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The five-dimensional universe and wave mechanics (*)

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The first part of the present work is dedicated to a systematic study of the five-dimensional universe that was considered by O. Klein [**1**], Th. De Donder [**2**], and L. de Broglie [**3**]. One knows that it was the latter that whose succeeded in making the concept of a five-dimensional universe satisfactory by showing that it is possible to define the metric of such a universe independently of the ratio $e : m$ that characterizes an electrified particle. We shall first show that concept can be deduced rigorously and very simply from Einsteinian gravitation. Then, by generalizing the work of Gordon [**4**] and Schrödinger [**5**], we will show that introducing the de Broglie-Schrödinger function Ψ will permit us to reduce the laws of gravitation, electromagnetism, and quantization (viz., the equation in Ψ) to a single variational principle in a five-dimensional universe. One likewise finds the conservation laws of energy, impulse, and electricity united into a single statement. Finally, an approximate formula is established in order to calculate the gravitational and electromagnetic potentials of a field that differs only slightly from a Minkowski field as a function of Ψ. The calculation is developed in the cases of a static charge and a charge that is animated with a uniform motion with a small velocity. Upon comparing the values thus-found to the classical values of the potentials, one will find that the amplitude of the function Ψ that represents the charge must have a constant value inside a finite volume and be zero outside of it. That result can be understood by means of the beautiful interpretation of the function Ψ that De Donder gave recently [**6**.f]. On the other hand, it seems to be irreconcilable with L. de Broglie's opinion that the charge would be a point-like singularity of the function Ψ.

In the second part, we shall say a few words about the extension of the preceding considerations to Th. De Donder's continuous systems [**6**].

This work was carried out under the direction of L. de Broglie and Th. De Donder, who did not cease to assist me with their advice and were so helpful as to send me manuscripts of their own work on the subject. I am pleased to be able to express my deepest gratitude to them here.

I. – Point-like systems.

In what follows, we shall generally adopt the notations, units, and sign conventions of De Donder's *Théories des champs gravifiques* [**7**]; the main exceptions are the following:

(*) Presented by De Donder at the session on 3 May 1927.

1. We shall write x^μ and not x_μ .

2. We shall use the well-known "summation convention." The Latin indices are supposed to vary from 1 to 4, while the Greek indices vary from 0 to 4.

3. We shall sometimes use the notations of covariant derivatives:

$$u_{,\mu} = \frac{\partial u}{\partial x^\mu},$$

$$u_{\nu,\mu} = \frac{\partial u_\nu}{\partial x^\mu} - \begin{Bmatrix} \nu\ \mu \\ \rho \end{Bmatrix} u_\rho,$$

$$u^\nu{}_{,\mu} = \frac{\partial u^\nu}{\partial x^\mu} + \begin{Bmatrix} \rho\ \mu \\ \nu \end{Bmatrix} u_\rho,$$

$$u_{,\mu\nu} = \frac{\partial^2 u^\nu}{\partial x^\mu\, \partial x^\nu} - \begin{Bmatrix} \rho\ \mu \\ \nu \end{Bmatrix} \frac{\partial u}{\partial x^\mu}, \qquad \text{etc.}$$

We further remark that one cannot distinguish between the contravariant components that relate to the ds^2 of space-time and the ones that relate to the $d\sigma^2$ on the five-dimensional universe, so no confusion should arise in practice. Only the components g^{ik}, R^{ik}, H^{ik}, S^{ik}, $F^{\,i}$, Φ^i, which will be used later on, refer to the ds^2.

1. Introduction of the variable x^0. – We start with the Jacobi equation [**2**] [equation (15) of the second communication]:

$$g^{ik}\,(S_{,\,i} - \Phi_i)\,(S_{,\,k} - \Phi_k) - \mu^2 = 0, \tag{1}$$

in which μ is an invariant of the system:

$$\mu = \frac{\tau^m}{\tau^e} \tag{2}$$

that will become:

$$\mu = \frac{m_0\, c^2}{e}, \tag{3}$$

in particular, in the case of a charge e of rest mass m_0 ; the Jacobi function S has the form:

$$S = -\,\frac{\mu}{2}\,s\ + S'(x^1, x^2, x^3, x^4)\,. \tag{4}$$

Now set:

$$\boxed{x^0 = -\frac{\mu\alpha}{2} s} \tag{5}$$

and replace the independent variable s by the new variable thus-defined; α is a *universal constant.* The function S takes the form:

$$S = \frac{x^0}{\alpha} + S'(x^1, x^2, x^3, x^4) . \tag{6}$$

We infer that:

$$S_{,0} = \frac{1}{\alpha}, \tag{7}$$

in such a way that the Jacobi equation (1) can be written:

$$g^{ik} S_{,i} S_{,k} - 2\alpha \Phi_i S_{,k} S_{,0} - \alpha^2 \Phi^i \Phi_i S_{,0} S_{,0} - \mu^2 = 0.$$

That will then give (upon introducing a second *universal* constant ξ for more generality):

$$\boxed{\begin{aligned} \gamma^{ik} &= g^{ik}, \\ \gamma^{0i} &= \gamma^{i0} = -\alpha\Phi^i, \\ \gamma^{00} &= \alpha^2\Phi^i\Phi_i - \frac{1}{\xi}, \end{aligned}} \tag{8}$$

and

$$\boxed{\xi\alpha^2 = 2\chi,} \tag{9}$$

so we can put equation (1) into the form:

$$J \equiv \gamma^{\mu\nu} S_{,\mu} S_{,\nu} - \left(\mu^2 - \frac{1}{2\chi}\right) = 0. \tag{10}$$

2. Interpretation of the preceding transformation. – Up to now, we have been dealing with only a purely-analytical transformation of the Jacobi equation. We shall now interpret the transformed equation (10) by giving a geometric meaning to the variable x^0.

In order to do that, it is first of all important to point out that the relation (5) that served as the starting point is indeed *sufficient* (and even particularly convenient) for us to define our transformation, but it is not *necessary*. Indeed, the necessary and sufficient relation between S and x^0 is the relation (6) or its equivalent (7). That relation will imply the following properties for x^0 :

1. x^0 is *invariant* under all transformations of the coordinates x^1, x^2, x^3, x^4.

2. x^0 enters into S linearly.

Having said that, we interpret the variable x^0 as a fifth parameter that is necessary for us to determine an "event"; i.e., a fifth dimension of the universe. Its invariance under the transformation that we can perform explains how that fifth dimension has escaped our direct observation.

By means of that meaning of the variable x^0, one will see that equation (10) has the form of a Jacobi equation for a five-dimensional gravitational field that is *only massive.* Thus, the trajectories of particles – even charged ones – will be *geodesics* in the five-dimensional universe. The quadri-dimensional trajectories that one observes will be the projections of those geodesics onto space-time; they will not generally be geodesics in space-time any more.

It is easy to calculate the *inclination* of a five-dimensional trajectory over space-time. Indeed, if S is a complete integral of equation (1) or (10) then, from (10), one will have:

$$\gamma^{\mu\nu} S_{,\nu} = \sqrt{\mu^2 - \frac{1}{2\chi}} \frac{dx^\mu}{d\sigma} \tag{11}$$

along a trajectory, and from (1), (8), (7), one will have:

$$\gamma^{m\nu} S_{,\nu} = \mu \frac{dx^m}{ds}; \tag{12}$$

$d\sigma$ is the five-dimensional line element. One will then infer:

$$\frac{d\sigma}{ds} = \sqrt{1 - \frac{1}{2\chi\mu^2}}. \tag{13}$$

One will then see that this inclination is determined by only the ratio μ. That is the geometric interpretation of the ratio μ, which is at the basis of L. de Broglie's arguments [**3**].

3. Metric on the five-dimensional universe. – We shall start from formulas (8) and calculate the $d\sigma^2$ and the curvature tensor of the five-dimensional universe as functions of the four-dimensional gravitational and electromagnetic potentials.

One first easily finds that:

$$\left.\begin{aligned} \gamma_{ik} &= g_{ik} - 2\chi\Phi_i\Phi_k, \\ \gamma_{0i} &= \gamma_{i0} = -\xi\alpha\Phi_i, \\ \gamma_{00} &= -\xi. \end{aligned}\right\} \tag{14}$$

If g is the determinant of the g_{ik} then the determinant γ of the $\gamma_{\mu\nu}$ will become:

$$\gamma = -\xi g\,. \tag{15}$$

Introduce the following quadri-dimensional tensor:

1. Electromagnetic field:

$$H_{ik} = \Phi_{i,\,k} - \Phi_{k,\,i}$$
$$= \frac{\partial \Phi_i}{\partial x^k} - \frac{\partial \Phi_k}{\partial x^i}\,. \tag{16}$$

Set:

$$H = H_{ik}\,H^{ik}\,, \tag{17}$$

and let F_i denote the divergence of H_{ik} :

$$\left.\begin{aligned} F_i &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^m}\left(H_i^{\ m}\sqrt{-g}\right) - \begin{Bmatrix} i\,m \\ n \end{Bmatrix}_4 H_n^{\ m} \\ &= -\frac{\partial H^m{}_i}{\partial x^m} + H^i{}_k \begin{Bmatrix} i\,l \\ k \end{Bmatrix}_4 - H^n{}_i \begin{Bmatrix} m\,r \\ r \end{Bmatrix}_4, \end{aligned}\right\} \tag{18}$$

$$\left.\begin{aligned} F^i &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^m}\left(H^{im}\sqrt{-g}\right) \\ &= -g^{ik}\frac{\partial H^m{}_k}{\partial x^m} + H^m{}_k\, g^{in} \begin{Bmatrix} m\,n \\ k \end{Bmatrix}_4 + H^{im} \begin{Bmatrix} m\,r \\ r \end{Bmatrix}_4. \end{aligned}\right\} \tag{19}$$

One has set:

$$\begin{Bmatrix} i\,k \\ l \end{Bmatrix}_4 = \frac{1}{2}g^{lm}\left(\frac{\partial g_{mi}}{\partial x^k} + \frac{\partial g_m}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^m}\right), \tag{20}$$

in order to distinguish it from:

$$\begin{Bmatrix} \mu\,\rho \\ \sigma \end{Bmatrix}_5 = \frac{1}{2}\gamma^{\sigma\tau}\left(\frac{\partial \gamma_{\tau\mu}}{\partial x^\rho} + \frac{\partial \gamma_{\tau\rho}}{\partial x^\mu} - \frac{\partial \gamma_{\mu\rho}}{\partial x^\tau}\right). \tag{21}$$

2. Electromagnetic energy-impulse tensor:

$$S_{ik} = \tfrac{1}{2}\,g_{ik}\,H - H_{il}\,H_k{}^l\,. \tag{22}$$

3. Curvature tensor:

$$R_{ik} = \frac{\partial}{\partial x^k}\begin{Bmatrix} i\,m \\ m \end{Bmatrix}_4 - \frac{\partial}{\partial x^m}\begin{Bmatrix} i\,k \\ m \end{Bmatrix}_4 + \begin{Bmatrix} i\,m \\ n \end{Bmatrix}_4 \begin{Bmatrix} k\,n \\ m \end{Bmatrix}_4 - \begin{Bmatrix} i\,m \\ m \end{Bmatrix}_4 \begin{Bmatrix} m\,n \\ n \end{Bmatrix}_4, \tag{23}$$

with the curvature:

$$R = g^{ik} R_{ik} . \tag{24}$$

We would like to calculate the five-dimensional curvature tensor:

$$P_{\mu\nu} = \frac{\partial}{\partial x^{\nu}} \begin{Bmatrix} \mu\,\rho \\ \rho \end{Bmatrix}_5 - \frac{\partial}{\partial x^{\rho}} \begin{Bmatrix} \mu\,\nu \\ \rho \end{Bmatrix}_5 + \begin{Bmatrix} \mu\,\rho \\ \sigma \end{Bmatrix}_5 \begin{Bmatrix} \nu\,\sigma \\ \rho \end{Bmatrix}_5 - \begin{Bmatrix} \mu\,\nu \\ \rho \end{Bmatrix}_5 \begin{Bmatrix} \rho\,\sigma \\ \sigma \end{Bmatrix}_5 \tag{25}$$

and the curvature:

$$P = \gamma^{\mu\nu} P_{\mu\nu} . \tag{26}$$

We first have:

$$\left\{ \begin{aligned} &\begin{Bmatrix} r\,s \\ i \end{Bmatrix}_5 = \begin{Bmatrix} r\,s \\ i \end{Bmatrix}_4 - \chi(\Phi_r H^{i}{}_{s} + \Phi_s H^{i}{}_{r}), \\ &\begin{Bmatrix} r\,s \\ 0 \end{Bmatrix}_5 = \chi\alpha\Phi_l(\Phi_r H^{l}{}_{s} + \Phi_s H^{l}{}_{r}) + \frac{\alpha}{2}\left(\frac{\partial\Phi_r}{\partial x^s} + \frac{\partial\Phi_s}{\partial x^r}\right), \\ &\begin{Bmatrix} r\,0 \\ 0 \end{Bmatrix}_5 = \chi\Phi_l\, H^{l}{}_{r}, \\ &\begin{Bmatrix} 0\,s \\ i \end{Bmatrix}_5 = -\frac{\alpha\xi}{2} H^{i}{}_{s}, \\ &\begin{Bmatrix} 0\,0 \\ \mu \end{Bmatrix}_5 = 0, \end{aligned} \right. \tag{27}$$

and then:

$$\left\{ \begin{aligned} &P^{ik} = R^{ik} - \chi H^{mi} H_m{}^{k}, \\ &P^{0i} = \frac{\alpha}{2} F^{i} - \alpha\Phi_m P^{im}, \end{aligned} \right.$$

$$\left\{ \begin{aligned} &P_i^{\,k} = R_i^{\,k} - \chi H^{m}{}_{i} H_m{}^{k} \\ &P_0^{\,i} = -\frac{\alpha\xi}{2} F^{i}, \\ &P_0^{\,0} = \frac{\chi}{2} H + \chi\Phi^{i} F_i, \end{aligned} \right. \tag{28}$$

$$\left\{ \begin{aligned} &P_{ik} = R_{ik} - \chi^2 H \Phi_i \Phi_k - \chi(\Phi_k F_i + \Phi_i F_k + H^{m}{}_{i} H_{mk}), \\ &P_{0i} = -\frac{\xi\alpha}{2} F_i - \frac{\xi\alpha}{2} \chi H \Phi_i, \\ &P_{00} = -\frac{\xi\chi}{2} H, \end{aligned} \right.$$

and finally:

$$P = R - \frac{\chi}{2} H. \tag{28′}$$

4. Introduction of the de Broglie-Schrödinger function Ψ. – We set:

$$\Psi = \Psi^* (x^1, x^2, x^3, x^4)\, e^{kS}, \tag{29}$$

in which k is a *purely-imaginary* constant, which will amount to:

$$k = i \cdot \frac{2\pi}{h} \cdot \frac{e}{c} \tag{30}$$

in the case of a charged particle; S is *real*. As for the amplitude Ψ^*, it will be generally considered to be a *complex* function of the form:

$$\Psi^* = A + i\, B. \tag{31}$$

The conjugate of a complex quantity u will be denoted by $\bar{u}$.

Suppose, for the moment, that $\Psi^* \equiv$ constant (real or imaginary). We will then infer from (29) that:

$$S_{,\,\mu} = \frac{1}{k} \frac{\Psi_{,\mu}}{\Psi} = -\frac{1}{k} \frac{\bar{\Psi}_{,\mu}}{\Psi},$$

and the Jacobi equation (10) will be written:

$$L \equiv \gamma^{\mu\nu}\, \Psi_{,\,\mu}\, \Psi_{,\,\nu} + k^2 \left(\mu^2 - \frac{1}{2\chi}\right) \Psi\, \bar{\Psi} = 0.$$

We then see the appearance of the "world-function":

$$\boxed{L \equiv \gamma^{\mu\nu} \Psi_{,\mu} \Psi_{,\nu} + k^2 \left(\mu^2 - \frac{1}{2\chi}\right) \Psi\, \bar{\Psi},} \tag{32}$$

which will play a very important role: It is the generalization of an analogous function that was considered by Gordon [**4**] and Schrödinger [**5**]. The link between that equation and the Jacobi equation was pointed out by De Donder [**6**.b].

