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Introduction

The problem of constructing a unified theory of electromagnetism and gravitation arose almost immediately after the appearance of Einstein's theory of gravitation. It seemed natural and tempting to look for equations for gravitation and electrodynamics on the basis of a single common principle, and it seemed that the construction of a unified field theory would lead to a deeper understanding of nature and enable one to predict and discover new and specialized electro-gravitational effects.

These hopes have not yet been realized. All of the numerous variants that have been proposed to date for a unified field theory have led to the formal unification of the Einstein and Maxwell equations, but not to any new knowledge about nature. The failure to achieve any progress along that direction led to a cooling-off of interest among the physicists that had contributed to our heritage of pre-quantum physics, and they largely continued to engage in more mathematical-geometric endeavors than physical ones.

The reason for this is that experimental physicists have significantly different capabilities regarding the study of the properties of electromagnetic and gravitational fields.

In the case of electrodynamics, one can create changes in spacetime fields and investigate their properties in a laboratory environment. Maxwell's equations are the mathematical formulation of the results of Faraday's experiments. The electromagnetic waves that were predicted by the theory were discovered experimentally and have found widespread applications in engineering.

In the case of gravitation, experimenters are denied the opportunity to create gravitational fields that are variable in space-time and observable in experiments. The only field that is in their possession is the constant gravitational field of the Earth and the Sun. That should be regarded not as unfortunate, but as an opportunity to observe relativistic effects, such the secular motion of the perihelion of Mercury and the bending of light rays near the Sun. The detection of gravitational fields that are predicted by the theory goes far beyond the capabilities of experiments. Therefore, unlike the equations of electrodynamics, the equations of the theory of gravitation were not obtained as the result of the mathematical formulation of laws that were found experimentally, but were based mainly upon the possibility of a covariant formulation of the laws of nature in arbitrary coordinate systems.

Therefore, it was natural that all attempts to generalize the theory of gravitation in the direction of a unified theory of gravitation and electromagnetism inevitably took on a formal mathematical character and could not rely upon experiment.

From the attempts to construct a unified field theory, it immediately emerged that there were two directions in which to proceed.

The first one was to abandon the Riemannian metrics of the four-dimensional space-time continua of the general theory of relativity and make a transition to a general non-Riemannian geometry. It was found that mathematical researchers had opened up a vast expanse of possible geometries from which to choose. With almost any choice, it was possible to obtain geometrical quantities that could be interpreted as the existence of potential electromagnetic fields. The ambiguity in the choice of a non-Riemannian geometry and the lack of a general physical principle that would make this choice unambiguous deprived this direction of any interest and physical content.

The second direction was to introduce an extra fifth dimension while maintaining a Riemannian metric in five-dimensional space. In order to come to an agreement with experience, which shows no dependence of macroscopic fields upon an extra fifth coordinate, the additional rigid requirement is imposed upon metrics in five-dimensional space that the metric potentials must be independent of the introduction of the extra fifth coordinate, which is the so-called *cylindricality condition*.

The inadequacy of this direction is obvious. Indeed, the introduction of an extra fifth dimension into space, whose existence, however, cannot be detected due to the postulated independence of all fields of the extra fifth coordinate, seems highly artificial.

However, as was shown by Einstein and Bergmann in 1938, you can introduce the concept of five-dimensional space without entering into conflict with experimentally detectable four-dimensional macroscopic fields and without imposing cylindricality conditions upon the metric. In order to do that, it is sufficient to assume that five-dimensional space is topologically closed in the fifth dimension, and that the period of the fifth coordinate has a microscopic value that we can take to be zero in the first approximation.

One visual two-dimensional model of a space that is topologically closed in one of its dimension might be the surface of an infinitely-extended cylinder whose radius will be denoted by $b / 2\pi$. Suppose that we define a scalar field $W(x, S)$ on the surface of that cylinder that is periodic in the coordinate S with period b. Consider two limiting cases:

1. *Macroscopic fields*, when the changes in the field over distances of order b can be neglected. Such an approximate field will satisfy cylindricality, and it will manifest itself as a one-dimensional field that depends upon only the coordinate x.

2. *Ultra-microscopic fields*, when the radius of the cylinder $b / 2\pi$ is large in comparison to the distances at which the field varies significantly. The topological closure of the surface can be neglected at such distances for the field (on a small part of the cylinder surface). Such a field will manifest itself as a two-dimensional one that depends upon both the coordinates x and S.

Microscopic fields represent intermediate cases that vary considerably at distances of order b. In those cases, the topological closure of the surface would be essential.

Let us return to five-dimensional Einstein-Bergmann spaces and denote the fifth coordinate by S and its period by b. The components of any field in such a space are assumed to functions of all five coordinates: $W(x, y, z, t, S)$, and periodic in the fifth coordinate with period b.

When we deal with macroscopic fields, we can neglect the periodic dependence on the fifth coordinate and treat them as four-dimensional.

Accounting for the periodic dependence upon the fifth coordinate will become significant under the transition to the consideration of microscopic fields. Mainly, there should exist effects in which that periodicity is manifested. In essence, we are talking about the introduction of a new universal constant b into the field theory, and the classical theory will be produced only in the limit as $b \to 0$.

However, the physical meaning and dimension of the extra fifth coordinate still remain open questions, so the entire theory retains its formal character.

One can also approach the notion of a topologically-closed, five-dimensional space entirely from "the other end" – i.e., without attempting to construct a unified theory of gravitation and electromagnetism. That path will lead to the discovery that it is possible to ascribe the physical meaning of action to the fifth coordinate S, while its period b is rooted in the value of Planck's constant h, and will lead to a profound synthesis of geometric ideas that were established in the general theory of relativity with ideas from quantum theory. In modern physics, it is customary to distinguish between the "macroscopic" and the "microscopic," which is related to the value of Planck's constant h, which finds its geometric analogue in the form of the distinction between "four-dimensional" and "five-dimensional," resp.

The path to the "five-dimensional" that will be chosen in this book is to look for some still-undiscovered, far-reaching symmetry in the equations of relativistic mechanics in space, time, and action. At the same time, a three-dimensional Einsteinian formulation and a four-dimensional Minkowskian formulation will make possible a new five-dimensional formulation of the equations of relativistic mechanics.

The problem of finding a five-dimensional formulation of classical relativistic mechanics for the motion of a charged, material point in given external gravitational and electromagnetic fields is equivalent to the problem in geometrical optics of the propagation of light rays in a five-dimensional Riemannian space of coordinates, time and action, whose metric has the cylindricality condition imposed upon it. Therefore, the entire field theory that will be discussed in this book will be called five-dimensional optics.

However, it would be wrong to view five-dimensional optics as just one of the various unified field theories. Rather, its main content is the geometrically fundamental concept of quantum physics, since it is found to be a manifestation of the quantization of the periodic dependence of physical fields on the fifth coordinate of action. Insofar as the "fifth dimension" is itself a quantum effect, it becomes an obvious failure of all previous attempts to construct five-dimensional unified field theories that are based upon the classical representation alone with no significant involvement of quantum concepts.

We shall now move on to a brief description of the contents of this book.

In the first chapter, one finds far-reaching symmetry equations for classical, relativistic mechanics in space, time, and action, and the expediency of interpreting action as the extra fifth coordinate of space is shown.

In the second chapter, the classical, relativistic mechanics of a charged, material point is described as geometrical optics in a five-dimensional space of coordinates, time, and action.

In the third chapter, a classical ("macroscopic") unified theory of the gravity and electricity fields is described that assumes that the periodic dependence of the components of the metric 5-tensor on the fifth coordinate of action can be neglected.

The first three chapters establish the parts of five-dimensional optics than can be loosely called "classical," because they can be defined only the limiting cases when $h \to 0$.

In the fourth and sixth chapters, quantum mechanics is described as wave optics in five-dimensional spaces of coordinates, time, and action that are topologically closed in the action coordinate with a period that is equal to *h*. In that regard, we point out the problem of the propagation of waves in multi-dimensional spaces that is often considered in mathematical optics. However, no one ever set out to investigate the problem of wave propagation in spaces that are topologically closed in one of their dimensions. Anyone that would have gone on to such a study would have been amazed to encounter the characteristically "quantum" phenomena that arise from the topological closure of the space in the course of investigating wave motion.

In the fifth chapter, we will study some special mathematical tools that are convenient for the covariant formulation of field equations in five-dimensional Riemann spaces. Those tools are equivalent to the usual tensor analysis, but they are useful in the sense that they allow one to write down wave equations for five-dimensional optics in a gradient-invariant form.

The following problems that we pose for ourselves in this monograph while we are presenting quantum mechanics as five-dimensional wave optics will always be regarded as special cases in which the dependency of the metric field on the fifth coordinate of action can be neglected. The transition to the general case, in which one takes that dependency into account, will have to be the subject of a later investigation.

Five-dimensional optics is a new way of defining couplings between space, time, and action. It should be emphasized especially that the five-dimensional space of space, time, and action is not the space of the general theory of relativity (when extended by one more dimension), but a configuration space for considering the motion of particles. We shall discuss that detail in § 7. Having said that, we will still be far from presenting a full explanation of the issues that have emerged along the way. Five-dimensional optics gives a new, purely geometric, rationale for quantum mechanics, and some philosophical and methodological questions will arise from that study that will require careful analysis and in-depth examination. The author hopes to return to these issues in a more specialized work.

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CHAPTER I

OPTICO-MECHANICAL ANALOGIES

§ 1. Historical background.

In 1891, F. Klein wrote, in regards to the work of Hamilton on optics and mechanics:

"Hamilton found that from the form of the corpuscular theory, light rays, which are defined to be trajectories that pass through some inhomogeneous (but isotropic) medium, can serve as special cases of the usual mechanical problems that are concerned with the motion of material points.

At the same time, we might presently add that the restrictions that are present at this point in special cases are inessential and that in every mechanical problem that is concerned with the motion of material points, one can determine the path of a light ray that passes through a suitable medium with the help of a space with a higher number of dimensions" [**1**].

In a footnote, Klein indicated, as he had said in a lecture in 1891 at Göttingen, that he had derived all of Hamilton–Jacobi theory from a quasi-optical representation in a space with a higher number of dimensions.

Ten years later, he bitterly noted that this idea, which he had presented to the Congress of Natural Scientists in France, "has not found the general recognition that I had hoped for." [**2**].

Note that Klein's words were spoken many years before the appearance of the theories of relativity and quantum mechanics.

At the beginning of our century, Klein was interested in this sphere of ideas, but his hopes starting cooling, and up until the end of his life they certainly did not increase. Neither the advent of relativistic mechanics nor the appearance and development of light quanta induced him to return to cultivating this abandoned sphere of ideas.

These ideas were developed further in our century. That development was close to the time in which Klein worked, but not, as sometimes happens in the history of science, close enough to attract his attention and make him abandon his apathy. We have subsequently seen the development of a "five-dimensional" unified theory of gravity and electricity.

At the same time, what touches upon the possibilities that were revealed by Klein of interpreting mechanics as quasi-optics in spaces with a higher number of dimensions are the forgotten and unnoticed authors of the numerous five-dimensional generalizations of the general theory of relativity. We will see, however, that there are important deep and intimate connections between the ideas of "five-dimensional spaces" and optico-mechanical analogies.

Our problem in this volume is now that of constructing the fundamental equations of geometrical optics and classical relativistic mechanics in such a form that they display similarities and differences with the aforementioned possibilities and to perceive in what sense the problems of optics can serve as particular cases of the problems of mechanics. We shall start with optics.

§ 2. The principle of least time in optics

Consider the problem of the trajectory of a light ray in an inhomogeneous (but isotropic) optical medium, whose index of refraction is $N(\mathbf{r})$ and does not depend upon time. According to the principle of least time, the trajectory of a light ray that passes between two points $\underline{\mathbf{r}}$ and $\overline{\mathbf{r}}$ is distinguished amongst the family of infinitely-close perturbed paths that connect these two particular points and whose elapsed time is $T(\underline{\mathbf{r}}, \overline{\mathbf{r}})$ by the fact that as the ray follows the true trajectory it will render a minimum to:

$$\delta T(\underline{\mathbf{r}}, \overline{\mathbf{r}}) = \delta \int_{\underline{r}}^{\overline{r}} \frac{N}{c} d\sigma . \tag{1.1}$$

In this, $d\sigma$ is the element of arc length and c is the speed of light *in vacuo*. Here, the symbol n_i will denote a unit vector that is tangent to the trajectory and satisfies:

$$d\sigma = n_i\, dx^i, \qquad n_i = dx^i / ds, \qquad \sigma_{ik}\, n_i\, n_k = 1, \tag{1.2}$$

and therefore:

$$d\sigma^2 = \delta_{ik}\, dx^i\, dx^k, \qquad \delta\, d\sigma = n_i\, d\delta x^i.$$

If one performs the variation then one will get:

$$\begin{aligned}\delta T(\underline{\mathbf{r}}, \overline{\mathbf{r}}) &= \frac{1}{c}\int_{\underline{r}}^{\overline{r}} (\delta N\, d\sigma + N\delta\, d\sigma) = \frac{1}{c}\int_{\underline{r}}^{\overline{r}} (\delta N\, n_k\, dx^k + N n_i\, d\delta x^i) \\ &= \frac{1}{c}\int_{\underline{r}}^{\overline{r}} \left\{ \frac{\partial N}{\partial x^i} n_k dx^k - d(Nn_i) \right\} \delta x^i + \frac{1}{c}\{\overline{N}\overline{n}_i \delta \overline{x}^i - \underline{N}\underline{n}_i \delta \underline{x}^i\} ,\end{aligned} \tag{1.3}$$

where $\underline{f}$ and $\overline{f}$ denote the values of the function f at the points $\underline{\mathbf{r}}$ and $\overline{\mathbf{r}}$.

If the variation vanishes at the endpoints: $\delta \overline{x}^i = \delta \underline{x}^i = 0$ then the extremum condition will give the equations of motion for the ray in Lagrangian form:

$$\frac{d}{d\sigma}(Nn_i) - \frac{\partial N}{\partial x^i} = 0 , \tag{1.4}$$

$$n_i = \frac{dx^i}{d\sigma} . \tag{1.4′}$$

One can derive two independent equations from the three equations (1.4). If one multiplies (1.4) by n_i then one will obtain:

$$n_i \frac{d}{d\sigma}(Nn_i) - \frac{\partial N}{\partial x^i} n^i = \frac{dN}{d\sigma} + N n_i \frac{dn_i}{d\sigma} - \frac{dN}{d\sigma} \equiv 0 . \tag{1.5}$$

We now consider the time $T(\underline{\mathbf{r}},\overline{\mathbf{r}})$ that elapses along the light ray as it passes from the point $\underline{\mathbf{r}}$ to the point $\overline{\mathbf{r}}$, which will be a function of the six coordinates of $\underline{\mathbf{r}}$ and $\overline{\mathbf{r}}$. From (1.3), since the line integral vanishes along the actual trajectory, we will have:

$$\delta T(\underline{\mathbf{r}},\overline{\mathbf{r}}) = \frac{1}{c}\{\overline{N}\,\overline{n}_i\,\delta\overline{x}^i - \underline{N}\,\underline{n}_i\,\delta\underline{x}^i\}\ , \tag{1.6}$$

and therefore:

$$\frac{\partial T}{\partial\overline{x}^i} = \frac{\overline{N}\,\overline{n}_i}{c}, \tag{1.7}$$

from which we conclude that for $\overline{\mathbf{r}}$ = const. the function $T(\underline{\mathbf{r}},\overline{\mathbf{r}})$ will satisfy the equation:

$$\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2+\left(\frac{\partial T}{\partial z}\right)^2 = \frac{N^2}{c^2}, \tag{1.8}$$

which is called the *3-eikonal equation.*

We now transform equation (1.8) and, as is customary in the theory of first-order differential equations that depend upon the variable *T*, introduce a fourth additional variable, i.e., we look for a single dependent parameter Σ_0 that makes the solution to equation (1.8) take the form:

$$\Sigma(x, y, z, T) = \Sigma_0\ . \tag{1.9}$$

Therefore:

$$\frac{\partial\Sigma}{\partial x^i}+\frac{\partial\Sigma}{\partial T}\cdot\frac{\partial T}{\partial x^i} = 0, \qquad \frac{\partial T}{\partial x^i} = -\frac{\partial\Sigma}{\partial x^i}\Big/\frac{\partial\Sigma}{\partial T}, \tag{1.10}$$

and if we substitute this in equation (1.8) then we will obtain the following differential equation for the function Σ:

$$\left(\frac{\partial\Sigma}{\partial x}\right)^2+\left(\frac{\partial\Sigma}{\partial y}\right)^2+\left(\frac{\partial\Sigma}{\partial z}\right)^2-\frac{N^2}{c^2}\left(\frac{\partial\Sigma}{\partial T}\right)^2 = 0, \tag{1.11}$$

which is called the *4-eikonal equation*.

In problems that are concerned with the propagation of lightlike rays, equation (1.11) will display a relativistic character that is hidden in the equivalent equation (1.8).

Note that although we have introduced time in the form of a fourth coordinate in our presentation, what is going on here is independent of any considerations of relativistic symmetry and what is usually methodologically acceptable, but it is well known in the theory of first-order partial differential equations.

On the other hand, our method of exposition is valid independently (but of course, far from physically, and rather artificially) of the methods that are introduced into the description of optical phenomena in order to represent relativistic symmetries of space-time.

From the standpoint of the theory of gravitation, and for the sake of problems that are concerned with the propagation of light rays in optically-inhomogeneous media, one formulates the 4-eikonal equation (1.11) in the particular cases in terms of the propagation of light rays in four-dimensional Riemann spaces:

$$x = x^1, \qquad y = y^1, \qquad z = z^1, \qquad icT = x^4, \tag{1.12}$$

in which the metric tensor takes on the special form:

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & N^2 \end{pmatrix}, \tag{1.13}$$

and does not depend upon the fourth coordinate x^4.

One can formulate problems of geometrical optics that are concerned with the propagation of light rays in arbitrary gravitational fields by means of the associated 4-eikonal equation:

$$g_{ik} \frac{\partial \Sigma}{\partial x^i} \frac{\partial \Sigma}{\partial x^k} = 0. \tag{1.14}$$

.

We shall make two remarks about this:

I. The 4-eikonal equation (1.14) is *homogeneous* in the metric potentials g^{ik}. This means that one does not have ten metric potentials g^{ik} in the problems of geometrical optics, but only nine of them, due to the relation that exists between them. However, this does not play much of a role, since in passing to the geometrical optics approximation one will simultaneously formulate problems in wave optics using Maxwell's equations in gravitational fields:

$$\frac{\partial \sqrt{-g}\, F^{ik}}{\partial x^k} = 0\,, \qquad \frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{li}}{\partial x^k} = 0, \tag{1.15}$$

which are certainly inhomogeneous in the potentials g^{ik}.

II. The 4-eikonal equation arises from the principle of least time. This conclusion is coupled with the assumption that the index of refraction N (or, in the general case, the metric potentials g^{ik}) does not manifestly depend upon time. However, it is well-known that the 4-eikonal equation loses its legitimacy in the case where the metric potentials g^{ik} do depend upon the fourth coordinate. In the general case, the 4-eikonal equation can be deduced from differential equations for the characteristic manifolds $\Sigma(x^1, x^2, x^3, x^4) = \Sigma_0$ of the system of Maxwell equations.

This conclusion is justified in the appendix.

