{\chi} - \frac{1}{m+2}\chi_{\mathrm{I}}\mathrm{I}\right) \tag{5.29}$$

(G = shear modulus, m = transverse contraction number) that is derived from the stress function tensor, in place of the latter tensor. The **Kröner-Marguerre** auxiliary condition then reads:

$$\nabla \cdot \underline{\gamma} = 0 \tag{5.30}$$

when one employs the bipotential equation, and will be guaranteed with certainty when one prescribes (5.30) as the boundary condition and extends it to the broader boundary condition:

$$\mathfrak{n} \cdot \nabla\nabla \cdot \overset{\downarrow}{\underline{\gamma}} = 0. \tag{5.31}$$

Since (5.30) and (5.31) are homogeneous, they can also be prescribed on each individual outer surface of a multiply-bounded region (*). From the validity of (5.30), **Kröner** ([**2**], (II.23)) gives the simplified formula for the stresses:

$$\underline{\sigma} = 2G\left[\Delta\underline{\gamma} + \frac{m}{m-1}(\nabla\nabla\gamma_{\mathrm{I}} - \Delta\gamma_{\mathrm{I}}\mathrm{I})\right], \tag{5.32}$$

instead of (1.2), and writes out those equations in cylindrical coordinates ([**2**], § 32), in addition. As one easily sees, it is sufficient that:

$$\underline{\chi}^0 = \mathrm{Def}\,\mathfrak{U} = 2G\ \underline{\gamma}^0 \tag{5.33}$$

for a null-stress function when the generating vector:

$$\mathfrak{A} = \nabla A + \nabla\times\vec{A} \tag{5.34}$$

can be derived from harmonic potentials A and $\vec{A}$. Such potentials also suffice for the generation of each prescribed spatial multi-valuedness of the vector $\mathfrak{A}$. If the jump on any simply-connected, double-edged cut-surface [cf., (5.6)] is given by:

$$-\ \vec{M}(\mathfrak{r}) = \mathfrak{A}_2 - \mathfrak{A}_1 = -\ \vec{M}_0 - \vec{P}\times(\mathfrak{r}-\mathfrak{r}_0) \tag{5.35}$$

then one prescribes, e.g., the boundary value 0 on edge 1 and the boundary values:

$$\begin{aligned} A &= -\,\mathfrak{r}\cdot\vec{M}_0, \\ \vec{A} &= \tfrac{1}{3}(\mathfrak{r}-\mathfrak{r}_0)\times[\vec{P}\times(\mathfrak{r}-\mathfrak{r}_0)] \end{aligned} \tag{5.36}$$

on edge 2, and solves the associated harmonic boundary-value problem in the infinite external space to the two edges of the cut. The null-stress function (5.33) is continuous in the cut-surface then, as one can easily derive from the theorem on equivalent cuts in **Volterra** ([**43**], Chap. III). The **Poisson** equation is only a special case of the general equation, which is satisfied by the displacements of a **Volterra** distortion. For more than double connection, one must superimpose the solutions for the different cut-surfaces.

In conclusion, we shall give the necessary and sufficient conditions for $\mathfrak{A}$. We see directly from (5.30) and (5.32) that $\mathfrak{A}$ must be harmonic, and div $\mathfrak{A}$ must be at most a constant.

(*) One sees this, e.g., when one reduces the domain to a simply-bounded one by drilling holes in it and lets the diameter of the connecting hoses go to zero. Fictitious supplementary stresses that remain in the connecting hose (§ **4**) will then have no effect, since, by definition, they leave the outer surface of the connecting hose force-free.

4. The **Schaefer** Ansatz in space [**10**, **11**]. With the Ansatz [**11**] (*):

$$\begin{aligned}
&\underline{\chi} = \underline{\Theta} - \Theta_{\mathrm{I}}\mathrm{I} + \Omega\mathrm{I}, \\
&\Delta\Omega = \frac{m}{m-1}(\nabla\cdot\underline{\Theta}\cdot\nabla + \Delta\Theta_{\mathrm{I}}), \\
&\Delta\underline{\Theta} = 0
\end{aligned} \tag{5.40}$$

(m = transverse contraction number)

Schaefer [**10**] presented, e.g., the stress functions for the problems of **Boussinesq** and **Cerutti** on the half-space. Here, we prefer a somewhat more advantageous form that is an extension to proper stress problems that was also given by **Schaefer** (**) and will emerge from (5.40) by the substitution:

$$\underline{\Theta} = \underline{\psi} - \tfrac{1}{2}\,\psi_{\mathrm{I}}\,\mathrm{I} \qquad [\underline{\psi} = \underline{\Theta} - \Theta_{\mathrm{I}}\,\mathrm{I}, \text{ resp.}]. \tag{5.41}$$

One will then have:

$$\begin{aligned}
&\underline{\chi} = \underline{\psi} + \Omega\,\mathrm{I}, \\
&\Delta\Omega = \frac{m}{m-1}(\nabla\cdot\underline{\psi}\cdot\nabla - \Delta\,\psi_{\mathrm{I}}), \\
&\Delta\underline{\psi} = 0.
\end{aligned} \tag{5.42}$$

One will get the stress from this:

$$\begin{aligned}
\underline{\sigma} &= \mathrm{Ink}\,\underline{\chi} = \mathrm{Ink}\,\underline{\psi} - \Delta\Omega\,\mathrm{I} + \nabla\nabla\,\Omega \\
&= -\nabla\nabla\underline{\psi} - \underline{\psi}\cdot\nabla\nabla + \nabla\nabla\psi_{\mathrm{I}} - \frac{1}{m-1}\nabla\cdot\underline{\psi}\cdot\nabla\mathrm{I} + \nabla\nabla\,\Omega.
\end{aligned} \tag{5.44}$$

That expression must vanish for a null-stress function $\underline{\psi}^0$. One must next be able to represent such a null-stress function as a deformator, since otherwise, from (5.42), Ω would be harmonic, and the first equation in (5.44) would give a contradiction. We then set:

$$\underline{\psi}^0 = \mathrm{Def}\,\mathfrak{A} \tag{5.45}$$

and obtain the condition:

$$\mathrm{Ink}\,\underline{\chi} = -\Delta\,\mathrm{Def}\,\mathfrak{A} - \frac{1}{m-1}\Delta\nabla\cdot\mathfrak{A}\,\mathrm{I} + \nabla\nabla\,\Omega = 0 \tag{5.46}$$

from (5.44), and with the second equation in (5.42), one will get:

(*) One should notice the partially-opposite sign conventions for $\underline{\chi}$ in the various papers. The one used here agrees with [**2**] and [**10**], while it is opposite to [**11**] and [**27**].

(**) Oral communication.

$$(\text{Ink}\,\underline{\chi})_{\text{I}} = -\frac{m+2}{m-1}\Delta\nabla\cdot\mathfrak{A} + \Delta\Omega = \frac{-2}{m-1}\Delta\nabla\cdot\mathfrak{A} = 0, \tag{5.47}$$

as well as:

$$\Delta\Omega = 0. \tag{5.48}$$

It is then necessary and sufficient that the deformator of the generating vector of the null-stress function should be harmonic; i.e., one arrives at essentially the same criterion as one arrives at for **Kröner-Marguerre** stress functions. Additionally, from (5.46), one must demand that Ω must be at most a linear function in Cartesian coordinates. In practice, one will usually set it equal to zero.

6. Planar annulus with a resultant dyname on its boundary.

Föppl [**23**] has given the **Airy** stress functions for numerous examples of this type. From § **4**, no representation by **Airy** stress functions is possible in such cases, unless a fictitious proper stress state appears, at least, on a singular line, which will be suppressed from the stress calculation. In the simple example of the isolated force, we shall show that this fictitious proper stress state can actually verified using known methods involving the stress function itself, and its extension to more complicated examples will be possible with no difficulty then.

As is known, one obtains the **Airy** stress function by restricting the stress function tensor to a *zz*-component that is independent of *z* and setting:

$$\chi_{zz} = -F(x, y) = -F(\rho, \varphi); \qquad \text{all other } \chi_{ij} = 0, \tag{6.1}$$

or with the employment of the basis vectors $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$:

$$\underline{\chi} = -F\,\mathfrak{k}\,\mathfrak{k}. \tag{6.2}$$

One will then have:

$$\underline{\chi}\times\nabla = \mathfrak{k}\left(\frac{\partial F}{\partial y}\mathfrak{i} - \frac{\partial F}{\partial x}\mathfrak{j}\right) \tag{6.3}$$

in Cartesian coordinates and:

$$\underline{\chi}\times\nabla = \mathfrak{k}\left(\frac{1}{\rho}\frac{\partial F}{\partial\varphi}\mathfrak{e}_\rho - \frac{\partial F}{\partial\rho}\mathfrak{e}_\varphi\right) \tag{6.4}$$

in cylindrical coordinates, with the basis vectors $\mathfrak{e}_\rho$, $\mathfrak{e}_\varphi$, $\mathfrak{k}$. The circuit of the **Peretti** line integrals (1.5-6) degenerates into an infinitely-long trail (*Schleifen*) that is perpendicular to the stress plane; in a spatial context, it then yields a force and a moment per unit length of a cylindrical strip that is spanned by two parallels to the *z*-axis, while in a two-dimensional context, it will yield a force and a moment along an arbitrary connecting line between the points of intersection P_1 and P_2. In Cartesian coordinates, one gets (*):

$$\vec{M}_0 = \{F(x_2, y_2) - F(x_1, y_1) - \left.\frac{\partial F}{\partial x}\right|_{P_2}(x_2 - x_0) - \left.\frac{\partial F}{\partial y}\right|_{P_2}(y_2 - y_0) + \left.\frac{\partial F}{\partial x}\right|_{P_1}(x_1 - x_0) + \left.\frac{\partial F}{\partial y}\right|_{P_1}(y_1 - y_0)\} \tag{6.5}$$

and

$$\vec{P} = \left\{\left.\frac{\partial F}{\partial y}\right|_{P_2} - \left.\frac{\partial F}{\partial y}\right|_{P1}\right\}\mathfrak{i} - \left\{\left.\frac{\partial F}{\partial x}\right|_{P_2} - \left.\frac{\partial F}{\partial x}\right|_{P1}\right\}\mathfrak{j}, \tag{6.6}$$

and in cylindrical coordinates, one will get (*):

$$\begin{aligned}\vec{M}_0 = \{F(\rho_2, \varphi_2) - F(\rho_1, \varphi_1) &- \left.\frac{\partial F}{\partial \rho}\right|_{P_2}[\rho_2 - \rho_0\cos(\varphi_2 - \varphi_0) - \frac{1}{\rho_2}\left.\frac{\partial F}{\partial \varphi}\right|_{P_2}\rho_0\sin(\varphi_2 - \varphi_0) \\ &- \left.\frac{\partial F}{\partial \rho}\right|_{P_1}[\rho_1 - \rho_0\cos(\varphi_1 - \varphi_0) + \frac{1}{\rho_1}\left.\frac{\partial F}{\partial \varphi}\right|_{P_1}\rho_0\sin(\varphi_1 - \varphi_0)\}\,\mathfrak{k}\end{aligned} \tag{6.7}$$

and

$$\vec{P} = \frac{1}{\rho_2}\left.\frac{\partial F}{\partial \varphi}\right|_{P_2}\mathfrak{e}_{\rho 2} - \frac{1}{\rho_1}\left.\frac{\partial F}{\partial \varphi}\right|_{P_1}\mathfrak{e}_{\rho 1} - \left.\frac{\partial F}{\partial \rho}\right|_{P_2}\mathfrak{e}_{\varphi 2} + \left.\frac{\partial F}{\partial \rho}\right|_{P1}\mathfrak{e}_{\varphi 1}, \tag{6.8}$$

resp. As usual, the point on the boundary line to the right of the outward-pointing normal will be provided with the index 1; on the internal cut-lines, the normal that points to the other cut-edge will enter in place of it.