If one considers the functions Ψ and $\bar{\Psi}$ to be independent and one annuls the variational derivatives of $L\sqrt{-g}$ with respect to those functions then one will get their equations of propagation:

$$\gamma^{\mu\nu}\,\Psi_{\mu\nu} - k^2\left(\mu^2 - \frac{1}{2\chi}\right)\Psi = 0, \tag{33}$$

$$\gamma^{\mu\nu}\,\bar{\Psi}_{,\mu\nu} - k^2\left(\mu^2 - \frac{1}{2\chi}\right)\bar{\Psi} = 0, \tag{33′}$$

which is a generalization of the de Broglie-Schrödinger equation [**3**].

In the general case where Ψ^* is arbitrary, the function L will no longer be zero along the trajectory. It is interesting to perform the calculation in the case of a *real* amplitude A. Upon setting:

$$\theta = \log A, \tag{34}$$

one will then have:

$$\left\{\begin{array}{llll} \Psi & = e^{kS+\theta}, & \bar{\Psi} & = e^{-kS+\theta}, \\ \Psi_{,\mu} & = \Psi(k\,S_{,\mu} + \theta_{,\mu}), & \bar{\Psi}_{,\nu} & = \bar{\Psi}(-k\,S_{,\nu} + \theta_{,\nu}), \end{array}\right. \tag{35}$$

so, upon taking (10) into account:

$$L = \Psi\,\bar{\Psi}\,\gamma^{\mu\nu}\theta_{,\mu}\,\theta_{,\nu}\,,$$

or rather:

$$L = \gamma^{\mu\nu}A_{,\,\mu}\,A_{,\,\nu}\,. \tag{36}$$

The complex equation (33) is equivalent to two partial differential equations in A, B, S. It is easy to establish those equations. To abbreviate the notation, we introduce the d’Alembertian notation:

$$\Box\,u = \gamma^{\mu\nu}\,u_{,\,\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\;\gamma^{\mu\nu}u_{,\nu}\right). \tag{37}$$

In the case of functions such as A, B, S, for which $u_{,\,0}$ = const., upon taking the Maxwell equation:

$$D \equiv \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^m}\left(\sqrt{-g}\;\Phi^m\right) = 0 \tag{38}$$

into account, it is easy to see that one has:

$$\Box\,u = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^m}\left(\sqrt{-g}\;g^{mn}\,u_{,n}\right). \tag{39}$$

One easily finds that:

$$\Box\,\Psi - k^2\left(\mu^2 - \frac{1}{2\chi}\right)\Psi \equiv e^{kS}[\Box\,\Psi^* + 2k\gamma^{\mu\nu}S_{,\nu}S_{,\mu} + k\Psi^*\Box\,S + k^2\Psi^*\mathbf{J}] = 0\,;$$

i.e., upon taking (10), (12), and $\Psi^*_{,0} = 0$ into account:

$$\square\Psi + 2k\mu\frac{d\Psi^*}{ds} + k\Psi^*\square S = 0\ .$$

Upon setting:

$$k = i\,K \tag{40}$$

and separating the real and imaginary parts, one will have:

$$\begin{cases} \square A - 2K\mu\dfrac{dB}{ds} - KB\square S = 0, \\ \square B + 2K\mu\dfrac{dA}{ds} + KA\square S = 0. \end{cases} \tag{41}$$

In the case of a *real* amplitude (B = 0), those relations reduce to:

$$\begin{cases} \square A = 0, \\ \square S = -\mu\dfrac{d\log A^2}{ds}. \end{cases} \tag{42}$$

5. Variational principle. – We shall show that *equations* (33), (33′) *in* Ψ *and* $\bar{\Psi}$, *the equations of gravitation*, and *the Maxwell equations* are *unified by the variational principle:*

$$\boxed{\delta\int(P + 2\chi L)\sqrt{-g}\,dx^0\cdots dx^4 = 0.} \tag{43}$$

We have already shown that equations (33), (33′) result from:

$$\frac{\delta L\sqrt{-g}}{\delta\Psi} = 0, \qquad \frac{\delta L\sqrt{-g}}{\delta\bar{\Psi}} = 0,$$

or, what amounts to the same thing, from:

$$\frac{\delta(P + 2\chi L)\sqrt{-g}}{\delta\Psi} = 0, \qquad \frac{\delta(P + 2\chi L)\sqrt{-g}}{\delta\bar{\Psi}} = 0\ .$$

It remains for us to write down the variational equations with respect to the $\gamma_{\mu\nu}$ (one must observe that, from (14), $\delta\gamma_{00} \equiv 0$). In order to do that, set:

$$^*P_{\mu\nu} \equiv P_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}P, \tag{44}$$

and then:

$$\tau_{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \frac{\partial L\sqrt{-g}}{\partial \gamma^{\mu\nu}}, \tag{45}$$

and as a result:

$$\tau^{\mu\nu} \equiv -\frac{1}{\sqrt{-g}} \frac{\partial L\sqrt{-g}}{\partial \gamma_{\mu\nu}}, \tag{46}$$

and finally:

$$\boxed{T_{\mu\nu} \equiv \tau^{\mu\nu} + \tau^{\nu\mu} = \Psi_{,\mu}\bar{\Psi}_{,\nu} + \bar{\Psi}_{,\mu}\Psi_{,\nu} - \gamma_{\mu\nu}L,} \tag{47}$$

which is the symmetric part of:

$$\frac{\delta\left(2L\sqrt{-g}\right)}{\delta\gamma^{\mu\nu}}.$$

One will then have the equations:

$$^{*}P^{\mu\nu} = -\chi T^{\mu\nu}, \tag{48}$$

which one can further put into the form:

$$^{*}P_{\mu}^{n} = -\chi T_{\mu}^{n}. \tag{49}$$

The covariant form is a little less simple; we shall not use it. For the sake of symmetry and ease of calculation, we shall employ the contravariant form for μ = 1, 2, 3, 4 and the mixed form for μ = 0; hence, one has the following system:

$$\boxed{\begin{aligned} P^{mn} - \tfrac{1}{2}\gamma^{mn}P &= -\chi T^{mn}, \\ P_{0}^{i} &= -\chi T_{0}^{i}, \end{aligned}} \tag{50}$$

which is equivalent to the system (49).

Thanks to formulas (28) and (28′), one immediately verifies that equations (50) are *formally* identical to the equations of gravitation and the Maxwell equations:

$$\begin{cases} R^{mn} - \tfrac{1}{2}g^{mn}R = -\chi(S^{mn} + T^{mn}), \\ F^{i} = \alpha T_{0}^{i}. \end{cases} \tag{51}$$

One first sees that the constant χ, which has been undetermined up to now, is nothing but Einstein's gravitational constant:

$$\chi = \frac{8\pi G}{c^4}, \qquad G = 6.7 \times 10^{-8} \qquad \text{CGS.} \tag{52}$$

In order for T_{0}^{i} to be interpreted as an electric current quadri-vector, it is further necessary that it must satisfy the condition of the conservation of electricity. Before showing that this is indeed the case, we shall see that appearance that this condition takes on in a five-dimensional universe.

6. Conservation of energy and electricity. – We must distinguish between the divergence of a tensor that is taken in the five-dimensional universe and the divergence that is taken in space-time; we propose:

$$\left\{\begin{aligned} T_{\nu,\mu}^{\ \ \mu} &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(T_{\nu}^{\ \mu}\sqrt{-g}\right)-\begin{Bmatrix}\nu\ \mu\\ \alpha\end{Bmatrix}_5 T_{\alpha}^{\ \mu},\\ T^{\nu\mu}_{\ \ ,\mu} &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(T^{\mu\nu}\sqrt{-g}\right)-\begin{Bmatrix}\alpha\ \mu\\ \nu\end{Bmatrix}_5 T^{\alpha\mu},\end{aligned}\right. \tag{53}$$

$$\left\{\begin{aligned} {}^4T_{i,m}^{\ \ m} &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{m}}\left(T_{i}^{\ m}\sqrt{-g}\right)-\begin{Bmatrix}i\ m\\ l\end{Bmatrix}_5 T_{l}^{\ m},\\ {}^4T^{im}_{\ \ ,m} &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{m}}\left(T^{im}\sqrt{-g}\right)-\begin{Bmatrix}l\ m\\ i\end{Bmatrix}_5 T^{im}.\end{aligned}\right. \tag{54}$$

We deduce the conservation of energy-impulse from this and the conservation of electricity can be deduced from the fundamental equations (48) and (50), thanks to the well-known *identity:*

$${}^*P^{\nu\mu}_{\ \ ,\mu} = 0, \tag{55}$$

which one can also write:

$${}^*P_{\nu,\mu}^{\ \ \mu} = 0\ ; \tag{56}$$

we take:

$$\boxed{\begin{aligned} {}^*P^{n\mu}_{\ \ ,\mu} &= 0,\\ {}^*P_{0,\mu}^{\ \ \mu} &= 0.\end{aligned}} \tag{57}$$

Upon combining (50) and (57), we will get:

$$\left\{\begin{aligned} {}^*P^{n0}_{\ \ ,0} - \chi T^{nm}_{\ \ ,m} &= 0,\\ {}^*P_{0,0}^{\ \ 0} - \chi T_{0,m}^{\ \ m} &= 0.\end{aligned}\right.$$

We shall transform these two equations in succession. Upon observing that ${}^*P^{\mu\nu}$ and g do not depend upon x^0, and upon taking (53) and (27) into account, one can first write them:

$$\left\{\begin{aligned} &\chi T^{nm}_{\ \ ,m} - \begin{Bmatrix}k\ 0\\ n\end{Bmatrix}_5 {}^*P^{k0} = 0,\\ &T_{0,m}^{\ \ m} = 0.\end{aligned}\right. \tag{58}$$

By means of (48), (58) can be initially written:

$$T^{nm}_{\ \ ,m} + \begin{Bmatrix} k\,0 \\ n \end{Bmatrix}_5 T^{k0} = 0\ ;$$

from (53), (54), and (27) that will become:

$$\begin{aligned}0 &= {}^4T^{nm}_{\ \ ,m} + \begin{Bmatrix} 0\,m \\ n \end{Bmatrix}_5 T^{0m} + \left[\begin{Bmatrix} m\,l \\ u \end{Bmatrix}_5 - \begin{Bmatrix} m\,l \\ n \end{Bmatrix}_4\right] T^{ml} + \begin{Bmatrix} k\,0 \\ n \end{Bmatrix}_5 T^{k0} \\ &= {}^4T^{nm}_{\ \ ,m} - 2\cdot\frac{\alpha\xi}{2} H^n_{\ m} T^{0m} - \frac{\alpha^2\xi}{2}\cdot 2\Phi_l\, H^n_{\ m} T^{ml} \\ &= {}^4T^{nm}_{\ \ ,m} + \alpha H^n_{\ m}(\gamma_{00}\, T^{0m} + \gamma_{0l} T^{ml}) \\ &= {}^4T^{nm}_{\ \ ,m} + H^n_{\ m}\,\alpha T^m_{\ 0}\,.\end{aligned}$$

Now, from (51), one has:

$$H^n_{\ m}\,\alpha T^m_{\ 0} = H^n_{\ m}\, F^m,$$

and one knows that the latter quantity is nothing but:

$${}^4S^{nm}_{\ \ ,m};$$

one will finally have the expression for the *conservation of "material" and electromagnetic energy:*

$${}^4S^{nm}_{\ \ ,m} + {}^4T^{nm}_{\ \ ,m} = 0. \tag{60}$$

As for equation (59), using (53) and (27), it can be written:

$$\begin{aligned}\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^m}\left(T^m_n \sqrt{-g}\right) &= \begin{Bmatrix} 0\,m \\ \alpha \end{Bmatrix}_5 T^m_{\ \alpha} \\ &= \frac{\xi\alpha^2}{2}\Phi_l\, H^n_{\ m} T^m_{\ 0} - \frac{\alpha\xi}{2} H^n_{\ m} T^m_{\ l} \\ &= -\frac{\xi\alpha}{2} H_{im}(\gamma^{0l} T^m_{\ 0} + \gamma^{lr} T^m_{\ r}) \\ &= -\frac{\xi\alpha}{2} H_{im} T^{lm},\end{aligned}$$

or finally, since H_{lm} is skew-symmetric and T^{lm} is symmetric:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^m}\left(T_n^{\ m}\sqrt{-g}\right) = 0, \tag{61}$$

which expresses the *conservation of electricity*.

We now arrive at the *verification* of equation (61), when $T_0^{\ m}$ is expressed as a function of Ψ. It will suffice for us to show that equation (59) is verified identically. In order to do that, we first establish the identity:

$$T_{0,\mu}^{\ \mu} \equiv 0, \tag{62}$$

which is true for any function Ψ that satisfies equation (33). Indeed, upon applying the rules of covariant differentiation:

$$\begin{aligned}
T_{0,\mu}^{\ \mu} &= (\gamma^{\mu\nu}\,\Psi_{,0}\,\bar{\Psi}_{,\nu} + \gamma^{\mu\nu}\,\bar{\Psi}_{,0}\,\Psi_{,\nu} - \gamma_0^0\,L)_{,\mu} \\
&= \gamma^{\mu\nu}\,\bar{\Psi}_{,\nu}\,\Psi_{,0\mu} + \gamma^{\mu\nu}\,\Psi_{,\nu}\,\bar{\Psi}_{,0\mu} + \Psi_{,0}\,\gamma^{\mu\nu}\,\bar{\Psi}_{,\mu\nu} + \bar{\Psi}_{,0}\,\gamma^{\mu\nu}\,\Psi_{,\mu\nu} - L_{,0} \\
&= (\gamma^{\mu\nu}\,\bar{\Psi}_{,\nu}\,\Psi_{,\mu})_{,0} + k^2\left(\mu^2 - \frac{1}{2\chi}\right)(\Psi_{,0}\bar{\Psi} + \bar{\Psi}_{,0}\Psi) - L_{,0} \\
&= \left[\gamma^{\mu\nu}\,\Psi_{,\mu}\,\bar{\Psi}_{,\nu} + k^2\left(\mu^2 - \frac{1}{2\chi}\right)\Psi\,\bar{\Psi} - L\right] \\
&\equiv 0, \qquad \text{from (32).}
\end{aligned}$$

It will then result that:

$$T_{0,m}^{\ m} = T_{0,\mu}^{\ \mu} - T_{0,0}^{\ 0} = -\,T_{0,0}^{\ 0} = -\,\frac{\partial T_0^0}{\partial x^0}.$$

Now, it is clear that (29) and (7) imply that one will have:

$$\frac{\partial T_0^0}{\partial x^0} \equiv 0$$

identically, and as a result:

$$T_{0,m}^{\ m} \equiv 0.$$

7. Mass density. – From (28), one will have:

$$P_i^i - \tfrac{1}{2}\gamma_i^i P = P - P_0^0 - 2P$$
$$= -R - \chi\Phi^i F_i$$
$$= -R - \chi\Phi_i F^i ,$$

and from (49):

$$P_i^i - \tfrac{1}{2}\gamma_i^i P = -\chi T_i^i .$$

Consequently, if we define the *mass density* ${}^4T$ by the equality:

$$\boxed{R = \chi\, {}^4T} \tag{63}$$

then we will have:

$${}^4T = T_i^i - \Phi_i T_0^i . \tag{64}$$

From (51), one can write:

$$\begin{aligned} {}^4T &= T - T_0^0 - \alpha\Phi_i T_0^i \\ &= T + \frac{1}{\xi}(\gamma_{00}T_0^0 + \gamma_{0i}T_0^i) \\ &= T + \frac{1}{\xi}T_{00} \\ &= T + L + \frac{2}{\xi}\Psi_{,0}\bar{\Psi}_{,0} \\ &= T + L - 2k^2\frac{1}{2\chi}\Psi\bar{\Psi} . \end{aligned}$$