§ 3. The principle of least action in mechanics

We now consider some problems regarding the trajectories of electrons [mass m, charge $(-e)$] in electromagnetic fields. All of these calculations will be carried out in four-dimensional form.

According to the principle of least action, the trajectory of an electron (which is a piecewise-defined curve) that passes through two events $\underline{R} = (\underline{x}, \underline{y}, \underline{z}, \underline{t})$ and $\bar{R} = (\bar{x}, \bar{y}, \bar{z}, \bar{t})$ is distinguished amongst the family of infinitely-close piecewise-defined curves that connect these two events by the fact that the action $S(\underline{R}, \bar{R})$ will be rendered a minimum along the actual trajectory (*):

$$\delta S(\underline{R}, \bar{R}) = -\,\delta mc \int_{\underline{R}}^{\bar{R}} \{ds + g_i\, dx^i\} = 0\,. \tag{1.16}$$

In this, ds is the element of arc-length and $g_i = \frac{e}{mc^2} A_i$, where A_i is the electromagnetic potential.

If u_i denotes the 4-velocity vector then that will give:

$$ds = -\,u_i\, dx^i, \qquad u^i = \frac{dx^i}{ds}, \qquad \delta_{ik}\, u^i\, u^k = -\,1, \tag{1.17}$$

and therefore:

$$ds^2 = -\,\delta_{ik}\, dx^i\, dx^k, \qquad \delta\, ds = -\,u_i\, d\,\delta x^i.$$

If we perform the variations then we will obtain:

$$\begin{aligned}
\delta S(\underline{R}, \bar{R}) &= -\,mc \int_{\underline{R}}^{\bar{R}} \{\delta\, ds + \delta(\mathrm{g}_i\, dx^i)\} \\
&= mc \int_{\underline{R}}^{\bar{R}} \left\{ u_i\, d\delta x^i - \frac{\partial g_i}{\partial x^k}\, \delta x^k\, dx^i - \mathrm{g}_i\, d\delta x^i)\right\} \\
&= mc \int_{\underline{R}}^{\bar{R}} \left\{ -\,du_i + \left(\frac{\partial g_i}{\partial x^k} - \frac{\partial g_k}{\partial x^i}\right) dx^k \right\} \delta x^i + \\
&\qquad + mc\, \{(\bar{u}_i - \bar{g}_i)\delta \bar{x}^i - (\underline{u}_i - \underline{g}_i)\delta \underline{x}^i\ \}.
\end{aligned} \tag{1.18}$$

This variation will vanish at the endpoints: $\delta \underline{x}^i = \delta \bar{x}^i = 0$, so the extremum condition will give the equations of motion for the electron in the Lorentz form:

$$\frac{du^i}{ds} = \left(\frac{\partial g_i}{\partial x^k} - \frac{\partial g_k}{\partial x^i}\right) u_k\,, \tag{1.19}$$

(*) Cf. [**12**], pp. 52.

$$\frac{dx^i}{ds} = u^i \,. \tag{1.19′}$$

Of the four equations (1.19), only three of them are independent. Upon multiplying both sides by u^i one will obtain:

$$u^i \frac{du_i}{ds} \equiv 0. \tag{1.20}$$

We shall now consider the action to be a function of the eight coordinates $\underline{x}^i$ and $\overline{x}^i$ when it is evaluated along the actual trajectory.

When the integral in (1.18) vanishes this gives us:

$$\delta S(\underline{R}, \overline{R}) = mc \, \{ (\overline{u}_i - \overline{g}_i) \, \delta \overline{x}^i - (\underline{u}_i - \underline{g}_i) \, \delta \underline{x}^i \}, \tag{1.21}$$

and therefore:

$$\frac{\partial S}{\partial \overline{x}^i} + \overline{g}_i \, mc = mc \, \overline{u}_i \,, \tag{1.22}$$

In this case, with the assumption that $\delta_{ik} \, u^i \, u^k = -1$, the function $S(\underline{R}, R)$ will be regarded as a function of R when $\overline{R}$ = const., and which satisfies the equation:

$$\frac{b_{ik}}{m^2c^2} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} + 2 g_i \frac{1}{mc} \frac{\partial S}{\partial x^i} + (1 + \delta_{ik} \, u^i \, u^k) = 0, \tag{1.23}$$

which is called the *relativistic Hamilton-Jacobi equation*.

We are utilizing the possibility that was suggested by Klein, viz., that every mechanical problem concerning the motion of material particles can be defined by the paths of light rays that pass through suitable media with the help of spaces of higher dimensionality.

Toward that end, we introduce the function $S(\underline{R}, \overline{R})$, which depends upon five additional independent variables, i.e., we will seek solutions of equation (1.23) of the form:

$$\Sigma \, (x^1, x^2, x^3, x^4, S) = \Sigma_0 \tag{1.24}$$

that depend upon the single parameter Σ_0. That will give:

$$\frac{\partial \Sigma}{\partial x^i} + \frac{\partial \Sigma}{\partial S} \cdot \frac{\partial S}{\partial x^i} = 0 \,; \qquad \frac{\partial S}{\partial x^i} = -\frac{\partial \Sigma}{\partial x^i} \Big/ \frac{\partial \Sigma}{\partial S} \,, \tag{1.25}$$

and when we substitute this in equation (1.23) we will obtain the differential equation for this function Σ:

$$\delta_{ik} \frac{\partial \Sigma}{\partial x^i} \cdot \frac{\partial S}{\partial x^k} - 2 g_i \frac{\partial \Sigma}{\partial x^i} \frac{\partial \Sigma}{\partial S} mc + (1 + g_i g_k \delta_{ik}) m^2 c^2 \left(\frac{\partial \Sigma}{\partial S} \right)^2 = 0, \tag{1.26}$$

which is called the *5-eikonal equation*.

Equation (1.26) is easily generalized to the case in which an external gravitational field is present, as long as we insure its general covariant form, namely:

$$g^{ik}\frac{\partial\Sigma}{\partial x^i}\cdot\frac{\partial S}{\partial x^k}-2g^{ik}g_k\frac{\partial\Sigma}{\partial x^i}\frac{\partial\Sigma}{\partial S}mc+(1+g^{ik}g_ig_k)m^2c^2\left(\frac{\partial\Sigma}{\partial S}\right)^2=0\,. \qquad (1.26')$$

From the standpoint of geometrical optics, the formulation of equation (1.26′) will serve as a special case of a general problem in five-dimensional geometrical optics, which is formulated as the 5-eikonal equation (*):

$$G^{\mu\nu}\frac{\partial\Sigma}{\partial x^\mu}\frac{\partial\Sigma}{\partial x^\nu}=0, \qquad (1.27)$$

which is concerned with the propagation of light rays in five-dimensional Riemannian spaces with whose coordinates are space, time, and action:

$$x=x^1, \qquad y=x^2, \qquad z=x^3, \qquad ict=x^4; \qquad \frac{S}{mc}=x^5. \qquad (1.28)$$

The contravariant metric 5-tensor in this space takes the following special form:

$$G^{\mu\nu}=\begin{pmatrix} g^{11} & g^{12} & g^{13} & g^{14} & -g^1 \\ g^{21} & g^{22} & g^{23} & g^{24} & -g^2 \\ g^{31} & g^{32} & g^{33} & g^{34} & -g^3 \\ g^{41} & g^{42} & g^{43} & g^{44} & -g^4 \\ -g^1 & -g^2 & -g^3 & -g^4 & 1+g^{ik}g_ig_k \end{pmatrix} \qquad (1.29)$$

and does not depend upon the fifth coordinate x^5 of action.

We re-write (1.29) in the abbreviated form:

$$G^{\mu\nu}=\begin{pmatrix} g^{ik} & -g^i \\ -g^k & 1+g^{ik}g_ig_k \end{pmatrix} \qquad (1.29')$$

and compute the covariantly-constructed metric tensor $G_{\mu\nu}$:

$$G_{\mu\nu}=\begin{pmatrix} g_{ik}+g_ig_k & g_i \\ g_k & 1 \end{pmatrix}. \qquad (1.30)$$

One easily verifies from expression (1.30), by computation, that:

(*) We denote 5-tensor indices by Greek symbols, and 4-tensor indices by Latin ones.

$$G_{\mu\nu}\, G^{\mu\nu} = \delta^{\sigma}_{\mu}\,.$$

Note that according to (1.30):

$$G_{55} = 1. \tag{1.31}$$

We must make two remarks:

I. The 5-eikonal equation is *homogeneous* in the metric potentials $G^{\mu\nu}$. This means that in the problems of geometrical optics (i.e., classical relativistic mechanics) it is not all fifteen of the metric potentials $G_{\mu\nu}$ that are meaningful, but only fourteen of them, since there is one relation between them.

Therefore the restriction that $G_{55} = 1$ is inessential, so problems in 5-optics that are concerned with the propagation of light rays in five-dimensional spaces whose coordinates are space, time, and action, and are endowed with a metric 5-tensor $G_{\mu\nu}$ that is restricted by one condition (viz., that it does not depend upon on the fifth coordinate of action) will be equivalent to associated problems in classical relativistic mechanics that are concerned with the motion of a particle with a charge-to-mass ratio of $e\,/\,m$ in a gravitational and electromagnetic field:

$$g_{ik} = \frac{G_{ik}}{G_{55}} - \frac{G_{i5}}{G_{55}} \cdot \frac{G_{k5}}{G_{55}}\,, \qquad A_i = \frac{mc^2}{e} \cdot \frac{G_{i5}}{G_{55}}\,. \tag{1.32}$$

II. The restriction we imposed (viz., that the metric tensor should be independent of the fifth coordinate of action) is more essential. We obtained the 5-eikonal equation from the variational principle of least action. The introduction of the restriction was connected with the assumption that the Lagrangian function did not depend upon the fifth coordinate of action. The fact that this demand still remains apparently points to physical considerations. It is always the character of macroscopic gravitational and electromagnetic fields in four dimensions that they do not exhibit any dependency upon the additional fifth coordinate.

Therefore, we shall retain this requirement for the present and pass on to a historical survey of some five-dimensional generalizations that are associated with the theory of relativity. At this point, we do not pose the problem of giving a systematic exposition of the numerous associated variational generalizations, but we only exhibit their connection with the notions of optico-mechanical analogies.

§ 4. Five-dimensional generalizations of the theory of gravity

After the appearance of Einstein's theory of gravity, *T. Kaluza* [**3**] (1921) was the first to show the possibility of constructing an approximate unified theory of gravity and electromagnetism by extending the four-dimensional spacetime continuum of the general theory of relativity with one additional dimension. He dared to show that the trajectory of a charged particle could be approximately interpreted as a geodesic line in a five-dimensional Riemannian space whose metric depends essentially upon the mass of the particle considered, but not upon the additional fifth coordinate (i.e., the *cylindricality condition*). For the sake of what we directly established above for the fifteen metric potentials $G_{\mu\nu}$ in a fifteen-dimensional space and the fourteen gravitational and electromagnetic potentials (g_{ik}, A_i), following T. Kaluza, we assume:

$$G_{ik} = g_{ik}\,, \qquad G_{i5} = \frac{e}{mc^2}A_i\,, \qquad G_{55} = 1. \tag{1.33}$$

Independently of T. Kaluza, *G. A. Mandel* [**4**] (1926) developed the idea of a five-dimensional generalization of the theory of gravity noticeably further than T. Kaluza.

In 1926, in connection with the advent of wave mechanics, two similar works by *O. Klein* [**5**] and *V. A. Fock* [**6**] appeared independently of each other that made a considerable step forward. Note that Klein borrowed the idea of a fifth dimension from Kaluza, whereas Fock borrowed it from Mandel. We summarize what was suggested in the author's introduction, viz., that the trajectory of a charge particle can be rigorously interpreted as a null-length geodesic line (i.e., a geometric ray) in a five-dimensional Riemannian space with a metric tensor that has the form (1.30), viz.:

$$G_{ik} = g_{ik} + \frac{e^2}{m^2c^4}A_i\,A_k\,, \qquad G_{i5} = \frac{e}{mc^2}A_i\,, \qquad G_{55} = 1. \tag{1.33′}$$

In fact, the aforementioned authors established the equivalence of classical mechanics with five-dimensional geometrical optics, i.e., independently of the fact that F. Klein showed the possibility of formulating mechanics as quasi-optics in spaces of a higher number of dimensions.

What is more, they revealed that suitable problems in wave mechanics that are concerned with the motion of spin-zero particles can be formulated as problems in wave optics that are concerned with the propagation of scalar waves in five-dimensional spaces, as long as the function W of five coordinates upon which the scalar waves depend satisfies the *cyclicality condition:*

$$W(x^1, x^2, x^3, x^4, x^5) = U(x^1, x^2, x^3, x^4)\,\exp\left(\frac{imcx^5}{\hbar}\right). \tag{1.34}$$

If the expression (1.34) is substituted in the wave equation for 5-spaces, namely:

$$\sum_{\alpha=1}^{\alpha=5}\frac{\partial^2 W}{\partial x^\alpha \partial x^\alpha} = 0, \tag{1.35}$$

then it is well-known that the function $U(x^1, x^2, x^3, x^4)$ that one obtains will satisfy the equation for matter waves:

$$\left\{\Box - \left(\frac{mc}{\hbar}\right)^2\right\}U(x^1, x^2, x^3, x^4) = 0\,. \tag{1.36}$$

However, the further development of this idea will not reveal the physical meaning of the fifth additional coordinate, which will still remain unknown. For the same reason, we cannot hope to understand the cylindricality condition that was imposed on the metric potentials in five-dimensional spaces and the cyclicality condition that was imposed on the wave functions. Moreover, we very quickly point out that the successes of the development of wave mechanics and its numerous applications make the introduction of five-dimensional spaces seem gratuitous.

Five-dimensional theory embraces this formalism completely. To be sure, it is a unified formalism, and it is unified in an essential way, but it does not imply anything beyond its own framework, and does not seem to predict any new specific electro-gravitational effects that would be accessible to experimental verification, even if only in principal.

We find considerable progress being made in the work of *Einstein* and *Bergmann* [**7**] (1938), in which the fifth dimension takes on some physical sense. In that work, the authors rejected the cylindricality condition that was imposed on the metric potentials in the previous works. However, as far as what is observed in the nature of macroscopic four-dimensional gravitational and electromagnetic fields (which do not exhibit any dependency upon the additional fifth coordinate) is concerned, it is necessary to assume that five-dimensional space obeys cylindrical relativity in the fifth dimension, at least approximately. From these considerations, the authors assumed that five-dimensional space is topologically closed in the fifth dimension, and that the period of the fifth coordinate (which is denoted by the symbol b) has a microscopic magnitude, which can be assumed to be equal to zero in the first approximation. In their work, such a form for the cylindricality condition weakened and substituted for the demand that the metric potentials should have microscopic periodicity in the fifth coordinate.

The infinitely-extended surface of a cylinder of radius of $b / 2\pi$ serves to define a two-dimensional model for the consideration of five-dimensional spaces. A spatial stratum of thickness b that is bounded by two parallel planes (when one has a suitable way of identifying each with the other) will give a three-dimensional model for the space of Einstein and Bergmann.

Rejecting strict cylindricality entails far-reaching physical consequences; principally, important effects in which the fields that are constructed display the postulated periodic dependency upon the fifth coordinate. On the important matter of how one goes about introducing the new arbitrary constant b into field theory, by reason of classical field theory, one should obtain an absence of new results after passing to the limit $b \to 0$.

All of the existing work leaves open the following questions:

1. The question of the physical meaning and units of the additional fifth dimension.

2. The question of the physical sense of the cylindricality condition on the metric potentials and the cyclicality condition on the wave functions (Kaluza, Mandel, O. Klein, Fock).

3. Unless one rejects the cylindricality condition and replaces it, following Einstein and Bergmann, with the demand of microscopic periodicity, then the question will arise: In what natural phenomena does this postulated periodicity manifest itself?

§ 5. The geometrical meaning of Planck's constant

From the presentation in the foregoing paragraphs, we can regard it as being established that there is work to do in giving the fifth additional coordinate a precise physical meaning in reality, and that in that discussion our own developments seemed to ignore the ideas of F. Klein on mechanics as quasi-optics in spaces of a higher number of dimensions.

At the same time, we posed and answered the question of what problems of optics might serve as particular cases of mechanical problems. We then displayed those problems of optics, formulated the 4-eikonal equation, which served as a four-dimensional particular case of the general problem in five-dimensional mechanical problems, and formulated the 5-eikonal equation.

Note especially that from the work of Einstein and Bergmann, we arrived at the conclusion that the new arbitrary constant b – viz., the period of the fifth coordinate – has real units and that it is reasonable to identify it with Planck's constant h.

Because of this and other work of the author on 5-optics [**8**], we fundamentally maintain that conclusion in substantiating and developing the following assumptions.

Problems of wave optics that are concerned with the propagation of (tensorial and spinorial) wave fields in Riemannian spaces with five coordinates of space, time, and action that are topologically closed in the action coordinate with period h are equivalent to problems of quantum mechanics that are concerned with particular motions with a charge-to-mass ratio of e / m (integer or half-integer spin) in an external field.

In such a form, 5-optics implies a somewhat unexpected synthesis of ideas from quantum mechanics with geometrical ideas at the foundations of the general theory of relativity by saying that Planck's constant obtains a precise geometric meaning as the period of the fifth coordinate of action.

5-optics gives the following answers to the questions that were left open in the work of the foregoing authors:

1. The fifth coordinate has the physical meaning of action, as well as the same units.

2. The cylindricality condition on the metric potentials and the cyclicality condition on the wave function are replaced with the single demand of the microscopic periodicity of those physical fields (gravitation, electromagnetism, and quantum ψ-fields) that always depend on the action coordinate. The period of the fifth coordinate of action has the universal magnitude of Planck's constant h.

3. The physical fields of Nature that appear in quantum mechanics always exhibit a periodicity dependency on the fifth coordinate of action. By passing to the limit $h \to 0$, the influence of the physical field will always be independent of the action coordinate, i.e., they will satisfy the cylindricality condition.

§ 6. Gradient invariance

The possibility and expediency of its interpretation demand that the additional fifth coordinate must be especially distinct from the other ones, unless we examine the group of general point-like transformations of all five coordinates:

$$\left.\begin{aligned} x^i &= \bar{x}^i + f^i(\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4, \bar{S}), \\ S &= \bar{S} + f(\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4, \bar{S}). \end{aligned}\right\} \quad (1.37)$$

The topological closure of a 5-space in the coordinate S imposes an important restriction upon the group elements (1.37), namely, that admissible unknown functions f^i and f must be periodic in the coordinate S with period h:

$$\left.\begin{aligned} f^i(x^1, x^2, x^3, x^4, S+h) &= f^i(x^1, x^2, x^3, x^4, S), \\ f(x^1, x^2, x^3, x^4, S+h) &= f(x^1, x^2, x^3, x^4, S). \end{aligned}\right\} \quad (1.38)$$

In connection with this, we observe that the “five-dimensional Lorentz group” is the group of linear transformations of all five coordinates that leave the quadratic form:

$$dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dS^2$$

invariant. These transformations do not define a subgroup of the group (1.37), since these elements do not satisfy the additional condition (1.38).

However, it is easy to see that the true group of Lorentz transformations in four coordinates x^1, x^2, x^3, x^4 does define a subgroup of the group (1.37).