The mathematically simplest cavities in the plane are (in agreement with **Föppl** [**23**], pp. 94) the ones that one can contract to a point, and therefore infinitely-small recesses around the point of application of single forces, force-couples, with and without moments, singular angular moments, etc., whose stress field was determined by **Föppl** with the help of **Airy** stress functions. It is precisely **Föppl**'s derivation of the stress function of a single force in complete plane that shows very intuitively the meshing of multiple-connectivity and multiple-bounding in the plane (cf., § **4**c). **Föppl** started with the stress function of the single force on a wedge and pulled it apart on the slotted plane. When viewed spatially, this corresponds to the cut away hollow cylinder – i.e., to a doubly-connected spatial domain with a separating surface, which is the best known case in the theory of isolated dislocations [**3**, **43**]. In fact, **Föppl** also found an edge dislocation at the point of application for the force in that domain whose stress field must

(*) The other components of the moment diverge to undetermined ones due to the infinite length of the cylindrical strip. For that reason, we have referred them to the "center plane" of the infinite cylinder and replaced them with a "principal **Cauchy** value" of zero.

be subtracted. It ultimately follows for an isolated force K in the negative x-direction at the origin (Fig. 9) that one will have the stress function:

$$F = \frac{K}{2\pi}\left(\rho\varphi\sin\varphi - \frac{m-1}{2m}\rho\ln\rho\cos\varphi\right). \tag{6.9}$$

After discarding the dislocation, we now consider the complete plane that is punctured at the point of application for the force to be a simply-connected, but doubly-bounded region, and must therefore demand spatial single-valuedness of the stress function. That is ensured for the second terms in (6.9) with no further assumptions. However, in the first term, we must reduce the angle φ for every value of ρ to lie between two values that are at most 2π apart, so the barriers must be at least piecewise-differentiable functions of ρ. That happens most simply when one can bend the angle φ back by 2π into a piecewise-smooth curve that cuts any circle around O only once; the connecting hose (§ **4**) is then distinguished by a place where φ jumps. Moreover, the connecting hose can be chosen arbitrarily, in contrast to the spatial examples of the following section. One can also introduce several connecting hoses (jump locations for φ) without altering the form of the solution (6.9) in any way.

That singularity also appears in the derivatives:

$$\begin{aligned}
\frac{\partial F}{\partial\rho} &= \frac{K}{2\pi}\left\{\varphi\sin\varphi - \frac{m-1}{m}(\ln\rho+1)\cos\varphi\right\},\\
\frac{1}{\rho}\frac{\partial F}{\partial\varphi} &= \frac{K}{2\pi}\left\{\varphi\cos\varphi + \sin\varphi + \frac{m-1}{2m}\rho\ln\rho\sin\varphi\right\}.
\end{aligned} \tag{6.10}$$

It first vanishes in the stresses that are smoothed-out at the singular location.

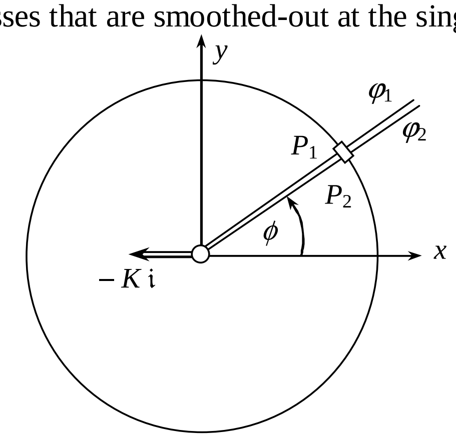


Figure 9. The isolated force in the plane.

Now, **Föppl** showed that the equilibrium condition was fulfilled by integration over the stresses on a circle around the point of application of the force and found that the resultant of all forces that act upon the circular region in the neighborhood of it was the force that is equal and opposite to – K i (Fig. 9), as it also must be, since the jump singularity would go away by differentiation ([**23**], § 23, (15)). By contrast, equations (6.7-8), which are derived from the **Peretti-Günther** integrals, must give zero for a closed circle when the **Airy** stress function is continuous, along with its first derivatives (Fig. 9), since the contributions of P_1 and P_2 will then cancel when one moves them together on the circle. In reality, one will get finite limiting values when one lets the points P_1 and P_2 in Fig. 9 move together into the singularity that lies on the ray $\varphi = \Phi$.

The sign of these limiting values depends upon whether one regards them as boundary points of the dashed circle (Fig. 9) or the cuts on the singular line. In one case, one will get the forces that are exerted on the dashed line by its neighborhood in the interior, as integrating over the stresses as in **Föppl** [**23**], while in the other case, one will get the dyname that is carried through the connecting hose in order to preserve the isolated force at the origin. The numbering of P_1 and P_2 in Fig. 9 corresponds to the second way of looking at things. One will get:

$$\varphi_2 - \varphi_1 = -2\pi, \tag{6.11}$$

and for a moment reference point $\mathfrak{x}_0$ midway between P_1 and P_2 on the singular line:

$$\vec{M}_0 = -K\,\rho_0 \sin\Phi\ \mathfrak{k}, \tag{6.12}$$

just as, from (6.8):

$$\vec{P} = K\,(-\cos\Phi\ \mathfrak{e}_{\rho 0} + \sin\Phi\ \mathfrak{e}_{\Phi}) = -K\,\mathfrak{i}, \tag{6.13}$$

so the dyname that is statically equivalent to the applied force.

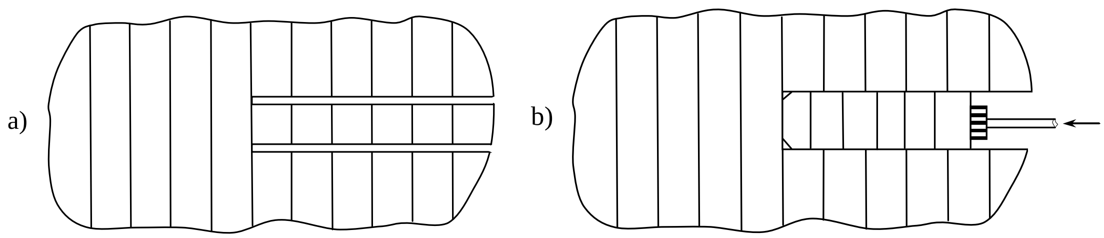


Fig. 10. The fictitious supplementary stress to the isolated force
(The elastic deformation outside of the singularity is suppressed)

The associated fictitious proper stress state (§ **4**) is especially simply to interpret for $\Phi = 0$, so along the line of action for the given force, where the moment vanishes. The fictitious supplementary stress on the singular line, which replaces the real external force for the stress function, will then correspond to a simple compression. One can think of it as arising, e.g., from the cutting process that was described *à la* **Eshelby** [**24**] in perhaps the following way: First, a moderately-wide slit is removed from both sides of the singularity line (Fig. 10a). Now, all of the material, with the exception of the strip between the slits, is stiffened up to complete rigidity, and from Fig. 10b, the strip is compressed by a rigid plunger from the outside until the aforementioned slit will be filled up completely as a result of the transverse strain. It is welded in that position. The compressive stress that arises in the strip is equal to the fictitious supplementary stress

$\underline{\sigma}^{K*}$. If one now once more solidifies the remaining material then the same stress state will exist there as if a force were acting upon the end of the singular strip. However, the compressive stress will continue to prevail in the singular strip itself, since we have verified that in (6.13).

A similar simple result follows for $\Phi = \pi$; in this case, the singularity lies on the negative x-axis, and in place of the pressure in the singular strip, one will now find a tension, such that to compensate, material must be replenished on both sides of the transverse contraction. The singular will be more complicated for all of the remaining Φ-values. Along with the simple, constant compression or tension, one will also have shearing stresses, and a double-covering of tension and compression that increases linearly outward, which, on the one hand, cancels the internal forces that are required by the shearing stresses on the outer surface of the connection hose, and on the other hand, produces the moment (6.12). The fictitious supplementary stress then corresponds to the stress state in a thin bar that can be curved arbitrarily in order to adapt it to the more complicated problems, in addition. In the next section, we shall employ the principle of constructing a fictitious supplementary stress that was described here to the problem of exhibiting the stress function tensor of an isolated force in space.

A term that is proportional to the polar angle $\varphi = \arctan y / x$ also appears in Schaefer's [**19**] stress functions Φ_1 and Φ_2 for a plate loaded with a dyname. An interpretation of the kind that was described is generally impossible, since Φ_1 and Φ_2 are introduced as components of the generating vector of the null-stress functions in external space, and will – in contrast to the **Airy** stress functions – first become the components of a spatial stress function tensor in an infinitely-long cylinder when one multiplies by $\frac{1}{2}z^2$. However, since that way of looking at things leads to stresses that diverge linearly at infinity, it is less suitable for the spatial representation of the plate (cf., § **4**c).

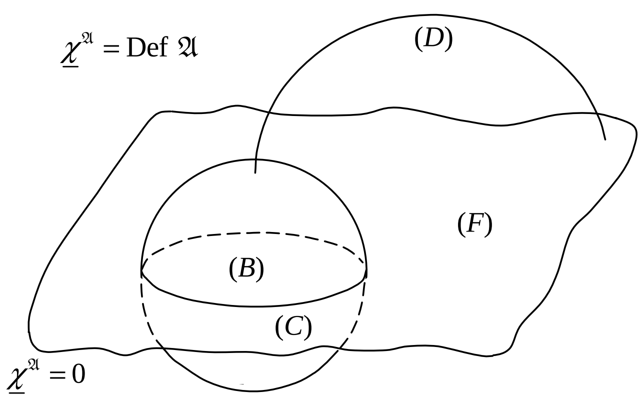


Fig. 11. The "bar model" for a load singularity on the outer surface.

By contrast, if one adopts the viewpoint of **Schaefer** [**19**] then the stress function will be represented symbolically by (5.27). It will consist of the product of the deformator and a **Heaviside** step function, and will then be spatially single-valued. The fictitious proper stress state is now first defined in the body (B) that is extended by the bar (D) and the surface (F) (Fig. 11), so it is no longer to be regarded as a purely planar problem (*). From **Schaefer** [**19**], the **Airy** stress function can also be regarded as a component of the

(*) Cf., on this, pp. 30, no. 2.

vector $\mathfrak{A}$ in (5.27). The interpretation that is employed here of the *zz*-component of a stress function tensor is, however, simpler, and for that reason it should be preferred for topological considerations.

7. The *Kröner-Marguerre* stress function tensor for an isolated force in infinite space.

For this problem, all of infinite space is unperturbed, except for the point of application of the force at the origin, where an infinitely-small cavity of unperturbed region has been removed. The unperturbed domain – viz., the punctured space – is therefore doubly-bounded, and the load on a boundary will have a non-vanishing resultant, such that from § **4**, that load stress problem cannot be solved directly with stress functions, but must be replaced with an equivalent proper stress problem. From § **4**, the associated fictitious supplementary stress will be housed in a singular connecting hose at the point of application of the force, and the stress function will only be valid in the "drilled-out space" then.

The simplest fictitious proper stress state for that case is clarified by Fig. 10 when one no longer regards the figure as a picture of a planar arrangement, but as a cross-section through a rotationally-symmetric structure. Furthermore, its mechanical production will result in precisely the same way as was described in conjunction with (6.13). The mathematical structure proceeds by the following steps (cf., Fig. 10, page 37):

1. According to **Kröner** [**2**], the stress functions of a simple displacement dipole are determined from the stress function tensor of a dislocation ring by passing to the limit.

2. The stress function tensor of a dilatation center will be constructed from the stress functions of three mutually-perpendicular displacement dipoles.

3. The simple displacement dipole and the dilatation center will be combined into am (internal) force-dipole [**2**].

4. A uniform distribution of force-dipoles along the positive *z*-axis will yield the desired stress function field of the isolated force. The singularity of the fictitious proper stress state in the connecting hose will be verified with the help of the **Peretti-Günther** integral.

5. Finally, one can also derive the stress function field of a double force with a moment and higher singularities from that by differentiation. The singularity in the connecting hose can also be determined from the limiting value of the contour integral here.

The exclusion of higher singularities (cf., **Sternberg** and **Eubanks** [**40**]) is guaranteed for the isolated force by the type of passage to the limit that was described in Para. **1**. We shall employ the form for the **Kröner-Marguerre** stress function tensor

(5.29) in cylindrical coordinates that was given by **Kröner** in ([**2**], § **32**). The agreement with the stress field of the isolated force that is known in the literature and was written out explicitly by **Trefftz** [**26**] can be verified with no difficulty with the help of the formula that **Kröner** gave in ([**2**], in conjunction with (V.30″)). We shall now perform the steps in detail.