Moreover:

$$\begin{aligned} 2L &= \gamma^{\mu\nu}(T_{\mu\nu} + \gamma_{\mu\nu}L) + 2k^2\left(\mu^2 - \frac{1}{2\chi}\right)\Psi\bar{\Psi} \\ &= T + 5L + 2k^2\left(\mu^2 - \frac{1}{2\chi}\right)\Psi\bar{\Psi} ; \end{aligned}$$

hence:

$${}^4T = -2\,(L + k^2\mu^2\,\Psi\bar{\Psi}) . \tag{65}$$

One deduces from (48), (63), (64), (28) that:

$$P^{\mu\nu} = -\chi\left(T^{m\nu} - \tfrac{1}{2}\gamma^{m\nu}T'\right), \tag{66}$$

with:

$$T' = {}^4T - \frac{H}{2} = T_i^i - \left(\Phi_i F^i + \frac{H}{2}\right), \tag{67}$$

and in particular:

$$R^{mn} = -\chi\left(T^{mn} + S^{mn} - \tfrac{1}{2}g^{mn}\,{}^4T\right) . \tag{68}$$

8. Approximate calculation of a gravitational and electromagnetic field that differs slightly from a Minkowski field. – Take the Cartesian spatial coordinates to be x^1, x^2, x^3, and set:

$$x^4 = ct.$$

Set:

$$\gamma_{\mu i} = \delta_{\mu i} + \varepsilon_{\mu i}\,, \tag{69}$$

in which the $\delta_{\mu i}$ are the Galilean values:

$$\begin{cases} \delta_{11} = \delta_{22} = \delta_{33} = -1, \\ \delta_{44} = +4, \\ \delta_{\mu i} = 0, \quad \mu \neq i. \end{cases} \tag{70}$$

Upon starting with (66) and (67), one will easily see by a well-known process that the corrections $\varepsilon_{\mu i}$ are given by:

$$\varepsilon_{\mu i} = -\frac{\chi}{2\pi}\int \left\{ {}^*T_{\mu i} \right\}_{t-\frac{v}{c}} \frac{dx^1\, dx^2\, dx^3}{r}\,, \tag{71}$$

with:

$${}^*T_{\mu i} = T_{\mu i} - \tfrac{1}{2}\delta_{\mu i} T_l^l\,. \tag{72}$$

r is the (Euclidian) distance of the potential point from the integration point, and $\{u\}_{t-\frac{v}{c}}$ denotes the "retarded" value of the function u.

From (70) and the fact that:

$$T_l^l = T_{44} - T_{11} - T_{22} - T_{33}\,, \tag{73}$$

one has:

$$\begin{cases} {}^4T_{\mu i} = T_{\mu i} \quad \mu \neq i, \\ {}^4T_{11} = \tfrac{1}{2}(T_{11} - T_{22} - T_{33} - T_{44}), \\ {}^4T_{22} = \tfrac{1}{2}(-T_{44} + T_{22} - T_{33} + T_{11}), \\ {}^4T_{33} = \tfrac{1}{2}(-T_{11} - T_{22} + T_{33} + T_{44}), \\ {}^4T_{44} = \tfrac{1}{2}(T_{11} + T_{22} + T_{33} + T_{44}). \end{cases} \tag{74}$$

However, in the present, (32) will give:

$$L = k^2 \mu^2\ \Psi\bar{\Psi} - \Psi_{,1}\bar{\Psi}_{,1} - \Psi_{,2}\bar{\Psi}_{,2} - \Psi_{,3}\bar{\Psi}_{,3} - \Psi_{,4}\bar{\Psi}_{,4}\,,$$

in such a way that:

$$\begin{cases} {}^4T_{\mu i} = \Psi_{,\mu}\bar{\Psi}_{,i} + \bar{\Psi}_{,\mu}\Psi_{,i}, \quad \mu \neq i, \\ {}^4T_{ii} = \delta_{ii}(k^2\mu^2\Psi\bar{\Psi} + 2\delta_{ii}\bar{\Psi}_{,i}\Psi_{,i}). \end{cases} \tag{75}$$

Suppose that the amplitude of Ψ is real:

$$\Psi = A\, e^{kS}.$$

By means of (34) and (35), we will then have:

$$\left\{\begin{aligned} {}^4T_{\mu i} &= -2k^2 A^2 S_{,\mu} S_{,i} + 2A_{,\mu} A_{,i}, \\ {}^4T_{ii} &= \delta_{ii} k^2 A^2 (\mu^2 - 2\delta_{ii} S_{,i} S_{,i}) + 2A_{,i} A_{,i}. \end{aligned}\right. \tag{76}$$

Case of a static charge. – Set:

$$S = \frac{x^0}{2} + \mu\, x^4. \tag{77}$$

Formulas (76) will then become:

$$\left\{\begin{aligned} {}^4T_{0i} &= 0, \quad i = 1, 2, 3, \\ {}^4T_{04} &= -2k^2 \frac{\mu}{\alpha}, \\ {}^4T_{ik} &= 2A_{,i} A_{,k}, \quad i \neq k, \\ {}^4T_{ii} &= -2k^2 A^2 \mu^2 + 2A_{,i} A_{,i}, \end{aligned}\right. \tag{78}$$

and as a result, if we take the values (3) and (30) for μ and k, resp., and set:

$$I = -\frac{2k^2\mu^2}{m_0} \int \left\{A^2\right\}_{t-\frac{r}{c}} \frac{dx^1\, dx^2\, dx^3}{r}, \tag{79}$$

then, from (74), we will get:

$$\left\{\begin{aligned} \varepsilon_{0i} &= 0, \quad i = 1, 2, 3, \\ \varepsilon_{04} &= -\xi\alpha \cdot \frac{1}{c^2} \frac{e}{4\pi} I, \\ \varepsilon_{ik} &= -\frac{\chi}{\pi} \int \left\{A_{,i} A_{,k}\right\}_{t-\frac{r}{c}} \frac{dx^1\, dx^2\, dx^3}{r}, \quad i \neq k, \\ \varepsilon_{ii} &= -\frac{\chi m_0}{4\pi} I - \frac{\chi}{\pi} \int \left\{(A_{,i})^2\right\}_{t-\frac{r}{c}} \frac{dx^1\, dx^2\, dx^3}{r}. \end{aligned}\right. \tag{80}$$

We now let r_0 denote the distance from the potential point to the point O where we finds the charge. When we take only the terms in $1 / r_0$, we will have:

$$\begin{cases} \varepsilon_{0i} = 0, \quad i = 1, 2, 3, \\ \varepsilon_{04} = -\xi\alpha \cdot \dfrac{e}{4\pi\, r_0}, \\ \varepsilon_{ik} = 0, \quad i \neq k, \\ \varepsilon_{ii} = -\dfrac{\chi m_0\, c^2}{4\pi\, r_0}. \end{cases} \tag{81}$$

The identification of the values (80) and (81) leads to the conditions:

$$\begin{cases} A_{,i} = 0, \\ I = \dfrac{c^2}{r_0}. \end{cases} \tag{82}$$

The first condition indicates that the static charge is represented by a *stationary* phase wave of *constant* amplitude.

From (79), the second condition will then be satisfied if one imagines that the amplitude is non-zero only in a finite volume around the point *O*. If one calls that volume υ then one can define r_0 by the theorem of the mean by setting:

$$\frac{\upsilon}{r_0} = \int \frac{dx^1\, dx^2\, dx^3}{r};$$

one must then have:

$$\frac{8\pi^2}{h^2} m_0\, A^2\, \upsilon = 1. \tag{83}$$

Observe that the conditions (42) are indeed satisfied then.

Case of a moving charge with a very low uniform velocity. – Suppose that the charge moves with the velocity βc along x^1, and that one can neglect β^2 in comparison to unity. An analogous calculation, to the same approximation, will lead to the same amplitude *A* and a phase velocity c / β; i.e.:

$$S = \frac{x^0}{\alpha} + \mu\, x^4 - \mu\, \beta\, x^1. \tag{84}$$

Other than the values (81) for ε_{ii} and ε_{04}, one will find the potential vector:

$$\begin{cases} \varepsilon_{01} = +\xi\, \alpha \cdot \dfrac{e\beta}{4\pi\, r_0}, \\ \varepsilon_{02} = \varepsilon_{03} = 0, \end{cases} \tag{85}$$

and the gravitational potential:

$$\begin{cases} \varepsilon_{i4} = \dfrac{\chi m_0 c^2}{4\pi r_0} \cdot 2\beta, \\ \varepsilon_{ik} = 0, \quad i \neq k, \text{except for } (1, 4). \end{cases} \tag{86}$$

II. – CONTINUOUS SYSTEMS.

9. Case of *N* point-like particles. – In the case where the system is composed of N point-like particles, one can [**6**.a, b] study the motion of each of them separately by the preceding method. One will then get N geodesics in the five-dimensional universe that might have differing inclinations over space-time. We pass directly to the general case.

10. General case. – Let a system with f degrees of freedom be defined by f parameters q^n. We let the Latin indices vary from 1 to f and the Greek indices from 0 to f. Nonetheless, the *underlined* indices will vary from 1 to 4 when they are Latin symbols and from 0 to 4 when they are Greek ones. We set:

$$x^{\underline{i}} = x^{\underline{i}}(x'^1, x'^2, x'^3, x'^4, q^1, \ldots, q^f), \tag{87}$$

by supposing that:

$$dx^{\underline{i}} = 0 \tag{88}$$

when s varies by ds, but that, on the other hand:

$$\delta x^{\underline{i}} \neq 0, \qquad \delta q^n = 0, \qquad \delta s = 0. \tag{89}$$

If $\sigma_{(m)}$, $\sigma_{(e)}$ are the mass and electromagnetic densities then we will set:

$$\begin{cases} \delta\tau_{(m)} = \sigma_{(m)}\, \delta x'^1\, \delta x'^2\, \delta x'^3\, \delta x'^4, \\ \delta\tau_{(e)} = \sigma_{(e)}\, \delta x'^1\, \delta x'^2\, \delta x'^3\, \delta x'^4, \\ \tau_{(m)} = \int \delta\tau_{(m)}, \\ \tau_{(e)} = \int \delta\tau_{(e)}, \\ \mu = \dfrac{\tau_{(m)}}{\tau_{(e)}}. \end{cases} \tag{90}$$

The integrations extend over the system.

We define the g^*_{mn} and the Φ^*_n by the equalities:

$$\begin{cases} ds^2 = g_{\underline{ik}}\, dx^{\underline{i}}\, dx^{\underline{k}} = g^*_{mn} dq^m dq^n, \\ \Phi_{\underline{i}}\, dx^{\underline{i}} = \Phi^*_n\, dq^n, \end{cases} \tag{91}$$

and we will introduce the "mean" values:

$$\begin{cases} G_{mn} = \dfrac{1}{\tau_{(m)}} \displaystyle\int g^*_{mn}\, \delta\tau_{(m)}, \\ Q_m = \dfrac{1}{\tau_{(e)}} \displaystyle\int \Phi^*_m\, \delta\tau_{(e)}. \end{cases} \tag{92}$$

We then have the Jacobi equation:

$$G^{mn}\left(\frac{\partial S}{\partial q^m} - Q_m\right)\left(\frac{\partial S}{\partial q^n} - Q_n\right) - \mu^2 = 0, \tag{93}$$

in which the contravariant components are taken in the configuration space $G_{mn}\, dx^m\, dx^n$.

If we introduce the fifth variable x^0 then, at the same time, we must introduce an $(f + 1)^{\text{th}}$ parameter q^0 ; we can take:

$$x^0 = q^0, \tag{94}$$

and if:

$$S = \frac{q^0}{\alpha} + S'(q^1, \ldots, q^f) \tag{95}$$

then the Jacobi equation (93) will take the form:

$$\Gamma^{\mu\nu} \frac{\partial S}{\partial q^\mu} \frac{\partial S}{\partial q^\nu} = \mu^2 - \frac{1}{2\chi} \tag{96}$$

in the $(f + 1)$-dimensional configuration space that is defined by:

$$\begin{cases} \Gamma^{mn} = G^{mn}, \\ \Gamma^{0m} = -\alpha Q^m, \\ \Gamma^{00} = \alpha^2 Q_m Q^m - \dfrac{1}{\xi}. \end{cases} \tag{97}$$

The quantization condition will take the form:

$$\Gamma^{\mu\nu}\Psi_{,\mu\nu} - k^2\left(\mu^2 - \frac{1}{2\chi}\right)\Psi = 0. \tag{98}$$

In the case of a real amplitude:

$$\Psi = A\, e^{kS}, \tag{99}$$

that equation will be equivalent to the two real equations:

$$\begin{cases} \Box A = 0, \\ \Box S = -\dfrac{d \log A^2}{ds}. \end{cases} \tag{100}$$

Suppose, moreover, that A is *invariant* during the motion of the system. Equations (100) will then become:

$$\begin{cases} \Box A = 0, \\ \Box S = 0. \end{cases} \tag{101}$$

De Donder showed [**6**.*f*] that under those conditions, one can interpret A as an *internal stress potential*.

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The five-dimensional universe and wave mechanics (*)

By L. ROSENFELD

(*Second communication*)

Translated by D. H. Delphenich (†)

On continuous systems that have an internal tension potential

This communication is a continuation of the second part of my preceding work and is concerned, more particularly, with continuous systems that admit an internal tension potential. The calculations are developed in the general case of a system with f degrees of freedom, but they will be applied, in particular, to a continuous system in four dimensions (viz., space-time), or, if one prefers, to a continuum in five-dimensions, as it was defined in the first communication.

De Donder recently introduced two important ideas into wave mechanics: viz., the notion of the *permanence* [**1**] of a system and the interpretation [**2**] of the amplitude A of the Schrödinger function Ψ as the *internal tension potential* of the system. In the first part of the present article, I shall study the conditions under which those ideas are introduced, and I shall establish a deep relationship that links them: *Any permanent system admits an internal tension potential, and conversely.*

In the second part, I shall indicate the "world-function" L, which permits one to recover the internal tensions by starting from the fundamental variational principle (43) of my first communication. My results permit one to better account for the meanings of the mass and charge densities $\rho_{(m)}$, $\rho_{(e)}$ that are defined by means of the function Ψ; viz., they are *mean* densities. The *true densities* $\sigma_{(m)}$, $\sigma_{(e)}$ *serve only to define a "mean" configuration space that is equivalent from a spectral standpoint.* I must thank De Donder for the very important remark that this is a particularly instructive aspect of the *correspondence principle;* I hope to return soon to that question in more detail and from a more general viewpoint.

Finally, in the third part, I shall develop some cosmological consequences of the notion of internal tension. Those remarks are only provisional, moreover. One will still be quite far from a solution to the fundamental cosmological problems.

I must express my deepest gratitude to De Donder, who did not cease to exhibit a very active interest in my work.

(*) Presented by De Donder.

(†) Translator's note: The erratum that was described on pp. 580 of this volume was incorporated into the text.