In the classical limit $h \to 0$, the group (1.37) will go to the subgroup of transformations:

$$\left.\begin{aligned} x^i &= \bar{x}^i + f^i(\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4), \\ S &= \bar{S} + f(\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4), \end{aligned}\right\} \quad (1.39)$$

which, in turn, will subdivide into:

1. The subgroup of general transformations of the four coordinates x^1, x^2, x^3, x^4:

$$\left.\begin{aligned} x^i &= \bar{x}^i + f^i(\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4), \\ S &= \bar{S}. \end{aligned}\right\} \quad (1.40)$$

2. The subgroup of gradient transformations:

$$\left.\begin{aligned} x^i &= \bar{x}^i, \\ S &= \bar{S} + f(\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4). \end{aligned}\right\} \quad (1.41)$$

We see that in 5-optics the group of gradient transformations does not stand alone, but is combined with the group of general transformations of the four coordinates into the group of general transformations of all five coordinates.

From this particularly clear presentation, it follows that the action serves as a "coordinate-like" magnitude, insofar as it, like each of the other four coordinates, is defined to be point-like, up to an arbitrary additive function.

This property of action is certainly well known in non-relativistic classical mechanics. At the same time, this property, which is displayed by each of the four coordinates, exhibits a possibility that is lacking in the general theory of relativity.

If one formulates equations between 5-tensors in covariant form then they will obviously be gradient invariant, since the group of gradient transformations defines a subgroup of the group of general transformations of the five coordinates. By passing to four-dimensional notation for the equations, the action coordinate must nevertheless disappear, in order that what remains should be gradient covariant. One deduces general formulas for the transformation of 5-tensors by the gradient transformations (1.41).

From the general formulas:

$$A^{\mu} = \bar{A}^{\nu} \frac{\partial x^{\mu}}{\partial \bar{x}^{\nu}}, \qquad A_{\mu} = \bar{A}_{\nu} \frac{\partial \bar{x}^{\nu}}{\partial x^{\mu}} \tag{1.42}$$

one will get:

$$\left.\begin{aligned} A^{i} &= \bar{A}^{i}; & A_{i} &= \bar{A}_{i} - \bar{A}_{5} \frac{\partial f}{\partial x^{i}}; \\ A^{5} &= \bar{A}^{5} + \bar{A}^{i} \frac{\partial f}{\partial x^{i}}; & A_{5} &= \bar{A}_{5}, \end{aligned}\right\} \tag{1.43}$$

from which, one can conclude the following rule:

The gradient-invariant construction of 5-tensors will produce tensors that are contravariant in the index i (= 1, 2, 3, 4) and covariant in the index $\mu = 5$.

At the beginning of this chapter, we examined the construction of the metric tensor $G_{\mu\nu}$. In accord with the requirement of gradient invariant construction, we can set:

$$G^{ik} = g^{ik}, \qquad G_{55} \text{ arbitrary.} \tag{1.44}$$

If we demand that G_{i5} should transform according to formula (1.43) then:

$$G_{i5} = \bar{G}_{i5} - \bar{G}_{55} \frac{\partial f}{\partial x^{i}}. \tag{1.45}$$

Hence, if one makes note of (1.32) then one will deduce the formula for the transformation of electromagnetic potentials:

$$A_{i} = \bar{A}_{i} - \frac{\partial f}{\partial x^{i}}. \tag{1.46}$$

§ 7. The physical meaning of 5-space

We now direct our attention to formula (1.32). We see the connection between the gravitational potentials g_{ik} that figure in the theory of gravity and the metric potentials $G_{\mu\nu}$ of 5-space, which define *universals*, in the same way that the ratio e / m enters into the expression that connects the electromagnetic potentials A_i with $G_{\mu\nu}$ for a particular motion that is being considered.

This shows that the 5-space of 5-optics does not define the universal space of the general theory of relativity (when it is extended with one additional dimension), but the *configuration space* for the particular motion which we consider.

Let us consider the question of the physical meaning of configuration 5-space in detail. Unless we consistently refuse to introduce metaphysical concepts such as absolute space and absolute time into physics, which are torn from matter and are in contradiction with it, like some sort of independent essence, we cannot, in principal, consider motion separately from its interaction with the rest of matter in the universe.

Physics created a successful method for dealing with the behavior of radiation that is assumed to exist separately from the rest of matter and interact with it: It is the method of field theory. In field theory, one singles out a particle that plays the role of test particle, while all of the rest of matter in the universe interacts with it through their roles as the sources of force fields.

The methodical division of all matter into test particles plus external fields gives us the possibility of associating test particles with their coordinates in four-dimensional manifolds (three space dimensions and one time dimension). We will call this association the *coordinate-wise* test-particle relativity of all matter in the universe.

In the detailed description of the theory of relativity, it is necessary to operate with and adapt the coordinates of material particles; for example, the emission of a light signal, a measuring device, a rigid ruler, a clock. To be sure, only someone who constructs an abstract picture that has no meaning can speak of coordinates and particles that are isolated in time, as if they were torn from the rest of matter.

The question arises of the metric and topological properties of four-dimensional coordinate manifolds, which, for us, will be independent configuration spaces for the test particles in question.

An answer to this question can be given only by experiment. Moreover, we can make the answer to the question more precise by refining our knowledge. In all cases, *a priori*, there is no fundamental reason whatsoever for postulating that the metric and topological properties of configuration space should be independent of the physical nature of the individual particles (for instance, mass, charge, etc.).

Finding the answers would then successively give way to the questions of the special theory of relativity, the theory of gravity, and 5-optics.

1. *Special theory of relativity.* In the absence of any dependency upon physical properties (mass, charge, etc.), the configuration spaces of test particles will define Minkowski spaces.

This allows us to introduce the space that one is concerned with in the special theory of relativity in place of the configuration space for the test particle in question.

The interactions of test particles with the rest of matter in the universe will take into account the way that this introduction was made. From a single feature, or from each feature, one ascribes a number of characteristics (mass, charge, etc.) with a test particle, the magnitude of which will determine the behavior of the test particle in a given external field.

2. *Theory of gravitation.* In the absence of any dependency of a test particle upon physical properties, this four-dimensional configuration space will define a Riemannian metric space whose metric determines the character of the influence of gravity on test particles due to the rest of all matter in the universe.

There is an equivalence principle for gravitational fields that one expresses in terms of fundamental properties of gravitational fields. It can be formulated in the following way:

Gravity, and only it, defines the universal concept – in the sense of the present work – that all uncharged bodies with a sufficiently small mass will move according to the same laws.

Thanks to this equivalence principle, the notions in the theory of gravitation that are concerned with the universal four-dimensional spaces of the general theory of relativity will still remain.

3. *5-optics*. The five-dimensional configuration space whose coordinates are space, time, and action for a given test particle will define a Riemannian metric space whose metric depends upon the ratio of charge to mass for the given test particle and will determine the gravitational and electromagnetic character of the influence of the rest of matter in the universe on the test particle. Five-dimensional configuration space will be topologically closed in the action coordinate, while the period of the fifth coordinate will have the universal unit of the Planck const h.

Thus, in 5-optics, it is no longer possible to maintain the notion of universal space, as one does in the theory of gravity and which is, however, quite unnecessary from a physical point of view.

It is hard to overestimate the role and significance of the equivalence principle in the history of physics. If one formulates the fundamental properties of gravity from it then one will allow the geometrization of the gravitational field, which will remain by its customary introduction into universal space. Although it is endowed with a non-Euclidian metric that depends upon the distribution of gravitating mass, the universal space of the general theory of relativity will still retain one characteristic trait of its own prototype – i.e., absolute Newtonian space – namely, that the theory of gravity can never give a satisfying answer to the eternal and inescapable question: What form does gravitating matter take when it simultaneously distorts the space in which it is localized?

Below, the equivalence principle will prove to be an obstacle that impedes the development of the unified theory of fields. The physics media implant the erroneous opinion that the construction of a unified theory of gravity and electricity will conclude in logical contradictions.

It is truly astonishing that Einstein himself spent the last 25 years of his existence on problems of unified field theory without ever noticing any contradiction.

5-optics shows that the four-dimensional space of gravity – i.e., the configuration space for a given neutral test particle – reflects the character of gravitation that it is the influence of all the remaining matter in the universe on a given test particle in its own metric properties. Nevertheless, in this book this question and the question of the manner by which matter distorts the space in which it is localized will be devoid of meaning.

MATHEMATICAL APPENDIX

Derivation of the 4-eikonal equation from Maxwell's equations

Let some initial manifold $\Sigma(x^1, x^2, x^3, x^4) = \Sigma_0$ be given, along which the fundamental tensor F_{ik} is given, with the meaning of a Cauchy initial condition.

Perform the coordinate transformation:

$$\left.\begin{array}{ll} \bar{x}^1 = \bar{x}^1(x^1, x^2, x^3, x^4), & \bar{x}^2 = \bar{x}^2(x^1, x^2, x^3, x^4), \\ \bar{x}^3 = \bar{x}^3(x^1, x^2, x^3, x^4), & \bar{x}^4 = \Sigma(x^1, x^2, x^3, x^4), \end{array}\right\} \tag{1.47}$$

and note that when one singles out the coordinate $\bar{x}^4$: ($\alpha, \beta, \gamma = 1, 2, 3$), equations (1.15) will take the form:

$$\frac{\partial \bar{F}_{\alpha\beta}}{\partial \bar{x}^4} + \frac{\partial \bar{F}_{\beta 4}}{\partial \bar{x}^\alpha} + \frac{\partial \bar{F}_{4\alpha}}{\partial \bar{x}^\beta} = 0 , \tag{1.48}$$

$$\frac{\partial \sqrt{-\bar{g}}\,\bar{F}^{\sigma\beta}}{\partial \bar{x}^\beta} + \frac{\partial \sqrt{-\bar{g}}\,\bar{F}^{\sigma 4}}{\partial \bar{x}^4} = 0 , \tag{1.49}$$

$$\frac{\partial \bar{F}_{\alpha\beta}}{\partial \bar{x}^\gamma} + \frac{\partial \bar{F}_{\beta\gamma}}{\partial \bar{x}^\alpha} + \frac{\partial \bar{F}_{\gamma\alpha}}{\partial \bar{x}^\beta} = 0 , \qquad \frac{\partial \sqrt{-\bar{g}}\,\bar{F}^{4\beta}}{\partial \bar{x}^\beta} = 0 \tag{1.50}$$

in the new coordinates.

In the new coordinates, the given manifold $\Sigma(x^1, x^2, x^3, x^4) = \Sigma_0$ will define the coordinates of the surface $\bar{x}^4 = \Sigma_0$. If one can calculate the derivatives in the normal direction $\frac{\partial \bar{F}_{\alpha\beta}}{\partial \bar{x}^4}, \frac{\partial \bar{F}_{4\beta}}{\partial \bar{x}^4}$ from the Maxwell equation and deduce the manifold Σ then one will call it *ordinary*; in the contrary case, it will be *characteristic*. Calculate $\frac{\partial \bar{F}_{\alpha\beta}}{\partial \bar{x}^4}$ from equation (1.48). Next, calculate $\partial \bar{F}_{4\beta} / \partial \bar{x}^4$ and substitute in (1.49):

$$\bar{F}^{4\alpha} = \bar{g}^{4\beta}\bar{g}^{\alpha\gamma}\bar{F}_{\beta\gamma} + (\bar{g}^{44}\bar{g}^{\alpha\beta} - \bar{g}^{4\beta}\bar{g}^{4\alpha})\bar{F}_{4\beta} \tag{1.51}$$

and thus obtain the determination of $\dfrac{\partial \bar{F}_{4\beta}}{\partial \bar{x}^4}$ from the system of three equations of the form:

$$(\bar{g}^{44}\bar{g}^{\alpha\beta} - \bar{g}^{4\beta}\bar{g}^{4\alpha})\,\frac{\partial \bar{F}_{4\beta}}{\partial \bar{x}^4} = S^\alpha . \tag{1.52}$$

In order that the manifold Σ should be characteristic, it will therefore be necessary that:

$$\mathrm{Det}\,(\bar{g}^{44}\bar{g}^{\alpha\beta} - \bar{g}^{4\beta}\bar{g}^{4\alpha}) = 0. \tag{1.53}$$

The roots of this equation do not depend upon the fundamental tensor $\bar{g}^{\alpha\beta}$ and give:

$$\bar{g}^{44} = 0, \tag{1.54}$$

which will give the desired condition – viz., that the manifold $\bar{x}^4 = \Sigma_0$ should be characteristic – in the $\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4$ coordinate system. If one returns to the initial coordinate system then one will get:

$$\bar{g}^{44} = g_{ik}\frac{\partial \bar{x}^4}{\partial x^i}\frac{\partial \bar{x}^4}{\partial x^k} = g^{ik}\frac{\partial \Sigma}{\partial x^i}\frac{\partial \Sigma}{\partial x^k} = 0, \tag{1.55}$$

i.e., the 4-eikonal equation for the characteristic manifold $\Sigma(x^1, x^2, x^3, x^4) = \Sigma_0$.

CHAPTER II

GEOMETRICAL 5-OPTICS

§ 8. 5-eikonal equation

In this chapter, we shall present relativistic point mechanics as geometrical 5-optics in a Riemannian configuration space whose coordinates are space, time, and action. Naturally, we will not, by any means, obtain any new results that would enter into the framework of relativistic mechanics. Nonetheless, when the general formulas are formulated in five-dimensional "optics," they will acquire more elegance and a more symmetric form than they have in their conventional four-dimensional form.

We showed in § 3 that general problems in geometrical 5-optics that are concerned with the propagation of light in five-dimensional Riemann space, when they are formulated in terms of the 5-eikonal equation:

$$G^{\mu\nu}\frac{\partial\Sigma}{\partial x^{\mu}}\cdot\frac{\partial\Sigma}{\partial x^{\nu}}=0\,, \tag{2.1}$$

will be equivalent to problems of classical relativistic mechanics that are concerned with the motion of a charged particle with a charge-to-mass ratio of e / m in the gravitational and electromagnetic fields:

$$g_{ik}=\frac{G_{ik}}{G_{55}}-\frac{G_{i5}}{G_{55}}\cdot\frac{G_{k5}}{G_{55}}\,,\qquad\qquad A_i=\frac{mc^2}{e}\frac{G_{i5}}{G_{55}}\,. \tag{2.2}$$

Moreover, in the classical limit $h\to 0$, we will be obliged to burden the potentials $G_{\mu\nu}$ with the cylindricality condition.

As far as the 5-eikonal equation (2.1), which is homogeneous in the potentials $G^{\mu\nu}$, is concerned, we can, with no loss of generality, and for the remainder of this chapter, suppose that G_{55} = 1, and then take the tensor $G_{\mu\nu}$ to havethe form that is expressed by formulas (1.29) and (1.30):

$$G_{\mu\nu}=\begin{pmatrix} g_{ik}+g_i g_{k,} & g_i \\ g_k & 1\end{pmatrix};\qquad\qquad G^{\mu\nu}=\begin{pmatrix} g^{ik}, & -g^i \\ -g^k & 1+g^{ik}g_i g_{k,}\end{pmatrix}. \tag{2.3}$$

If we substitute equation (2.3) into (2.1) then we will obtain the 5-eikonal equation in four-dimensional form:

$$g^{ik}\frac{\partial\Sigma}{\partial x^i}\cdot\frac{\partial\Sigma}{\partial x^k}-2g^{ik}g_k\frac{\partial\Sigma}{\partial x^i}\cdot\frac{\partial\Sigma}{\partial x^5}+(1+g^{ik}g_i g_k)\left(\frac{\partial\Sigma}{\partial x^5}\right)^2=0\,. \tag{2.4}$$

Since fields should not depend upon the x^5 coordinate in the classical limit $h \to 0$, we will pass on to the "truncated" 4-eikonal $S(x^1, x^2, x^3, x^4)$ by means of the formula:

$$\Sigma(x^1, x^2, x^3, x^4) = \Pi_5\, x^5 + S(x^1, x^2, x^3, x^4), \tag{2.5}$$

in which Π_5 is constant. If we substitute (2.5) into (2.4) then we will obtain the equation for the function $S(x^1, x^2, x^3, x^4)$:

$$g^{ik}\left(\frac{\partial S}{\partial x^i} - \Pi_5 g_i\right)\left(\frac{\partial S}{\partial x^k} - \Pi_5 g_k\right) + \Pi_5^2 = 0, \tag{2.6}$$

which coincides with the Hamilton-Jacobi equation for a particle of mass $|Z|\, m$ and charge $\pm Ze$ (in which Z is an arbitrary number), as long as the constant Π_5 takes the value:

$$\Pi_5 = \pm Zmc, \tag{2.7}$$

$$g^{ik}\left(\frac{\partial S}{\partial x^i} \pm \frac{Ze}{c} A_i\right)\left(\frac{\partial S}{\partial x^k} \pm \frac{Ze}{c} A_k\right) + (Zmc)^2 = 0\,. \tag{2.8}$$

We see that the 5-eikonal equation (2.1) will describe the motion of all families of particles whose charge-to-mass ratio is e / m, as long as every particle of the family in question has a distinct charge, and therefore a mass, which consists of the fundamental characteristic constant Π_5. In the particular case for which Π_5 has the value zero, the 5-eikonal equation will describe the motion of particles of zero mass and charge that move in accordance with the 4-eikonal equation for gravitational fields:

$$g^{ik}\left(\frac{\partial S}{\partial x^i}\right)\left(\frac{\partial S}{\partial x^k}\right) = 0\,. \tag{2.9}$$

§ 9. Hamilton's canonical equations

We now introduce the wave 5-vector $\Pi_\mu = \dfrac{\partial \Sigma}{\partial x^\mu}$ and the optical Hamiltonian function H^* by the formula:

$$H^* = \tfrac{1}{2} G^{\mu\nu}\, \Pi_\mu\, \Pi_\nu\,, \tag{2.10}$$

which, by virtue of (2.1), and in light of reality, equals zero. We note the system of ten differential equations that are associated with the characteristic equations (2.1), and obtain:

$$\frac{dx^\mu}{d\tau} = \frac{\partial H^*}{\partial \Pi_\mu} = \Pi^\mu\,, \tag{2.11}$$

$$\frac{d\Pi_\mu}{d\tau} = -\frac{\partial H^*}{\partial x^\mu} = -\frac{1}{2}\frac{\partial G^{\alpha\beta}}{\partial x^\mu}\Pi_\alpha\Pi_\beta, \tag{2.12}$$

which is a system of ten canonical Hamiltonian equations for light rays in a 5-space. The parameter τ that enters into this equation is defined point-wise, up to an additional constant.

In this construction, we have, according to (2.3):

$$\left.\begin{aligned} G_{\mu\nu}\Pi^\mu\Pi^\nu &= g_{ik}\Pi^i\Pi^k + (g_i\Pi^i + \Pi^5)^2 = 0, \\ \Pi_5 &= G_{5i}\Pi^i + G_{55}\Pi^5 = g_i\Pi^i + \Pi^5, \end{aligned}\right\} \tag{2.13}$$

and therefore, taking into account (2.11):

$$g_{ik}\,\Pi^i\,\Pi^k + \Pi_5^2 = g_{ik}\,\frac{dx^i}{d\tau}\frac{dx^k}{d\tau} + \Pi_5^2 = 0\,. \tag{2.14}$$

In this, $g_{ik}\,dx^i\,dx^k = -\,ds^2$, where ds is the 4-interval element. Therefore, (cf., 2.7):

$$\frac{ds}{d\tau} = |\,Z\,|\,mc\,; \qquad \tau = \tau_0 + \frac{s}{|\,Z\,|\,mc}, \tag{2.15}$$

with the condition that the parameter τ must range along the world line.