1. The stress function tensor of a displacement dipole. From **Kröner** ([**2**], (II.107), the **Kröner-Marguerre** stress function tensor (5.29) is that of a closed dislocation loop:

$$\underline{\gamma} = -\frac{1}{8\pi}\{\mathfrak{b} \times \nabla \oint |\mathfrak{r}' - \mathfrak{r}|\, d\,\mathfrak{r}'\}^{\text{sym.}}, \tag{7.1}$$

in which $\mathfrak{r}'$ is the position vector of a point on the dislocation loop, and $\mathfrak{b}$ means the **Burgers** vector with the sign convention in **Kröner** [**2**]. For a circular dislocation loop with radius R around the origin in the xy-plane and the **Burgers** vector:

$$\mathfrak{b} = b\,\mathfrak{k}, \tag{7.2}$$

one will have, in particular ([**2**], (V.38)):

$$\begin{aligned} \underline{\gamma} &= -\frac{b}{8\pi}\{\mathfrak{k} \times \nabla \oint |\mathfrak{r}' - \mathfrak{r}|\, \mathfrak{e}_{\varphi'} d\,\mathfrak{r}'\}^{\text{sym.}}, \\ &= -\frac{b}{8\pi}\{\mathfrak{k} \times \nabla f(\rho, z)\, \mathfrak{e}_{\varphi}\}, \end{aligned} \tag{7.3}$$

in which, from **Franz** and **Kröner** ([**41**], (8)) (*):

$$f(\rho, z) = \frac{8}{3}\frac{R\sqrt{\rho R}}{k^3}\{2\,(1 - k^2)\,\mathbf{K}\,(k) - (2 - k^2)\,\mathbf{E}\,(k), \tag{7.4}$$

with the complete elliptic integrals $\mathbf{E}\,(k)$ and $\mathbf{K}\,(k)$, and:

$$k^2 = \frac{4\rho R}{(R+\rho)^2 + z^2}. \tag{7.5}$$

We now contract the dislocation loop around the origin. At the same time, k also tends to zero with R, and one will get:

$$f(\rho, z) = \pi R^2 \frac{\rho}{r} = F\frac{\rho}{r}; \qquad r = \sqrt{\rho^2 + z^2} \tag{7.6}$$

(*) In ([**2**], (V.39)), the pre-factor is incorrect, and equations ([**2**], (V.41)) must be multiplied by – 1/2 on the right.

from a development of the elliptic integral up to the fourth power in k ([**42**], pp. 75), in which F is the planar surface that is enclosed by the dislocation loop.

The stress function tensor of a zz-displacement dipole of strength bF follows from that:

$$\underline{\gamma}^{zz} = \frac{bF}{8\pi}\frac{1}{r}\left\{\mathfrak{e}_\rho\mathfrak{e}_\rho + \frac{z^2}{r^2}\mathfrak{e}_\varphi\mathfrak{e}_\varphi\right\}. \tag{7.7}$$

From the sign convention of **Kröner** [**2**] for the **Burgers** vector of the circuit-pair in the supporting sketch in Fig. 8a, pp. 24, a positive b will imply a displacement dipole that can be generated by the removal of material from the dislocation surface, and for negative b, a displacement dipole that can be generated by filling in material in the dislocation surface ["negative" ("positive," resp.) displacement dipole].

2. The stress function tensor of a dilatation center. With the transformation formulas:

$$\begin{aligned} \rho^{*2} &= z^2 + \rho^2 \sin^2\varphi, \\ \cot\varphi^* &= \frac{\rho\sin\varphi}{z}, \\ x &= \rho\cos\varphi, \end{aligned} \tag{7.8}$$

one will get the unit vectors of a cylindrical coordinate system with the x-axis as an axis:

$$\begin{aligned} \mathfrak{e}_{\rho^*} &= \frac{\nabla\rho^{*2}}{|\nabla\rho^{*2}|} = \frac{\mathfrak{e}_\rho\rho\sin^2\varphi + \mathfrak{e}_\varphi\rho\sin\varphi\cos\varphi + \mathfrak{e}_\rho z}{\sqrt{z^2+\rho^2\sin^2\varphi}}, \\ \mathfrak{e}_{\varphi^*} &= \frac{-\nabla\cot\varphi^*}{|\nabla\cot\varphi^*|} = \frac{-\mathfrak{e}_\rho z\sin\varphi - \mathfrak{e}_\varphi z\cos\varphi + \mathfrak{e}_\rho\rho\sin\varphi}{\sqrt{z^2+\rho^2\sin^2\varphi}}, \end{aligned} \tag{7.9}$$

and making the substitution:

$$(\sin\varphi, \cos\varphi) \parallel (\cos\varphi, -\sin\varphi) \tag{7.10}$$

in that will give the unit vectors $\mathfrak{e}_{\rho^{**}}, \mathfrak{e}_{\varphi^{**}}$ of a cylindrical coordinate system with the y-axis as its axis. The stress function tensor of the dilatation center will then follow by addition:

$$\underline{\gamma}^d = \underline{\gamma}^{xx} + \underline{\gamma}^{yy} + \underline{\gamma}^{zz} = \frac{bF}{8\pi}\frac{1}{r}\left\{\mathfrak{e}_{\rho^*}\mathfrak{e}_{\rho^*} + \frac{x^2}{r^2}\mathfrak{e}_{\varphi^*}\mathfrak{e}_{\varphi^*} + \mathfrak{e}_{\rho^{**}}\mathfrak{e}_{\rho^{**}} + \frac{y^2}{r^2}\mathfrak{e}_{\varphi^{**}}\mathfrak{e}_{\varphi^{**}} + \mathfrak{e}_\rho\mathfrak{e}_\rho + \frac{z^2}{r^2}\mathfrak{e}_\varphi\mathfrak{e}_\varphi\right\}. \tag{7.11}$$

An elementary intermediate calculation, which will be skipped here, will yield:

$$\underline{\gamma}^d = \frac{bF}{8\pi}\frac{1}{r}\begin{pmatrix} 1+\dfrac{\rho^2}{r^2} & 0 & \dfrac{\rho z}{r^2} \\ 0 & 1 & 0 \\ \dfrac{\rho z}{r^2} & 0 & 2-\dfrac{\rho^2}{r^2} \end{pmatrix} \tag{7.12}$$

in matrix notation with the coordinate sequence ρ, φ, z. Negative b will yield a positive dilatation center, while positive b will produce a negative one (viz., a "compression center"). That stress function tensor obviously represents a counterexample to the suggestion that **Kröner** ([**2**], pp. 157) expressed in a provisional form about the stress states that can be expressed in terms of only $\gamma_{\rho\rho}$ and $\gamma_{\varphi\varphi}$. Since the **Kröner-Marguerre** stress function tensor is determined uniquely, up to a constant part (i.e., up to constant components in Cartesian coordinates), from the uniqueness theorems in bipotential theory, the remaining components of (7.12) cannot be made to vanish. **Kröner**'s suggestion is not applicable.

3. The stress function tensor of the force dipole. From **Kröner** ([**2**], (II.151)), the displacement dipole can be defined by passing to limit from a dislocation loop that is defined by the tensor:

$$\underline{Q} = -\lim (F_i\, b_j), \tag{7.13}$$

which we can restrict to its symmetric part for the present problem. From **Kröner** ([**2**], (II.153)), in a homogeneous medium, the (internal) force-dipole can be replaced with:

$$\underline{P} = \underline{c} \cdot\cdot \underline{Q}, \tag{7.14}$$

and by isotropy, that will give:

$$\underline{P} = 2G\left(\underline{Q} + \frac{Q_{\mathrm{I}}\mathrm{I}}{m-2}\right), \tag{7.15}$$

We now require a force-dipole of the form:

$$\underline{P} = P\, \mathfrak{k}\, \mathfrak{k}, \tag{7.16}$$

and in order to do that, form a combination of a simple displacement-dipole and a dilatation center:

$$\underline{Q} = A\, \mathfrak{k}\, \mathfrak{k} + B\, \mathrm{I}, \tag{7.17}$$

which will lead to the force-dipole:

$$\underline{P} = 2G\left(A\mathfrak{k}\mathfrak{k} + B\mathrm{I} + \frac{A+3B}{m-2}\mathrm{I}\right). \tag{7.18}$$

By a comparison with (7.16), we will get:

$$A = \frac{P}{2G}, \qquad B = -\frac{P}{2G(m+1)}. \tag{7.19}$$

From (7.13), we now set $-bF = A$ in (7.7) and $-bF = B$ in (7.12) and add them. That implies the stress function tensor of the force-dipole (7.16):

$$\underline{\gamma}^P = \frac{P}{16\pi G}\frac{1}{r}\begin{pmatrix} \frac{1}{m+1}\frac{\rho^2}{r^2} - \frac{m}{m+1} & 0 & \frac{1}{m+1}\frac{\rho z}{r^2} \\ 0 & \frac{\rho^2}{r^2} - \frac{m}{m+1} & 0 \\ \frac{1}{m+1}\frac{\rho z}{r^2} & 0 & \frac{1}{m+1}\left(2 - \frac{\rho^2}{r^2}\right) \end{pmatrix}. \tag{7.20}$$

4. The stress function tensor of the isolated force. A definition of the displacement-dipole that is equivalent to (7.13) upon restricting to the symmetric tensor follows from the supplementary strain $\underline{\varepsilon}^Q$ [**5**, **6**, **7**, **33**, **34**]:

$$\underline{Q} = \lim_{V \to 0} \int_{(V)} \underline{\varepsilon}^Q \, d\tau, \tag{7.21}$$

from which, one can derive a corresponding definition for the force-dipole from [**5**] ([**7**], (I.5)) from the supplementary stress $\underline{\sigma}^Q$:

$$\underline{P} = -\lim_{V \to 0} \int_{(V)} \underline{\sigma}^Q \, d\tau. \tag{7.22}$$

If we now replace the supplementary compression-stress in the proper stress state in the excised cylinder of Fig. 10, pp. 37, with the sign changed, with a (fictitious) supplementary tension-stress:

$$\underline{\sigma}^{K*} = \sigma\, \text{ŧ}\ \text{ŧ} \tag{7.23}$$

in a cylinder with a base surface F around the positive z-axis then that will correspond to a distribution of force-dipoles:

$$\frac{d\underline{P}}{dz} = -\,\underline{\sigma}^{K*} F = -\,\sigma F\, \text{ŧ}\ \text{ŧ}. \tag{7.24}$$

The internal force:

$$K\, \text{ŧ} = \sigma F\, \text{ŧ} = -\,\text{ŧ} \cdot \frac{d\underline{P}}{dz} \tag{7.25}$$

will then appear on the base surface F in the xy-plane. The isolated force of magnitude K in the positive z-direction can then be replaced with a distribution of infinitesimal fictitious force-dipoles:

$$dP = -K\,dz \tag{7.26}$$

along the positive z-axis. One must now calculate the stress function field of such a dipole at the location z on the z-axis with the help of (7.20) and integrate over the entire positive z-axis. We then make the following substitutions in (7.20):

$$\begin{gathered} -K\,dz \text{ for } P, \\ z - z' \text{ for } z, \\ r' = \sqrt{\rho^2 + (z - z')^2} \text{ for } r, \end{gathered} \tag{7.27}$$

and integrate over all positive z'. That will show that the integral of $1\ /\ r'$ is logarithmically divergent. However, from **Kröner**'s formulas ([**2**], pp. 156), it is also permissible in cylindrical coordinates to add arbitrary constants to the components of the principal diagonal, except that these constants must be the same for $\gamma_{\rho\rho}$ and $\gamma_{\varphi\varphi}$. A glimpse at (7.20) will then show that one can replace:

$$\int_0^\infty \frac{1}{r'}\,dz' \qquad \text{with} \qquad \int_0^\infty \left(\frac{1}{r'} - \frac{1}{z' - \frac{1}{2}}\right) dz',$$

although the constants are also essential for the other integrals. With:

$$\begin{aligned} &\int_0^\infty \left(\frac{1}{r'} - \frac{1}{z' - \frac{1}{2}}\right) dz' = -\ln(r - z), \\ &\int_0^\infty \frac{\rho^2}{r'^3}\,dz' = \frac{z}{r} + 1, \\ &\int_0^\infty \frac{\rho(z - z')}{r'^3}\,dz' = -\frac{\rho}{r}, \end{aligned} \tag{7.28}$$

one will then get the stress function tensor of the isolated force K in the positive z-direction in cylindrical coordinates:

$$\underline{\gamma}^K = \frac{K}{16\pi G} \begin{pmatrix} -\frac{1}{m+1}\left(\frac{z}{r}+1\right) - \frac{m}{m+1}\ln(r-z) & 0 & \frac{1}{m+1}\frac{\rho}{r} \\ 0 & -\left(\frac{z}{r}+1\right) - \frac{m}{m+1}\ln(r-z) & 0 \\ \frac{1}{m+1}\frac{\rho}{r} & 0 & \frac{1}{m+1}\left(\ln(r-z) + \frac{z}{r}\right) \end{pmatrix}, \tag{7.29}$$

in which only the singularity along the positive z-axis (connecting hose) is to be adopted. Naturally, one must be able to lay the singularity along a different curve, but the tensor (7.29), whose symmetry is suited to the problem, is certainly the simplest solution.