I. – PERMANENCE AND INTERNAL TENSION POTENTIAL.

1. Quantization of systems with $f + 1$ degrees of freedom. – Take the second part of our preceding article [**3**] to be the starting point. One defines an $(f + 1)$-dimensional *mean* configuration space metric $\Gamma^{\mu\nu}$ with the aid of the given distribution of mass and charge densities $\sigma_{(m)}$, $\sigma_{(e)}$, resp. In that space, the Jacobi equation is:

$$J \equiv \Gamma^{\mu\nu} \frac{\partial S}{\partial x^\mu} \cdot \frac{\partial S}{\partial x^\nu} - \left(\mu^2 - \frac{1}{2\chi} \right) = 0, \tag{1}$$

and the quantization condition is:

$$\Gamma^{\mu\nu} \Psi_{\mu\nu} - k^2 \left(\mu^2 - \frac{1}{2\chi} \right) \Psi = 0. \tag{2}$$

If one sets:

$$\Psi = A\, e^{kS}, \tag{3}$$

to simplify, then equation (2) will give, separately:

$$\left.\begin{aligned} \Box A &= 0, \\ \Box S &= -\mu \frac{d \log A^2}{ds}, \end{aligned}\right\} \tag{4}$$

in which:

$$\Box f \equiv \Gamma^{\mu\nu} f_{\mu\nu} . \tag{5}$$

The function S that enters into (3) and (4) always has the form:

$$S = \frac{q^n}{2} + S'(q^1, \ldots, q^f), \tag{6}$$

but the same thing will not be true *a priori* for the Jacobi function that enters into (1) when the system is subjected to internal tensions. Meanwhile, De Donder has shown [**2**] that even in that case, equation (1) will persist with the significance that (6) gives to S. For the reader's convenience, we shall rapidly summarize De Donder's presentation.

2. Invariance conditions for A. – If we introduce the tensions ([1]):

([1]) De Donder had set:

$$\sqrt{-g}\, \Pi_{\underline{\alpha}} = \sigma_{(m)} \frac{\partial A}{\partial x^{\underline{\alpha}}} .$$

The difference is just a question of homogeneity; De Donder's A does not have the same dimensions as mine.

$$\sqrt{-g}\,\Pi_{\underline{\alpha}} = \sigma_{(e)}\frac{\partial A(x^1, x^2, x^3, x^4)}{\partial x^{\underline{\alpha}}}\,, \tag{7}$$

which is derived from a potential A then the conservation of mass will demand ([2]) that one must have:

$$\Pi_{\underline{\alpha}}\frac{dx^{\underline{\alpha}}}{ds} = 0\;; \tag{8}$$

i.e.:

$$\boxed{\frac{dA}{ds} = 0.} \tag{9}$$

The tension potential is invariant under the motion of the system.

The Hamiltonian function of the system without tension is:

$$H(p) = \frac{1}{2\mu}\left\{\Gamma^{\mu\nu}p_\mu p_\nu + \frac{1}{2\chi}\right\}; \tag{10}$$

it was obtained by setting:

$$p_\mu = \frac{\partial S}{\partial q^\mu}$$

in (96) of our preceding note.

The Hamiltonian function of the system with internal tensions will be:

$$H^* = H - A, \tag{11}$$

and the Jacobi equation will have the form:

$$J^* \equiv \Gamma^{\mu\nu}\frac{\partial S^*}{\partial q^\mu}\cdot\frac{\partial S^*}{\partial q^\nu} - \left(\mu^2 - \frac{1}{2\chi}\right) = 0, \tag{12}$$

with

$$S^* = As + \frac{q^n}{2} + S'(q^1, \ldots, q^f). \tag{13}$$

Upon observing that $\partial A / \partial q^0 = 0$, one will infer from (13), (6), (12), and (1) that:

$$J^* \equiv J + 2s\,\Gamma^{\mu\nu}\frac{\partial A}{\partial q^\mu}\frac{\partial S}{\partial q^\nu} + s^2\,\Gamma^{\mu\nu}\frac{\partial A}{\partial q^\mu}\frac{\partial A}{\partial q^\nu} = 0. \tag{14}$$

Now, from (9):

([2]) See equation (339) in *Théorie des Champs gravifiques* [**4**].

$$\left.\begin{aligned} 0 = \mu\frac{dA}{ds} &= \frac{\partial A}{\partial q^\nu}\cdot\mu\frac{dq^\nu}{ds} = \frac{\partial A}{\partial q^\nu}\Gamma^{\mu\nu}\frac{\partial S^\bullet}{\partial q^\mu} \\ &= \Gamma^{\mu\nu}\frac{\partial A}{\partial q^\nu}\frac{\partial S}{\partial q^\mu} + s\,\Gamma^{\mu\nu}\frac{\partial A}{\partial q^\nu}\frac{\partial A}{\partial q^\mu}. \end{aligned}\right\} \tag{15}$$

Since the two double sums in (15) are independent of *s*, one will have, separately:

$$\Gamma^{\mu\nu}\frac{\partial A}{\partial q^\nu}\frac{\partial S}{\partial q^\mu} = 0, \tag{16}$$

$$\Gamma^{\mu\nu}\frac{\partial A}{\partial q^\nu}\frac{\partial A}{\partial q^\mu} = 0, \tag{17}$$

and (14) or (12) reduce to $J = 0$.

Remark. – From the second equation (4), the fundamental invariance condition (9) is *equivalent* to the condition:

$$\Box S = 0. \tag{18}$$

The condition (9) or (18) is *necessary and sufficient* for one to be able to interpret A as an internal tension potential.

The quantization of systems that have an internal tension potential is then determined by the equations:

$$\boxed{\Box A = 0, \quad \Box S = 0.} \tag{19}$$

A satisfies the condition (9).

3. Permanence. – One says that the system considered is *permanent* when one has the integral invariant:

$$\boxed{\frac{d}{ds}\int\sqrt{|G|}\,dq^1\cdots dq^f = 0.} \tag{20}$$

When one takes the complementary Maxwell equation into account, one will easily see that equation (20) is *equivalent* to the equation:

$$\Box S^* = 0\,; \tag{21}$$

i.e.:

$$\Box S + s\Box A = 0,$$

or rather, since will always have $\Box A = 0$, from (4):

$$\Box S = 0\,;$$

the permanence condition is then *equivalent* to the fundamental condition (9). In other words:

Any permanent system admits an internal tension potential, and conversely.

II. – THE FUNCTION L OF SYSTEMS THAT HAVE AN INTERNAL TENSION POTENTIAL.

4. The function L. – We first introduce the notations:

$$\rho_{(m)} = -\,2\,k^2\,\mu^2\,A^2, \tag{22}$$

$$\rho_{(e)} = -\,2\,k^2\,\mu\,A^2, \tag{23}$$

in such a way that:

$$\frac{\rho_{(m)}}{\rho_{(e)}} = \mu = \frac{\tau_{(m)}}{\tau_{(e)}}\,; \tag{24}$$

we shall have to discuss the meaning of these quantities in a moment.

Having said that, we define the function L by:

$$L = \Gamma^{\mu\nu}\,\frac{\partial\Psi}{\partial q^\mu}\cdot\frac{\partial\bar{\Psi}}{\partial q^\nu} - \left(\mu^2 - \frac{1}{2\chi}\right)\Psi\,\bar{\Psi} - \rho_{(e)}\,A\,; \tag{25}$$

i.e., we simply add the "tension function" $-\,\rho_{(e)}\,A$ to the function L of the system without tensions. Since the former function does not contain Ψ or $\bar{\Psi}$ explicitly, the quantum equation of the continuous system (2) will not be modified. Furthermore, one also has:

$$T_{\mu\nu} = \frac{\partial\Psi}{\partial q^\mu}\frac{\partial\bar{\Psi}}{\partial q^\nu} + \frac{\partial\bar{\Psi}}{\partial q^\mu}\frac{\partial\Psi}{\partial q^\nu} - \Gamma_{\mu\nu}\,L\,. \tag{26}$$

We shall develop the calculation of that expression.

In order to do that, first recall that equation (36) of the first communication [**3**]:

$$\Gamma^{\mu\nu}\,\frac{\partial\Psi}{\partial q^\mu}\frac{\partial\bar{\Psi}}{\partial q^\nu} - \left(\mu^2 - \frac{1}{2\chi}\right)\Psi\,\bar{\Psi} = \Gamma^{\mu\nu}\,\frac{\partial A}{\partial q^\mu}\frac{\partial A}{\partial q^\nu}\,;$$

as a result, due to (17), the function L will have the value:

$$L = -\,\rho_{(e)}\,A\,. \tag{27}$$

Now, by virtue of the Jacobi equation, upon setting:

$$u^\mu = \frac{dq^\mu}{ds}, \tag{28}$$

we will have:

$$\mu\, u^\mu = \Gamma^{\mu\nu} \frac{\partial S}{\partial q^\nu}, \tag{29}$$

or, in *covariant components with respect to the* (f + 1)*-dimensional form* $\Gamma_{\mu\nu}\, dq^\mu\, dq^\nu$:

$$\mu\, u_\mu = \frac{\partial S}{\partial q^\mu}. \tag{30}$$

Furthermore:

$$T_{\mu\nu} = -\,2k^2 A^2 + \frac{\partial S}{\partial q^\mu}\frac{\partial S}{\partial q^\nu} + 2\frac{\partial A}{\partial q^\mu}\frac{\partial A}{\partial q^\nu} + \Gamma_{\mu\nu}\, \rho_{(e)}\, A,$$

which can be written, by means (30) and (22):

$$T_{\mu\nu} = \rho_{(m)}\, u_\mu\, u_\nu + \Pi_{\mu\nu}, \tag{31}$$

with

$$\Pi_{\mu\nu} = 2\frac{\partial A}{\partial q^\mu}\frac{\partial A}{\partial q^\nu} + \Gamma_{\mu\nu}\, \rho_{(e)}\, A. \tag{32}$$

5. The mean densities. – We shall now compare the expression (31)-(32) for the *material tensor* $T_{\mu\nu}$ to the classical expression for *Einsteinian gravity*, in such a way that we can specify the significance of the magnitudes $\rho_{(m)}$, $\rho_{(e)}$, and $\Pi_{\mu\nu}$.

First observe that since A is independent of q^0, one will have:

$$\Pi_0^i = 0 ;$$

on the other hand, from (30) and (6):

$$u_0 = \frac{1}{\mu\alpha}.$$

If one introduces those results into (31) for μ = 0, ν = 1, after passing to mixed components, then upon taking (24) into account, one will find that:

$$\alpha T_0^i = \rho_{(e)}\, u^i . \tag{33}$$

For the components relative to space-time, one will have simply:

$$T^{mn} = \rho_{(m)}\, u^m\, u^n + \Pi^{mn}. \tag{33′}$$

The material tensor T^{mn} is decomposed into a *dynamical tensor* $\rho_{(m)}\, u^m\, u^n$ and a *massive tensor* Π^{mn} (cf., equation (41), pp. 11, of *Théorie des Champs gravifiques* [**4**]).

We first address $\rho_{(m)}$ and $\rho_{(e)}$; formulas (33)-(33′) show that $\rho_{(m)}$ and $\rho_{(e)}$ must be interpreted as the mass and charge densities of the system, or better (*), the ones that *correspond* to the densities of the system in configuration space:

Indeed, it is essential to observe that $\rho_{(m)}$ and $\rho_{(e)}$ *are not* generally the densities $\sigma_{(m)}$ and $\sigma_{(e)}$ that were given originally; that will exhibit the relation (24) immediately:

$$\frac{\rho_{(m)}}{\rho_{(e)}} = \frac{\int \sigma_{(m)}\, \delta x'^1\, \delta x'^2\, \delta x'^3\, \delta x'^4}{\int \sigma_{(e)}\, \delta x'^1\, \delta x'^2\, \delta x'^3\, \delta x'^4}.$$

From that relation, one can set:

$$\rho_{(m)} = \frac{\tau_{(m)}}{\upsilon}, \qquad \rho_{(e)} = \frac{\tau_{(e)}}{\upsilon}, \tag{34}$$

and one can always “normalize” A in such a way that:

$$\int \frac{1}{\upsilon}\sqrt{|G|}\, dq^1 \cdots dq^f = 1, \tag{35}$$

which will show that $\rho_{(m)}$ and $\rho_{(e)}$ are (ponderable) *mean densities.*

One then sees that the *true* densities $\sigma_{(m)}$ and $\sigma_{(e)}$ serve to simply define a *mean* configuration space metric $\Gamma^{\mu\nu}$:

$$\left(\text{recall that } G_{mn} = \frac{1}{\tau_{(m)}} \int g^*_{mn}\, \delta\tau_{(m)}, \quad Q_m = \frac{1}{\tau_{(e)}} \int \Phi^*_m\, \delta\tau_{(e)}\right).$$

In that *mean* configuration space, one determines a function Ψ with the aid of which one defines the *mean* densities $\rho_{(m)}$ and $\rho_{(e)}$, which permit one to calculate the “global” gravitational and electromagnetic actions of the system.

6. Internal tensions. – Finally, we move on to consider the tensor $\Pi_{\mu\nu}$. The tensions Π_μ that are defined by $\Pi_{\mu\nu}$ are:

$$\Pi_\mu = \Pi^\nu_{\mu,\nu}. \tag{36}$$

We first confirm that the first term $2\Gamma^{\rho\nu}\dfrac{\partial A}{\partial q^\rho}\dfrac{\partial A}{\partial q^\nu}$ of Π^ν_μ makes no contribution to the tension Π_μ . Indeed:

(*) Remark by De Donder.

$$\left. \begin{aligned} 2\left(\Gamma^{\rho\nu}\frac{\partial A}{\partial q^\rho}\cdot\frac{\partial A}{\partial q^\nu}\right)_{,\nu} &= 2\Gamma^{\mu\nu}(A_{,\rho}A_{,\mu})_{,\nu} = 2\Gamma^{\mu\nu}(A_{,\rho\nu}A_{,\mu} + A_{,\rho}A_{,\mu\nu}), \\ &= 2A_{,\mu}\Gamma^{\mu\nu}A_{,\rho\mu} + 2\Gamma^{\mu\nu}A_{,\rho}A_{,\nu\mu}, \\ &= 2A_{,\mu}\Box A + (\Gamma^{\mu\nu}A_{,\rho}A_{,\nu})_{,\mu}, \\ &= 0, \end{aligned} \right\} \tag{37}$$

due to (4) and (17).

It then results that the system behaves like a "massive perfect fluid" from the standpoint of Π_μ . (*Théorie des Champs gravifiques* [**4**], pp. 15). One has:

$$\Pi_\mu = \Gamma_\mu^\nu (\rho_{(e)} A)_{,\nu} \, ;$$

i.e.:

$$\Pi_\mu = \frac{\partial(\rho_{(m)} A)}{\partial q^\mu} \, . \tag{38}$$

It is easy to see that one will indeed arrive at that value of Π_μ when one starts from the value (7) of $\Pi_{\underline{\alpha}}$.

III. – COSMOLOGICAL CONSIDERATIONS IN REGARD TO INTERNAL TENSIONS.

It is interesting to develop some of the consequences of introducing an internal tension function, especially from the cosmological standpoint. In order to do that, we shall place ourselves in the five-dimensional universe and repeat the calculation of the scalar ${}^4T$ that was developed in number **7** of our preceding article [**3**] with the new value of L. We effortlessly find that:

$${}^4T = -\,2\,(L + k^2\,\mu^2\,\Psi\bar{\Psi}) + 2\rho_{(e)}\,A = \rho_{(m)} + 4\rho_{(e)}\,A \, , \tag{39}$$

in such a way that the curvature of space-time will be:

$$R = \chi\,\rho_{(m)} + 4\lambda_{(e)} \, , \tag{40}$$

with

$$\lambda_{(e)} = \chi\,\rho_{(e)}\,A \, . \tag{41}$$

We will then be led to a "cosmic" term with a curvature that is *radically different* from Einstein's, since it will depend upon the distribution of electricity.