Equation (2.11) serves as the definition of the ray vector Π^μ. We construct the optical Lagrangian function L^*:

$$L^* = \Pi_\mu\,\Pi^\mu - H^* = \tfrac{1}{2}\,G_{\alpha\beta}\,\Pi^\alpha\,\Pi^\beta, \tag{2.16}$$

which will then equal zero, in light of reality. We easily verify the formula:

$$\frac{\partial H^*}{\partial x^\alpha} = -\frac{\partial L^*}{\partial x^\alpha}, \tag{2.17}$$

which follows from:

$$G_{\nu\tau}\,G^{\mu\tau} = \delta^\mu_\nu\,, \qquad G_{\nu\tau}\,\frac{\partial G^{\mu\tau}}{\partial x^\alpha} = -\,G^{\mu\tau}\,\frac{\partial G_{\nu\tau}}{\partial x^\alpha}, \tag{2.18}$$

for the change in both parts of $\Pi_\mu\,\Pi^\mu$.

From (2.16), we have:

$$\Pi_\mu = \frac{\partial L^*}{\partial \Pi^\mu}, \tag{2.19}$$

and if we substitute this in (2.12), while taking (2.17) into account, then we will obtain:

$$\frac{d}{d\tau}\left(\frac{\partial L^*}{\partial \Pi^\mu}\right) - \frac{\partial L^*}{\partial x^\mu} = 0 \,, \tag{2.20}$$

which is the ray equation in Lagrangian form.

We now return to the systems of equations (2.11) and (2.12), while singling out the action coordinate x^5:

$$\frac{dx^i}{d\tau} = \Pi^i, \tag{2.11a}$$

$$\frac{dx^5}{d\tau} = \Pi^5, \tag{2.11b}$$

$$\frac{d\Pi_i}{d\tau} = -\frac{1}{2}\frac{\partial G^{\alpha\beta}}{\partial x^i}\Pi_\alpha \Pi_\beta \,, \tag{2.12a}$$

$$\frac{d\Pi_5}{d\tau} = -\frac{1}{2}\frac{\partial G^{\alpha\beta}}{\partial x^5}\Pi_\alpha \Pi_\beta = 0 \,, \tag{2.12b}$$

and elucidate the physical meaning of each group of equations individually.

1. We have:

$$\frac{dx^i}{d\tau} = \Pi^i = G^{ik}\,\Pi_k + G^{i5}\,\Pi_5 = g^{ik}\,(\Pi_k - g_k\,\Pi_5),$$

$$|\,Z\,|\,mc\frac{dx^i}{ds} = g^{ik}\left(\Pi_k \mp \frac{Ze}{c}A_k\right),$$

$$\Pi_k = |\,Z\,|\,mc\,g_{ki}\,\frac{dx^i}{ds} \pm \frac{Ze}{c}A_k \,. \tag{2.11′a}$$

Thus, equation (2.11a) expresses the usual connection between the 4-momentum Π_k and the 4-velocity $\dfrac{dx^i}{d\tau}$ in the presence of external fields.

2. We have (cf., 2.13):

$$\frac{dx^5}{d\tau} = \Pi^5 = \Pi_5 - g_i\,\Pi^i,$$

$$|\,Z\,|\,mc\frac{dx^5}{ds} = \pm\,|\,Z\,|\,mc - g_i\,|\,Z\,|\,mc\,\frac{dx^i}{ds}\,,$$

$$\mp\,dS = |\,Z\,|\,mc\,ds \pm \frac{Ze}{c}A_i\,dx^i \,. \tag{2.11′b}$$

Thus, equation (2.11b) gives (for a choice of the sign in front of dS) the usual definition of the element of action for a test particle in the presence of an external field.

In 5-optics, the action coordinate algebraically defines a magnitude and a sign that is connected with the sign of the charge; by changing the sign of action, one will change the sign of the charged particle.

3. We have, taking (2.17) into account:

$$\frac{d\Pi_i}{d\tau} = -\frac{1}{2}\frac{\partial G^{\alpha\beta}}{\partial x^i}\Pi_\alpha \Pi_\beta = \frac{1}{2}\frac{\partial G_{\alpha\beta}}{\partial x^i}\Pi^\alpha \Pi^\beta =$$
$$= \frac{1}{2}\left\{\left(\frac{\partial g_{mn}}{\partial x^i} + \frac{\partial g_m g_n}{\partial x^i}\right)\Pi^m \Pi^n + 2\frac{\partial g_m}{\partial x^i}\Pi^m \Pi^5\right\}.$$

Substituting the expressions for Π_i and Π_5 in formulas (2.11′a) and (2.13) will give:

$$\frac{d}{d\tau}\left\{g_{ik}\frac{dx^k}{d\tau} \pm g_i\right\} = \frac{1}{2}\left\{\frac{\partial g_{mn}}{\partial x^i}\frac{dx^m}{d\tau}\frac{dx^n}{d\tau}\Pi^m \Pi^n + 2\,|\,Z\,|\,mc\frac{\partial g_m}{\partial x^i}\frac{dx^m}{d\tau}\right\}.$$

We introduce the notation:

$$\Gamma^i_{kl} = \frac{1}{2}g^{in}\left(\frac{\partial g_{nk}}{\partial x^l} - \frac{\partial g_{kl}}{\partial x^n} + \frac{\partial g_{ln}}{\partial x^k}\right),$$

$$F_{ik} = \frac{mc^2}{e}\left(\frac{\partial g_k}{\partial x^i} - \frac{\partial g_i}{\partial x^k}\right),$$

which will have the obvious consequence of transforming the equation of motion into Lorentz form:

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} \pm \frac{e}{mc^2}g^{ni}F_{nk}\frac{dx^k}{ds} = 0\,. \tag{2.12′a}$$

4. We have:

$$\frac{d\Pi_5}{d\tau} = 0\,, \qquad \Pi_5 = \text{const.}$$

Thus, equation (2.12b) expresses the law of conservation of rest mass (and therefore charge) for fields that do not depend upon x^5, i.e., for any field in the classical limit $h \to 0$.

§ 10. The canonical equations in asymmetrical form

The canonical equations can be put into an asymmetrical form by eliminating the parameter τ. We then single out one of equations (2.11) (for instance, the one that relates to a coordinate that we shall denote by x^0):

$$\frac{dx^0}{d\tau} = \Pi^0 \tag{2.21}$$

and construct the rest of the system (2.11, 2.12) from it.

Taking into account that:

$$\frac{\partial H^*}{\partial x^\alpha} + \frac{\partial H^*}{\partial \Pi_0}\frac{\partial \Pi_0}{\partial x^\alpha} = 0\,, \qquad \frac{\partial H^*}{\partial \Pi_n} + \frac{\partial H^*}{\partial \Pi_0}\frac{\partial \Pi_0}{\partial \Pi_n} = 0\,, \tag{2.22}$$

this will give the system of ten canonical equations:

$$\frac{dx^n}{dx^0} = -\frac{\partial \Pi_0}{\partial \Pi_n} = \frac{\Pi^n}{\Pi^0}\,, \tag{2.23}$$

$$\frac{d\Pi_n}{dx^0} = \frac{\partial \Pi_0}{\partial x^n}\,, \tag{2.23a}$$

$$\frac{d\Pi_0}{dx^0} = \frac{\partial \Pi_0}{\partial x^0}\,, \tag{2.23b}$$

in which the field will be independent of any changes in the distinguished coordinate x^0, and will be linked with the "mechanical Hamiltonian function" Π_0, which is defined by solving the equation:

$$2H^* = G^{mn}\,\Pi_m\,\Pi_n + 2G^{n0}\,\Pi_n\,\Pi_0 + G^{00}\,\Pi_0^2 = 0\,, \tag{2.24}$$

namely:

$$\Pi_0 = -\frac{G^{0n}}{G^{00}}\Pi_n \pm \frac{1}{G^{00}}\sqrt{(G^{0n}G^{0m} - G^{00}G^{mn})\Pi_m\Pi_n}\,. \tag{2.25}$$

If we choose a completely arbitrary $d\Pi_0 / dx^0$ and take (2.23b) into account then we will find that:

$$\frac{d\Pi_0}{dx^0} = \frac{\partial \Pi_0}{\partial x^0} + \frac{\partial \Pi_0}{\partial x^n}\frac{\partial x^n}{\partial x^0} + \frac{\partial \Pi_0}{\partial \Pi^n}\frac{\partial \Pi^n}{\partial x^0} = \frac{\partial \Pi_0}{\partial x^0} \tag{2.26}$$

and conclude that equation (2.23b) implies equations (2.23) and (2.23a) as a consequence.

If we multiply equation (2.23) by Π_n then, taking into account that $\Pi_n\, dx^n + \Pi_0\, dx^0 \equiv 0$, we will find that:

$$\Pi_n\left(\frac{dx^n}{dx^0} + \frac{d\Pi_0}{d\Pi_n}\right) = -\,\Pi_0 - \Pi_n\frac{\Pi^n}{\Pi^0} \equiv 0\,. \tag{2.27}$$

Thus, only eight of the nine equations (2.23) are independent.

In mechanics, it is customary, although it is not necessary, to single out the time variable $x^0 = x^4 = ict$ to be the independent variable. In this case, $\Pi_0 = \Pi_4 = iH / c$, where

H is the mechanical Hamiltonian function. In this case, the system of eight independent equations (2.23) will take the form:

$$\left.\begin{array}{lll} \dfrac{dx^1}{dt}=\dfrac{\partial H}{\partial \Pi_1}, & \dfrac{dx^2}{dt}=\dfrac{\partial H}{\partial \Pi_2}, & \dfrac{dx^3}{dt}=\dfrac{\partial H}{\partial \Pi_3}, \\ \dfrac{d\Pi_1}{dt}=-\dfrac{\partial H}{\partial x^1}, & \dfrac{d\Pi_2}{dt}=-\dfrac{\partial H}{\partial x^2}, & \dfrac{d\Pi_3}{dt}=-\dfrac{\partial H}{\partial x^3}, \end{array}\right\} \tag{2.28}$$

which is well known in mechanics.

§ 11. The law of conservation of 5-impulse

In 5-optics, the laws of conservation of energy, impulse and charge combine into one law of conservation of 5-impulse, which is formulated in the following form:

If the external fields do not depend upon the fifth coordinate x^0 then that will be associated with the conservation of the impulse coordinate Π_0.

The law of conservation of 5-impulse immediately implies equation (2.12) as a consequence. In the classical limit ($h \to 0$), fields must not depend upon the fifth coordinate; therefore, charge can be taken to be a universal constant, although it is not an integral of the motion.

As we shall see, the law of conservation of charge in the usual sense has no place in wave-like 5-optics, in which it is necessary to consistently take into account fields that are periodically dependent upon the action coordinate.

We shall further see that the business of establishing new fundamental properties of particles (with a conserved e / m) for a single charge relates to whether it emits or absorbs a charged, massive quantum.

§ 12. Variational principles of mechanics

The geometrical meaning of action as the fifth coordinate of a distinguished test particle can be formulated in terms of the variational principles of mechanics. We show this in the following proposition.

As long as the fields $G_{\mu\nu}$ do not depend upon one of the five coordinates x^0, which will then define the first group of canonical equations:

$$\frac{dx^\mu}{d\tau}=\frac{\partial H^*}{\partial \Pi_\mu}=\Pi^\mu, \tag{2.29}$$

from the variational principle:

$$\delta\int dx^0 = 0, \tag{2.30}$$

we will obtain the second group of equations:

$$\frac{d\Pi_n}{d\tau} = -\frac{\partial H^*}{\partial x^n} . \tag{2.31}$$

We have:

$$\delta \int \frac{dx^0}{d\tau} d\tau = \delta \int \Pi^0 \, dt , \tag{2.31a}$$

where the selected coordinate x^0 (i.e., "the mechanical Lagrangian function," Π^0) is defined by the solution to the quadratic equation:

$$G_{mn} \Pi^m \Pi^n + 2G_{m0} \Pi^m \Pi^0 + G_{00} (\Pi^0)^2 = 0 , \tag{2.32}$$

$$\Pi^0 \equiv -\frac{G_{0n}}{G_{00}} \Pi^n \pm \frac{1}{G_{00}} \sqrt{(G_{n0} G_{m0} - G_{mn} G_{00}) \Pi^m \Pi^n} . \tag{2.33}$$

Compare formulas (2.25) and (2.33). The Euler equation (2.30) for the variational problem takes the form:

$$\frac{d}{d\tau}\left(\frac{\partial \Pi^0}{\partial \Pi^n}\right) - \frac{\partial \Pi^0}{\partial x^n} = 0 . \tag{2.34}$$

Note that:

$$\left.\begin{aligned} \frac{\partial L^*}{\partial x^n} + \frac{\partial L^*}{\partial \Pi^0}\frac{\partial \Pi^0}{\partial x^n} &= 0, \\ \frac{\partial L^*}{\partial \Pi^n} + \frac{\partial L^*}{\partial \Pi^0}\frac{\partial \Pi^0}{\partial \Pi^n} &= 0, \end{aligned}\right\} \tag{2.35}$$

gives:

$$\left.\begin{aligned} \frac{\partial \Pi^0}{\partial x^n} &= -\frac{1}{\Pi_0}\frac{\partial L^*}{\partial x^n} = \frac{1}{\Pi_0}\frac{\partial H^*}{\partial x^n}, \\ \frac{\partial \Pi^0}{\partial \Pi^n} &= -\frac{1}{\Pi_0}\frac{\partial L^*}{\partial \Pi^n} = \frac{1}{\Pi_0}\frac{\partial H^*}{\partial \Pi^n}. \end{aligned}\right\} \tag{2.36}$$

Substituting this expression in (2.34) and taking into account the condition that Π_0 = const. will give (2.31). The assertion is thus proved.

If the distinguished coordinate is the action $x^0 = x^5$ then (2.30) will indeed be Hamilton's principle; if the distinguished coordinate is time $x^0 = x^4 = ict$ then (2.30) will be Fermat's principle. Since all fields in geometrical 5-optics are independent of x^5 when $h \to 0$, Hamilton's principle can be regarded as a universal principle of classical mechanics, and Fermat's particular principle will be justifiably absent from this particular volume, in which the fields will not depend upon time.

CHAPTER III

CLASSICAL FIELD THEORY

§ 13. Metric field equations

In the last chapter, we showed that problems of five-dimensional geometrical optics that are concerned with the propagation of rays in a Riemannian configuration 5-space whose coordinates are space, time, and action, with a metric 5-tensor $G_{\mu\nu}$ that is subject to one condition – viz., that it should not depend upon the fifth coordinate – are equivalent to problems of classical relativistic mechanics for charged particles whose charge-to-mass ratio is e / m that move in a gravitational field and an electromagnetic field:

$$g_{ik} = \frac{G_{ik}}{G_{55}} - \frac{G_{i5}}{G_{55}} \cdot \frac{G_{k5}}{G_{55}}, \qquad A_i = \frac{mc^2}{e} g_i = \frac{mc^2}{e} \frac{G_{i5}}{G_{55}}. \tag{3.1}$$

In this chapter, we shall consider problems that involve a particular metric field $G_{\mu\nu}$ whose source field is $Q_{\mu\nu}$. We recall that in the general theory of metric fields $G_{\mu\nu}$ there are special cases for which the unified theory of gravity and electricity is summarized by the Einstein equation for 5-spaces whose coordinates are space, time, and action:

$$P_{\lambda\mu} - \tfrac{1}{2} G_{\lambda\mu} P = \kappa Q_{\lambda\mu} \ , \tag{3.2}$$

in which $P_{\lambda\mu}$ is the Riemann curvature 5-tensor.

In contrast to the 5-eikonal equation, the fifteen equations (3.2) are inhomogeneous in the potentials $G_{\mu\nu}$, and thus, in order to be consistent, we cannot arbitrarily assume that $G_{55} = 1$, as we did in the previous chapter.

If we introduce the notation $G_{55} = N$ then we will, in turn, substitute the metric tensor that is defined by the formula:

$$G_{\mu\nu} = \begin{pmatrix} N(g_{ik} + g_i g_k) & N g_k \\ N g_i & N \end{pmatrix}; \qquad G^{\mu\nu} = \begin{pmatrix} \dfrac{g_{ik}}{N} & -\dfrac{g^{ik} g_k}{N} \\ -\dfrac{g^{ik} g_i}{N} & \dfrac{1}{N} + \dfrac{1}{N} g^{ik} g_i g_k \end{pmatrix} \tag{3.3}$$

in (3.2), and then determine the electromagnetic fields g_i that figure in electromagnetism from equations (3.2), as well as the gravitational fields g_{ik} that figure in the theory of gravity, and all of them will be normalized by the factor $N = G_{55}$.

Furthermore, in problems of geometrical 5-optics, we can describe the motion of material particles in an external field as an optical process that involves the propagation of rays in a five-dimensional *configuration* space whose coordinates are space, time, and action.

For that matter, we see that in order for us to come into agreement with experiment in the solution of the field equations (3.2), we must multiply the magnitude $\frac{mc^2}{e}$ that figures in the expression (3.1) by the universal constant $\sqrt{\frac{2\pi}{\kappa}}$, where κ is the gravitational constant.

Therefore, in equations (3.2), with source $Q_{\lambda\mu}$ and metric potentials $G_{\lambda\mu}$, we apply the formulas:

$$g_{ik} = \frac{G_{ik}}{G_{55}} - \frac{G_{i5}}{G_{55}} \cdot \frac{G_{k5}}{G_{55}}, \qquad A_i = \sqrt{\frac{2\pi}{\kappa}}\, g_i = \sqrt{\frac{2\pi}{\kappa}}\, \frac{G_{i5}}{G_{55}} \tag{3.1a}$$

in order to determine the gravitational metric field and the electromagnetic field for the excitation source $Q_{\lambda\mu}$.

For that matter, in order to clearly emphasize the situation that we have created, we note that we have introduced two details in the form of a configuration space with a metric that combines the potentials g_{ik} and A_i by means of formulas (3.1) and an additional *fundamental* space with a metric that combines the potentials g_{ik} , A_i by means of formulas (3.1a).

Since the two problems:

a) determining the motion of material particles in an external field (configuration space)

and

b) determining the construction of fields with given sources (fundamental space) in the considered approximation

are sufficiently distinct, we will not introduce any special notation for the metric potentials in configuration space and fundamental space, and we shall preserve the notations $g_i = \frac{e}{mc^2} A_i$ or $g_i = \sqrt{\frac{\kappa}{2\pi}}\, A_i$, resp., depending on which problem we are solving.

In this chapter, we shall consider the resulting classical theory of fields, and then in the limiting case $h \to 0$ we will impose the usual cylindricality on the metric potentials $G_{\mu\nu}$ in equations (3.2), viz., to account for its independence of the fifth action coordinate.