In order to examine the singularity with the line integrals (1.5) and (1.6), one will require the stress function tensor $\underline{\chi}$ itself. From the inversion of (5.29) that was given by **Kröner** ([**2**], (II.18)), it will follow from (7.29) that:

$$\underline{\chi}^K = \frac{K}{16\pi G}\begin{pmatrix} -\dfrac{m+2}{m+1}\ln(r-z)-\dfrac{2m}{m^2-1}\left(\dfrac{z}{r}+1\right) & 0 & \dfrac{1}{m+1}\dfrac{\rho}{r} \\ 0 & -\dfrac{m+2}{m+1}\ln(r-z)-\dfrac{m}{m-1}\left(\dfrac{z}{r}+1\right) & 0 \\ \dfrac{1}{m+1}\dfrac{\rho}{r} & 0 & \dfrac{2}{m^2-1}\left(\dfrac{z}{r}-1\right) \end{pmatrix}. \tag{7.30}$$

We choose a circle r, z = const. around the z-axis to be our integration path. Only the components of the second row will enter into the line integral, and of them, only $\chi_{\varphi\varphi}$ is, in turn, non-zero, and since the integrand does not depend upon φ, it will follow directly that:

$$\oint d\mathfrak{r}\cdot\underline{\chi} = \int_0^{2\pi}\rho\, d\varphi\, \mathfrak{e}_\varphi\cdot\underline{\chi} = \int_0^{2\pi}\rho\, d\varphi\, \chi_{\varphi\varphi}(r,z)\,\mathfrak{e}_\varphi = 0. \tag{7.31}$$

We will further require the integral:

$$\oint d\mathfrak{r}\cdot\underline{\chi}\times\nabla = \int_0^{2\pi}\rho\, d\varphi\,\mathfrak{e}_\varphi\cdot(\underline{\chi}\times\nabla), \tag{7.32}$$

and, with $\mathfrak{r}_0 = z\,\mathfrak{k}$:

$$\oint (d\mathfrak{r}\cdot\underline{\chi}\times\nabla)\times(\mathfrak{r}-\mathfrak{r}_0) = \int_0^{2\pi}\rho\, d\varphi\,\mathfrak{e}_\varphi\cdot(\underline{\chi}\times\nabla)\times\rho\,\mathfrak{e}_\rho\,. \tag{7.33}$$

After a somewhat lengthy intermediate calculation that will be omitted here, it will follow in cylindrical coordinates that:

$$\underline{\chi}\times\nabla = \begin{pmatrix} -\dfrac{1}{\rho}\dfrac{\partial\chi_{\rho z}}{\partial\varphi}+\dfrac{\partial\chi_{\rho\varphi}}{\partial z}+\dfrac{1}{\rho}\chi_{\varphi z} & -\dfrac{\partial\chi_{\rho\rho}}{\partial z}+\dfrac{\partial\chi_{\rho z}}{\partial\rho} & -\dfrac{1}{\rho}\dfrac{\partial(\rho\,\chi_{\rho z})}{\partial\rho}+\dfrac{1}{\rho}\dfrac{\partial\chi_{\rho\rho}}{\partial\varphi}-\dfrac{1}{\rho}\chi_{\varphi\rho} \\ -\dfrac{1}{\rho}\dfrac{\partial\chi_{\varphi z}}{\partial\varphi}+\dfrac{\partial\chi_{\varphi\varphi}}{\partial z}-\dfrac{1}{\rho}\chi_{\rho z} & -\dfrac{\partial\chi_{\varphi\rho}}{\partial z}+\dfrac{\partial\chi_{\varphi z}}{\partial\rho} & -\dfrac{1}{\rho}\dfrac{\partial(\rho\,\chi_{\varphi\varphi})}{\partial\rho}+\dfrac{1}{\rho}\dfrac{\partial\chi_{\varphi\rho}}{\partial\varphi}+\dfrac{1}{\rho}\chi_{\rho\rho} \\ -\dfrac{1}{\rho}\dfrac{\partial\chi_{zz}}{\partial\varphi}+\dfrac{\partial\chi_{z\varphi}}{\partial z} & -\dfrac{\partial\chi_{z\rho}}{\partial z}+\dfrac{\partial\chi_{zz}}{\partial\rho} & -\dfrac{1}{\rho}\dfrac{\partial(\rho\,\chi_{z\varphi})}{\partial\rho}+\dfrac{1}{\rho}\dfrac{\partial\chi_{z\rho}}{\partial\varphi} \end{pmatrix}, \tag{7.34}$$

and after substituting (7.30) in this:

$$\mathfrak{e}_\rho \cdot (\underline{\chi} \times \nabla)$$

$$= \frac{K}{8\pi}\left\{\left(-\frac{1}{r} + \frac{m}{m+1}\frac{z^3}{r^3}\right)\mathfrak{e}_\rho + \left[\frac{m}{m+1}\frac{1}{\rho}\left(\frac{z}{r}+1\right) + \frac{m+2}{m+1}\frac{\rho}{r(r-z)} - \frac{m}{m+1}\frac{\rho z}{r^3}\right]\mathfrak{k}\right\}. \qquad (7.35)$$

One infers the vanishing of (7.33) from that directly. All that remains then is the contribution from (7.32), and from the dyname on the edge that faces the origin of a circular cut surface through the z-axis (the normal to the cut-edge is + $\mathfrak{k}$), the resultant force is, from (1.5):

$$\vec{P} = \oint d\mathfrak{r} \cdot \underline{\chi} \times \nabla = \frac{K\rho}{4}[\ldots]\,\mathfrak{k}. \qquad (7.36)$$

If one now contracts the circumference over the positive z-axis then that will yield:

$$\lim_{\rho\to 0}\frac{1}{\rho}\left(\frac{z}{r}+1\right) = \frac{2}{\rho}; \qquad \lim_{\rho\to 0}\frac{\rho}{r(r-z)} = \frac{2}{\rho}; \qquad \lim_{\rho\to 0}\frac{\rho z}{r^3} = 0 \qquad (7.37)$$

for the only term in the square brackets. All terms will vanish upon contracting over the negative z-axis. It will then follow that:

$$\lim_{\rho\to 0}\vec{P} = \begin{cases} K\,\mathfrak{k} & \text{for} \quad z > 0 \\ 0 & \text{for} \quad z < 0 \end{cases} \qquad (7.38)$$

The interpretation as a singularity of a fictitious proper stress state in analogy to (6.12-13) and Fig. 9, pp. 36 is then obvious.

5. The stress function tensor of a double force with moment. We imagine a force $K\,\mathfrak{k}$ at the origin and a force $-K\,\mathfrak{k}$ at the location:

$$-\delta\mathfrak{l} = -i\,\delta l. \qquad (7.39)$$

The double force has the moment:

$$-K\,\delta l\,\mathfrak{j} = -M\,\mathfrak{j}. \qquad (7.40)$$

One obtains the stress functions of that double force for $|\mathfrak{r}| \leq |\delta\mathfrak{l}|$ from:

$$\underline{\chi}^{-M^*\mathfrak{j}} = -\underline{\chi}^K(\mathfrak{r} + \delta\mathfrak{l}) + \underline{\chi}^K(\mathfrak{r}) = -\delta\mathfrak{l}\cdot\nabla\,\underline{\chi}^K(\mathfrak{r})$$

$$= -\delta l\,(\mathfrak{e}_\rho\cos\varphi - \mathfrak{e}_\varphi\sin\varphi)\cdot\left(\mathfrak{e}_\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho}\mathfrak{e}_\varphi\frac{\partial}{\partial\varphi}\right)\underline{\chi}^K(\mathfrak{r})\,(\mathfrak{r})$$

$$= \delta l \left(-\cos\varphi \frac{\partial}{\partial\rho} + \frac{1}{\rho}\sin\varphi \frac{\partial}{\partial\varphi} \right) \underline{\chi}^{K}(\mathfrak{r})\,. \tag{7.41}$$

The asterisk in the index shall then remind one that one is by no means dealing with a pure moment singularity here. Such a thing can be first arrived at by adding a composed force-dipole:

$$\underline{P}^{*} = -\tfrac{1}{2}\,\delta l\, K\,(\mathfrak{k}\;\mathfrak{i} + \mathfrak{i}\;\mathfrak{k}) \tag{7.42}$$

whose stress functions can, in turn, be arrived at by superimposing two simple force-dipoles of opposite signs with $P = \pm\frac{K}{2}\delta l$ in 2 and 1 medians in the zx-plane, so by superimposing two tensors of type (7.20) after rotating by $\pm\ \pi / 4$ around the y-axis. Another possibility for arriving at a pure moment singularity consists of adding a further double-force with moment that has been rotated by $\pi / 2$ around the y-axis (in the plane, cf. [**23**]). In order to do that, the tensor (7.41) must be rotated, or when a second singular line should be avoided, a new stress function tensor that is composed of the stress functions of the isolated force in the x-direction with singular connecting hose on the positive z-axis. We shall dispense with that extension and then get (*):

$$\underline{\chi}^{-M^{*}\mathfrak{j}} = \frac{M}{8\pi}\begin{pmatrix} \cos\varphi\left(\frac{m+2}{m+1}\frac{\rho}{r(r-z)} - \frac{2m}{m^2-1}\frac{\rho z}{r^3}\right) & \frac{\sin\varphi}{\rho}\frac{m}{m+1}\left(\frac{z}{r}+1\right) & -\frac{\cos\varphi}{m+1}\frac{z^2}{r^3} \\ \frac{\sin\varphi}{\rho}\frac{m}{m+1}\left(\frac{z}{r}+1\right) & \cos\varphi\left(\frac{m+2}{m+1}\frac{\rho}{r(r-z)} - \frac{m}{m^2-1}\frac{\rho z}{r^3}\right) & \frac{\sin\varphi}{m+1}\frac{1}{r} \\ -\frac{\cos\varphi}{m+1}\frac{z^2}{r^3} & \frac{\sin\varphi}{m+1}\frac{1}{r} & \frac{2\cos\varphi}{m^2-1}\frac{\rho z}{r^3} \end{pmatrix} \tag{7.43}$$

from (7.41) by calculation.

For the determination of the contour integrals (1.5) and (1.6) around circles around the z-axis, one next calculates (7.31). One gets from (7.43), with (7.37), after a brief calculation:

$$\lim_{\rho\to 0}\oint d\mathfrak{r}\cdot\underline{\chi} = \begin{cases} M/2\,\mathfrak{j} & \text{for} \quad z>0 \\ 0 & \text{for} \quad z<0 \end{cases}. \tag{7.44}$$

For the two integrals (7.32-33), one will get from (7.34) and (7.37):

(*) Obviously, the basis vectors must also be differentiated in this, or – what amounts to the same thing – the scalar components must be covariantly differentiated.

$$\lim_{\rho\to 0} \mathfrak{e}_\varphi \cdot (\underline{\chi}^{-M^*\mathfrak{j}} \times \nabla) = \begin{cases} \dfrac{M}{4\pi}\cos\varphi\left(\dfrac{1}{r^2}\dfrac{m^2+2m-2}{m^2-1}+\dfrac{2}{\rho^2}\right)\mathfrak{k} & \text{for} \quad z>0 \\ -\dfrac{M}{4\pi}\cos\varphi\dfrac{1}{r^2}\dfrac{m^2+m-1}{m^2-1}\mathfrak{k} & \text{for} \quad z<0 \end{cases}. \tag{7.45}$$

Integration before the ultimate passage to the limit then shows that (7.32) will vanish. Substitution in (7.33) will finally give:

$$\lim_{\rho\to 0} \oint d\mathfrak{r} \cdot (\underline{\chi} \times \nabla) \times (\mathfrak{r} - \mathfrak{r}_0) = \begin{cases} M/2\,\mathfrak{j} & \text{for} \quad z>0 \\ 0 & \text{for} \quad z<0 \end{cases}. \tag{7.46}$$

The result of adding (1.5) and (1.6) will be:

$$\lim_{\rho\to 0} \vec{P} = 0 \tag{7.47}$$

and

$$\lim_{\rho\to 0} \vec{M}_0 = \begin{cases} -M\,\mathfrak{j} & \text{for} \quad z>0 \\ 0 & \text{for} \quad z<0 \end{cases}. \tag{7.48}$$

The singular supplementary stress then corresponds roughly to the stress state of a beam under a constant bending moment. No further statements about the details of this singular stress process are possible or meaningful. In particular, the distribution of the moment in the integrals (7.44) and (7.46) can be altered arbitrarily by the addition of a deformator. It will then depend upon a special Ansatz for the stress functions and will have no physical meaning beyond that.