In order to see what happens, consider the *three fundamental formulas:*

$$\left. \begin{aligned} \rho_{(m)} &= -2k^2A^2\mu^2, \\ \rho_{(e)} &= -2k^2A^2\mu, \\ \lambda_{(e)} &= \chi\rho_{(e)}A, \end{aligned} \right\} \tag{42}$$

which one can write:

$$\boxed{\left.\begin{aligned} \rho_{(m)} &= \zeta^2 (eA)^2 \mu^2, \\ \rho_{(e)} &= \zeta^2 (eA)^2 \mu, \\ \lambda_{(e)} &= \chi(\rho_{(e)} A), \end{aligned}\right\}} \tag{43}$$

when one takes into account the meaning of *k*. In those formulas, ζ and χ are two *universal constants*, and e, μ, A are the *three fundamental magnitudes* that determine the state of the universe at each point. The magnitudes e and μ are such that their product $e\mu$ is a quantity that is independent of e. The fundamental formulas then define the *three auxiliary quantities* $\rho_{(m)}$, $\rho_{(e)}$, $\lambda_{(e)}$ that suffice to describe the massive and electromagnetic systems in equilibrium, such as electrons, protons, light quanta (from the *microscopic* viewpoint), or even molecules and systems of molecules (from the *macroscopic* viewpoint), or finally stars and star systems (from the *ultra-macroscopic*, or *cosmic*, viewpoint). A system for which $\rho_{(e)} = 0$, $\rho_{(m)} \neq 0$ at each point is called a *neutral material system*. From the microscopic viewpoint, there exist no neutral material systems except for possibly light quanta, if one (like L. de Broglie) would like to attribute a non-zero mass to them. A system for which one has $\rho_{(m)} = \rho_{(e)} = 0$ at each point is called *vacuous*.

A discussion of the first two fundamental formulas will give the following result:

The system (e, A, μ) is vacuous only if at least one of the three quantities e, A, μ is zero at each point. Meanwhile, there is one important exceptional case: It is the one where $A \neq 0$, where e tends to zero and μ tends to infinity in such a manner that the product $e\mu$ has a finite limit, which is necessarily non-zero then. The system will then be a neutral material system, and that is the only case in which one can have such a system.

The third fundamental formula, in turn, shows that $\lambda_{(e)}$ is zero only for a vacuous system or a neutral material system.

We see that a vacuous system or a neutral material system can be under tension, but those tensions will not lead to any supplementary curvature of the universe.

A general (massive or electromagnetic) system is necessarily under tension, and those tensions will produce a supplementary curvature of the universe.

If one adopts the cosmic point of view and one remarks that the universe, taken globally, is a neutral material system then one will arrive at the conclusion that the cosmic curvature $\lambda_{(e)}$ is zero: One knows how many arguments exist against that concept. It then seems necessary to introduce, as Einstein did, a *third fundamental universal constant*, namely, the curvature of the vacuum $4\lambda_0$. That introduction can be achieved with no difficulty; it suffices to replace the variational principle:

$$\delta \int (P + 2\chi L)\sqrt{-g}\, dx^0 \cdots dx^4 = 0 \tag{44}$$

with the principle:

$$\boxed{\delta \int (P - 4\lambda_0 + 2\chi L)\sqrt{-g}\, dx^0 \cdots dx^4 = 0.} \tag{45}$$

However, the curvature $4\lambda_0$ will no longer play precisely the same role that it did in Einstein's theory. It will enter only *in part* to ensure the equilibrium of massive and electromagnetic systems.

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The five-dimensional universe and wave mechanics

By L. ROSENFELD (*)

(*Third communication*)

Translated by D. H. Delphenich

THE CORRESPONDENCE PRINCIPLE

The goal of the present article has been touched upon already in my second communication (**). I shall now take up the question again systematically by referring back to my first communication (***), and the present one constitutes an indispensible complement to it. All of the arguments will be developed for a point-like system in the five-dimensional universe, but the extension to continuous (f + 1)-dimensional systems is immediate.

Wave mechanics, as contained in the *variational principle* [U^1, (43)], is a formal theory. In order to give a physical interpretation, one is guided by the *correspondence principle*, with the meaning that O. Klein (IV) gave to it, and in a more general and precise manner, Th. De Donder (V).

The compatibility of the two principles – i.e., the possibility of establishing such a correspondence – is assured by a *fundamental theorem* whose proof will be the goal of number **2** below.

Having laid those general foundations, it will essentially remain for us to establish the correspondence in question. Thanks to some extremely important consideration that that are contained in a letter from L. de Broglie, I have succeeded in generalizing paragraph **8** in his recent paper (VI). The details are presented in number **3.**

De Donder played an essential part in this article by suggesting the fundamental idea to me. I also have much to thank Louis de Broglie for, who was kind enough to continue a correspondence with me that was greatly profitable.

(*) Presented by Th. De Donder.

(**) Bull. Acad. roy. Belg. (5) **13** (1927), session on 2 July. In that communication, which is dedicated to the study of the internal tension potential, I kept to the approximation that is called "geometrical optics." The existence of an internal tension potential implies a restriction on the function Ψ. In the general case (cf., no. **3** of that communication), one will see internal tensions of a different type appear.

(***) Bull. Acad. roy. Belg. (5) **13** (1927), session on 3 May. In what follows, that work will be denoted by U^1. Similarly, De Donder's *Théorie des champs gravifiques* [Mém. des Sc. math., fasc. XIV (1926)] will be denoted by G. Hence, formula (12) in *Champs gravifiques* will be denoted by "[G, (12)]."

(IV) O. KLEIN, Zeit. Phys **41** (1927), 407-442.

(V) TH. DE DONDER, Bull. Acad. roy. Belg. (5) **13** (1927), session on 2 August.

(VI) L. DE BROGLIE, J. de Phys. (5) (May 1927).

1. Correspondence principle. – Wave mechanics, which is contained entirely in the variational principle [U^1, (43)], *formally* realizes the fusion of the general theory of relativity and quantum theory. Along with the *field equations* that describe gravitational and electromagnetic phenomena, one also has the *quantization equation* [U^1, (33)], which governs the quantum exchanges of energy. That latter equation involves a *fundamental quantity* Ψ, and the fusion of the two theories consists precisely of the fact that the five-dimensional material tensor that appears in the field equations is defined by means of the fundamental quantity Ψ. On the contrary, in *pure* Einsteinian gravity, it is a function of some other fundamental quantities of system: viz., the *mass density* $\sigma_{(m)}$ and *charge density* $\sigma_{(e)}$. (For the moment, I shall abstract from the massive-electromagnetic tensor $H_{\mu\nu}$ that Th. De Donder introduced *a priori* into gravitation.)

The new definition [U^1, (47)] of the material tensor as a function of Ψ then implies a modification of our new conception of the role of the fundamental quantities $\sigma_{(m)}$ and $\sigma_{(e)}$. In Einsteinian gravitation, those quantities enter *directly* into the field equations in order to determine the gravitational and electromagnetic potentials that correspond to the given distribution ($\sigma_{(m)}$, $\sigma_{(e)}$). In wave mechanics, they enter directly into only the quantization equations by which they determine the quantity Ψ. It is therefore the latter quantity that one must introduce into the field equations in order to obtain potentials.

A little reflection will show that the material tensor $T^{\mu\nu}$, when defined as a function of Ψ, is not necessarily identical to the material tensor of *pure* gravity, which is defined as a function of $\sigma_{(m)}$ and $\sigma_{(e)}$. Moreover, it seems desirable to analyze the behavior of the tensor $T^{\mu\nu}$ a little more closely in such a way as to exhibit the possible modifications that the introduction of the quantization quantity Ψ can afford to gravitation; that is precisely the role of the *correspondence principle.* It comes down to interpreting the tensor $T^{\mu\nu}$ as an ordinary gravitational tensor, which is a function of certain mass and charge "densities" $\rho_{(m)}$, $\rho_{(e)}$, which naturally depend upon Ψ. A comparison of $\rho_{(m)}$, $\rho_{(e)}$ with $\sigma_{(m)}$, $\sigma_{(e)}$ will indicate how quantization modifies gravitational and electromagnetic phenomena. One cannot give a more precise *general* statement of the correspondence principle. One will see later on how one can effectively define $\rho_{(m)}$ and $\rho_{(e)}$ as functions of $\sigma_{(m)}$, $\sigma_{(e)}$, and Ψ. One will see that one must introduce a *massive tensor* Π^{ij} that determines the *internal quantum tensions*. Those defining formulas constitute the correspondence principle in the strict sense. It establishes the identification of the formal schema of wave mechanics with the gravitational schema of Th. De Donder [*G*, Chapter VI], which then illustrates the manner in which wave mechanics enlarges the scope of pure gravitation in order to introduce quantum phenomena into it.

2. The fundamental theorem. – Before going further, it would be appropriate to turn our attention to a very remarkable property of the tensor $T^{\mu\nu}$: In order for that tensor to be capable of being interpreted as an ordinary gravitational tensor, conforming to the correspondence principle, it must satisfy the conservation equations for energy-impulse and electricity [U^1, (60)] and [U^1, (61)]. That condition is entirely essential to ensure the compatibility of the two principles: viz., the variational principle [U^1, (43)] and the correspondence principle, which constitute wave mechanics.

We shall show precisely that *equations* [U^1, (60)], [U^1, (61)] *are verified identically for any solution of the quantization equations* [U^1, (33)], [U^1, (33′)]. That is the *fundamental compatibility equation:* It is a *consequence* of the variational principle. The latter point is particularly remarkable. The variational principle is "intrinsically" compatible with the correspondence principle with no restriction or supplementary condition.

The proof is extremely simple. It has been performed already [U^1, no. **6**] in the context of electricity, moreover. From the calculations in [U^1, no. **6**], equations [U^1, (61)], [U^1, (60)] are equivalent to:

$$\begin{cases} T_{0,m}^{\ \ m} = 0, \\ T^{nm}_{\ \ ,m} + \begin{Bmatrix} k\ \sigma \\ n \end{Bmatrix} T^{k0} = 0, \end{cases}$$

however, from [U^1, (20)] and [U^1,(7)], one will have:

$$\frac{\partial T_{\mu\nu}}{\partial x^0} \equiv 0,$$

in such a way that by means of [U^1, (53)], the preceding equations are equivalent to:

$$\begin{cases} T_{0,\mu}^{\ \ \mu} = 0, \\ T^{n\mu}_{\ \ ,\mu} = 0, \end{cases}$$

or rather, it will suffice to show that one will have:

$$\boxed{T_{\nu,\mu}^{\ \ \mu} \equiv 0} \tag{1}$$

for any solution Ψ, $\bar{\Psi}$, of [U^1, (33)], [U^1, (33′)].

Formula (1) can be verified by direct calculation [U^1, no. **6**]. However, as De Donder and Nuyens suggested to me, it is more elegant to resort to the fundamental identities of gravitation when they are applied to the invariant $2L\sqrt{-g}$. Upon observing that $T_{\mu\nu}$ precisely the symmetric part of $\dfrac{\delta\left(2L\sqrt{-g}\right)}{\delta\gamma^{\mu\nu}}$, those identities can be written:

$$2T_{\nu,\mu}^{\ \ \mu} + \frac{\delta\left(2L\sqrt{-g}\right)}{\delta\Psi}\Psi_{,\nu} + \frac{\delta\left(2L\sqrt{-g}\right)}{\delta\bar{\Psi}}\bar{\Psi}_{,\nu} \equiv 0,$$

but [U^1, (33)], [U^1, (33′)] are equivalent to:

$$\frac{\delta\left(2L\sqrt{-g}\right)}{\delta\Psi} = 0, \qquad \frac{\delta\left(2L\sqrt{-g}\right)}{\delta\bar{\Psi}} = 0,$$

which result immediately from formula (1).

3. The correspondence principle and Schrödinger-de Broglie dynamics. – It now remains for us to construct a system ($\rho_{(m)}$, $\rho_{(e)}$) that *corresponds* to the given system ($\sigma_{(m)}$, $\sigma_{(e)}$) in the manner that was specified above. It is presently impossible for me to resolve the question of knowing whether the correspondence that I established is the only acceptable one; that seems quite probable. In what follows, I have made much use of the letter from De Broglie to which I alluded in the Introduction.

Set:

$$\Psi = A' e^{k S'}; \tag{2}$$

it is easy to see [cf., U^1, no. 4] that the quantization equation [U^1, (33)] is equivalent to the system of two real equations:

$$\gamma^{\mu\nu} S'_{,\mu} S'_{,\nu} = \mu^2 - \frac{1}{2\chi} \frac{\Box A'}{K^2 A}, \tag{3}$$

$$2\mu \gamma^{\mu\nu} S'_{,\mu} A'_{,\nu} + A' \Box S' = 0. \tag{4}$$

Set:

$$\boxed{\mu'^2 = \mu^2 + \frac{\Box A'}{K^2 A'}.} \tag{5}$$

By means of that notation, we confirm that equation (3) takes the form [cf., U^1, (10)] of the Jacobi equation:

$$J' \equiv \gamma^{\mu\nu} S'_{,\mu} S'_{,\nu} - \left(\mu'^2 - \frac{1}{2\chi}\right) = 0 \tag{6}$$

that relates to a system that is characterized by a quantity μ' (which differs slightly from μ): *We shall take that new system to be the basis for the "corresponding" system.* It is important to note that the Jacobi equation of that corresponding system is a consequence of the quantization condition. In the case of a particle of mass m_0 and charge e, the only modification that wave mechanics brings with it is to replace the mass m_0 with *L. de Broglie's mass:*

$$M_0 = \sqrt{m_0^2 + \frac{h^2}{4\pi^2 c^2} \frac{\Box A'}{A'}}. \tag{7}$$

Now, the correspondence, properly speaking, is easy to establish. By virtue of the Jacobi equation (6), we will have:

$$\mu' u^\nu = \gamma^{\mu\nu} S'_{\cdot\mu} \tag{8}$$

for the velocity vector u^ν of the corresponding system or:

$$\left.\begin{aligned} \mu' u_i &= S'_{\cdot i}, \\ u_0 &= \frac{1}{\mu'\alpha} \end{aligned}\right\} \tag{9}$$

for the covariant vector u_ν *with respect to the* $d\sigma^2$.

If one sets:

$$\boxed{\begin{aligned} \rho_{(m)} &= 2K^2 A'^2 \mu'^2, \\ \rho_{(e)} &= 2K^2 A'^2 \mu' \end{aligned}} \tag{10}$$

then one will have:

$$\boxed{T_{\mu\nu} = \rho_{(m)} u_\mu u_\nu + 2A'_{\cdot\mu} A'_{\cdot\nu} - \gamma_{\mu\nu} L,} \tag{11}$$

in which L is the world-function that was defined in our first communication; upon taking (3) into account, one will have:

$$\boxed{L = \gamma^{\mu\nu} A'_{\cdot\mu} A'_{\cdot\nu} + A' \Box A'} \tag{12}$$

here.

In particular, one will have:

$$T^{mn} = \rho_{(m)} u^m u^n + \Pi^{mn}, \tag{13}$$

with the notation:

$$\boxed{\Pi_{\mu\nu} = 2A'_{\cdot\mu} A'_{\cdot\nu} - \gamma_{\mu\nu} L.} \tag{14}$$

Upon taking (9) and (10) into account, one will have, in addition, that:

$$\alpha T_0^i = \rho_{(e)} u^i. \tag{15}$$

We will then recover the material tensor [*G*, (38)] in (13), on the condition that we must interpret $\rho_{(m)}$ as a (mean) mass density of the corresponding system and Π^{mn} as a *mass tensor* that determines the *internal tensions:*

$$\Pi^i = {}^4\Pi^{in}_{\cdot n}. \tag{16}$$

Similarly, (15) gives us the current quadri-vector of we interpret $\rho_{(e)}$ as a (mean) charge density of the corresponding system. The correspondence principle is then expressed by formulas (5), (10), (14), and (12).