We then pass on to the four-dimensional form of the field equations (3.2) and single out the action coordinate with due care, in order that the remaining equations should be gradient invariant. In accordance with what we decreed in § 6 concerning the rules of gradient-invariant formulas, the field equations (3.2) will take the following form:

$$\left.\begin{aligned} P^{ik} - \tfrac{1}{2} G^{ik} P &= \kappa Q^{ik}, \\ P_5^k \qquad\quad &= \kappa Q_5^k, \\ P_{55} - \tfrac{1}{2} G_{55} P &= \kappa Q_{55}. \end{aligned}\right\} \tag{3.4}$$

§ 14. True and effective gravitational potentials; χ-field

We have the following identity for the differential of arc length $d\sigma$ in a 5-space:

$$
\begin{aligned}
+ d\sigma^2 &= - G_{\mu\nu}\, dx^\mu\, dx^\nu \\
&= - G_{ik}\, dx^i\, dx^k - 2G_{i5}\, dx^i\, dx^5 - G_{55}\, (dx^5)^2 \\
&= - G_{55} \left(\frac{G_{ik}}{G_{55}} - \frac{G_{i5}}{G_{55}} \cdot \frac{G_{5k}}{G_{55}} \right) dx^i\, dx^k - \frac{1}{G_{55}} (G_{i5}\, dx^i + G_{55}\, dx^5)^2 \\
&= - N\, g_{ik}\, dx^i\, dx^k - (dx^5)^2,
\end{aligned}
\tag{3.5}
$$

in which we have made use of the formula:

$$dx_5 = G_{5i}\, dx^i + G_{55}\, dx^5 \, .$$

The expression (3.5) gives a gradient-invariant decomposition of 5-space into a 4-space of position-time coordinates and a linear space that is orthogonal to it. The formula:

$$ds^2 = - N\, g_{ik}\, dx^i\, dx^k \tag{3.6}$$

establishes a metric in the gradient-invariant 4-subspace. We introduce the *true* gravitational potentials into 4-space by way of the formulas:

$$\bar{g}_{ik} = N\, g_{ik} , \qquad \bar{g}^{ik} = \frac{1}{N} g^{ik} , \tag{3.7}$$

which determines the true interval between two events, and the conventional gravitational potentials g_{ik} , g^{ik} that figure in the theory of gravity, which we will call the *effective* gravitational potentials.

For the purposes of this volume, the true and effective gravitational potentials will coincide in the particular case when $N = G_{55} = 1$.

The effective gravitational potentials g_{ik} that figure in the theory of gravity do not have a simple geometrical meaning in a five-dimensional space, and therefore we expect that the field equation (3.2) will acquire a clearer sense as long as we make use of the true gravitational potentials $\bar{g}_{ik}$ and $\bar{g}^{ik}$. Therefore, we introduce the notation:

$$G_{55} = N = 1 + \chi , \tag{3.8}$$

and note, while paying attention to (3.7), that the expression for the metric potentials $G_{\mu\nu}$ in formula (3.3) takes the form:

$$\left.\begin{aligned} G_{\mu\nu} &= \begin{pmatrix} \overline{g}_{ik} + (1+\chi) g_i g_k & (1+\chi) g_k \\ (1+\chi) g_i & 1+\chi \end{pmatrix}; \\ G^{\mu\nu} &= \begin{pmatrix} \overline{g}^{ik} & -\overline{g}^{ik} g_k \\ -\overline{g}^{ik} g_i & \dfrac{1}{1+\chi} + \overline{g}^{ik} g_i g_k \end{pmatrix}. \end{aligned}\right\} \tag{3.9}$$

We see that in 5-optics, along with gravitational and electromagnetic fields of contemporary physics, there appears a new scalar χ-field that is linked with the potential G_{55} by formula (3.8).

At the same time, why is this χ-field not evident in nature? As far as that is concerned, in the limit of geometrical 5-optics (i.e., classical mechanics) the influences of the true gravitational field $\tilde{g}_{ik}$ and the χ-field on test particles cannot be separated from each other, since they both influence the expression for the effective gravitational potential $g^{ik} = N\tilde{g}^{ik}$, which is the only one that figures in the formulas of classical mechanics. In other words, in the problems of geometrical 5-optics (i.e., classical mechanics), the metric fields $\{\tilde{g}^{ik}, g_k, \chi\}$ are completely equivalent to the metric fields $\{g^{ik}, g_k, 0\}$. However, this equivalence will not have much significance when we pass to wave-like 5-optics, since the equations of wave-like 5-optics are certainly inhomogeneous in the potentials $G_{\mu\nu}$.

What is more, we see that disregarding the χ-field in classical field theory will give the conventional theory of gravity, for instance, in problems that are concerned with the fields of charged point-like masses, which is unacceptable.

§ 15. Harmonic coordinate systems

We present equation (3.4) in a special harmonic coordinate system, with the same success that many researchers (cf. [**9**]) have found.

In a harmonic coordinate system, we assume that the metric 5-tensor $G^{\mu\nu}$ satisfies the following condition:

$$\frac{\partial(\Delta G^{\mu\nu})}{\partial x^{\nu}} = 0\,, \tag{3.10}$$

into which we have introduced the notation:

$$\Delta = \sqrt{|\,\mathrm{Det}(G_{\mu\nu})\,|}\,. \tag{3.11}$$

We remark that $\Delta = \sqrt{|\,\tilde{g}\,|\,(1+\chi)}$, and if we single out the action coordinate then we can write condition (3.10) in the form:

$$\frac{\partial(\sqrt{(1+\chi)}\sqrt{|\tilde g|}\tilde g^{ik})}{\partial x^k} = 0\ , \tag{3.12a}$$

$$\frac{\partial(\sqrt{(1+\chi)}\sqrt{|\tilde g|}\tilde g^{ik} g_k)}{\partial x^i} = 0\ . \tag{3.12b}$$

Making note of (3.12a), the condition (3.12b) can be expressed in the form:

$$\tilde g^{ik}\frac{\partial g_k}{\partial x^i} = 0\ . \tag{3.12c}$$

We remark that the introduction of harmonic coordinate systems into 5-space will provide a generalization of the Lorentz normalization of the electromagnetic potentials that is used in electrodynamics to the case of the metric field $G_{\mu\nu}$.

We will give formulas for P, P^{ik}, P_5^i, P_{55} in a harmonic coordinate system in a mathematical appendix. If we direct our attention to fields that are independent of the action coordinate then we will get these formulas:

$$\left.\begin{aligned}
P &= G^{ik}\frac{\partial^2 \ln\sqrt{|\tilde g|(1+\chi)}}{\partial x^i \partial x^k} + \frac{1}{2}G^l_{ik}\frac{\partial G^{ik}}{\partial x^l} + G^l_{i5}\frac{\partial G^{i5}}{\partial x^l} + \frac{1}{2}G^l_{55}\frac{\partial G^{55}}{\partial x^l},\\
P^{ik} &= -\frac{1}{2}G^{mn}\frac{\partial^2 G^{ik}}{\partial x^i \partial x^k} + G^l_{ik}G^{k,mn} + 2G^l_{m5}G^{k,m5} + G^l_{55}G^{k,55},\\
P_5^i &= G^{mn}\frac{\partial G^i_{m5}}{\partial x^n} + G^{k5}\frac{\partial G^i_{55}}{\partial x^k},\\
P_{55} &= \frac{1}{2}G^{mn}\frac{\partial^2 G_{55}}{\partial x^m \partial x^n} - G^{i\alpha}G^\beta_{\alpha 5}\frac{\partial G_{5\beta}}{\partial x^i}.
\end{aligned}\right\} \tag{3.13}$$

We introduce the Christoffel symbols for 5-space by way of:

$$G^\alpha_{\mu\nu}\ ; \quad G^{\alpha,\mu\nu} = G^{\mu\tau}G^{\nu\sigma}G^\alpha_{\tau\sigma}\ .$$

§ 16. Form of the field equations

The calculation of the expressions in formula (3.13) is associated with straightforward, but tedious, computations. We can make an essential simplification, as long as the calculation of the Christoffel symbols preserves the character of the terms that are known to be gradient-invariant – i.e., the terms that do not involve the quantities g_i, but only their derivatives – since the remaining terms, which we will not write out explicitly, will all cancel each other when they are substituted in the expression (3.13) (*).

(*) Cf., the analogous method in [**12**] (§ 94, problems).

If we introduce the symbol "≅" to mean "point-wise equal to a gradient-invariant term," then we will have:

$$\left.\begin{array}{lll} G^i_{kl} \cong \tilde{\Gamma}^i_{kl}; & G^i_{m5} \cong \frac{1}{2}\tilde{g}^{ik} f_{mk}(1+\chi); & G^i_{55} \cong -\frac{1}{2}\tilde{g}^{ik}\frac{\partial\chi}{\partial x^k}, \\ G^{i,kl} \cong \tilde{\Gamma}^{i,kl}; & G^{i,m5} \cong \frac{1}{2} f^{mi}; & G^{i,55} \cong \frac{1}{2(1+\chi)^2}\tilde{g}^{ik}\frac{\partial\chi}{\partial x^k}, \\ G^5_{i5} \cong \frac{1}{2(1+\chi)}\tilde{g}^{ik}\frac{\partial\chi}{\partial x^k} & G^5_{i5} \cong 0; & f_{ik} = \frac{\partial g_k}{\partial x^i} - \frac{\partial g_i}{\partial x^k}. \end{array}\right\} \tag{3.14}$$

If we introduce the Riemann curvature tensor and scalar curvature for 4-space by way of $\tilde{R}^{ik}$ and $\tilde{R}$, which are constructed from $\tilde{g}_{ik}$, substitute them in the expression (3.14) and formula (3.13), and again discard the resulting terms that contain the non-gradient invariant quantities g_i then we will obtain (cf., Mathematical Appendix, note 5):

$$\left.\begin{aligned} P &= \left\{\tilde{R} + \frac{\partial\tilde{\Gamma}^i}{\partial x^i} + \tilde{\Gamma}^i_{ik}\tilde{\Gamma}^k\right\} + \frac{1}{4}(1+\chi) f_{ik} f^{ik} + \tilde{g}^{ik}\frac{\partial^2 \ln\sqrt{1+\chi}}{\partial x^i \partial x^k} + \frac{1}{4}\frac{\tilde{g}^{ik}}{(1+\chi)^2}\frac{\partial\chi}{\partial x^i}\frac{\partial\chi}{\partial x^k} \\ &= \tilde{R} + \frac{1}{4}(1+\chi) f_{ik} f^{ik} + \frac{\tilde{g}^{ik}}{1+\chi}\frac{\partial^2\chi}{\partial x^i\partial x^k} - \frac{\tilde{g}^{ik}}{(1+\chi)^2}\frac{\partial\chi}{\partial x^i}\frac{\partial\chi}{\partial x^k}, \\ P^{ik} &= \left\{\tilde{R}^{ik} + \tilde{g}^{is}\left(\frac{\partial\tilde{\Gamma}^i}{\partial x^i} + \tilde{\Gamma}^i_{ik}\tilde{\Gamma}^k\right)\right\} + \frac{1}{2}(1+\chi)\tilde{g}_{mn} f^{im} f^{kn} - \frac{1}{4}\frac{\tilde{g}^{in}\tilde{g}^{im}}{(1+\chi)^2}\frac{\partial\chi}{\partial x^i}\frac{\partial\chi}{\partial x^k} \\ &= \tilde{R}^{ik} + \frac{1}{2}(1+\chi)\tilde{g}_{mn} f^{im} f^{kn} + \frac{1}{2}\frac{\tilde{g}^{in}\tilde{g}^{im}}{1+\chi}\left(\frac{\partial^2\chi}{\partial x^i\partial x^k} - \frac{3}{2}\frac{1}{(1+\chi)}\frac{\partial\chi}{\partial x^n}\frac{\partial\chi}{\partial x^m}\right) - \frac{1}{2}\frac{\tilde{\Gamma}^{m,ik}}{(1+\chi)}\frac{\partial\chi}{\partial x^m}, \\ P^i_5 &= \frac{1}{2}\tilde{g}^{mn}\frac{\partial}{\partial x^n}(1+\chi)\tilde{g}^{ik} f_{mk}, \\ P_{55} &= \frac{1}{2}\tilde{g}^{ik}\frac{\partial^2\chi}{\partial x^i\partial x^k} - \frac{1}{4}(1+\chi)^2 f_{ik} f^{ik} - \frac{1}{2}\frac{\tilde{g}^{ik}}{(1+\chi)}\frac{\partial\chi}{\partial x^i}\frac{\partial\chi}{\partial x^k}. \end{aligned}\right\}$$

(3.13′)

If we substitute this expression into the field equation (3.4) then we will obtain, using formula (3.12):

$$\left\{\tilde{R}^{ik} - \frac{1}{2} g^{ik}\tilde{R}\right\} + \frac{1}{2}(1+\chi)\left\{\tilde{g}_{mn} f^{im} f^{kn} - \frac{1}{4}\tilde{g}^{ik} f_{mn} f^{mn}\right\}$$
$$+ \frac{1}{2(1+\chi)}\left\{\left[\tilde{g}^{in}\tilde{g}^{km}\left(\frac{\partial^2\chi}{\partial x^n\partial x^m} - \frac{3}{2(1+\chi)}\frac{\partial\chi}{\partial x^n}\frac{\partial\chi}{\partial x^m}\right) - \tilde{\Gamma}^{m.ik}\frac{\partial\chi}{\partial x^m}\right]\right.$$

$$-g^{ik}\left[\frac{\partial^2\chi}{\partial x^i\partial x^k}-\frac{1}{(1+\chi)}\frac{\partial\chi}{\partial x^m}\frac{\partial\chi}{\partial x^n}\right]\Bigg\} = \kappa Q^{ik}\,, \qquad (3.15^{\dagger})$$

$$\frac{1}{2}\frac{1}{\sqrt{|\tilde{g}|(1+\chi)}}\frac{\partial}{\partial x^k}\left\{\sqrt{|\tilde{g}|}(1+\chi)^{\frac{3}{2}}f^{ki}\right\}=\kappa Q_5^i\,, \qquad (3.16)$$

$$-\frac{1}{2}(1+\chi)\left\{\tilde{R}+\frac{3}{4}(1+\chi)\,f_{ik}\,f^{ik}\right\}= \kappa Q_{55}. \qquad (3.17)$$

These equations justify the introduction of a harmonic coordinate system, with additional normalization conditions in the form of equations (3.12a) and (3.12b).

§ 17. Comparison with classical field theory

For the sake of comparing the field equations that we obtained with the field equations of gravity and electromagnetism in contemporary physics, which disregard the χ-field, it will follow from the assumption $\chi = 0$ that $\tilde{g}_{ik} = g_{ik}$, $\tilde{R}^{ik} = R^{ik}$.

We obtain:

$$R^{ik} - \tfrac{1}{2}g^{ik}\,R + \tfrac{1}{2}\left\{g_{mn}\,f^{im}\,f^{kn} - \tfrac{1}{4}g^{ik}\,f_{mn}\,f^{mn}\right\} = \kappa Q^{ik}, \qquad (3.15a)$$

$$\frac{1}{2}\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^k}\left(\sqrt{|g|}\,f^{ki}\right)= \kappa Q_5^i\,, \qquad (3.16a)$$

$$-\frac{3}{8}\,f_{ik}\,f^{ik}-\frac{1}{2}R= \kappa Q_{55}\,. \qquad (3.17a)$$

We add the normalization conditions:

$$\frac{\partial(\sqrt{|g|}\,g^{ki})}{\partial x^k} = 0, \qquad \frac{\partial(\sqrt{|g|}\,g^{i})}{\partial x^i}= 0 \qquad (3.18)$$

to these results.

In essence, equations (3.15a) and (3.16a) are precisely the Maxwell-Einstein equations of the unified theory of gravity and electricity when we set:

$$g_i = \sqrt{\frac{\kappa}{2\pi}}\,A_i\,, \qquad Q_5^i = \frac{1}{c}\sqrt{\frac{2\pi}{\kappa}}\,s^i, \qquad (3.19)$$

in which s^i is the current 4-vector, while A^i is the electromagnetic potential 4-vector.

(†) Translator’s note: [*sic*] (The indices in the last term are not consistent with the ones on the right-hand side of the equation.)

Equation (3.17a) serves to subsequently determine the source 5-tensor component Q_{55}, which cannot be posed independently. Thus, equation (3.17a) is indeed the condition for neglecting the χ-field.

We convince ourselves that it is necessary for the solution of problems in the determination of fields with a given source to go over to the fundamental space, in which the metric tensor takes the universal form:

$$G^{\mu\nu} = \begin{pmatrix} \tilde{g}^{ik} & -\sqrt{\dfrac{\kappa}{2\pi}} A^i \\ -\sqrt{\dfrac{\kappa}{2\pi}} A^k & \dfrac{1}{1+\chi} + \dfrac{\kappa}{2\pi} A_i A^i \end{pmatrix},$$

whereas in the case of configuration space, in which that tensor will depend upon the ratio *e* / *m* for a material point, that motion will describe the optical process that takes the form of the propagation of rays in configuration 5-space.

§ 18. Energy and impulse of a field source

The characteristic feature of 5-optics is the possibility of duplicating the general formulas that were obtained in the theory of gravity without reaching any new deductions; in particular, one duplicates all of the formulas that were obtained as mathematical consequences of Einstein's equations for gravitational fields.

In the theory of gravity ([**12**], § 98), it is proved that the 4-impulse of matter (including electromagnetic fields) that is contained inside of a closed surface is determined by the values of the gravitational potentials and their first derivatives on this surface and can be expressed by means of surface integrals:

$$P^k = \frac{1}{2\kappa c} \int \frac{\partial}{\partial x^i} \{| g | (g^{k\alpha} g^{4i} - g^{k4} g^{\alpha i} \} \, df_\alpha , \tag{3.20}$$

in which α takes on the values $\alpha = 1, 2, 3$ and df_α signifies the surface element. As far as formula (3.20) is concerned, it is a mathematical consequence of the Einstein equations, so we can carry it over into 5-optics and write it in the form:

$$\begin{aligned} P^k &= \frac{1}{2\kappa c} \int \frac{\partial}{\partial x^i} \{| G | (G^{k\alpha} G^{4i} - G^{k4} G^{\alpha i} \} \, df_\alpha \\ &= \frac{1}{2\kappa c} \int \frac{\partial}{\partial x^i} \{| g | (1 + \chi) (g^{k\alpha} g^{4i} - g^{k4} g^{\alpha i} \} \, df_\alpha . \end{aligned} \tag{3.21}$$

We retain the four-dimensional divergence in formula (3.21), in place of the correct five-dimensional one, since the potentials will not depend upon the fifth action coordinate in the classical approximation.

We see that in that case, since the χ-field is absent, the 5-optics formula (3.21) will agree with formula (3.20) from the theory of gravity. In the general case, the 4-impulse that is contained within a closed surface is determined by not only the values of the gravitational potentials and their derivatives, but also the values of the χ-potential and its derivatives on the surface.

§ 19. The Schwarzschild problem for 5-space

We find static, spherically-symmetric solutions to the equations for the metric field *in vacuo*, so $Q_{\mu\nu} = 0$. The objects (G_{14}, G_{24}, G_{34}) and (G_{15}, G_{25}, G_{35}) can be regarded as two 3-vectors that must be equal to zero, for symmetry reasons. Therefore, the remaining components can be written as the tensor $G_{\mu\nu}$ in the form:

$$G_{\mu\nu} = \begin{pmatrix} \delta_{ik} + (e^{\nu} - 1) n_i n_k & 0 & 0 \\ 0 & e^{\mu} + e^{\lambda} g^2 & e^{\lambda} g \\ 0 & e^{\lambda} g & e^{\lambda} \end{pmatrix}. \tag{3.22}$$

In this section, Latin indices will take on the values 1, 2, 3, n_i will be a unit 3-vector, and λ, μ, ν, g will be four functions of the radius vector that vanish at infinity; moreover, g will be completely imaginary.