8. The *Schaefer* stress function tensor for the isolated force in infinite space.

On the same grounds as in § **7**, the actual load stress state can also be replaced with a fictitious proper stress state that we choose (likewise with the replacement of the supplementary compression with a supplementary tension) in precisely the way that was described at the beginning of § **7**. Up to now, nothing has been published on the solution of the proper stress problems with **Schaefer** stress functions, which is why some general remarks on that subject shall next be made. As is known (cf., e.g., [**2**, **5**, **7**, **21**, **30**, **31**]), when a strain that is not produced by elastic stresses is present – namely, a so-called supplementary strain $\underline{\varepsilon}^G$ – the equation:

$$\underline{\varepsilon}^G = \operatorname{Def} \mathfrak{u} = \underline{\varepsilon}^Q + \underline{\varepsilon}^E \tag{8.1}$$

will be true. $\underline{\varepsilon}^Q$ can originate in, e.g., plastic deformation, heating, magnetostriction, etc. Furthermore, **Hooke**'s law:

$$\underline{\sigma}^G = \underline{c} \cdot\cdot\, \underline{\varepsilon}^E \tag{8.2}$$

is true for elastic strain $\underline{\varepsilon}^E$ and the total stress $\underline{\sigma}^G$, and the equilibrium condition (1.3) for $\underline{\sigma}^G$ can be fulfilled identically, since it can be derived from (1.2) by differentiating a stress function tensor, However, what now enters in place of the differential equation (2.10) is:

$$\text{Ink}\;(\underline{s}\cdot\cdot\text{Ink}\,\underline{\chi}) = \underline{\eta}, \tag{8.3}$$

with

$$\underline{\eta} = -\,\text{Ink}\;\underline{\varepsilon}^Q. \tag{8.4}$$

Now, according to **Schaefer** (*), (8.3) will be fulfilled identically in an isotropic medium, when one replaces (5.42) with:

$$\begin{aligned}
&\underline{\chi} = \underline{\psi} + \Omega\,\text{I},\\
&\Delta\Omega = \frac{m}{m-1}(\nabla\cdot\underline{\psi}\cdot\nabla - \Delta\,\psi_{\text{I}}),\\
&\Delta\underline{\psi} = -\,2G\;\underline{\varepsilon}^Q.
\end{aligned} \tag{8.5}$$

We prove this with the help of the known development of the operator Ink (cf., e.g., [**12**], (10)):

$$\begin{aligned}
&\text{Ink}\,\underline{\chi} = \Delta\underline{\chi} - \nabla\nabla\cdot\underline{\chi} - \underline{\chi}\cdot\nabla\nabla + \nabla\cdot\underline{\chi}\cdot\nabla\text{I} + \nabla\nabla\,\chi_{\text{I}} - \Delta\,\chi_{\text{I}}\,\text{I},\\
&(\text{Ink}\,\underline{\chi})_{\text{I}} = \nabla\cdot\underline{\chi}\cdot\nabla - \Delta\,\chi_{\text{I}}.
\end{aligned} \tag{8.6}$$

Applying this to the first equation (8.5) yields:

$$\begin{aligned}
\underline{\sigma}^G &= \text{Ink}\,\underline{\chi} = \text{Ink}\,\underline{\psi} + (\nabla\nabla - \Delta\text{I})\,\Omega\\
&= -\,2G\left(\underline{\varepsilon}^Q + \frac{\varepsilon_{\text{I}}^Q}{m-1}\text{I}\right) - 2\text{Def}\;\nabla\underline{\psi} - \frac{1}{m-1}\nabla\cdot\underline{\psi}\cdot\nabla\text{I} + \nabla\nabla\,\psi_{\text{I}} + \nabla\nabla\Omega
\end{aligned} \tag{8.7}$$

and

$$\begin{aligned}
\underline{\varepsilon}^E &= \underline{s}\cdot\cdot\;\text{Ink}\,\underline{\chi} = \frac{1}{2G}\left[\text{Ink}\,\underline{\chi} - \frac{1}{m+1}(\nabla\cdot\underline{\chi}\cdot\nabla - \Delta\chi_{\text{I}})\text{I}\right]\\
&= -\;\underline{\varepsilon}^Q + \frac{1}{2G}(-\,2\text{Def}\;\nabla\cdot\underline{\psi} + \nabla\nabla\,\psi_{\text{I}} + \nabla\nabla\Omega)\,.
\end{aligned} \tag{8.8}$$

A pure deformator appears in the last parentheses; hence, the fulfillment of (8.3) will be obvious. A comparison with (8.1) will yield (**):

(*) Oral communication.

(**) The agreement will follow directly from the substitution (5.41). Due to the change in the sign convention from [**12**], ([**11**], (1.6) will be true with the opposite signs.

$$\mathfrak{u} = \frac{1}{G}\left(-\nabla\cdot\underline{\psi}+\tfrac{1}{2}\nabla\psi_{\mathrm{I}}+\tfrac{1}{2}\nabla\Omega\right) \tag{8.9}$$

as the extension of **Schaefer**'s formulas ([**11**], (1.6)) and ([**12**], (7.18)) to the case of proper stresses, up to a rigid motion.

We now assume that the fictitious supplementary stress for the force $K\,\mathfrak{k}$ is a line singularity along the positive z-axis. In the symbolic notation of the theory of distributions, one will then have:

$$\underline{\sigma}^{K^*} = K\,\delta^{(1)}(x)\;\delta^{(1)}(y)\;\delta^{(1)}(z)\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} \tag{8.10}$$

in cylindrical and Cartesian coordinates. One calculates the associated fictitious supplementary strain from this by using the negative of **Hooke**'s law [**5**], ([**7**], (1.5)):

$$\underline{\varepsilon}^{K^*} = \frac{K}{2G}\,\delta^{(1)}(x)\;\delta^{(1)}(y)\;\delta^{(1)}(z)\begin{pmatrix}\dfrac{1}{m+1}&0&0\\0&\dfrac{1}{m+1}&0\\0&0&-\dfrac{m}{m+1}\end{pmatrix}. \tag{8.11}$$

In order to determine $\underline{\psi}$, from the third equation in (8.5), one will then require the potential $U(\mathfrak{r})$ of the unit-source distribution along the positive z-axis. From (7.28), one will then easily derive that it is:

$$U(\mathfrak{r}) = \frac{1}{4\pi}\ln(r-z), \tag{8.12}$$

and find that:

$$\underline{\psi} = \frac{K\ln(r-z)}{4\pi(m+1)}\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&m\end{pmatrix}. \tag{8.13}$$

In order to calculate Ω, one will then require:

$$\nabla\cdot\underline{\psi}\cdot\nabla-\Delta\psi_{\mathrm{I}} = \frac{K}{4\pi}\left\{\frac{z}{r^3}-\frac{m-1}{m+1}\Delta\ln(r-z)\right\}, \tag{8.14}$$

and one will then get:

$$\Omega = \frac{K}{4\pi}\left\{-\frac{m}{2(m-1)}\frac{z}{r}-\frac{m-1}{m+1}\Delta\ln(r-z)\right\} \tag{8.15}$$

from the second equation in (8.5), as one easily verifies. By combining this with the first equation in (8.5), one will finally get the **Schaefer** stress function tensor of the isolated force in infinite space:

$$\underline{\chi}^K = \frac{K}{4\pi}\left\{\begin{matrix} -\ln(r-z)-\dfrac{m}{2(m-1)}\dfrac{z}{r} & 0 & 0 \\ 0 & -\ln(r-z)-\dfrac{m}{2(m-1)}\dfrac{z}{r} & 0 \\ 0 & 0 & \dfrac{m}{2(m-1)}\dfrac{z}{r} \end{matrix}\right\}, \tag{8.16}$$

and indeed, only the main diagonal will figure in cylindrical, as well as Cartesian, coordinates.

In order to investigate the singularity, one must once more calculate the integrals (7.31-33). One sees immediately that (7.31) must vanish. With (7.34), it will then follow that:

$$\mathfrak{e}_\varphi\cdot(\underline{\chi}^K\times\nabla) = \frac{K}{4\pi}\left\{\mathfrak{e}_\rho\left(\frac{1}{r}-\frac{m}{2(m-1)}\frac{\rho^2}{r^3}\right)+\mathfrak{k}\left(\frac{\rho}{r(r-z)}-\frac{m}{2(m-1)}\frac{\rho z}{r^3}\right)\right\}. \tag{8.17}$$

(7.33) also vanishes in the integration independently of the limiting value of the contents inside the brackets in (8.17), and from (1.5) and (7.32), along with (7.37), what will remain is:

$$\lim_{\rho\to 0}\vec{P} = \lim_{\rho\to 0}\oint d\mathfrak{r}\cdot\underline{\chi}\times\nabla = \begin{cases} K\,\mathfrak{k} & \text{for} \quad z>0 \\ 0 & \text{for} \quad z<0. \end{cases} \tag{8.18}$$

One will then get the singularity (7.38), as expected. We shall omit the calculation of the stress function tensor for the double force here.

We shall reach our goal noticeably faster with **Schaefer**'s Ansatz than with the **Kröner-Marguerre** stress functions. Naturally, that is based in part upon the fact that we have employed the results of the previous paragraph in it and in part upon the fact that we have saved ourselves one integration step by the immediate use of the supplementary strain that the **Schaefer** Ansatz makes possible. That advantage will be in effect everywhere that the supplementary strain itself is given or it can be integrated very easily from the dislocation density (up to an inessential deformator). However, normally for a given dislocation density $\underline{\alpha}$, the integration along a contour using the equation that **Kröner** gave ([**2**], (II.105)) or the integral formula:

$$\underline{\gamma} = \frac{1}{8\pi}\left\{\int\underline{\alpha}(\mathfrak{r}')\times\frac{\mathfrak{r}'-\mathfrak{r}}{|\mathfrak{r}'-\mathfrak{r}|}d\tau\right\}^{sym} \tag{8.19}$$

that is extended over all of infinite space and that is easily derived from the **Kröner** formula, like (2.20) and (2.26), will lead to the goal much faster.

Which Ansatz one prefers to employ for the stress functions will then depend entirely upon the details of the special case.

Added in proof: The stress functions (8.16) were found already by **Indenbom** ([**46**], eq. (17)), up to the different positions of the singularities and an error in the logarithm. The asymmetric first-order stress function tensor that is given there ([**46**], eq. (14-15)) is also interesting. Its examination with the formulas:

$$\vec{P} = \oint d\mathfrak{r} \cdot \underline{\varphi} \tag{8.20}$$

and

$$\vec{M}_0 = -\oint d\mathfrak{r} \cdot \underline{\varphi} \times (\mathfrak{r} - \mathfrak{r}_0) + \oint \underline{\varphi} \cdot d\mathfrak{f}\,, \tag{8.21}$$

which correspond to the **Peretti-Günther** integrals (1.5-6), from (2.21), reveal singular proper stresses along the six coordinate half-axes, which the isolated force at the origin "produce" in equal parts. The meaning of these singularities for the problem that was studied by **Indenbom** and the influence function for the proper stress sources on the displacement shall be reported on another occasion [**47**].

9. The *Schaefer* stress function tensor for the load singularity in half-space.

(The **Boussinesq-Ceruti** problem)

Schaefer [**11**] derived the outer surface conditions for the stress function tensor for a given outer surface loading on the basis of differential-static considerations and then solved the boundary-value problem for a general load singularity on the half-space, in particular. We seek to achieve the same goal in a different way, whose basic ideas are likewise found in **Schaefer** in a different place [**19**], and there they were employed in order to exhibit the stress functions of the two-dimensional, planar continuum [cf., (5.27-28)]. The load-stress problem was also replaced with a fictitious proper stress problem in it. However, the fictitious supplementary stress no longer lies in the domain (*B*) that was investigated in Fig. 11, pp. 38, but in the bar (*D*) and the surface (*F*) that are put in it, which carry forces and moments from one part of the outer surface of (*B*) to another. The singular stress state in the bar will be given from (5.6) with a suitable multi-valuedness of the generating vector $\mathfrak{A}$ for the null-stress functions in external space.