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The five-dimensional universe and wave mechanics

By L. ROSENFELD (*)

(*Fourth communication*)

Translated by D. H. Delphenich

THE PRINCIPLES OF WAVE MECHANICS

The present article has the main goal of presenting a synthesis of my first and third communications (**), and to include De Donder's important note on the correspondence principle (***).

Wave mechanics is initially based upon two *formal* principles:

1. The *variational principle* that was established in my first communication [U^1, (43)] and which permits one to write out partial differential equations by means of the introduction of a complex auxiliary function Ψ.

2. The *eigenfunction principle* or *Schrödinger's principle*, which gives the boundary conditions for the function Ψ.

One must add a *physical* principle to those formal principles: viz., the *correspondence principle*, which was stated most precisely in the cited note by De Donder, and which indicates the physical meaning of the formal operations that were performed by virtue of the first two principles.

A *compatibility theorem* establishes the link between the two groups of principles.

Having posed those fundamentals, one can *construct* various corresponding systems. I shall give two examples: The first one, which is entirely new, introduces a *quantum current*. The second one, which was treated already in my third communication, is the system that L. de Broglie adopted (IV). By now, it is well-ensconced in the general framework of the theory. It will take on some complements here that relate to *internal tensions* especially.

In the first part of this paper, everything will be developed in detail for the case of an *electrically-charged point particle*. In the second part, I shall indicate how one extends that to the *holonomic systems* that were considered in my first communication. Finally,

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(*) Presented by Th. De Donder.

(**) Bull. Acad. roy. de Belg. (5) **13** (1927), sessions on 3 May and 2 August 1927, which will be cited as U^1 and U^2, respectively. One should refer to them for the notations that are not explained here.

(***) *Ibid.*, session on 2 August 1927.

(IV) J. de Phys. (May 1927), 225-241, and above all, § **8**. C. R. Acad. Sci. Paris 8 August 1927.

in the third part, I will study certain *systems of N points* by a method that is due to De Donder (*). I will show how the present synthesis encompasses the *statistical conceptions* of Born and his school by means of the results obtained. I will conclude with some brief remarks on non-holonomic systems of *N* points.

In the editing of my present communication, I have been able to profit from numerous instructive conversations with De Donder, as well as from work done at his remarkable institute, which he placed at my disposal with indefatigable helpfulness.

I. POINT-LIKE SYSTEM.

1. – Wave mechanics is based upon three principles:

1. A *variational principle*, which permits one to write down the field equations *formally*.

2. A *correspondence principle*, which gives the physical meaning of those equations.

3. An *eigenfunction principle*, which determines the quantization of the system considered.

We shall study the first two principles in detail, and first of all in the case of point-like system in the five-dimensional universe.

2. The variational principle. – Let m_0 be the rest mass, and let e be the charge of the point considered. The ratio:

[U^1, (3)]
$$\mu = \frac{m_0\,c^2}{e} \tag{1}$$

is then a *constant of the point-like system* under study.

The motion of the point is determined by five parameters $q^0, q^1, \ldots, q^4$ that forms a *configuration space:*

$$d\sigma^2 = \gamma_{\mu\nu}\, dq^\mu\, dq^\nu \qquad (\mu, \nu = 0, 1, \ldots, 4); \tag{2}$$

one can, for example, set:

$$x^\mu = q^\mu. \tag{3}$$

Our problem consists of calculating the $\gamma_{\mu\nu}$ in the form (2); i.e., the gravitational and electromagnetic field of the point-like system (m_0 , e) *in the configuration space.* The field potentials are given as functions of the $\gamma_{\mu\nu}$ by the formulas:

(*) DE DONDER, C. R. Acad. Sci. Paris **184** (1927), 698-700 (presented at the session on 20 September 1927).

[U^1, (14)]
$$\begin{cases} \gamma_{ik} = g_{ik} - 2\chi\Phi_i\Phi_k, \\ \gamma_{0i} = \gamma_{i0} = -\xi\alpha\Phi_i, \qquad (i, k = 1, 2, \ldots, 4), \\ \gamma_{00} = -\xi, \end{cases} \tag{4}$$

or, in contravariant components:

[U^1, (8)]
$$\begin{cases} \gamma^{ik} = g^{ik}, \\ \gamma^{0i} = \gamma^{i0} = -\alpha\Phi^i, \\ \gamma^{00} = \alpha^2\Phi^i\Phi_i - \dfrac{1}{\xi}. \end{cases} \tag{5}$$

In these formulas, α, ξ, χ are *universal constants*, which satisfy the relations:

[U^1, (9)]
$$\xi\,\alpha^2 = 2\chi,$$

[U^1, (52)]
$$\chi = \frac{8\pi G}{c^4}, \qquad G = 6.7 \times 10^{-8} \qquad \text{CGS.} \tag{6}$$

The *formal* solution to that problem is given by the following *variational principle:* Introduce an (unknown) complex function Ψ, as well as the conjugate function $\bar{\Psi}$, and set:

[U^1, (32)]
$$\begin{cases} W = P + 2\chi L, \\ L = \gamma^{\mu\nu}\Psi_{,\mu}\Psi_{,\nu} + k^2\left(\mu^2 - \dfrac{1}{2\chi}\right)\Psi\,\bar{\Psi}. \end{cases} \tag{7}$$

P is the five-dimensional curvature invariant; it is a function of the $\gamma_{\mu\nu}$. k is a *system constant*, that is given by:

[U^1, (30), (40)]
$$k = i\,K = i \cdot \frac{2\pi}{h} \cdot \frac{e}{c}. \tag{8}$$

Having said that, *the partial differential equations give the* $\gamma_{\mu\nu}$ *and the auxiliary functions* Ψ, $\bar{\Psi}$ *are deduced from the variational equation:*

[U^1, (43)]
$$\delta\int W\sqrt{-g}\,dq^0 \cdots dq^4 = 0. \tag{9}$$

One will thus obtain the two *quantization conditions:*

$$\frac{\delta W\sqrt{-g}}{\delta\bar{\Psi}} = 0, \qquad \frac{\delta W\sqrt{-g}}{\delta\Psi} = 0,$$

and the fourteen *field equations:*

$$\frac{\delta W\sqrt{-g}}{\delta\gamma^{\mu\nu}} = 0 \qquad \text{(except for } \mu = \nu = 0\text{)}.$$

They are written explicitly as:

$$[U^1, (33), (37)] \qquad \Box\Psi - k^2\left(\mu^2 - \frac{1}{2\chi}\right)\Psi = 0,$$

$$[U^1, (33')] \qquad \Box\bar{\Psi} - k^2\left(\mu^2 - \frac{1}{2\chi}\right)\bar{\Psi} = 0, \tag{10}$$

and

$$[U^1, (51)] \qquad \left\{\begin{array}{c} R^{mn} - \frac{1}{2}g^{mn}R = -\chi(S^{mn} + T^{mn}), \\ {}^4H^{im}_{\ ,m} = \alpha T_0^i, \end{array}\right. \qquad (m, n, i = 1, \ldots, 4) \tag{11}$$

by means of the notations:

$$[U^1, (22)] \qquad S^{mn} = \tfrac{1}{4}g^{mn}H_{ik}H^{ik} - H^m{}_l H^{nl} \qquad (i, k, l, m, n = 1, \ldots, 4) \tag{12}$$

and

$$[U^1, (47)] \qquad T_{\mu\nu} = \Psi_{,\mu}\bar{\Psi}_{,\nu} + \bar{\Psi}_{,\mu}\Psi_{,\nu} - \gamma_{\mu\nu}L \qquad (\mu, \nu = 0, 1, \ldots, 4)\,. \tag{13}$$

3. The correspondence principle. – As one sees, the problem is solved *formally* by the sixteen equations (10), (11) in the sixteen unknowns $\gamma_{\mu\nu}$ (except γ_{00}), Ψ, $\bar{\Psi}$. The boundary conditions for the functions Ψ, $\bar{\Psi}$ are given by the third principle (*the eigenfunction principle* or *Schrödinger's principle*), which we shall not discuss here. However, it is important now to extract the physical sense of those equations: That is the role of the *correspondence principle,* which we shall examine.

First, let us introduce some useful terminology. Imagine a system of charged masses that are defined by the mass and charge *densities* $\rho_{(m)}$ and $\rho_{(e)}$, resp. Suppose that the system also includes a *quantum current* Λ_i, and that it is subject to *internal tensions* H^{mn}. Having done that, we will say that the system in question is *Maxwellian* if it satisfies the four equations:

$${}^4H^{im}_{\ ,m} = \rho_{(m)}\,u^i + \Lambda^i\,. \tag{14}$$

We say that it is *Einsteinian* if it satisfies the ten equations:

$$R^{mn} - \tfrac{1}{2}g^{mn}R = -\chi(S^{mn} + \rho_{(m)}\,u^m u^n + H^{mn}), \tag{15}$$

when we use the notation (12). In the formulas (14) and (15), u^i represents the contravariant *velocity* at the point (q^0, q^1, q^2, q^3, q^4). In particular, a Maxwellian system for which $\Lambda^i = 0$ will be called *pure Maxwellian.* An Einsteinian system for which $H^{mn} = 0$ will be called *pure Einsteinian*.

One knows that for a Maxwellian system, one will have the equation for *the conservation of total electric current:*

$$^4(\rho_{(e)}\, u^i + \Lambda^i)_{,\,i} = 0. \tag{16}$$

For an Einsteinian system, one will have the equations of *the conservation of energy-impulse*, which give the *dynamics* of the system in question:

$$^4S^{mn}_{\ \ ,n} + {}^4(\rho_{(m)}\, u^m\, u^n + H^{mn})_{,\,n} = 0. \tag{17}$$

Having said that, we return to equations (11). We observe that they have the *form* of the Einstein equations (15) and the Maxwell equations (14). The vector $\alpha T_0^{\ i}$ and the tensor T^{mn}, which are both functions of Ψ, $\bar{\Psi}$, by virtue of (13), take the places of the total current $\rho_{(e)}\, u^i + \Lambda^i$ and the material current $\rho_{(m)}\, u^m\, u^n + H^{mn}$ in them. The *correspondence principle* consists precisely of asserting that this analogy is not just formal, but also physical. More precisely: *The two functions* $\alpha T_0^{\ i}$ *and* T^{mn} *of* Ψ, $\bar{\Psi}$ *define a system that is both Einsteinian and Maxwellian in the configuration space*, which will be called the *corresponding system* to the given point-like one. In formulas:

$$\boxed{\begin{aligned} \rho_{(e)}\, u^i + \Lambda^i &= \alpha T_0^{\ i}(\Psi, \bar{\Psi}), \\ \rho_{(m)}\, u^m\, u^n + H^{mn} &= T^{mn}(\Psi, \bar{\Psi}). \end{aligned}} \tag{18}$$

It is essential to remark, first of all, that the correspondence thus-established takes place in *configuration space.* In the second place, it is necessary to *prove* the compatibility of the correspondence principle with the variational principle, because if the system $\alpha T_0^{\ i}$, T^{mn} is both Maxwellian and Einsteinian then it *must* give rise to the conservation equations (16) and (17), in the form:

$$\begin{cases} (\alpha T_0^{\ i})_{,i} = 0, \\ ^4S^{mn}_{\ \ ,n} + {}^4T^{mn}_{\ \ ,n} = 0. \end{cases} \tag{19}$$

The left-hand sides of formulas (19) are functions of Ψ, $\bar{\Psi}$ that must be *identically zero for any solution* Ψ, $\bar{\Psi}$ *of the quantization equations* (10). One verifies that this is, in fact, the case (viz., the *fundamental compatibility theorem*). One will then see that the correspondence principle has a different character from the other two principles. They are *postulates*, in the sense of formal logic, whereas the correspondence principle is a *physical principle*. The compatibility of the two groups is ensured by the intermediary of formulas (18), thanks to the compatibility theorem.

It now remains for us to *effectively* determine the corresponding systems; i.e., to calculate $\rho_{(e)}$, $\rho_{(m)}$, Λ^i, and H^{mn} as functions of Ψ, $\bar{\Psi}$, in such a manner that they satisfy formulas (18). As we have posed that problem, it is obviously indeterminate.

We shall give two particularly remarkable solutions, which we will then compare briefly.

4. Corresponding system with quantum current Λ^i. – We can always put the function Ψ into the form:

[U^1, (2)]
$$\Psi = A' \, e^{kS'}, \tag{20}$$

in which A' and S' are two *real* functions, and k has the meaning that it had in (8): A' is the *modulus* or *amplitude* or *amplitude* of Ψ; its *argument* or *phase* is KS'. Thanks to formula (20), one can replace the two *complex* quantization equations (10) with two *real* equations by replacing, for example, Ψ with the value (20) in the first equation (10) and separating the real and imaginary parts. One will get:

$$\gamma^{\mu\nu} S'_{,\mu} \, S'_{,\nu} = \mu^2 - \frac{1}{2\chi} + \frac{\Box A'}{k^2 A}, \tag{21}$$

[U^3, (3)](*)

$$(\gamma^{\mu\nu} S'_{,\mu} \, A'^2)_{,\nu} = 0 \tag{22}$$

by a simple calculation.

If one performs the same substitution in the expressions (7) and (13) of L and $T_{\mu\nu}$ then, upon taking (21) into account, one will get:

[U^3, (12)]
$$L = \gamma^{\mu\nu} S'_{,\mu} A'_{,\nu} + A' \Box S' \; = \tfrac{1}{2} \Box (A'^2) \tag{23}$$

and

$$\boxed{T_{\mu\nu} = 2K^2 A'^2 S'_{,\mu} \, S'_{,\nu} + 2 A'_{,\mu} \, A'_{,\nu} - \gamma_{\mu\nu} L.} \tag{24}$$

Up to now, we have performed only some absolutely general formal transformations. Before going further, we point out two more general formulas that will be useful for us: They will permit one to pass from a divergence that is taken in the five-dimensional universe to a divergence that is taken in the space-time. If u^μ is an arbitrary five-dimensional vector, and $\tau^{\mu\nu}$ is an arbitrary five-dimensional symmetric tensor then one will have:

$$u^\mu_{,\mu} \; = \; {}^4 u^m_{,m} + \frac{\partial u^0}{\partial x^0}, \tag{25}$$

(*) The second equation [U^3, (3)] must be written $2\gamma^{\mu\nu} S'_{,\mu} A'_{,\nu} + A' \Box S' = 0$. The form (22) is found in LONDON, Zeit. Phys. **42** (1927), pp. 385. Cf., also L. DE BROGLIE, C. R. Acad. Sci. Paris, 8 August 1927.

$$\tau^{m\nu}_{\;\;,\nu} = {}^4\tau^{mn}_{\;\;,n} + \frac{\partial \tau^{mn}}{\partial x^0} + H^m_{\;n}\,\alpha\,\tau^n_0 . \tag{26}$$

the proofs are accomplished very easily thanks to formulas [U^1, (27)], [U^1, (53)], [U^1, (54)].

Having established those preliminaries, we arrive at our first example of a corresponding system (*). The hypothesis that characterizes that system is that the Jacobi equation is true in its *classical* form:

[U^1, (10)]
$$\gamma^{\mu\nu} S_{,\mu} S_{,\nu} = \mu^2 - \frac{1}{2\chi} . \tag{*27}$$

One then deduces that the velocity vector in components that are *covariant with respect to $d\sigma^2$*:

$$\mu\, u_\mu = S_{,\mu} ; \tag{*28}$$

hence, since:

[U^1, (7)]
$$S_{,0} = \frac{1}{\alpha} ,$$

one will have:

$$u_0 = \frac{1}{\mu\alpha} \tag{*29}$$

and

[U^1, (22)]
$$\mu\, u^1 = \gamma^{i\mu} S_{,\mu} = g^{im} (S_{,m} - \Phi_m) . \tag{*30}$$

Now set:

$$S' = S + C. \tag{*31}$$

C is a function that is independent of q^0, as is A'.