Calculation of the contravariant object gives:

$$G^{\mu\nu} = \begin{pmatrix} \delta_{ik} + (e^{-\nu} - 1) n_i n_k & 0 & 0 \\ 0 & e^{-\mu} & -e^{-\mu} g \\ 0 & -e^{-\mu} g & -e^{-\lambda} + -e^{-\mu} g^2 \end{pmatrix}. \tag{3.23}$$

We can derive the field equation from the variational principle:

$$\delta \int L \, 4\pi r^2 \, dr = 0 , \tag{3.24}$$

in which L is the Lagrangian function, which we imitate from the theory of gravity:

$$\begin{aligned} L &= \sqrt{|G|}\, G^{\alpha\beta} (G^{\sigma}_{\alpha\beta} G^{\tau}_{\sigma\tau} - G^{\sigma}_{\alpha\tau} G^{\tau}_{\sigma\beta}) \\ &= \sqrt{|G|} \left(\frac{1}{2} G^{i}_{\alpha\beta} \frac{\partial G^{\alpha\beta}}{\partial x^i} - G^{ik} \frac{\partial \ln \sqrt{|G|}}{\partial x^i} \frac{\partial \ln \sqrt{|G|}}{\partial x^k} - \frac{\partial \ln \sqrt{|G|}}{\partial x^i} \frac{\partial G^{ik}}{\partial x^k} \right). \end{aligned} \tag{3.25}$$

Calculation gives:

$$\left.\begin{aligned}
G^i_{kl} &= \left[\frac{1}{2}\nu' n_k n_l + \frac{1-e^{-\nu}}{r}(\delta_{kl} - n_k n_l)\right] n_i, \\
G^i_{44} &= -\frac{1}{2}e^{\lambda-\nu}(e^{\mu-\nu}\mu' + g^2\lambda' + 2gg')n_i, \\
G^i_{45} &= -\frac{1}{2}e^{\lambda-\nu}(\lambda' g + g')n_i, \\
G^i_{55} &= -\frac{1}{2}e^{\lambda-\nu}n_i, \\
\sqrt{|G|} &= \exp(\lambda+\mu+\nu).
\end{aligned}\right\} \tag{3.26}$$

Substituting these equations into (3.25) will give, after some tedious calculations:

$$L = \exp\left(\frac{\lambda+\mu-\nu}{2}\right)\left\{\frac{e^{\nu}-1}{r}(\lambda'+\mu'+\nu') - \frac{1}{2}\lambda'\mu' + \frac{1}{2}e^{\lambda-\mu}g'^2\right\}. \tag{3.27}$$

If we put this into (3.24) with the new variable $u = \frac{1}{r}$ then we will obtain:

$$\delta\int_0^\infty \exp\left(\frac{\lambda+\mu-\nu}{2}\right)\left\{\frac{e^{\nu}-1}{u}(\dot\lambda+\dot\mu+\dot\nu) + \frac{1}{2}\dot\lambda\dot\mu - \frac{1}{2}e^{\lambda-\mu}\dot g^2\right\}du = 0\,, \tag{3.28}$$

in which a dot denotes differentiation by u.

The field equations are the Euler equations for the variational problem (3.28) and take the form:

$$\left.\begin{aligned}
\frac{e^{\nu}-1}{u^2} + \frac{\dot\lambda+\dot\mu}{u} - \frac{\dot\lambda\dot\mu}{4} + \frac{1}{4}e^{\lambda-\mu}\dot g^2 &= 0, \\
\frac{e^{\nu}-1}{u^2} - \frac{\dot\nu}{u} + \frac{\dot\nu\dot\lambda}{4} - \frac{\ddot\lambda}{2} - \frac{\dot\lambda^2}{4} + \frac{1}{4}e^{\lambda-\mu}\dot g^2 &= 0, \\
\frac{e^{\nu}-1}{u^2} - \frac{\dot\nu}{u} + \frac{\dot\nu\dot\mu}{4} - \frac{\ddot\mu}{2} - \frac{3}{4}e^{\lambda-\mu}\dot g^2 &= 0, \\
\frac{d}{du}\left[\dot g\exp\left(\frac{3\lambda-\mu-\nu}{2}\right)\right] &= 0.
\end{aligned}\right\} \tag{3.29}$$

Integrating the resulting equations gives:

$$\dot g = i\varepsilon\exp\left(\frac{\mu+\nu-3\lambda}{2}\right), \tag{3.30}$$

in which ε is a material constant, and has the unit of length. Eliminating the function $\dot{g}$ from equations (3.29) and (3.30) will give the following three equations for the determination of the three functions λ, μ, ν:

$$\left.\begin{aligned} C_1 &\equiv \frac{e^\nu - 1}{u^2} + \frac{\dot\lambda + \dot\mu}{u} - \frac{\dot\lambda\dot\mu}{4} - \frac{\varepsilon^2}{4} e^{\nu - 2\lambda} = 0, \\ C_2 &\equiv \frac{e^\nu - 1}{u^2} - \frac{\dot\nu}{u} + \frac{\dot\nu\dot\lambda}{4} - \frac{\varepsilon^2}{4} e^{\nu-2\lambda} - \frac{\ddot\lambda}{2} - \frac{\dot\lambda^2}{4} = 0, \\ C_3 &\equiv \frac{e^\nu - 1}{u^2} - \frac{\dot\nu}{u} + \frac{\nu\dot\lambda}{4} - \frac{3}{4}\varepsilon^2 e^{\nu-2\lambda} - \frac{\ddot\mu}{2} - \frac{\dot\mu^2}{4} = 0. \end{aligned}\right\} \tag{3.31}$$

For the sake of solving this system more easily, we transform to the following linear combinations:

$$\left.\begin{aligned} C_1 - C_2 &\equiv \frac{\dot\lambda + \dot\mu + \dot\nu}{u} - \frac{\dot\lambda(\dot\nu + \dot\mu)}{4} + \frac{\ddot\lambda}{2} + \frac{\dot\lambda^2}{4} = 0, \\ 2C_1 + C_2 + C_3 &\equiv \frac{2(\dot\lambda + \dot\mu + \dot\nu)}{u} + \frac{\dot\nu(\dot\lambda + \dot\mu)}{4} - \frac{(\ddot\lambda + \ddot\mu)}{2} - \frac{(\dot\lambda + \dot\mu)^2}{4} + 4\left(\frac{e^\nu - 1}{u^2} - \frac{\dot\nu}{u}\right) = 0, \\ C_1 - C_3 &\equiv \frac{\dot\lambda + \dot\mu + \dot\nu}{u} - \frac{\dot\mu(\dot\lambda + \dot\nu)}{4} + \frac{\ddot\mu}{2} + \frac{\dot\mu^2}{4} - \varepsilon^2 e^{\nu-2\lambda} = 0. \end{aligned}\right\} \tag{3.32}$$

For the sake of later calculations, we choose a special system of coordinates (*), for which we impose the following condition on the metric potentials:

$$|\,G\,| = 1\,. \tag{3.33}$$

In such a special coordinate system, one will have, from (3.26):

$$\lambda + \mu + \nu = 0, \tag{3.33$'$}$$

and therefore, from formula (3.32):

$$\left.\begin{aligned} C_1 - C_2 &\equiv \frac{1}{2}(\ddot\lambda + \dot\lambda^2) = 0, \\ 2C_1 + C_2 + C_3 &\equiv 4\left(\frac{e^\nu - 1}{u^2} - \frac{\dot\nu}{u}\right) + \frac{1}{2}(\ddot\nu - \dot\nu^2) = 0, \end{aligned}\right\} \tag{3.34}$$

$$C_1 - C_3 \equiv \frac{1}{2}\{(\dot\lambda + \dot\nu)^2 - (\ddot\lambda + \ddot\nu)\} - \varepsilon^2 e^{\nu-2\lambda} = 0\,, \tag{3.35}$$

$$\dot g = i\varepsilon\, e^{-2\lambda}\,. \tag{3.36}$$

Integration of the system (3.34) gives:

(*) On the possibility of making such a choice, cf., [**15**], § 49.

$$e^{\lambda} = 1 + \alpha\mu\ ; \qquad e^{\nu} = \frac{1}{1-\beta u}, \tag{3.37}$$

in which α, β are two integration constants that each have the unit of length.

Substituting (3.37) into $(3.33')$ and (3.35) gives:

$$c^{\mu} = \frac{1-\beta u}{1+\alpha u}, \tag{3.38}$$

$$\alpha^2 + \alpha\beta - \varepsilon^2 = 0\ . \tag{3.39}$$

Integrating (3.36) gives:

$$g = \frac{i\varepsilon u}{1+\alpha u}\ . \tag{3.40}$$

We will then obtain the following solution, which is based upon (3.39), with its two arbitrary constants:

$$\left.\begin{aligned} & e^{\nu} = \frac{1}{1-\dfrac{\beta}{r}}, \qquad e^{\mu} = \frac{1-\dfrac{\beta}{r}}{1+\dfrac{\alpha}{r}}, \quad e^{\lambda} = 1+\frac{\alpha}{r}, \\ & A_4 = \sqrt{\frac{2\pi}{\kappa}}\,g = \sqrt{\frac{2\pi}{\kappa}}\,\frac{i\varepsilon}{r+\alpha}. \end{aligned}\right\} \tag{3.41}$$

We now go on to the calculation of the constants α, β, and ε. For large values of r, the potential A_4 must obey Coulomb's law $A_4 = \dfrac{e't}{r}$, in which e' is the source charge. Therefore, the constant ε will define the electronic radius of the source:

$$\varepsilon = \sqrt{\frac{\kappa}{2\pi}}\,e'. \tag{3.42}$$

Further calculation that is based upon formula (3.21) for the energy of a charged point-like mass and equating it with $m'c^2$ will give:

$$m'c^2 = icP^4 = -\frac{1}{2\kappa}\int \frac{\partial}{\partial x^{\alpha}}\ \{|\,G\,|\,G^{-\beta}\,G^{44}\}\ df_{\beta}\,. \tag{3.43}$$

We integrate this over a sphere of radius R, which then goes to infinity.

Formulas $(3.22')$, (3.26), (3.41) then give:

$$|\,G\,| = e^{\lambda+\mu+\nu} = 1\ ,$$

$$G^{-\beta} = \delta_{\alpha\beta} + (e^{-\nu} - 1)\, n_\alpha n_\beta = \delta_{\alpha\beta} - \frac{\beta}{R} n_\alpha n_\beta, \quad \frac{\partial G^{\alpha\beta}}{\partial x^\alpha} = -\frac{\beta}{R^2} n_\beta,$$

$$G^{44} = e^{-\mu} = \frac{1+\dfrac{\alpha}{R}}{1-\dfrac{\beta}{R}} \cong 1 + \frac{\alpha+\beta}{R}, \qquad \frac{\partial G^{44}}{\partial x^\alpha} = -\frac{(\alpha+\beta)}{R^2} n_\alpha,$$

and therefore:

$$m'c^2 = -\frac{1}{2\pi}\left\{4\pi R^2 \left[\frac{\partial}{\partial R}(|G|\, G^{\alpha\beta} G^{44}\right] n_\alpha n_\beta\right\}_{R\to\infty} = \frac{2\pi}{\kappa}(\alpha + 2\beta). \qquad (3.43')$$

We let:

$$\gamma = \frac{\kappa}{2\pi}\frac{m'c^2}{2}$$

denote the gravitational radius that corresponds to the mass m' and obtain from (3.43′) :

$$\alpha = 2(\gamma - \beta). \qquad (3.44)$$

Substituting this expression in (3.39) will give the determination of the constant β by way of the quadratic equation:

$$4(\gamma - \beta)^2 + 2(\gamma - \beta)\,\beta - \varepsilon^2 = 0\ . \qquad (3.45)$$

Solving this equation and substituting the expression for β in (3.44) will give the expressions for the constants α and β in terms of the gravitational radius γ and the electron radius ε:

$$\left.\begin{aligned} \alpha &= \left\{\sqrt{1+2\left(\frac{\varepsilon}{\gamma}\right)^2} - 1\right\}\gamma, \\ \beta &= \left\{\frac{3}{2} - \frac{1}{2}\sqrt{1+2\left(\frac{\varepsilon}{\gamma}\right)^2}\right\}\gamma. \end{aligned}\right\} \qquad (3.46)$$

The sign of the root was chosen so that when $\varepsilon \to 0$ the solution (3.41) would go to the classical Schwarzschild solution for the field of a point-like neutral mass:

$$e^\nu = \frac{1}{1-\dfrac{\gamma}{r}}, \qquad e^\mu = 1 - \frac{\gamma}{r}, \qquad e^\lambda = 1, \qquad A_4 = 0\ . \qquad (3.41')$$

This will allow us to neglect the gravitational radius γ in comparison with the electron radius ε when:

$$\left.\begin{aligned} \beta &= -\frac{1}{\sqrt{2}}|\varepsilon|; \\ \alpha &= \sqrt{2}|\varepsilon|, \end{aligned}\right\} \tag{3.46′}$$

and we will obtain the solution:

$$e^{\nu} = \frac{1}{1+\frac{|\varepsilon|}{\sqrt{2}\,r}}; \qquad e^{\mu} = \frac{1+\frac{|\varepsilon|}{\sqrt{2}\,r}}{1+\frac{\sqrt{2}|\varepsilon|}{r}};$$

$$e^{\lambda} = 1+\frac{\sqrt{2}|\varepsilon|}{r}; \qquad A_4 = \frac{ie'}{r+\sqrt{2}|\varepsilon|} \tag{4.41″}$$

that corresponds to the field of a point-like charge.

§ 20. Fields of charged, point-like masses in the theory of gravity

The problems that we examined in the previous section have been solved in the theory of gravity. We obtain solutions from the theory of gravity that give $\lambda = 0$ in equation (3.31) (i.e., we neglect the χ-field), and we throw out the equation $C_3 = 0$, which results from varying the function λ. We will then have:

$$\left.\begin{aligned} C_1 &\equiv \frac{e^{\nu}-1}{u^2}+\frac{\dot{\mu}}{u}-\frac{\varepsilon^2}{4}e^{\nu} = 0, \\ C_2 &\equiv \frac{e^{\nu}-1}{u^2}-\frac{\dot{\nu}}{u}-\frac{\varepsilon^2}{4}e^{\nu} = 0, \\ \dot{g} &= i\varepsilon \exp\left(\frac{\mu+\nu}{2}\right). \end{aligned}\right\} \tag{3.47}$$

From the equation $C_1 - C_2 = 0$, we get that $\mu + \nu = 0$. Integrating the equation $C_2 = 0$ will give:

$$e^{\nu} = \left[1-\gamma u+\frac{\varepsilon^2}{4}u^2\right]^{-1}; \qquad g = i\varepsilon u \tag{3.48}$$

or:

$$e^{\nu} = \left[1-\frac{\gamma}{r}+\frac{\varepsilon^2}{4r^2}\right]^{-1}; \quad e^{\mu} = \left[1-\frac{\gamma}{r}+\frac{\varepsilon^2}{4r^2}\right];$$

$$e^{\lambda} = 1; \qquad A_4 = \frac{ie'}{r}. \tag{3.48′}$$

We then see that 5-optics, when we take into account the χ-field, and the theory of gravity, which neglects the χ-field in problems concerning the fields of charged, point-like masses, arrive at different solutions. Note that in the theory of gravity, Coulomb potentials take the usual classical form, whereas for 5-optics, one does not have a singularity at the point $r = 0$.

§ 21. Generalized Kepler problem

As an exercise, we examine the problem of the motion of a test particle in the field of a charged point-like mass. The 5-eikonal equation takes the form:

$$\{(e^{-\nu}-1)\, n_i\, n_k + \delta_{ik}\} \frac{\partial\Sigma}{\partial x^i}\frac{\partial\Sigma}{\partial x^k} + e^{-\mu}\left(\frac{\partial\Sigma}{\partial x^4}\right)^2 + \\ +\left(e^{-\lambda} + e^{-\mu}\frac{e^2}{m^2c^4}A_4^2\right)\left(\frac{\partial\Sigma}{\partial x^5}\right)^2 - 2\frac{e}{mc^2}A_4\left(\frac{\partial\Sigma}{\partial x^4}\right)\left(\frac{\partial\Sigma}{\partial x^5}\right)e^{-\mu} = 0\,. \tag{3.49}$$

Passing to the reduced 3-eikonal $S(x^1, x^2, x^3)$ by the formula:

$$\Sigma = \Pi_5\, x^5 + \Pi_4\, x^4 + S(x^1, x^2, x^3) \tag{3.50}$$

will give the Hamilton-Jacobi equation for the function S:

$$e^{-\nu}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 + e^{-\mu}\left(\Pi_4 - \frac{e}{mc^2}A_4\Pi_5\right)^2 + e^{-\lambda}\Pi_5^2 = 0\,, \tag{3.51}$$

in which $\Pi_5 = \pm\, mc$, $E = -\,\Pi_4\, ic$ (energy).

For sake of generality, we put the solution of equation (3.51) into the form:

$$S = M\varphi + f(r)\,, \tag{3.52}$$

and arrive at the expression (M is the moment of momentum):

$$S = M\varphi + \int\sqrt{e^{-(\nu+\mu)}\left(\frac{E}{cM} + \frac{eA_4 i}{mc^2M}\Pi_5\right)^2 - \frac{e^{-\nu}}{r^2} - e^{-(\lambda+\mu)}\left(\frac{\Pi_5}{M}\right)^2}\, dr. \tag{3.53}$$

The trajectory equation takes the form $\partial S / \partial M = \text{const}$.

If we suppose, moreover, that $r = \dfrac{1}{u}$ then that will give the trajectory equation in the form:

$$\varphi = \int \frac{du}{\sqrt{e^{-(\nu+\mu)}\left(\frac{E}{cM}+\frac{eA_4 i}{mc^2 M}\Pi_5\right)^2 - u^2 e^{-\nu} - e^{-(\lambda+\mu)}\left(\frac{\Pi_5}{M}\right)^2}}\,. \tag{3.54}$$

Note the three special cases:

I. Substituting solution $(3.41')$ will give ($\Pi_5 = \pm\, mc$):

$$\varphi = \int \frac{du}{\sqrt{\left(\frac{E}{cM}\right)^2 - u^2(1-\gamma u) - \left(\frac{mc}{M}\right)^2(1-\gamma u)}}\,, \tag{3.55}$$

which is exactly the Keplerian orbit equation in Einstein's theory of gravity.

II. Substituting solution $(3.41'')$ will give ($\Pi_5 = \pm\, mc$), when one lets $\varepsilon \to 0$:

$$\varphi = \int \frac{du}{\sqrt{\left(\frac{E}{cM} \pm \frac{ee'u}{cM}\right)^2 - u^2 - \left(\frac{mc}{M}\right)^2}}\,, \tag{3.56}$$

which is the classical orbit for a charge $\pm\, e$ that is moving in a Coulomb field e'/r.