One obtains the case that **Schaefer** treated of the plane (disc, resp.) by flattening the region (*B*) and for the isolated dyname, by bending the bar until it is perpendicular to the plane of the plate (§ **5***c*, 2.). We also bend the bar in that way for the present problem, but separate the region (*B*) to the lower half-space. In order to employ **Schafer**'s stress function, the null-stress function (5.45) must also satisfy the derived auxiliary conditions. Here, we shall use the sufficient condition that the generated vector must be harmonic (*), so:

(*) Here, we understand a harmonic vector to be a vector whose components in Cartesian coordinates are harmonic. The absence of sources and vortices will not be postulated.

$$\Delta\mathfrak{A} = 0. \tag{9.1}$$

From (1.5) and (1.6), the transition to the interior of the body will then be mediated by the “static boundary conditions” (cf., also the footnote on pp. 11, eq. (2.11a,b)):

$$\mathfrak{n} \times \underline{\chi} = \mathfrak{n} \times \mathrm{Def}\,\mathfrak{A}, \tag{9.2}$$

$$\mathfrak{n} \cdot \nabla\mathfrak{n} \times \underline{\chi} \times \mathfrak{n} = \mathfrak{n} \cdot \nabla\mathfrak{n} \times \mathrm{Def}\,\mathfrak{A} \times \mathfrak{n}\,; \tag{9.3}$$

they are to be fulfilled by a harmonic tensor $\underline{\psi}$ and a scalar Ω using (5.42).

a) *The **Boussinesq** problem on a half-space.* – We choose our reference point $\mathfrak{r}_0$ in (5.6) to be the point of application *O* of the force $K_z\,\mathfrak{k}$. The required jump around a circuit around the singular bar in the *z*-axis will then be:

$$\mathfrak{A}_2 - \mathfrak{A}_1 = -K_z\,\rho\,\mathfrak{e}_\varphi\,. \tag{9.4}$$

One vector field that fulfills this jump condition is:

$$\mathfrak{A}' = -\frac{K_z}{2\pi}\rho\,\varphi\,\mathfrak{e}_\varphi\,. \tag{9.5}$$

However, it is not harmonic; one has:

$$\Delta\mathfrak{A}' = -\frac{K_z}{2\pi}\left(-\frac{2\,\mathfrak{e}_\rho}{\rho}\right), \tag{9.6}$$

moreover. Now, one must determine a single-valued vector field that cancels the right-hand side of (9.6) exactly; one finds such a thing in (*):

$$\mathfrak{A}'' = -\frac{K_z}{2\pi}\rho\ln\rho\,. \tag{9.7}$$

Furthermore, in order to eliminate a superfluous constant in the deformator, we add:

$$\mathfrak{A}''' = -\frac{K_z}{2\pi}(-\rho\,\mathfrak{e}_\rho) \tag{9.8}$$

(*) A generating vector of this form can lead to a stress singularity of order at most r^{-2}, if at all; however, such a thing would be excluded by its single-valuedness. According to **Sternberg** and **Eubanks** [**40**], the absence of higher singularities will then be guaranteed.

and thus obtain the ultimate generating vector field in the external space:

$$\mathfrak{A} = \mathfrak{A}' + \mathfrak{A}'' + \mathfrak{A}''' = -\frac{K_z}{2\pi}[\rho\,\varphi\,\mathfrak{e}_\rho + \rho(\ln\rho - 1)\,\mathfrak{e}_\rho]. \tag{9.9}$$

We then calculate the deformator:

$$\underline{\chi}^a = \text{Def}\,\mathfrak{A} = -\frac{K_z}{2\pi}\ln\rho\,(\mathfrak{e}_\rho\,\mathfrak{e}_\rho + \mathfrak{e}_\varphi\,\mathfrak{e}_\varphi), \tag{9.10}$$

and obtain the static boundary conditions from this using (9.2):

$$\underline{\chi}\Big|_{z=0} = -\frac{K_z}{2\pi}\begin{pmatrix}\ln\rho & 0 & 0\\ 0 & \ln\rho & 0\\ 0 & 0 & -\end{pmatrix}. \tag{9.11}$$

(The line means "no boundary condition"), and from (9.3):

$$\frac{\partial}{\partial z}\underline{\chi}\Big|_{z=0} = -\frac{K_z}{2\pi}\begin{pmatrix}0 & 0 & -\\ 0 & 0 & -\\ - & - & -\end{pmatrix}. \tag{9.12}$$

The boundary conditions (9.11) will now be fulfilled by the harmonic tensor:

$$\underline{\psi} = \begin{pmatrix}-\dfrac{K_z}{2\pi}\ln(r-z) & 0 & 0\\ 0 & -\dfrac{K_z}{2\pi}\ln(r-z) & 0\\ 0 & 0 & \psi_{zz}\end{pmatrix} \tag{9.13}$$

with ψ_{zz} undetermined. If one sets:

$$\Omega = -\frac{K}{2\pi}\frac{z}{r}, \tag{9.14}$$

in addition, which is then the product of z with a harmonic function, then the fulfillment of (9.12) will also be ensured. However, one has, on the other hand, from the second equation in (5.42):

$$\Delta\Omega = \frac{m}{m-1}\left(\frac{K_z}{2\pi}\frac{z}{r^3} + \frac{\partial^2\psi_{zz}}{\partial z^2}\right), \tag{9.15}$$

which leads to:

$$\frac{\partial^2 \psi_{zz}}{\partial z^2} = \frac{K_z}{2\pi}\frac{m-2}{m}\frac{z}{r^2}. \tag{9.16}$$

The harmonic solution of (9.16) that is less-than-linearly increasing at infinity is:

$$\psi_{zz} = \frac{K_z}{2\pi}\frac{m-2}{m}\ln(r-z). \tag{9.17}$$

For the **Boussinesq** problem, that yields **Schaefer**'s stress function tensor directly:

$$\underline{\chi}^{K_z} = \frac{K_z}{2\pi}\begin{pmatrix} -\ln(r-z) & 0 & 0 \\ 0 & -\ln(r-z) - \frac{z}{r} & 0 \\ 0 & 0 & \frac{m-2}{m}\ln(r-z) - \frac{z}{r} \end{pmatrix}, \tag{9.20}$$

which agrees with **Schaefer**'s result [**11**].

Added in proof: The same stress function tensor was found (up to an exchange of the upper and lower half-spaces and an error in the logarithm) recently by **Schaefer** [**11**], as well as **Indenbom** ([**46**], eq. (19)), and in order to calculate the displacement of the outer surface in the stress field, an edge dislocation that was perpendicular to it was employed.

b) *The **Cerutti** problem on a half-space.* – If one again places the reference point $\mathfrak{r}_0$ at the point of application O of the force with the components K_x and K_y then, from (5.6), the generating vector must jump by:

$$\mathfrak{A}_2 - \mathfrak{A}_1 = -(K_x\,\mathfrak{i} + K_y\,\mathfrak{j}) \times \mathfrak{r} = -K_y\,z\,\mathfrak{i} + K_x\,z\,\mathfrak{j} + (K_x\,y + K_y\,x)\,\mathfrak{k}, \tag{9.21}$$

after a circuit around the z-axis. Such a jump will possess the vector field:

$$\mathfrak{A}' = \frac{z\varphi}{2\pi}(-K_y\,\mathfrak{i} + K_x\,\mathfrak{j}) - \frac{y\varphi}{2\pi}K_x\,\mathfrak{k} + \frac{x\varphi}{2\pi}K_y\,\mathfrak{k}. \tag{9.22}$$

However, it is not harmonic; moreover, one has:

$$\Delta\mathfrak{A}' = \frac{1}{2\pi}\left(-\frac{2x}{\rho^2}K_x\mathfrak{k} - \frac{2y}{\rho^2}K_y\mathfrak{k}\right). \tag{9.23}$$

This will be balanced out by the single-valued vector field (*)(**):

(*) The component functions are known from the **Airy** stress function of the edge dislocation [**3**].
(**) Cf., footnote to (9.7), pp. 53.

$$\mathfrak{A}'' = \frac{1}{2\pi}(x \ln\rho\, K_x\, \mathfrak{k} + y \ln\rho\, K_y\, \mathfrak{k}) \,. \tag{9.24}$$

The gradient tensor in Cartesian coordinates follows from these two vector fields:

$$\nabla\,(\mathfrak{A}' + \mathfrak{A}'') = \frac{1}{2\pi}\begin{pmatrix} \frac{yz}{\rho^2}K_y & -\frac{yz}{\rho^2}K_x & (\ln\rho - 1)K_x + \varphi K_y \\ -\frac{xz}{\rho^2}K_y & \frac{xz}{\rho^2}K_x & (\ln\rho - 1)K_y - \varphi K_x \\ -\varphi K_y & \varphi K_x & 0 \end{pmatrix} \tag{9.25}$$

The boundary conditions that are derived from the symmetric part of this tensor lead to a contradiction for the harmonic $\psi_{xy} = \chi_{xy}$. On the one hand, from (9.2), χ_{xy} must vanish on the boundary then and therefore in the entire half-space; on the other hand, the boundary condition (9.3) with (9.25) requires a non-vanishing normal derivative. In order to eliminate this contradiction by setting the normal derivative equal to zero, we add another null-stress function tensor with the harmonic and single-valued generating vector (*):

$$\mathfrak{A}''' = \frac{1}{2\pi} z \ln\rho\,(K_x\, \mathfrak{i} + K_y\, \mathfrak{j}), \tag{9.26}$$

and in order to eliminate the superfluous constant, we add:

$$\mathfrak{A}'''' = -\frac{1}{2\pi}(x\, K_x + y\, K_y)\, \mathfrak{k} \,. \tag{9.26a}$$

Combining them leads to the tensor of the null-stress functions in external space:

$$\underline{\chi}^a = \mathrm{Def}\,(\mathfrak{A}' + \mathfrak{A}'' + \mathfrak{A}''' + \mathfrak{A}'''') = \frac{1}{2\pi}\begin{pmatrix} (xK_x + yK_y)\frac{z}{\rho^2} & 0 & K_x \ln\rho \\ 0 & (xK_x + yK_y)\frac{z}{\rho^2} & K_y \ln\rho \\ K_x \ln\rho & K_y \ln\rho & 0 \end{pmatrix} \tag{9.27}$$

and therefore to the boundary conditions:

$$\underline{\chi}\Big|_{z=0} = \frac{1}{2\pi}\begin{pmatrix} 0 & 0 & K_x \ln\rho \\ 0 & 0 & K_y \ln\rho \\ K_x \ln\rho & K_y \ln\rho & - \end{pmatrix} \tag{9.28}$$

(*) Cf., footnote to (9.7), pp. 53.

and

$$\frac{\partial}{\partial z}\underline{\chi}\bigg|_{z=0} = \frac{1}{2\pi}\begin{pmatrix} (xK_x + yK_y)\dfrac{1}{\rho^2} & 0 & - \\ 0 & (xK_x + yK_y)\dfrac{1}{\rho^2} & - \\ - & - & - \end{pmatrix}. \tag{9.29}$$

If follows directly from this that:

$$\underline{\psi} = \frac{1}{2\pi}\begin{pmatrix} 0 & 0 & K_x \ln(r-z) \\ 0 & 0 & K_y \ln(r-z) \\ K_x \ln(r-z) & K_y \ln(r-z) & 2\pi\psi_{zz} \end{pmatrix}, \tag{9.30}$$

and from the second equation in (5.42):

$$\Delta\Omega = \frac{1}{2\pi}\frac{m}{m-1}\left(2K_x\frac{\partial^2 \ln(r-z)}{\partial x\,\partial z} + 2K_x\frac{\partial^2 \ln(r-z)}{\partial y\,\partial z} + 2\pi\frac{\partial^2\psi_{zz}}{\partial z^2}\right), \tag{9.31}$$

or, after performing the partial differentiations:

$$\Delta\Omega = \frac{1}{2\pi}\frac{m}{m-1}\frac{\partial}{\partial z}\left(2K_x\frac{x}{r(r-z)} + 2K_y\frac{y}{r(r-z)} + 2\pi\frac{\partial\psi_{zz}}{\partial z}\right). \tag{9.31}$$

On the other hand, the remaining boundary conditions will be fulfilled by:

$$\Omega = \frac{1}{2\pi}\frac{z}{r(r-z)}(x\,K_x + y\,K_y). \tag{9.33}$$

If one applies the **Laplace** operator to this and compares with (9.32) then it will follow that:

$$\frac{\partial^2\psi_{zz}}{\partial z^2} = -\frac{2}{m}\frac{1}{2\pi}\frac{\partial}{\partial z}\left(K_x\frac{x}{r(r-z)} + K_y\frac{y}{r(r-z)}\right). \tag{9.34}$$

ψ_{zz} is then determined up to a product of z with a singularity-free harmonic function of x and y. However, should the stresses die off at infinity then it can be at most linear and will then have effect on the stresses, as one can easily verify. We then set them equal to zero and find:

$$\psi_{zz} = -\frac{1}{2m}\frac{1}{2\pi}\left(K_x \frac{\partial}{\partial x} + K_y \frac{\partial}{\partial y}\right)(z \ln(r-z) + r)$$
$$= -\frac{1}{2m}\frac{1}{2\pi}\left(x K_x + y K_y\right)\left(\frac{z}{r(r-z)} + \frac{1}{r}\right). \tag{9.35}$$

With that, one can assemble the stress function tensor of the **Cerutti** problem from the first equation in (5.42), in agreement with **Schaefer** [**11**]:

$$\underline{\chi}^{K_x,K_y} = \frac{1}{2\pi}\begin{pmatrix} (xK_x + yK_y)\dfrac{z}{r(r-z)} & 0 & K_x \ln(r-z) \\ 0 & (xK_x + yK_y)\dfrac{z}{r(r-z)} & K_y \ln(r-z) \\ K_x \ln(r-z) & K_y \ln(r-z) & (xK_x + yK_y)\left(\dfrac{2m-1}{2m}\dfrac{z}{r(r-z)} - \dfrac{1}{2m}\dfrac{1}{r}\right) \end{pmatrix}. \tag{9.36}$$

We shall forego the corresponding calculations for singular moments [**11**] here.

c) *Overview.* – The application of the fictitious proper stress state in external space to the problem of the singular load stresses that was described is closely related to the idea of extending to a continuous load distribution. In fact, one can now (at least, for the half-space) go backwards from the singularity solution and conclude with **Schaefer**'s equations ([**11**], (3.22)). However, in the general case, one must expand the bar (*D*) in Fig. 11, pp. 38 to a spatial structure or even to all of space; i.e., one will arrive at a second boundary-value problem for external space. The result of the procedure for singular loads is based upon just the exceptional simplicity of the "boundary-value problem" for the singular bar. In the other case, one must continue the continuous outer surface load in a suitable way to an equilibrium stress in external space, then determine a stress function tensor $\underline{\chi}^a$, and then connect that with the internal stress function field with matching conditions of the type (2.11a-b) on the outer surface. One might then expect that in many cases this problem could be simplified very much by a suitable choice of fictitious proper stress state in external space; the example of the half-space is also especially clear here.