Using (*31) and (*27), equation (21) will reduce to the following relation between C and A':

$$\gamma^{\mu\nu} C_{,\mu} C_{,\nu} + 2\gamma^{\mu\nu} S_{,\mu} C_{,\nu} = \frac{\Box A'}{K^2 A} . \tag{*32}$$

Introducing (*31) and (*28) in (24), we will get:

$$T_{\mu\nu} = 2K^2 A'^2 \mu^2 u_\mu u_\nu + 2K^2 A'^2 (S_{,\mu} C_{,\nu} + S_{,\nu} C_{,\mu} + C_{,\mu} C_{,\nu}) + 2A'_{,\mu} A'_{,\nu} - \gamma_{\mu\nu} L . \tag{*33}$$

Upon taking (*29) into account, we infer that:

(*) The numbers of the formulas that relate to this example are affected with an asterisk.

$$\left.\begin{aligned} T^{mn} &= 2K^2A'^2\mu^2u^m u^n \\ &\quad +\gamma^{\mu m}\gamma^{\nu n}\left[2K^2A'^2(S^{,\mu}C^{,\nu}+S^{,\nu}C^{,\mu}+C^{,\mu}C^{,\nu})+2A'_{,\mu}A'_{,\nu}-\gamma_{\mu\nu}L\right], \\ \alpha T_0^i &= 2K^2A'^2\mu u^i+2K^2A'^2\gamma^{i\nu}C_{,\nu}. \end{aligned}\right\} \qquad ({}^*34)$$

Now, if we compare formulas (*34) to formulas (18) then we will see that we can satisfy the correspondence principle by setting:

$$\left\{\begin{aligned} \rho_{(m)} &= 2K^2A'^2\mu^2, \\ \rho_{(e)} &= 2K^2A'^2\mu, \end{aligned}\right. \qquad ({}^*35)$$

as well as:

$$\left\{\begin{aligned} H^{mn} &= \gamma^{m\mu}\gamma^{n\mu}H_{\mu\nu}, \\ \Lambda^i &= \alpha H_0^i, \end{aligned}\right. \qquad ({}^*36)$$

by means of:

$$H_{\mu\nu} = 2K^2A'^2(S_{,\mu}C_{,\nu}+S_{,\nu}C_{,\mu}+C_{,\mu}C_{,\nu})+2A'_{,\mu}A'_{,\nu}-\gamma_{\mu\nu}L, \qquad ({}^*37)$$

in such a way that:

$$\Lambda^i = 2K^2A'^2g^{im}C_{,n}, \qquad ({}^*38)$$

or, in covariant components with respect to ds^2 :

$$\boxed{\Lambda_i = 2K^2A'^2C_{,i}.} \qquad ({}^*38')$$

The function C is therefore the *potential of the quantum current* Λ_i .

Equation (22) can now be put into the form:

$${}^4(\rho_{(e)}u^i+\Lambda^i)_{,i}=0, \qquad ({}^*39)$$

thanks to (*35), (*30), (*38), and (25): It expresses the conservation of the *total* current. That is a new aspect of one part of the compatibility theorem. We shall now pursue the dynamical study of the system, and we will begin with the aspect that is analogous to the other part of the theorem.

We first address the calculation of the *internal tensions* (*):

$$\Pi^i = {}^4\Pi^{in}_{,n}; \qquad ({}^*40)$$

from (26) and (*36), one will have:

(*) Of course, the internal tensions that will be at issue from now on have nothing in common with the ones in my note U^2, which would now be pointless to consider.

$$\Pi^i = \Pi^{i\mu}_{\ \ ,\mu} - \Pi^{\ i}_{n}\, \Lambda^i . \qquad (^*41)$$

On the other hand, due to (23), (*32), (22), and (*31), (*37) will give:

$$\begin{aligned}
\Pi_{\mu,\nu}^{\ \nu} &= \{2K^2 A'^2\, \gamma^{\nu\rho}\,(S_{,\mu}\, C_{,\rho} + S_{,\rho}\, C_{,\mu} + C_{,\mu}\, C_{,\rho})\}_{,\nu} + 2A'_{,\mu\nu}\, \gamma^{\nu\rho} A'_{,\rho} + 2A'_{,\mu}\, \gamma^{\nu\rho} A'_{,\nu\rho} \\
&\quad - 2A'_{,\mu\rho}\, \gamma^{\sigma\rho} A'_{,\sigma} - A'_{,\mu}\, \Box A' - A'(\Box A')_{,\mu} , \\
&= \{2K^2 A'^2\, \gamma^{\nu\rho}\,(S_{,\rho} + C_{,\rho})\, C_{,\mu}\}_{,\nu} + \{2K^2 A'^2\, \gamma^{\nu\rho}\, S_{,\mu}\, C_{,\rho}\}_{,\nu} - A'^2 \left(\frac{\Box A'}{A'}\right)_{,\mu} , \\
&= 2K^2 A'^2\, \gamma^{\nu\rho}\,(S_{,\rho} + C_{,\rho})\, C_{,\mu\nu} + \{2K^2 A'^2\, \gamma^{\nu\rho}\, C_{,\rho}\}_{,\nu}\, S_{,\mu} - 2K^2 A'^2\, \gamma^{\nu\rho}\, C_{,\rho}\, S_{,\mu\nu} \\
&\quad - K^2 A'^2\, \{\gamma^{\nu\rho}\, C_{,\rho}\, C_{,\nu} + 2\, \gamma^{\rho\nu}\, C_{,\nu}\, C_{,\rho}\}_{,\mu} , \\
&= 2K^2 A'^2\, \gamma^{\nu\rho}\,(S_{,\rho} + C_{,\rho})\, C_{,\mu\nu} + \{2K^2 A'^2\, \gamma^{\nu\rho}\, C_{,\rho}\}_{,\nu}\, S_{,\mu} - 2K^2 A'^2\, \gamma^{\nu\rho}\, C_{,\rho}\, S_{,\mu\nu} \\
&\quad - K^2 A'^2\, \{2\gamma^{\rho\nu}\, C_{,\rho}\, C_{,\mu\nu} + 2\, \gamma^{\rho\nu}\, S_{,\nu}\, C_{,\rho\mu}\} - 2K^2 A'^2\, \gamma^{\nu\rho}\, C_{,\rho}\, S_{,\mu\nu} , \\
&= \{2K^2 A'^2\, \gamma^{\nu\rho}\, C_{,\rho}\}_{,\nu}\, S_{,\mu} , \\
&= - S_{,\mu}\{2K^2 A'^2\, \gamma^{\nu\rho}\, S_{,\rho}\}_{,\nu} ,
\end{aligned}$$

or finally, from (25), (*28), and (*35):

$$\Pi_{\mu,\nu}^{\ \nu} = - u_\mu\, {}^4(\rho_{(m)}\, u^n)_{,n} . \qquad (^*42)$$

By means of (*42), (*41) will become:

$$\Pi^i = - u^i\, {}^4(\rho_{(m)}\, u^n)_{,n} - H^{\ i}_{n}\, \Lambda^n . \qquad (^*43)$$

Now, from the compatibility theorem [cf., U^2, no. **2**], the dynamical equations that we seek are written:

$$T^{m\mu}_{\ \ ,\mu} = 0,$$

or, by virtue of (26):

$${}^4T^{mn}_{\ \ ,n} + H^{\ m}_{n}\, \alpha T^{\ n}_{0} = 0,$$

or rather, by virtue of (18):

$${}^4(\rho_{(m)}\, u^n)_{,n} + {}^4\Pi^{mn}_{\ \ ,n} + H^{\ m}_{\ \ n}\, (\rho_{(m)}\, u^n + \Lambda^n) = 0,$$

or finally:

$$\varphi^m_{(m)} + \varphi^m_{(e)} + \Pi^m + u^m\, {}^4(\rho_{(m)}\, u^n)_{,n} + H^{\ m}_{\ \ n}\, \Lambda^n = 0,$$

from (*40) and upon setting:

$$\begin{cases} \varphi^m_{(m)} = \rho_{(m)}\ {}^4u^{\ n}_{,n}\, u^n, \\ \varphi^m_{(e)} = \rho_{(e)}\, H^{\ m}_{\ n}\, u^n. \end{cases} \qquad (^*45)$$

Now, due to (*43), (*44) will reduce to:

$$\varphi^m_{(m)} + \varphi^m_{(e)} = 0. \qquad (^*46)$$

$\varphi^m_{(m)}$ is the generalized Einstein *force of inertia*, $\varphi^m_{(e)}$ is the generalized Lorentz *electromagnetic force*. One then recovers Einsteinian dynamics; it is the novel aspect of the second part of the compatibility theorem.

One deduces the equation of continuity from (*44):

$${}^4(\rho_{(m)}\, u^n)_{,\,n} + \Pi_m\, u^m + H_{mn}\, \Lambda^n\, u^m = 0, \qquad (^*47)$$

with

$$\Pi_m = g_{mn}\, \Pi^m\ ;$$

that equation is satisfied *identically* thanks to (*43).

5. Corresponding system with quantum mass. – Recall equation (21). Since it has the form of the Jacobi equation, we can construct a second corresponding system that is characterized by the Jacobi equation (21), which we write:

[U^3, (6)]
$$\gamma^{\mu\nu}\, S'_{,\mu}\, S'_{,\nu} = \mu'^2 - \frac{1}{2\chi}, \qquad (27)$$

with the notation:

[U^3, (5)]
$$\mu'^2 = \mu^2 + \frac{\Box A'}{K^2 A'}, \qquad (28)$$

and upon keeping the same value for the charge e, that will amount to the replacement of the mass m_0 with the *L. de Broglie mass:*

[U^3, (7)]
$$M_0 = \sqrt{m_0^2 + \frac{h^2}{4\pi^2 c^2}\cdot\frac{\Box A'}{A'}}\ . \qquad (29)$$

The velocity vector is presently defined by:

[U^3, (9)]
$$\mu'\, u_\nu = S'_{,\,\nu}\ ; \qquad (30)$$

hence, in particular:

$$u_0 = \frac{1}{\mu'\alpha}, \qquad (31)$$

and

$$[U^3, (8)] \qquad \mu' u^n = \gamma^{n\nu} S'_{,\nu} = g^{mn} (S'_{,m} - \Phi_m) . \tag{32}$$

Formula (24) now gives:

$$T_{\mu\nu} = 2 K^2 A'^2 m'^2 u_\mu u_\nu + 2A'_{,\mu} A'_{,\nu} - \gamma_{\mu\nu} L ; \tag{33}$$

hence:

$$\left\{ \begin{aligned} T^{mn} &= 2K^2 A'^2 \mu'^2 u^m u^n + \gamma^{m\mu} \gamma^{n\nu} (2A'_{,\mu} A'_{,\nu} - \gamma_{\mu\nu} L), \\ \alpha T_0^i &= 2K^2 A'^2 \mu' u^i , \end{aligned} \right. \tag{34}$$

We see that we can satisfy the correspondence formulas (18) by setting:

$$[U^3, (10)] \qquad \left\{ \begin{aligned} \rho_{(m)} &= 2K^2 A'^2 \mu'^2, \\ \rho_{(e)} &= 2K^2 A'^2 \mu', \end{aligned} \right. \tag{35}$$

as well as:

$$\left\{ \begin{aligned} \Pi^{mn} &= \gamma^{m\mu} \gamma^{n\nu} \Pi_{\mu\nu}, \\ \Lambda^i &= 0, \end{aligned} \right. \tag{36}$$

by means of:

$$[U^3, (14)] \qquad \Pi_{\mu\nu} = 2A'_{,\mu} A'_{,\nu} - \gamma_{\mu\nu} L . \tag{37}$$

The corresponding system that we are studying now is therefore *pure* Maxwellian. The total current reduces to the convection current here. The conservation of that current is once more expressed by equation (22), which conforms to the compatibility theorem; indeed, thanks to (25), (32), and (35), that equation is written:

$$^4(\rho_{(m)} u^i)_{,i} = 0. \tag{38}$$

It now remains for us to study the dynamics of our second model, in parallel with what we did for the first system. From (26), (37), (23), we have immediately:

$$\Pi^i = {}^4\Pi^{in}_{,n} = {}^4\Pi^{i\nu}_{,\nu},$$

$$\Pi_{\mu,\nu}^{\ \nu} = -A'^2 g^{in} \left(\frac{\Box A'}{A'} \right);$$

hence:

$$\Pi^i = -A'^2 g^{in} \left(\frac{\Box A'}{A'} \right)_{,n} , \tag{39}$$

or in covariant components with respect to ds^2 :

$$\boxed{\Pi_i = -A'^2 \left(\frac{\Box A'}{A'} \right)_{,i} .} \tag{39′}$$

The function $\Box A' / A'$ is then the *potential of the internal tension* Π_i .

The dynamics of the system can be condensed into the equations:

$$\varphi^m_{(m)} + \varphi^m_{(e)} + \Pi^m + u^m \cdot {}^4(\rho_{(m)} u^n)_{,n} = 0, \tag{40}$$

with the notations:

$$\begin{cases} \varphi^m_{(m)} = \rho_{(m)} \, {}^4u^m_{,n} \, u^n , \\ \varphi^m_{(e)} = \rho_{(e)} \, H^m_{\ n} \, u^n . \end{cases} \tag{41}$$

One deduces the continuity equation from (40):

$$\Pi_m \, u^m + {}^4(\rho_{(m)} \, u^n)_{,n} = 0. \tag{42}$$

Upon introducing the values (39) and (35) for Π_m and $\rho_{(m)}$, resp., into that equation, one will get:

$$-A'^2 \left(\frac{\Box A'}{A'} \right)_{,m} u^m + {}^4(\rho_{(m)} \, u^n)_{,n} \, \mu' + 2K^2 A'^2 \mu' \mu'_{,n} \, u^n = 0,$$

or rather, from formula (28):

$${}^4(\rho_{(m)} \, u^n)_{,n} = 0,$$

which is nothing but the real equation of the quantization (38). Hence, (42) will be true identically by virtue of (38); one recognizes the compatibility theorem. Thanks to (42), one can again put (40) into the form:

$$\varphi^m_{(m)} + \varphi^m_{(e)} + \Pi_m - u^m \, \Pi_i \, u^i = 0. \tag{43}$$

One indeed recovers the various equations of Chapter VI of De Donder's *Théorie des Champs gravifiques*.

We conclude with a remark that relates to the tension potential.

Set:

$$\rho^*_{(m)} = \left(\frac{\mu}{\mu'} \right)^2 \rho_{(m)} , \tag{44}$$

or, by virtue of (35):

$$\rho^*_{(e)} = 2 \, K^2 \, A'^2 \, \mu^2 . \tag{44′}$$

In addition, set:

$$V = \frac{1}{2K^2 \mu^2} \cdot \frac{\Box A'}{A'} . \tag{45}$$

Formula (39′), which gives us the tension vector Π_i , is written:

$$\Pi_i = -\overset{*}{\rho}_{(m)} V_{,\,i} \,; \tag{46}$$

on the other hand, formula (28) will give, *in the first approximation:*

$$\mu' \sim \mu\,(1 + V), \tag{47}$$

or rather:

$$M_0\, c^2 \sim m_0\, c^2\,(1 + V)\,. \tag{48}$$

Formula (48) was pointed out by L. de Broglie in his cited paper in *Journal de Physique* [formula (64)]. He remarked that "everything happens as if there exists…a (supplementary) potential energy term" $m_0\, c^2\, V$. Our method gives us the interpretation of that potential energy now: From (46), it is nothing but the *energy of internal tensions.*

6. Remark concerning the preceding two examples. – The interest in the systems with quantum current is found, above all, in the fact that it preserved the completely classical Einsteinian dynamics. One will then see a quantum current appear that gets added to the convection current. The total current is conserved, but not the convection current. Charged moving bodies will not always keep their initial personalities then, but might possibly break up or coalesce into each other.