III. The case of the trajectory of a light ray in the field of a charged point-like mass will follow from (3.54) if we assume that $\Pi_5 = 0$. The trajectory equation will take the form:

$$\varphi = \int \frac{du}{\sqrt{\frac{e^{-(\nu+\mu)}}{R^2} - u^2 e^{-\nu}}}\,. \tag{3.57}$$

in which $R = \frac{Mc}{E}$ is the affine parameter of the ray. The integral (3.57) results in an ellipse. The differential equation of the trajectory takes the form:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{e^{-(\nu+\mu)}}{R^2} - u^2\, e^{-\nu}\,. \tag{3.58}$$

If we substitute the solution of 5-optics (3.41) then we will obtain:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{1+\alpha u}{R^2} - u^2\,(1-\beta u)\,. \tag{3.59}$$

An approximate integration will give the following formula for the angle Δ_φ between the asymptotes:

$$\Delta_\varphi = \frac{\alpha + 2\beta}{R} = \frac{2\gamma}{R}. \tag{3.60}$$

Substituting this into the solution for the theory of gravity (3.48) will give:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{1}{R^2} - u^2\left(1 - \gamma u + \frac{\varepsilon^2}{4}u^2\right), \tag{3.61}$$

and the angle between the asymptotes will be:

$$\Delta_\varphi = \frac{2\gamma}{R} + \frac{3\pi}{16}\frac{\varepsilon^2}{R^2}. \tag{3.62}$$

Upon comparing the formulas, we see that according to 5-optics, electrical charge does not influence the value of the angle between the asymptotes, whereas the theory of gravity gives an additional term that is quadratic in (ε / R).

MATHEMATICAL APPENDIX

Harmonic coordinate systems in Riemannian spaces

In this appendix, we shall consider the general case of an *n*-dimensional Riemannian space.

1. If the metric tensor g^{ik} satisfies the condition:

$$\frac{\partial \sqrt{|g|}\, g^{ik}}{\partial x^k} = 0 \tag{3.63}$$

then one will call that coordinate system *harmonic*.

The covariant derivative of the tensor g^{ik} is equal to zero:

$$\frac{\partial g^{ik}}{\partial x^s} + g^{ir}\Gamma^k_{rs} + g^{kr}\Gamma^i_{rs} = 0\,. \tag{3.64}$$

Contracting this over the indices *k*, *s*, while keeping in mind that $\Gamma^s_{is} = \dfrac{\partial \ln \sqrt{|g|}}{\partial x^i}$, will give:

$$\frac{\partial\left(\sqrt{|g|}\,g^{ik}\right)}{\partial x^k}+\sqrt{|g|}\,g^{rs}\Gamma^i_{rs}=0\ , \tag{3.65}$$

and therefore the harmonic condition can be expressed in the form:

$$\Gamma^i=\ g^{rs}\Gamma^i_{rs}\ =\ -\frac{1}{\sqrt{|g|}}\frac{\partial\left(\sqrt{|g|}\,g^{ik}\right)}{\partial x^k}\ =0\ , \tag{3.66}$$

in which Γ^λ is the anharmonicity pseudo-vector.

2. From the formula for the transformed Christoffel symbols:

$$\Gamma^\lambda_{\mu\nu}=\overline{\Gamma}^\alpha_{\beta\gamma}\frac{\partial x^\lambda}{\partial\overline{x}^\alpha}\frac{\partial\overline{x}^\beta}{\partial x^\mu}\cdot\frac{\partial\overline{x}^\gamma}{\partial x^\nu}+\frac{\partial^2\overline{x}^\alpha}{\partial x^\mu\partial x^\nu}\frac{\partial x^\lambda}{\partial\overline{x}^\alpha}\ , \tag{3.67}$$

one derives, upon multiplication by $g^{\mu\nu}$, the transformed formula for the pseudo-vector Γ^λ:

$$\Gamma^\lambda=\left(\overline{\Gamma}^\lambda+g^{\sigma\tau}\frac{\partial^2\overline{x}^\alpha}{\partial x^\sigma\partial x^\tau}\right)\frac{\partial x^\lambda}{\partial\overline{x}^\alpha}\ , \tag{3.68}$$

from which it will follow that the coordinate system remains harmonic, as long as the functions $f^{\,\alpha}$ in the transformed formula:

$$\overline{x}^\alpha=x^\alpha+f^{\,\alpha}\left(x^1,\,x^2,\,\dots,\,x^n\right) \tag{3.69}$$

satisfy the condition:

$$g^{ik}\,\frac{\partial^2 f^{\,\alpha}}{\partial x^i\partial x^k}=0\ . \tag{3.70}$$

Upon passing to a harmonic coordinate system, the structure tensor $G^{\mu\nu}$ will transform according to the formula:

$$\overline{G}^{\mu\nu}=G^{\mu\nu}+G^{\mu\alpha}\frac{\partial f^{\,\nu}}{\partial x^\alpha}+G^{\nu\alpha}\frac{\partial f^{\,\mu}}{\partial x^\alpha}+G^{\alpha\beta}\frac{\partial f^{\,\mu}}{\partial x^\alpha}\frac{\partial f^{\,\nu}}{\partial x^\beta}\ , \tag{3.71}$$

in which the functions $f^{\,\nu}$ satisfy the condition (3.70).

3. By simple calculations, along with the quantities Γ^i_{kl}, we introduce the quantities:

$$\Gamma^{i,\mu\nu}=g^{mk}\,g^{nl}\,\Gamma^i_{kl}\ . \tag{3.72}$$

Multiplying (3.64) by g^{st} gives:

$$g^{st}\frac{\partial g^{ik}}{\partial x^s}+\Gamma^{k,\,it}+\Gamma^{i,\,kt}=0\,. \tag{3.73}$$

If we solve this relativistic equation for $\Gamma^{k,\,it}$ then we will find that:

$$\Gamma^{k,\,it}=\frac{1}{2}\left\{g^{ks}\frac{\partial g^{it}}{\partial x^s}-g^{is}\frac{\partial g^{kt}}{\partial x^s}-g^{ts}\frac{\partial g^{ki}}{\partial x^s}\right\}. \tag{3.74}$$

4. We calculate the transformed contracted Riemann tensor, which is defined to be:

$$R^h_{kij}=\frac{\partial\Gamma^h_{kj}}{\partial x^i}-\frac{\partial\Gamma^h_{ki}}{\partial x^j}+\Gamma^h_{il}\Gamma^l_{kj}-\Gamma^h_{jl}\Gamma^l_{ik}\,.$$

Multiplying by g^{ik} and noting that:

$$\Gamma^h=g^{ik}\Gamma^h_{ik}\,,\qquad \Gamma^h_{lk}\left(\frac{\partial g^{lk}}{\partial x^j}+2g^{ik}\Gamma^l_{ij}\right)=0\,,$$

gives:

$$\begin{aligned}R^h_j&=g^{ik}\frac{\partial\Gamma^h_{kj}}{\partial x^i}-\frac{\partial\Gamma^h}{\partial x^j}+\Gamma^h_{lk}\left(\frac{\partial g^{lk}}{\partial x^i}+g^{ik}\Gamma^l_{ij}\right)-\Gamma^l\Gamma^h_{lj}=\\&=g^{ik}\frac{\partial\Gamma^h_{kj}}{\partial x^i}+\frac{1}{2}\Gamma^h_{lk}\frac{\partial g^{lk}}{\partial x^j}-\left(\frac{\partial\Gamma^h}{\partial x^j}+\Gamma^h_{lj}\Gamma^l\right).\end{aligned} \tag{3.75}$$

Contracting over the indices h, j gives:

$$R=g^{ik}\frac{\partial^2\ln\sqrt{|g|}}{\partial x^i\partial x^k}+\frac{1}{2}\Gamma^h_{lk}\frac{\partial g^{lk}}{\partial x^h}-\left(\frac{\partial\Gamma^h}{\partial x^h}+\Gamma^h_{lh}\Gamma^l\right). \tag{3.76}$$

Multiplying (3.75) by g^{sj} gives:

$$R^{sh}=g^{ik}\frac{\partial g^{sj}\Gamma^h_{kj}}{\partial x^i}-g^{ik}\Gamma^h_{kj}\frac{\partial g^{lk}}{\partial x^i}+\frac{1}{2}\Gamma^h_{lk}\frac{\partial g^{lk}}{\partial x^j}g^{sj}-g^{sj}\left(\frac{\partial\Gamma^h}{\partial x^j}+\Gamma^h_{lj}\Gamma^l\right).$$

Symmetrizing the first term in the indices h, s and the second term in the indices k, j, while keeping in mind formulas (3.64) and (3.73), gives:

$$R^{sh}=-\frac{1}{2}g^{ik}\frac{\partial^2 g^{sh}}{\partial x^i\partial x^k}+\Gamma^h_{kj}\Gamma^{s,kj}-g^{sj}\left(\frac{\partial\Gamma^h}{\partial x^j}+\Gamma^h_{lj}\Gamma^l\right). \tag{3.77}$$

Multiplying (3.75) by g_{sh} gives:

$$R_{sj} = g^{ik}\frac{\partial\Gamma_{s,kj}}{\partial x^i} - g^{ik}\Gamma^h_{kj}\frac{\partial g_{sh}}{\partial x^i} + \frac{1}{2}\Gamma_{s,lk}\frac{\partial g^{lk}}{\partial x^j} - g_{sh}\left(\frac{\partial\Gamma^h}{\partial x^j} + \Gamma^h_{lj}\Gamma^l\right).$$

Symmetrizing the first term in the indices *s, j* gives:

$$R_{sj} = \frac{1}{2}g^{ik}\frac{\partial^2 g_{sj}}{\partial x^i\partial x^k} - g^{ik}\Gamma^h_{kj}\frac{\partial g_{sh}}{\partial x^i} + \frac{1}{2}\Gamma_{s,lk}\frac{\partial g^{lk}}{\partial x^j} - g_{sh}\left(\frac{\partial\Gamma^h}{\partial x^j} + \Gamma^h_{lj}\Gamma^l\right). \qquad (3.78)$$

If we use a harmonic coordinate system in formulas (3.75), (3.76), (3.77), and (3.78) then we will obtain the formulas that were cited in the text (§ 15) from the assumption that $\Gamma^h = 0$.

5. In order to pass from formula (3.13) to formula (3.13′) as a result, recall that coordinate systems in 4-space cannot be harmonic. From the harmonic condition in 5-space (3.12a), that will give:

$$\frac{\partial\sqrt{1+\chi}}{\partial x^k}\sqrt{|\tilde{g}|}\,\tilde{g}^{ik} + \sqrt{1+\chi}\frac{\partial}{\partial x^k}\sqrt{|\tilde{g}|}\,\tilde{g}^{ik} = 0\,,$$

which, from formula (3.66), will give:

$$\tilde{\Gamma}^i = -\frac{1}{\sqrt{\tilde{g}}}\frac{\partial\sqrt{|\tilde{g}|}\,\tilde{g}^{ik}}{\partial x^k} = \tilde{g}^{ik}\frac{\partial\ln\sqrt{1+\chi}}{\partial x^k} = \frac{1}{2}\frac{\tilde{g}^{ik}}{1+\chi}\frac{\partial\chi}{\partial x^k}\,,$$

and furthermore:

$$\tilde{g}^{ks}\left(\frac{\partial\tilde{\Gamma}^i}{\partial x^s} + \tilde{\Gamma}^i_{sl}\Gamma^l\right) = \frac{1}{2}\frac{\tilde{g}^{in}\tilde{g}^{km}}{1+\chi}\left\{\frac{\partial^2\chi}{\partial x^n\partial x^m} - \frac{1}{(1+\chi)}\frac{\partial\chi}{\partial x^n}\frac{\partial\chi}{\partial x^m} - \Gamma^l_{mn}\frac{\partial\chi}{\partial x^n}\right\},$$

$$\frac{\partial\tilde{\Gamma}^i}{\partial x^i} + \tilde{\Gamma}^i_{ik}\tilde{\Gamma}^k = \frac{1}{2}\frac{g^{ik}}{(1+\chi)}\left\{\frac{\partial^2\chi}{\partial x^i\partial x^k} - \frac{3}{2(1+\chi)}\frac{\partial\chi}{\partial x^i}\frac{\partial\chi}{\partial x^k}\right\}. \qquad (3.79)$$

CHAPTER IV

WAVE-LIKE 5-OPTICS IN MINKOWSKI 5-SPACE

§ 22. Introduction

In the last two chapters, we clearly presented 5-optics, which we agreed to call "classical," since we always considered it in the limiting case $h \to 0$.

We presently set about presenting quantum mechanics as wave-like 5-optics in rigorous detail from the following fundamental assumptions:

Problems in wave-like optics that are concerned with the propagation of wave-fields in a five-dimensional Riemannian configuration space whose coordinates are space, time, and action, and which is topologically closed in the in the action coordinate with period h, are equivalent to problems of quantum mechanics that are concerned with the motion of particles whose charge-to-mass ratio is e / m in an external field.

In connection with this, we emphasize that we will be concerned with only configuration spaces in what follows. Fundamental space, which we dealt with in Chapter III, will not be considered at all.

We saw that in geometrical 5-optics, the motion of a test particle is described by the 5-eikonal equation. This equation serves as a universal concept in the sense that it describes the motion of any test particle and regardless of its spin. In the transition to 5-optics, we shall not give either a universal wave equation or a system of wave equations. That is connected with the fact that we will have different systems of wave equations for particles of different spins.

In wave-like 5-optics, the motion of test particles describes the optical process of the propagation of wave-fields in configuration 5-spaces with space, time, and action coordinates. The motion of particles of integer spin is described by tensorial fields, while the motion of particles of half-integer spin is described by spinorial fields.

In this chapter, we shall consider various examples of tensorial and spinorial wave-fields in 5-space. To that end, we will meanwhile propose that external fields should be absent; that is, that our metric 5-space should indeed be Minkowski 5-space. In Chapter VI, we will then consider fields in a Riemannian 5-space; i.e., we will study what happens when external fields are present.

Henceforth, we will utilize the Pauli units; i.e., $\hbar = 1$, $c = 1$. We denote the "quantum radius" of a particle by $\frac{1}{\mu} = \frac{\hbar}{mc}$; therefore μ will indeed be the mass of the test particle when it is measured in Pauli units. Following the Pauli convention, we will call particles of integer spin *mesons*.

Since we know that 5-space is periodic in the action coordinate, we will decompose each field that we construct into a Fourier series:

$$W^{(r)}(x^1, x^2, x^3, x^4, x^5) = \sum_{g=-\infty}^{g=+\infty} \exp(iZ\mu x^5)\, U(r)(Z \mid x^1, x^2, x^3, x^4) \qquad (4.1)$$

and regard:

$$U^{(r)}(Z \mid x^1, x^2, x^3, x^4)$$

as the set of Fourier components of $W^{(r)}$

Hereinafter, when we describe operations of the form $\int A(x^1, x^2, x^3, x^4, x^5)\, dx^5$, we will understand that to mean integrating over one period of the fifth coordinate:

$$\int A\, dx^5 = \frac{\mu}{2\pi}\int_0^{2\pi/\mu} A\ dx^5 = \overline{A}(x^1, x^2, x^3, x^4). \tag{4.2}$$

Let a field be assigned to the Lagrangian function:

$$L\left(W^{(r)}; \frac{\partial W^{(r)}}{\partial x^\sigma}\right). \tag{4.3}$$

With our notation, that will give:

$$\begin{aligned} &\int L\left(W^{(r)}; \frac{\partial W^{(r)}}{\partial x^\sigma}\right) dx^1\, dx^2\, dx^3\, dx^4\, dx^5 \\ &\qquad = \int \overline{L}\left(U^{(r)}(Z\mid\cdots); \frac{\partial U^{(r)}(Z\mid\cdots)}{\partial x^k}\right)(\, dx^1\, dx^2\, dx^3\, dx^4\,)\ . \end{aligned} \tag{4.4}$$

Therefore, the transition from functions of five coordinates $W^{(r)}(x^1, x^2, x^3, x^4, x^5)$ to the Fourier components $U^{(r)}(Z \mid x^1, x^2, x^3, x^4)$ that depend upon four coordinates x^1, x^2, x^3, x^4 is actually the transition from the q-representation that relates to the coordinates x^1, x^2, x^3, x^4, x^5 to the "mixed" representation for which the p-representation relates to the coordinate x^5 and the q-representation relates to the coordinates x, y, z. In that representation, all formulas will take on their usual four-dimensional form.

With the help of the canonical formulation, we can express the field equations in terms of the Lagrangian function:

$$\frac{\partial}{\partial x^\sigma}\left(\frac{\partial L}{\partial\left(\dfrac{\partial W^{(r)}}{\partial x^\sigma}\right)}\right) - \frac{\partial L}{\partial W^{(r)}} = 0\ , \tag{4.5}$$

and calculate the canonical 5-tensor:

$$T_{\mu\nu} = \sum_{(r)} \frac{\partial L}{\partial\left(\dfrac{\partial W^{(r)}}{\partial x^\nu}\right)} \cdot \frac{\partial W^{(r)}}{\partial x^\mu} - \delta_{\mu\nu} L\ , \tag{4.6}$$

which, by virtue of equation (4.5), will satisfy the equation:

$$\frac{\partial T_{\mu\nu}}{\partial x^{\nu}} = 0 . \tag{4.7}$$

In the sequel, when we consider special cases of fields in formulas (4.5)-(4.7), we will pass over to the representation in terms of Fourier components, which will permit us to give a physical interpretation to the five-dimensional formulation.

§ 23. Problems concerning the propagation of sound waves in plane-parallel media

Before we turn to the study of special cases of fields, it will be useful to illustrate our objective by considering some problems in the propagation of sound waves in inhomogeneous flat (plane)-parallel media. We will content ourselves with expressing the thickness of the medium l in terms of some fictitious mass m that is related to l by the formula:

$$m = \frac{h}{al}, \tag{4.8}$$

in which a is the velocity of sound. For a monochromatic wave of frequency ω, that will give the Helmholtz equation:

$$\Delta W + \frac{\omega^2}{a^2} W = 0 . \tag{4.9}$$

We choose the z-axis to be perpendicular to the medium and decompose the function $W(x, y, z)$ into a Fourier series:

$$W(x, y, z) = \sum_{n=-\infty}^{n=+\infty} U(n \mid x, y) \exp\left(\frac{inmaz}{\hbar}\right). \tag{4.10}$$

We will regard $W(x, y, z)$ as the set of Fourier components $U(n \mid x, y)$; i.e., we will use the relative coordinate z to pass from the q-representation to the p-representation.

We can now consider all problems to be two-dimensional. The functions $U(n \mid x, y) \equiv U_n$ satisfy the Helmholtz equation:

$$\frac{\partial^2 U_n}{\partial x^2} + \frac{\partial^2 U_n}{\partial y^2} + \left(\frac{\omega^2}{a^2} - \left(\frac{nma}{\hbar}\right)^2\right) U_n = 0 . \tag{4.11}$$

In order that the two-dimensional waves $U(n \mid x, y)$ should be non-vanishing, according to (4.11), it will be necessary that one must have:

$$\hbar\omega > nma^2 \qquad \text{or} \qquad \lambda < \frac{l}{n},$$

in which λ is the wavelength of the sound wave.

If we consider spatially non-vanishing two-dimensional plane waves in the medium then we will obtain characteristic equations for the wave 2-vector **k**:

$$k_n^2 + \left(\frac{nma}{\hbar}\right)^2 = \frac{\omega^2}{a^2}. \tag{4.12}$$

All 2-vectors with group velocity **v** and phase velocity **w** are determined by the formulas:

$$v_n = \frac{d\omega}{dk_n} = a^2 \frac{k_n}{\omega}, \tag{4.13}$$

$$w_n = \frac{\omega}{k_n}, \tag{4.13′}$$

$$v_n\, w_n = a^2. \tag{4.13″}$$

As long as the only sound fields that appear have wavelength $\lambda > l$, we will be dealing with two-dimensional waves with n = 0. One propagates without dispersion when $v_0 = w_0 = a$, which defines the usual corresponding two-dimensional phonons, for which $hk_0 = h\omega / a$, which we will, in turn, take into account, and two-dimensional fields with $c \neq 0$, for which we will have, when we substitute (4.13) in (4.12):

$$hk_n = \frac{(nm)v_n}{\sqrt{1-\left(\frac{v_n}{a}\right)^2}}; \qquad \hbar\omega_n = \frac{(nm)a^2}{\sqrt{1-\left(\frac{v_n}{a}\right)^2}}. \tag{4.14}$$

These two-dimensional fields propagate with dispersion, corresponding to two-dimensional "heavy phonons" with a discrete mass spectrum.