Appendix to § 2.

a) *Derivation of (2.14).* – When one substitutes (2.11) and (2.13) in the left-hand side of (2.14) and makes the concomitant repeated applications of the product rule and **Stokes**'s theorem, one will get:

$$\tfrac{1}{2}\int \underline{\chi}^- \cdot\cdot [d\mathfrak{f} \times (\underline{s} \cdot\cdot \operatorname{Ink} \underline{\chi}^-) \times \nabla] - \tfrac{1}{2}\int (\underline{\chi}^- \times \nabla) \cdot\cdot [(\underline{s} \cdot\cdot \operatorname{Ink} \underline{\chi}^-) \times d\mathfrak{f}]$$

$$= \tfrac{1}{8}\int(\nabla\mathfrak{A}^- + \mathfrak{A}^-\nabla)\cdot\cdot(d\mathfrak{f}\times\nabla\mathfrak{u}^-\times\nabla) - \tfrac{1}{8}\int(d\mathfrak{f}\times\nabla\mathfrak{A}^-\times\nabla)\cdot\cdot(\nabla\mathfrak{u}^- + \mathfrak{u}^-\nabla)$$

$$= \tfrac{1}{8}\int(d\mathfrak{f}\times\nabla)\cdot(\nabla\mathfrak{A}^- + \mathfrak{A}^-\nabla)\cdot\mathfrak{u}^-\times\nabla - \tfrac{1}{8}\int(d\mathfrak{f}\times\nabla)\cdot(\nabla\mathfrak{u}^- + \mathfrak{u}^-\nabla)\cdot(\mathfrak{A}^-\times\nabla)$$

$$= -\tfrac{1}{8}\oint d\mathfrak{r}\cdot(\nabla\mathfrak{A}^- + \mathfrak{A}^-\nabla)\cdot(\nabla\times\mathfrak{u}^-) + \tfrac{1}{8}\oint d\mathfrak{r}\cdot(\nabla\mathfrak{u}^- + \mathfrak{u}^-\nabla)\cdot(\nabla\times\mathfrak{A}^-)$$

$$+ \tfrac{1}{8}\int d\mathfrak{f}\cdot(\nabla\times\mathfrak{A}^-)(\nabla\times\mathfrak{u}^-)\cdot\nabla - \tfrac{1}{8}\int d\mathfrak{f}\cdot(\nabla\times\mathfrak{u}^-)(\nabla\times\mathfrak{A}^-)\cdot\nabla$$

$$= -\tfrac{1}{8}\oint d\mathfrak{r}\cdot(\cdots) + \tfrac{1}{8}\oint d\mathfrak{r}\cdot(\cdots) + \tfrac{1}{8}\oint d\mathfrak{r}\cdot\{\nabla\times[(\nabla\times\mathfrak{A}^-)\times(\nabla\times\mathfrak{u}^-)]\}$$

$$= \tfrac{1}{8}\oint d\mathfrak{r}\cdot\{-2\,(\mathrm{Def}\ \mathfrak{A}^-)\cdot(\nabla\times\mathfrak{u}^-) + 2\,(\mathrm{Def}\ \mathfrak{u}^-)\cdot(\nabla\times\mathfrak{A}^-) + (\nabla\times\mathfrak{A}^-)\times(\nabla\times\mathfrak{u}^-)\}\,,$$

which is the right-hand side of (2.14).

b) *Closing flux on the outer surface (to 2.20).* – We assume that the vector of the outer surface flux density has the form $\mathfrak{W}\times\mathfrak{n}$. The continuity condition for each component of the outer surface then demands that:

$$\oint d\mathfrak{r}\cdot\mathfrak{W} = \int d\mathfrak{f}\cdot\mathfrak{v}\,, \tag{2.20a}$$

and therefore, from **Stokes**'s theorem:

$$\int(d\mathfrak{f}\times\nabla)\cdot\mathfrak{W} = \int d\mathfrak{f}\cdot\mathfrak{v}\,. \tag{2.20b}$$

Now let:

$$\mathfrak{r} = \mathfrak{r}\,(u, v, n) \tag{2.20c}$$

be determined in such a way that one has:

$$n = 0; \qquad \nabla n = \mathfrak{n} \tag{2.20d}$$

on the outer surface. If we make the further Ansatz for $\mathfrak{W}$ that:

$$\mathfrak{W} = W^*\frac{\partial\mathfrak{r}}{\partial u} \tag{2.20e}$$

then that will establish the direction of the outer surface flux as perpendicular to the parameter lines v = const. on the outer surface. The ∇-vector on the outer surface can be decomposed ([**9**], § 225) into:

$$\nabla = \frac{1}{\sqrt{EG-F^2}}\left(\frac{\partial\mathfrak{r}}{\partial v}\times\mathfrak{n}\frac{\partial}{\partial u} + \mathfrak{n}\times\frac{\partial\mathfrak{r}}{\partial u}\frac{\partial}{\partial v}\right) + \mathfrak{n}\frac{\partial}{\partial n}\,, \tag{2.20f}$$

in which we have expressed the triple product of the three basis vectors:

$$\left| \frac{\partial \mathfrak{r}}{\partial u} \frac{\partial \mathfrak{r}}{\partial v} \mathfrak{n} \right| = \sqrt{EG - F^2} \tag{2.20g}$$

in terms of the first three fundamental quantities E, F, G ([**9**], § 48) of the outer surface, which is possible here with no further assumptions, since $\mathfrak{n} \perp \frac{\partial \mathfrak{r}}{\partial u}, \frac{\partial \mathfrak{r}}{\partial v}$. Substituting in the left-hand side of (2.20b) will now yield:

$$\begin{aligned}
\int (d\mathfrak{f} \cdot \nabla) \cdot \mathfrak{W} &= \int \left[\frac{d\mathfrak{f}}{\sqrt{EG - F^2}} \times \left(\frac{\partial \mathfrak{r}}{\partial v} \times \mathfrak{n} \frac{\partial}{\partial u} + \mathfrak{n} \times \frac{\partial \mathfrak{r}}{\partial u} \frac{\partial}{\partial v} \right) \right] \cdot W^* \frac{\partial \mathfrak{r}}{\partial u} \\
&= \int \frac{df}{\sqrt{EG - F^2}} \left(\frac{\partial \mathfrak{r}}{\partial v} \frac{\partial}{\partial u} - \frac{\partial \mathfrak{r}}{\partial u} \frac{\partial}{\partial v} \right) \cdot W^* \frac{\partial \mathfrak{r}}{\partial u} \\
&= \int \frac{df}{\sqrt{EG - F^2}} \left\{ F \frac{\partial W^*}{\partial v} - E \frac{\partial W^*}{\partial v} + W^* \left(\frac{\partial F}{\partial u} - \frac{\partial E}{\partial v} \right) \right\}.
\end{aligned} \tag{2.20h}$$

In order to eliminate the derivation with respect to u, we now specialize to an orthogonal system with $F = 0$ and upon introducing the geodetic curvature G_1 of the parameters lines v = const. ([**8**], § 49), we will obtain:

$$\int (d\mathfrak{f} \cdot \nabla) \cdot \mathfrak{W} = \int df \left(-\sqrt{\frac{E}{G}} \frac{\partial W^*}{\partial v} + 2W^* \sqrt{E} G_1 \right). \tag{2.20i}$$

That will be once more substituted in (2.20b). Now, since this equation must be true for any arbitrary surface patch, we can set the integrands equal to each other, and since $d\mathfrak{f} = \mathfrak{n}\, df$, we can thus arrive at the ordinary differential equation for W^*:

$$-\sqrt{\frac{E}{G}} \frac{\partial W^*}{\partial v} + 2W^* \sqrt{E} G_1 = \mathfrak{n} \cdot \mathfrak{v}, \tag{2.20k}$$

which can be solved by a quadrature along each individual flux line u = const. for arbitrarily given initial values for each of them. The dependency on u is arbitrary, to begin with. That quadrature will become especially simple when the parameter lines v = const. are geodetic lines, so the second term of (2.20k) will vanish.

The dependency of the outer surface flux on u is certainly meaningless then when all of the flux lines intersect; i.e., so for a polar coordinate system with two “poles” on the outer surface. However, one must generally abandon the use of geodetic lines ([**4**], pp. 196) then. Another possibility consists of overlapping the outer surface piecewise with geodetic parameter lines v = const. ([**45**], pp. 102, 106). However, since the productivity of the individual surface patches does not need to vanish, linear singular closing fluxes must be introduced into the boundaries of the surface patches. The same thing is true for surfaces of rotation when one lays the flux lines along the parallels. In general, a singular flux line along a meridian must then serve as a sort of "backbone" for the flux in order to balance between the other closed flux lines.

c) *Conversion of (2.25) into (2.26).* – In order to convert, we write the stress as a dyadic sum [**9**]:

$$\underline{\sigma}^G = \mathfrak{A}_\nu\,\mathfrak{B}_\nu \tag{2.25a}$$

and employ the auxiliary formula:

$$(\mathfrak{A}_\nu \times \mathrm{I} \times \mathfrak{B}_\nu)_{kl} = \sigma_{ij}\,\varepsilon_{kim}\,\varepsilon_{imj} = \sigma_{ij}\,\varepsilon_{kim}\,\varepsilon_{jlm} = -\,\sigma_{ij}\,(\delta_{kl}\,\delta_{ij} - \delta_{jk}\,\delta_{li}) = (\tilde{\sigma} - \sigma_{\mathrm{I}}\,\mathrm{I})_{kl}\,, \tag{2.25b}$$

which is easy to derive from ([**2**], (A.2)). It will then follow from (2.25) that:

$$\underline{\chi}(\mathfrak{r}) = -\,\frac{1}{8\pi}\int \frac{\mathfrak{r}-\mathfrak{r}'}{|\,\mathfrak{r}-\mathfrak{r}'\,|}\times\underline{\sigma}(\mathfrak{r}')\times\nabla\,d\tau'$$

$$= -\,\frac{1}{8\pi}\int \mathfrak{A}_\nu(\mathfrak{r}')\times\frac{\mathfrak{r}-\mathfrak{r}'}{|\,\mathfrak{r}-\mathfrak{r}'\,|}\nabla\times\mathfrak{B}_\nu(\mathfrak{r}')\,d\tau'$$

$$= -\,\frac{1}{8\pi}\int \left\{\frac{1}{|\,\mathfrak{r}-\mathfrak{r}'\,|}\mathfrak{A}_\nu(\mathfrak{r}')\times\mathrm{I}\times\mathfrak{B}_\nu(\mathfrak{r}') - \mathfrak{A}_\nu(\mathfrak{r}')\times(\mathfrak{r}-\mathfrak{r}')\frac{\mathfrak{r}-\mathfrak{r}'}{|\,\mathfrak{r}-\mathfrak{r}'\,|^2}\times\mathfrak{B}_\nu(\mathfrak{r}')\right\}d\tau'$$

$$= -\,\frac{1}{8\pi}\int \left\{\frac{1}{|\,\mathfrak{r}-\mathfrak{r}'\,|}[\tilde{\underline{\sigma}}(\mathfrak{r}') - \sigma_{\mathrm{I}}(\mathfrak{r}')\,\mathrm{I}] - \frac{1}{|\,\mathfrak{r}-\mathfrak{r}'\,|^3}(\mathfrak{r}-\mathfrak{r}')\times\underline{\sigma}(\mathfrak{r}')\times(\mathfrak{r}-\mathfrak{r}')\right\}d\tau'\,, \tag{2.25c}$$

which is then formula (2.26).

d) *Derivation of (2.7).* − After an application of the product rule from differential calculus, the second term of the second equation (2.6) will decompose into two summands, the first of which will yield the right-hand side of (2.7) directly. Since:

$$\nabla'\,(\mathfrak{r}' - \mathfrak{r}) = \mathrm{I}, \tag{2.6a}$$

one can now convert the second summand using (2.25b), when one replaces $\underline{\sigma}$ with the asymmetric tensor $\underline{\varepsilon}(\mathfrak{r}') \times \nabla'$. Since the first scalar of that tensor vanishes due to the symmetry of $\underline{\varepsilon}$, it will now follow that this second summand cancels the first term of (2.6), which proves (2.7). Inverting that calculation with $\underline{\sigma}$ in place of $\underline{\eta}$ will yield the derivation of the **Peretti-Günther** equation (1.6).