On the contrary, the second system, or *L. de Broglie system*, is pure Maxwellian, but its dynamics are somewhat complicated, due to the intervention of internal tensions. What is truly remarkable about such systems is the significance that the complex quantization equation takes on when it is put into the form of the two real equations (27) and (38): The first of those equations is the Jacobi equation of the system; it provides the dynamics. The second one gives the conservation of electricity.

II. – HOLONOMIC SYSTEMS.

7. Continuous holonomic system with (f + 1) **degrees of freedom** [cf., U^1, II]. – The motion of such a system is determined by (f + 1) parameters $q^0, q^1, \ldots, q^f$ that one considered to be the coordinates of a configuration space with metric $\Gamma_{\mu\nu}\, dx^\mu\, dx^\nu$ (μ, ν = 0, 1, …, f). If x'^1, x'^2, x'^3, x'^4 denote the coordinates of a point of the system with respect to a reference system that is linked with it then one will have a change of variables or holonomic constraint for an arbitrary system x^0, x^1, x^2, x^3, x^4 that takes the form:

[U^1, (87), (94)]

$$\begin{cases} x^{\underline{i}} = x^{\underline{i}}(x'^1, x'^2, x'^3, x'^4; q^1, \ldots, q^f) \quad (i = 1, 2, 3, 4) \\ x^{\underline{0}} = q^0. \end{cases} \tag{49}$$

More precisely, one supposes that there exist differentiations d and δ that enjoy the following properties: When $ds \neq 0$, $dq^n \neq 0$, one has $dx^{\underline{i}} = 0$; when $\delta s = 0$, $\delta q^n = 0$, one has $\delta x^{\underline{i}} \neq 0$.

Now, it is clear that all of the arguments of the Part I can be transposed to the new configuration space with $\Gamma_{\mu\nu}\, dx^\mu\, dx^\nu$. The constants $m_0\, c^2$ and e will be replaced by the constants $\tau_{(m)}$, $\tau_{(e)}$, resp., which can be calculated as functions of the given density factors by the formulas:

$$[U^1, (90)] \qquad \begin{cases} \tau_{(m)} = \int \sigma_{(m)}\, \delta x^1 \delta x^2 \delta x^3 \delta x'^4, \\ \tau_{(e)} = \int \sigma_{(e)}\, \delta x^1 \delta x^2 \delta x^3 \delta x'^4. \end{cases} \tag{50}$$

In these formulas, the symbol δ has the significance that was just recalled: $\tau_{(m)}$ is the total internal (rest) energy, and $\tau_{(e)}$ is the total charge of the continuous system (charged material particle). Apart from that, all of the formulas will remain the same: Of course, the Latin indices vary from 1 to f now, while the Greek indices vary from 0 to f.

On the other hand, that extension of our principles to the new configuration is *natural*, because if one abstracts those principles, for the moment, then one can calculate, just as one did in no. **10** of U^1, the $\Gamma_{\mu\nu}$, by starting with the $\gamma_{\mu\nu}\,(x^1, x^2, x^3, x^4)$ of the five-dimensional universe, and one will effectively find a Jacobi equation that will permit one to generalize the quantization equation in the configuration space, thus-determined. Of course, the field $\Gamma_{\mu\nu}$ that is determined by starting from the $\gamma_{\mu\nu}$ (the original field), as was just said, does not rigorously coincide with field $\Gamma_{\mu\nu}$ that is calculated from the method that was developed in Part I (viz., *the corresponding field*); the new element that wave mechanics brings with it is precisely that difference. The correspondence principle asserts that in configuration space, the field that has a true *physical* significance is not the original field, but, in fact, the corresponding field.

III. – SYSTEMS OF *N* POINTS. STATISTICS.

8. System of *N* points embedded in a given field. – Consider a field $g_{ik}\,(x^1, x^2, x^3, x^4)$, $\Phi_1\,(x^1, x^2, x^3, x^4)$ that is known at each. Introduce N bodies (*test bodies*) with masses and charges $\tau_\nu^{(m)}$, $\tau_\nu^{(e)}$ ($\nu = 1, 2, \ldots, N$). We shall treat the dynamics and quantization of that system by a method whose principle is due to De Donder.

We first argue in space-time. The system has $4N$ degrees of freedom. Take the parameters of motion to be the coordinates x_ν^i (i = 1, 2, 3, 4; ν = 1, 2, …, N) of the various points; let u_ν^i be the corresponding components of the velocities. In general, let f_ν denote the value of a function f for $x^i = x_\nu^i$, $u^i = u_\nu^i$, i = 1, 2, 3, 4. Set:

$$\begin{cases} U = \Phi_i\, u^i, \\ Y = \tfrac{1}{2}\, g_{ik} u^i u^k. \end{cases} \tag{51}$$

The fundamental theorem (*):

(*) TH. DE DONDER, *Théorie des Champs gravifiques*, pp. 38, equation (100).

$$\int \delta\tau^{(m)} \left[\frac{d}{ds}\left(\frac{\partial Y}{\partial u^i}\right) - \left(\frac{\partial Y}{\partial x^i}\right) \right] + \int \delta\tau^{(e)} \left[\frac{d}{ds}\left(\frac{\partial U}{\partial u^i}\right) - \left(\frac{\partial U}{\partial x^i}\right) \right] = 0 \tag{52}$$

is written:

$$\tau_\nu^{(m)} \left[\frac{d}{ds}\left(\frac{\partial Y_\nu}{\partial u_\nu^i}\right) - \left(\frac{\partial Y_\nu}{\partial x_\nu^i}\right) \right] + \tau_\nu^{(e)} \left[\frac{d}{ds}\left(\frac{\partial U_\nu}{\partial u_\nu^i}\right) - \left(\frac{\partial U_\nu}{\partial x_\nu^i}\right) \right] = 0 \tag{53}$$

in the present case for each point of the system.

Set:

$$A_\nu = \tau_\nu^{(m)} Y_\nu + \tau_\nu^{(e)} U_\nu \,; \tag{54}$$

we can write equations (53) in the Lagrangian form:

$$\frac{d}{ds}\left(\frac{\partial \Lambda_\nu}{\partial u_\nu^i}\right) - \left(\frac{\partial \Lambda_\nu}{\partial x_\nu^i}\right) = 0. \tag{55}$$

Upon setting:

$$p_{i,\,\nu} = \frac{\partial \Lambda_\nu}{\partial u_\nu^i} = \frac{\partial S_\nu}{\partial x_\nu^i}\,, \tag{56}$$

one will deduce the N Jacobi equations:

$$g_\nu^{ik} \left(\frac{\partial S_\nu}{\partial x_\nu^i} - \tau_\nu^{(e)} \Phi_{i,\nu} \right)\left(\frac{\partial S_\nu}{\partial x_\nu^k} - \tau_\nu^{(e)} \Phi_{k,\nu} \right) - (\tau_\nu^{(m)})^2 = 0 \tag{57}$$

from the N systems of equations (55), by the usual method.

In reality, the problem then splits into N independent equations. One can nonetheless introduce a *unique* 4N-dimensional configuration space to represent the states of the system. To that effect, set:

$$\Lambda = \sum_\nu \Lambda_\nu \tag{58}$$

and observe that:

$$\frac{\partial \Lambda}{\partial x_\nu^i} = \frac{\partial \Lambda_\nu}{\partial x_\nu^i}\,, \qquad \frac{\partial \Lambda}{\partial u_\nu^i} = \frac{\partial \Lambda_\nu}{\partial u_\nu^i}\,. \tag{59}$$

Moreover, equations (55) can be written:

$$\frac{d}{ds}\left(\frac{\partial \Lambda}{\partial u_\nu^i}\right) - \left(\frac{\partial \Lambda}{\partial x_\nu^i}\right) = 0. \tag{60}$$

Upon setting:

$$p_{i,\nu}^{*} = \frac{\partial \Lambda}{\partial u_\nu^i} = \frac{\partial S}{\partial x_\nu^i}\,, \tag{61}$$

one will deduce the *single* Jacobi equation from the system (60):

$$\sum_\nu \frac{1}{\tau_\nu^{(m)}}\left\{g_\nu^{ik}\left(\frac{\partial S}{\partial x_\nu^i}-\tau_\nu^{(e)}\Phi_{i,\nu}\right)\left(\frac{\partial S}{\partial x_\nu^k}-\tau_\nu^{(e)}\Phi_{k,\nu}\right)-(\tau_\nu^{(m)})^2\right\} = 0, \tag{62}$$

which is written:

$$ds_*^2 = \sum_\nu \frac{\tau_\nu^{(m)}}{\tau^{(m)}} g_{ik,\nu}\, dx_\nu^i\, dx_\nu^k \tag{63}$$

in configuration space, with:

$$\tau^{(m)} = \sum_\nu \tau_\nu^{(m)} .$$

Equation (62) is equivalent to N equations (57), moreover. In order to see that, it is sufficient to compare (56) and (61) using (59); one has $p_{i,\nu} = p_{i,\nu}^*$ and:

$$\frac{\partial S_\nu}{\partial x_\nu^i} = \frac{\partial S}{\partial x_\nu^i}; \tag{64}$$

hence:

$$S = \sum_\nu S_\nu(x_\nu^1, x_\nu^2, x_\nu^3, x_\nu^4) .$$

Thanks to (64), equation (62) can be written:

$$\sum_\nu \frac{1}{\tau_\nu^{(m)}}\left\{g_\nu^{ik}\left(\frac{\partial S_\nu}{\partial x_\nu^i}-\tau_\nu^{(e)}\Phi_{i,\nu}\right)\left(\frac{\partial S_\nu}{\partial x_\nu^k}-\tau_\nu^{(e)}\Phi_{k,\nu}\right)-(\tau_\nu^{(m)})^2\right\} = 0. \tag{65}$$

We shall construct the world-function L that enters into the variation principle by starting with that equation (65). It will depend upon N complex functions Ψ_ν and their conjugates $\bar{\Psi}_\nu$.

First, introduce the fifth dimension x^0, which takes the value x_ν^0 at the ν^{th} point. Set:

$$\frac{\partial S}{\partial x_\nu^0} = \frac{\partial S_\nu}{\partial x_\nu^0} = \frac{\tau_\nu^{(e)}}{\alpha}, \tag{66}$$

and further define the $\gamma^{\mu\nu}$, $\gamma_{\mu\nu}$ by (4) and (5), resp. Equation (65) is written:

$$\sum_\nu \frac{1}{\tau_\nu^{(m)}}\left\{\gamma_\nu^{\alpha\beta}\frac{\partial S_\nu}{\partial x_\nu^\alpha}\frac{\partial S_\nu}{\partial x_\nu^\beta}-\left[(\tau_\nu^{(m)})^2-\frac{(\tau_\nu^{(e)})^2}{2\chi}\right]\right\} = 0, \tag{67}$$

in the $5N$-dimensional configuration space with:

$$d\sigma_*^2 = \sum_\nu \frac{\tau_\nu^{(m)}}{\tau^{(m)}} \gamma_{\alpha\beta,\nu}\, dx_\nu^\alpha\, dx_\nu^\beta . \tag{68}$$

From (67), one must take:

$$L = \sum_\nu \frac{1}{\tau_\nu^{(m)}} \left\{ \gamma_\nu^{\alpha\beta} \frac{\partial \Psi_\nu}{\partial x_\nu^\alpha} \frac{\partial \bar{\Psi}_\nu}{\partial x_\nu^\beta} + k_\nu^2 \left(\mu_\nu^{(m)} - \frac{1}{2\chi} \right) \Psi_\nu\, \bar{\Psi}_\nu \right\} \tag{69}$$

to be the world-function L, upon setting:

$$\begin{cases} k_\nu = i\,\dfrac{2\pi}{h} \cdot \dfrac{\tau_\nu^{(m)}}{c}, \\ \mu_\nu = \dfrac{\tau_\nu^{(m)}}{\tau_\nu^{(e)}}. \end{cases} \tag{70}$$

In addition, one must observe that in the present problem, the Einstein and Maxwell equations play a role only *dynamically*, since the fields are assumed to be given. Other than the dynamical equations, which contain the tensor $T_\nu^{\alpha\beta}(\Psi_\nu, \bar{\Psi}_\nu)$, the variational principle will then yield $2N$ quantization equations in Ψ_ν, $\bar{\Psi}_\nu$. As one sees, the corresponding system is no longer determined by just one tensor $T^{\alpha\beta}$, but by a set of N tensors $T_\nu^{\alpha\beta}$ (or, if one prefers, by a tensor of rank $5N$ that has a very special form). In particular, the "densities" of the corresponding system are defined by two sets of N functions, that proportional to the squares of the amplitudes Ψ_ν, $\bar{\Psi}_\nu$, respectively.

9. Statistics. – Apply these results to a system of N *identical* points. In that case, one will get the *same* formal equations for *all* value of the index ν. One can then say that *formally* the problem is the same as the one that was treated in Part I in relation to a *single* point. One can even dispense with the *explicit* consideration of the $5N$-dimensional configuration space and say, more briefly, that *the system that corresponds to a single point* (as described in Part I) *is also the system that corresponds to an arbitrary system of points of the same nature* ***that is embedded in the given field***. The latter restriction is obviously essential: When one considers *just one* point, the corresponding system will permit one to calculate not only the motion of the point in the field, but also the field at each point. When one considers a "cloud" of points, the corresponding system will give only the motion; one will no longer have an "*in sich geschlossene Feldtheorie*" (*).

Now, as L. de Broglie ingeniously remarked in his aforementioned article, one can interpret the motion of a cloud of points in a field *statistically*. If one considers a particle that arrives at an arbitrary (unknown) point in the field with a given velocity then the density of the cloud at each point of the field will be proportional to the *probability of*

(*) "An intrinsically-closed field theory." Cf., Schrödinger, Ann. Phys. (Leipzig) **82** (1927), pp. 265, *et seq.*, to the end.

presence of the particle at that point. However, by virtue of the correspondence principle, the density that we must consider is that of the corresponding system. Now, that density is proportional to $\Psi\bar{\Psi}$. We thus arrive at the *statistical interpretation* of the function Ψ that Born had proposed. Born and his school placed that statistical interpretation at the basis of a remarkable theory that is radically different from the one that was presented in the present work. The considerations that were just developed seem to show that, given the present state of affairs concerning the question, the statistical aspect of atomic phenomena, as interesting and fruitful as it is, does not necessarily lead to the "indeterministic" attitude of the Göttingen school, however.

The initial idea of my argument was that of L. de Broglie. However, I arrived at the same conclusion as he along a very different path. One might say that whereas he sought to *superimpose* the individual waves in order to obtain the wave of the system, I have simply *juxtaposed* then. In that fashion, *thanks to the correspondence principle*, I could avoid the hypothesis of the "double solution" that one is obliged to introduce.

10. Observations about non-holonomic systems of *N* points that are embedded in their field. **–** On first glance, it seems that can extend the considerations of number **8** to a system of *N* points that interact with each other, because, at the end of the day, the field of such a system is likewise finite and well-defined at each point. However, that viewpoint is contrary to the spirit of the Lagrangian method; that is why I resorted to the mode of exposition that I adopted. If one would like to repeat the considerations of number **8** for the present case then one will be stopped by equation (58) or (59) in the application of that Lagrangian method.

In order to treat such systems, one must (*) introduce a $4N$-dimensional configuration space *a priori* with a metric of the form:

$$g_{ik}\,(q^1, q^1, \ldots, q^{4N})\, dq^i\, dq^k \qquad\qquad (i, k = 1, 2, \ldots, 4N).$$

I hope to be able to return to the matters above in more detail.

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(*) Cf., TH. DE DONDER, Bull. Acad. roy. de Belg. (5) **13**, pp. 509, (§ **5**).