Appendix to § 5: distributions.

Let:

$$\delta^{(0)}(z) = \begin{cases} 0 & \text{for} \quad z < 0 \\ 1 & \text{for} \quad z > 0 \end{cases} \tag{5.27a}$$

be the **Heaviside** step function. The DIRAC δ-function can then be regarded as a kind of derivative of (5.27a) ([**2**], § 9):

$$\delta^{(1)}(z) = \frac{d}{dz}\,\delta^{(0)}(z), \tag{5.27b}$$

and one can formally proceed to higher-order "distributions" by using the recursion formula:

$$\delta^{(n+1)}(z) = \frac{d}{dz}\,\delta^{(n)}(z). \tag{5.27c}$$

Naturally, one cannot be dealing with differentiation in the usual sense in this. Schwartz [**28**] has given a thorough analytic foundation of these computing operations, which correspond to differentiation only formally. However, in many cases, it will satisfy for one to regard distributions as limiting cases of a family of differentiable functions, so e.g., one replaces the **Heaviside** step function with **Gauss**ian error integrals with various parameters. In other cases, distributions can be regarded as formal ways of writing out the inverses of otherwise-defined integrals. In that sense then, e.g., in space, $\delta^{(1)}(z)$ will represent a simple assignment and $\delta^{(1)}(z)$ will represent a double assignment in the place $z = 0$, as is known from potential theory. The expressions for line and point singularities can then be represented with no difficulties as products of distributions of different variables. Hence, e.g.:

$$\delta^{(1)}(x-a)\;\delta^{(1)}(x-b)\;\delta^{(1)}(x-c) \tag{5.27d}$$

means a point singularity at the location (a, b, c), which is sort of spatial **Dirac** δ-function. Furthermore:

$$\delta^{(0)}(F(\mathfrak{r})) \tag{5.27e}$$

means a function that has the value zero at points on the surface:

$$F(\mathfrak{r}) = 0 \tag{5.27f}$$

and possesses the value unity in external space, and the expression:

$$\nabla\;\delta^{(0)}(F(\mathfrak{r})) = \delta^{(1)}(F)\,\nabla F \tag{5.27g}$$

means a singular assignment on the outer surface with vectors that are perpendicular to it. In particular, with:

$$F(\mathfrak{r}) = \mathfrak{n}\,(\mathfrak{r}), \quad \text{with} \quad |\,\nabla n\,| = 1 \qquad \text{for } n\,(\mathfrak{r}) = 0, \tag{5.27h}$$

one will arrive at representation of the normal unit vector $\mathfrak{n}$ that is sometimes preferable. If one formally defines the vector field:

$$\delta^{(0)}\,(n)\;\mathfrak{n}\,(\mathfrak{r}) \tag{5.27i}$$

and its divergence:

$$\nabla\cdot[\delta^{(0)}\,(n)\;\mathfrak{n}\,(\mathfrak{r})] = \mathfrak{n}\cdot\mathfrak{a}\,(\mathfrak{r})\;\delta^{(1)}\,(n) + \nabla\cdot\mathfrak{a}\,(\mathfrak{r})\;\delta^{(0)}\,(n) \tag{5.27j}$$

then the first terms on the right-hand side will no longer give the singular source-density on the outer surface correctly. The last example generally shows that one cannot carry over the "naïve" calculation with distributions to curvilinear coordinates without making further assumptions. In particular, one might also have higher derivatives of the function $n\,(\mathfrak{r})$ on the singular surface at one's disposal in a suitable way, should the need arise.

References

1. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Cambridge, 1952.
2. E. Kröner, "Kontinuumstheorie der Versetzungen und Eigenspannungen," Berlin-Göttingen-Heidelberg, 1958. "Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen," Archive Rat. Mech. Anal. **4** (1960), pp. 273.
3. F. R. N. Nabarro, *The Mathematical Theory of Stationary Dislocations*, Advances in Physics **1** (1952), pp. 271.
4. D. Hilbert and S. Cohn-Vossen, *Anschauliche Geometrie*, Berlin, 1932.
5. G. Rieder, "Spannungen und Dehnungen in gestörten elastischen Medium," Z. Naturforschung **11a** (1956), pp. 171, correction in [**6**].
6. G. Rieder, "Plastische Verformung und Magnetostriktion," Z. angewandte Physik **9** (1957), pp. 187.
7. G. Rieder, "Eigenspannungen in unendlichen geschichten und elastisch anisotropen Medien, insbesonder in *Weiss*schen Bezirken und geschichten Platten," Abh. d. Braunschw. Wiss. Ges. **11** (1959), pp. 20.
8. J. M. Burgers, "Some considerations of the Field of Stress connected with Dislocations in a Regular Crystal Lattice I, II," Proc. Kon. Nederl. Akad. Wetensch. **42** (1939), pp. 293, 378.
9. M. Lagally, *Vorlesungen über Vektor-Rechnung*, Leipzig, 1949.
10. H. Schaefer, "Die Spannungsfunktionen des dreidimesionalen Kontinuums; statische Deutung und Randwerte," Ing.-Archiv **28** (1959), pp. 291.
11. H. Schaefer, "Die Spannungsfunktionen des dreidimensional Kontinuums und des elastischen Körpers," Z. angew. Math. Mech. **33** (1953), pp. 356.
12. E. Kröner, "Die Spannungsfunktionen der dreidimensionalen isotropen Elastizitätstheorie," Z. Phys. **139** (1954), pp. 175; correction in Z. Phys. **143** (1955), pp. 374.

13. W. Günther, “Spannungsfunktionen und Verträglichkeitsbedingungen der Kontinuumsmechanik,” Abh. d. Braunschw. Wiss. Ges. **6** (1954), pp. 207.
14. G. Rieder, Examination paper, Stuttgart, 1955, unpublished.
15. E. Madelung, *Die mathematischen Hilfsmittel des Physikers*, Berlin, 1936.
16. J. H. Michell, Proc. London Math. Soc. (1) **31** (1899), pp. 100.
17. W. Prager, “On Plane Elastic Strain in Doubly-Connected Domains,” Quart. Appl. Math. **3** (1945), pp. 377.
18. H. Schaefer, “Die vollständige Analogie Scheibe-Platte,” Abh. d. Braunschw. Wiss. Ges. **8** (1956), pp. 142.
19. H. Schaefer, “Die drei Spannungfunktionen des zweidimensionalen ebenen Kontinuums,” Österr. Ing.-Archiv **10** (1956), pp. 267.
20. G. Colonetti, “Su di una reciprocita tra deformationi e distorsioni,” Atti Accad. naz. Lincei, Rend., Cl. Sci. fis. mat. natur., V Ser. **24/1** (1915), pp. 404.
21. W. Günther, “Zur Statik und Kinematik des *Cosserat*schen Kontinuum,” Abh. d. Braunschw. Wiss. Ges. **10** (1958), pp. 195.
22. G. Peretti, “Significato del tensore arbitrario che interviene nell’integrale generale delle equazione della statica dei continui,” Atti. Sem. Mat. Fis. Univ. Modena **3** (1949), pp. 77.
23. L. Föppl, *Drang und Zwang*, v. 3, Munich, 1947.
24. J. D. Eshelby, “The Determination of the Elastic Field of an Ellipsoidal Inclusion, and Related Problems,” Proc. Roy. Soc. London **A241** (1957), pp. 376.
25. K. Marguerre, “Ansätze zur Lösung der Grundgleichungen der Elastizitätstheorie,” Z. angew. Math. Mech. **35** (1955), pp. 242.
26. E. Trefftz, “Mathematische Elastizitätstheorie,” *Handbuch der Physik*, ed. by H. Geiger and K. Scheel, Bd. 6, Mechanik der elastischen Körper, pp. 46, Berlin, 1928.
27. C. Truesdell, “Invariant and Complete Stress Functions for General Continuum,” Archive Rat. Mech. Anal. **4** (1959), pp. 1.
28. L. Schwartz, *Théorie des distributions*, t. I, Paris, 1930.
29. M. Brdička, “On the General Form of the *Beltrami* Equation and *Papkovich*’s Solution of the Axially Symmetrical Problem of the Classical Theory of Elasticity,” Czech. J. Phys. **7** (1957), pp. 262.
30. K. Kondo, *RAAG Memoirs of the Unifying Study of Basic Problems in Engineering and Physical Sciences by Means of Geometry*, vol. 1, Tokyo, 1955. vol. II, Tokyo, 1958.
31. B. A. Bilby, “Continuous Distributions of Dislocations,” Progress in Solid Mechanics **1** (1960), pp. 329.
32. M. G. Slobodianskii, “General and Complete Solutions of the Equations of Elasticity,” PMM (J. Appl. Math. Mech.) **23** (1959), pp. 666.
33. J. D. Eshelby, “The Force on an Elastic Singularity,” Phil. Trans. Roy. Soc. London **A244** (1951), pp. 87. “The Continuum Theory of Lattice Defects,” Solid State Physics **III** (1956), pp. 79. New York, 1956.
34. G. Rieder, “Mechanische Arbeit bei plastischen Vorgängen,” Z. angew. Physik **10**(1958), pp. 140.
35. H. B. Phillips, *Vector Analysis*, New York-London, 1933.
36. O. D. Kellogg, *Foundations of Potential Theory*, Berlin, 1929.
37. W. Lietzmann, *Anschauliche Topologie*, Munich, 1955.

38. G. Rieder, “Der Beitrag des elastischen Feldes zur freien Energie nicht-elastischer Vorgänge,” Materialprüf. **2** (1960), no. 11, pp. 429.
39. H. Schaefer, “Die Spannungsfunktionen einer Dyname,” Abh. d. Braunschw. Wiss. Ges. **7** (1955), pp. 107.
40. E. Sternberg and R. A. Eubanks, “On the Concept of Concentrated Loads and an Extension of the Uniqueness Theorem in the Linear Theory of Elasticity,” J. Rat. Mech. Anal. **4** (1955), pp. 135.
E. Sternberg, “On Some Recent Developments in the Linear Theory of Elasticity,” Structural Mechanics (Proc. First Symp. Naval Struct. Mech.), J. N. Goodier and N. J. Hoff, eds., pp. 48, Oxford-London-New York-Paris, 1960.
41. H. Franz and E. Kröner, “Zur Stabilität der *Guinier-Preston* Zonen in Aluminum-Kupfer-Legierungen,” Z. Metallkunde **46** (1955), pp. 639.
42. Jahnke-Emde, *Tafeln Höherer Funktionen*, Leipzig, 1948.
43. V. Volterra, “Sur l’équilibre des corps élastiques multiplement connexes,” Ann. Éc. Norm. Sup. (3) **24** (1907), pp. 401.
44. J. C. Maxwell, *Lehrbuch der Electricität und des Magnetismus*, German trans. by B. Weinstein, v. I, Berlin, 1883,
45. W. Haack, *Differential-Geometrie*, Part I, Wolfenbüttel-Hannover, 1948.
46. V. L. Indenbom, “Teoremy vzaimnosti i funktsii vliianiia dlia tenzora plotnosti dislokazii i tenzora nesovmestnosti deformatsii,” Dokl. Akad. Nauk USSR **128**, no. 5 (1959), pp. 906.
(V. L. Indenbom, “Reciprocity theorems and influence functions for the tensors of dislocation density and incompatibility,” Dokl. Akad. Nauk USSR **128**, no. 5 (1959), pp. 906.)
47. G. Rieder, In preparation.