Note that by virtue of (4.10) we have:

$$\overline{W}(x,y) = \frac{1}{l}\int_0^l W(x,y,z)dz = U(0\mid x,y), \tag{4.15}$$

so we can apply equation (4.9) to the z-coordinate and obtain the two-dimensional Helmholtz equation:

$$\frac{\partial^2 \overline{W}}{\partial x^2} + \frac{\partial^2 \overline{W}}{\partial y^2} + \frac{\omega^2}{a^2}\overline{W} = 0 \tag{4.16}$$

for the function $\overline{W}(x,y)$.

Thus, as long as all of the sound fields in the medium are not short waves with $\lambda > l$, we can describe sound fields by means of equation (4.16) and ignore "heavy phonons." In the presence of short waves, we will have the following alternatives:

1) Disregard the two-dimensional description of sound fields and return to the initial three-dimensional equations.

2) Preserve the two-dimensional description of the sound fields but, at the same time, introduce the usual two-dimensional phonons that are called "heavy phonons."

We now consider an example that proves to be very helpful for our further understanding.

§ 24. Scalar mesons

We start by considering the simplest real scalar fields, which are the ones whose Lagrangian function takes the form:

$$L = \frac{1}{2}\frac{\partial W}{\partial x^{\nu}} \cdot \frac{\partial W}{\partial x^{\nu}} . \tag{4.17}$$

From formulas (4.5), (4.6), and (4.7), we will get:

The field equation:

$$\frac{\partial^2 W}{\partial x^{\nu} \partial x^{\nu}} = 0 . \tag{4.18}$$

The expression for the 5-tensor:

$$T_{\mu\nu} = \frac{\partial W}{\partial x^{\mu}} \cdot \frac{\partial W}{\partial x^{\nu}} - \delta_{\mu\nu} L, \tag{4.19}$$

which satisfies the equation:

$$\frac{\partial T_{\mu\nu}}{\partial x^{\nu}} = 0 . \tag{4.20}$$

We now convert formulas (4.17)-(4.20) into the form that they take in terms of Fourier components. Note that for real fields:

$$U(-Z \mid x^1, x^2, x^3, x^4) = U^*(Z \mid x^1, x^2, x^3, x^4) , \tag{4.21}$$

which will give, upon substituting into the development formula (4.1):

$$\begin{aligned} L &= \frac{1}{2}\sum_{Z}\sum_{Z'}\int \left\{ \frac{\partial U(Z')}{\partial x^k}\frac{\partial U(Z)}{\partial x^k} - Z'Z\mu^2 U(Z')U(Z) \right\} \exp[i(Z+Z')\mu x^5]\, dx^5 \\ &= \frac{1}{2}\sum_{Z=-\infty}^{Z=+\infty}\left\{ \frac{\partial U^*(Z)}{\partial x^k}\frac{\partial U(Z)}{\partial x^k} + Z^2\mu^2 U^*(Z)U(Z) \right\} \\ &= L(0) + \sum_{Z=1}^{Z=\infty} L(Z) , \end{aligned} \tag{4.17'}$$

into which we have introduced the abbreviated notations:

$$\left.\begin{aligned} L(0) &= \frac{1}{2}\frac{\partial U(0)}{\partial x^k}\frac{\partial U(0)}{\partial x^k}, \\ L(Z) &= \frac{\partial U(0)}{\partial x^k}\frac{\partial U(0)}{\partial x^k} + Z^2\mu^2 U^*(Z)U(Z). \end{aligned}\right\} \tag{4.22}$$

If we single out the action coordinate then, from (4.19), we will have:

$$\left.\begin{aligned} \overline{T}_{ik} &= \overline{T}_{ik}(0) + \sum_{Z=1}^{Z=\infty} \overline{T}_{ik}(Z), \\ \overline{T}_{5k} &= \sum_{1}^{\infty} i\,|\,Z\,|\,\mu \left\{ U(Z)\frac{\partial U^*(Z)}{\partial x^k} - U^*(Z)\frac{\partial U(Z)}{\partial x^k} \right\}, \end{aligned}\right\} \tag{4.19′}$$

in which:

$$\overline{T}_{ik}(0) = \frac{\partial U(0)}{\partial x^i}\frac{\partial U(0)}{\partial x^k} - L(0)\,\delta_{ik}\,,$$

$$\overline{T}_{ik}(Z) = \frac{\partial U^*(Z)}{\partial x^i}\frac{\partial U(Z)}{\partial x^k} + \frac{\partial U^*(Z)}{\partial x^k}\frac{\partial U(Z)}{\partial x^i} - \delta_{ik}\,L(0).$$

The field equations in Fourier components will then take the form:

$$\left.\begin{aligned} (\Box - Z^2\mu^2)U(Z\,|\ldots) &= 0; \\ (\Box - Z^2\mu^2)U^*(Z\,|\ldots) &= 0. \end{aligned}\right\} \tag{4.18′}$$

From (4.20), if we single out the action coordinate then averaging over the action coordinate will give:

$$\frac{\partial \overline{T}_{lk}}{\partial x^k} = 0\,, \qquad \frac{\partial \overline{T}_{5k}}{\partial x^k} = 0\,. \tag{4.20′}$$

We then conclude the transformation of the formulas into Fourier components. In order to do this, we shall make the assumption that only two terms should appear in the Fourier series (4.1), which will correspond to $Z = \pm\, 1$, so the field expressions will make physical sense:

$$W = U \exp(i\mu x^5) + U^* \exp(-i\mu x^5)\,. \tag{4.23}$$

In this special case, the system of equations will assume the form:

$$
\left.
\begin{aligned}
&\textit{Lagrangian function}:\\
&\qquad \bar{L} = \frac{\partial U^*}{\partial x^k}\frac{\partial U}{\partial x^k} + \mu^2 U^* U;\\
&\textit{Field equations}:\\
&\qquad (\Box - \mu^2)U = 0; \qquad (\Box - \mu^2)U^* = 0;\\
&\textit{Tensor components}\ T_{\mu\nu}:\\
&\qquad \bar{T}_{ik} = \frac{\partial U^*}{\partial x^i}\frac{\partial U}{\partial x^k} + \frac{\partial U^*}{\partial x^k}\frac{\partial U}{\partial x^i} - \delta_{ik}\bar{L};\\
&\qquad \bar{T}_{5k} = i\mu\left(\frac{\partial U^*}{\partial x^i}U - \frac{\partial U}{\partial x^k}U^*\right).
\end{aligned}
\right\}
\tag{4.24}
$$

The system of formulas (4.24) coincides precisely with the system of formulas that was described in the Pauli-Weisskopf theory of scalar mesons of mass $m = \mu\hbar / c$. The 4-tensor $\bar{T}_{ik}$ is the symmetric energy-impulse tensor, while the 4-vector $\bar{T}_{5k}$ is distinguished from the current 4-vector by the factor $mc/\hbar$. The Fourier components U and U^* describe particles that are endowed with charge; indeed, the current 4-vector $\bar{T}_{5k}$ will change sign when one replaces U with U^*.

Formula (4.20′) expresses the laws of conservation of energy and charge, which combine into one conservation law in 5-optics. Therefore, we will henceforth call the 5-tensor $T_{\mu\nu}$ the *energy-impulse-charge* 5-tensor.

Returning to the general case of a Fourier series (4.1), we conclude that a real scalar field $W(x^1, x^2, x^3, x^4, x^5)$ describes the entire family of scalar mesons of mass $|Z|\,m$ and charge Ze, in which Z is a positive or negative integer, including zero. In other words, the entire family of scalars mesons can be regarded as one multi-particle of distinct discrete charges in its construction.

As long as we are concerned with free particles (in the absence of external fields), or (as is incorrectly done in contemporary quantum mechanics) if we can neglect the periodic dependence of the external field upon the action coordinate (viz., the cylindricality condition) then the transition from one charged particle state to another would be forbidden. However, a consistent quantum theory must account for such a transition.

We especially consider the particular case of $Z = 0$, which are scalar mesons of zero charge and mass. In this case, the Fourier series $U(0)$ will be real, and we will have the system of formulas:

$$\left.\begin{aligned} \bar{L} &= \frac{1}{2}\frac{\partial U}{\partial x^k}\frac{\partial U}{\partial x^k}. & \Box U &= 0, \\ \bar{T}_{ik} &= \frac{\partial U}{\partial x^i}\frac{\partial U}{\partial x^k} - \bar{L}\delta_{ik}, & \bar{T}_{5k} &= 0, \end{aligned}\right\} \tag{4.25}$$

instead of (4.24), and these formulas describe a classical scalar field that is associated with a particle of zero mass, charge, and spin.

Contemporary physics makes a sharp distinction between complex Ψ-fields, which describe the behavior of charged particles with non-zero rest mass, and real classical fields, which describe the behavior of neutral particles with zero rest mass.

Such a sharp distinction implies something that is nevertheless fundamental in the viewpoint of contemporary physics, namely, that Ψ-fields are localized in a suitable configuration space, just as classical fields are localized in the fundamental 4-space of relativity theory.

It is natural that the principal distinction between quantum and classical fields would disappear in 5-optics, which is a radical departure from the concept of universal space.

§ 25. Vector mesons

1. *Field equations*. The foregoing example of a real scalar field serves to illustrate our viewpoint without introducing new results that are absent from the framework of contemporary theory. We will not obtain important new results until we pass on to the consideration of real vector fields.

Consider Maxwell's system of equations for 5-space. In a notation that does not require explanation, we have the following system of equations for the field components $W_{\lambda\mu} = - W_{\mu\lambda}$:

$$\frac{\partial W_{\lambda\mu}}{\partial x^\mu} = Q_\lambda\,, \tag{4.26}$$

$$\frac{\partial W_{\lambda\mu}}{\partial x^\nu} + \frac{\partial W_{\mu\nu}}{\partial x^\kappa} + \frac{\partial W_{\nu\lambda}}{\partial x^\mu} = 0\,. \tag{4.27}$$

The field components are expressed in terms of potential fields through the formula:

$$W_{\lambda\mu} = \frac{\partial W_\lambda}{\partial x^\mu} - \frac{\partial W_\mu}{\partial x^\lambda}\,. \tag{4.28}$$

The potentials satisfy the wave equation:

$$\frac{\partial^2 W_\lambda}{\partial x^\nu \partial x^\nu} - \frac{\partial}{\partial x^\lambda}\left(\frac{\partial W_\nu}{\partial x^\nu}\right) = - Q_\lambda\,, \tag{4.29}$$

and can be subjected to gauge transformations:

$$W_\lambda = W'_\lambda + \frac{\partial F}{\partial x^\lambda}. \tag{4.30}$$

The Lagrangian function and symmetrized energy-impulse-charge 5-tensor take the form:

$$L = \frac{1}{4} W_{\lambda\mu} W_{\lambda\mu}, \tag{4.31}$$

$$\Theta_{\lambda\mu} = W_{\lambda\mu} W_{\mu\nu} - \delta_{\lambda\mu} L. \tag{4.32}$$

In order to explain the physical sense of these five-dimensional equations, it is necessary to transform them to Fourier components. For the sake of simplicity, we should observe that only two components, namely, $Z = \pm 1$, will appear in the Fourier series for the present example. The generalization of that would then be effortless.

We have:

$$\left.\begin{aligned} W_\lambda &= U_\lambda \exp(i\mu x^5) + U^*_\lambda \exp(-i\mu x^5), \\ W_{\lambda\mu} &= U_{\lambda\mu} \exp(i\mu x^5) + U^*_{\lambda\mu} \exp(-i\mu x^5), \\ F &= f \exp(i\mu x^5) + f^* \exp(-i\mu x^5), \\ Q_\lambda &= q_\lambda \exp(i\mu x^5) + q^*_\lambda \exp(-i\mu x^5). \end{aligned}\right\} \tag{4.33}$$

In Fourier components, we have the following system of formulas:

$$\frac{\partial U_{mk}}{\partial x^k} + i\mu U_{m5} = q_m; \qquad \frac{\partial U_{5n}}{\partial x^n} = q_5, \tag{4.26$'$}$$

$$i\mu U_{nm} + \frac{\partial U_{n5}}{\partial x^m} + \frac{\partial U_{m5}}{\partial x^n} = 0; \qquad \frac{\partial U_{mk}}{\partial x^k} + \frac{\partial U_{nk}}{\partial x^m} + \frac{\partial U_{km}}{\partial x^n} = 0, \tag{4.27$'$}$$

$$U_{mn} = \frac{\partial U_n}{\partial x^m} - \frac{\partial U_m}{\partial x^n}; \qquad U_{m5} = \frac{\partial U_{5n}}{\partial x^n} - i\mu U_m, \tag{4.28$'$}$$

$$\left.\begin{aligned} (\Box - \mu^2)U_m - \frac{\partial}{\partial x^m}\left(\frac{\partial U_n}{\partial x^n} + i\mu U_5\right) &= -q_n, \\ \Box U_5 - i\mu\left(\frac{\partial U_n}{\partial x^n}\right) &= -q_5, \end{aligned}\right\} \tag{4.29$'$}$$

$$U_m = U'_m + \frac{\partial f}{\partial x^m}; \qquad U_5 = U'_5 + i\, m\, f. \tag{4.30$'$}$$

The corresponding system of complex-conjugate equations can be written down immediately:

$$\bar{L} = \frac{1}{2} U^*_{\lambda\mu} U_{\lambda\mu}, \tag{4.31'}$$

$$\left.\begin{aligned} \bar{\Theta}_{ik} &= U^*_{i\sigma} U_{k\sigma} + U^*_{k\sigma} U_{i\sigma} - \delta_{ik} \bar{L}, \\ \bar{\Theta}_{5k} &= U^*_{5\sigma} U_{k\sigma} + U_{5\sigma} U^*_{k\sigma}. \end{aligned}\right\} \tag{4.32'}$$

In equation $(4.32')$, it is convenient to convert to three-dimensional notation and utilize a time coordinate $x^0 = -\, i\, x^4$, instead of the usual coordinate x^4.

If we note that:

$$U_{0n} = -\left(\frac{\partial U_n}{\partial x^0} + \frac{\partial U_0}{\partial x^n}\right); \qquad U_{n0} = \frac{\partial U_0}{\partial x^n} + \frac{\partial U_n}{\partial x^0} \tag{4.34}$$

then we will get:

The energy density (α, β = 1, 2, 3)

$$\bar{\Theta}_{00} = \left\{\frac{1}{2} U^*_{\alpha\beta} U_{\alpha\beta} + U^*_{\beta 5} U_{\beta 5} + U^*_{\beta 0} U_{\beta} + U^*_{50} U_{50}\right\}; \tag{4.35}$$

The impulse density:

$$\bar{\Theta}_{\alpha 0} = \left\{U^*_{\alpha\beta} U_{0\beta} + U^*_{\alpha 5} U_{05} + U_{\alpha\beta} U^*_{0\beta} + U_{\alpha 5} U^*_{05}\right\};$$

The charge density:

$$\bar{\Theta}_{50} = \left\{U^*_{0\beta} U_{5\beta} + U_{0\beta} U^*_{5\beta}\right\}; \tag{4.36}$$

The current density:

$$\bar{\Theta}_{5\alpha} = \left\{U^*_{\alpha\beta} U_{5\beta} - U^*_{\alpha 0} U_{50} + U_{\alpha\beta} U^*_{5\beta} - U_{\alpha} U^*_{50}\right\}.$$

2. *Gauge potentials.* Up till now, we have not made use of the possibility of introducing gauge potentials. We now choose such a potential that satisfies:

$$\frac{\partial U'_i}{\partial x^i} = 0\ . \tag{4.37}$$

Equations $(4.29')$ will then assume the form:

$$\left.\begin{aligned}(\Box-\mu^2)U'_\alpha&=-q_\alpha+i\mu\frac{\partial U'_5}{\partial x^\alpha},\\(\Box-\mu^2)U'_0&=-q_0+i\mu\frac{\partial U'_5}{\partial x^0},\\\Box U'_5&=-q^5,\end{aligned}\right\}\tag{4.38}$$

from which it will follow that:

$$U'_5(r,x^0)=\frac{1}{4\pi}\int\frac{q_5(r',x^0-|r-r'|)}{|r-r'|}\,dV'+\tilde{U}'_5(r,x^0)\,,\tag{4.39}$$

in which $\tilde{U}'_5$ is a solution to the homogeneous wave equation $\Box\tilde{U}'_5=0$.

If q_λ = 0 in this space (i.e., if to be we can consider the field to be completely wavelike, so we can assume that $U'_5=0$) then we will choose $\tilde{U}'_5=0$. The system of equations $(4.26')$, $(4.27')$ will then assume the form (we now discard the prime symbol):

$$\left.\begin{aligned}&\frac{\partial U_{mn}}{\partial x^n}+\mu^2U_m=0, &&\frac{\partial U_n}{\partial x^n}=0,\\&U_{mn}=\frac{\partial U_n}{\partial x^m}-\frac{\partial U_m}{\partial x^n}, &&\frac{\partial U_{mn}}{\partial x^k}+\frac{\partial U_{nk}}{\partial x^m}+\frac{\partial U_{km}}{\partial x^n}=0,\end{aligned}\right\}\tag{4.40}$$

while the system of expressions (4.35), (4.36) will take the form:

$$\left.\begin{aligned}\bar{\Theta}_{00}&=\left\{\frac{1}{2}U^*_{\alpha\beta}U_{\alpha\beta}+U^*_{\beta0}U_{\beta0}+\mu^2(U^*_\beta U_\beta+U^*_0U_0)\right\},\\\bar{\Theta}_{0\alpha}&=\left\{U^*_{\alpha\beta}U_{0\beta}+U_{\alpha\beta}U^*_{0\beta}+\mu^2(U^*_\alpha U_0+U_\alpha U^*_0)\right\},\\\bar{\Theta}_{05}&=i\mu\{U^*_{0\beta}U_\beta-U_{0\beta}U^*_\beta\},\\\bar{\Theta}_{\alpha5}&=i\mu\{U^*_{\alpha\beta}U_\beta-U_{\alpha\beta}U^*_\beta+U^*_{\alpha0}U_0-U^*_{\alpha0}U_0\}.\end{aligned}\right\}\tag{4.41}$$

The system of formulas (4.40) and (4.41) coincides precisely with the Proca system of formulas that describe vector mesons.

Thus, we have shown that in the case of a vanishing source ($Q_\lambda\equiv0$) and for a suitable gauge potential (4.37), five-dimensional Maxwell fields will describe the entire family of Proca vector mesons of mass $|Z|\,m$ and charge Ze, in which Z is any positive or negative integer. The case of Z = 0 obviously includes classical electrodynamics, and we do not need to consider it at this point.

It is appropriate that we once again emphasize that the principal differences between complex ψ-fields and real classical fields will disappear in 5-optics.

In electrodynamics, as was pointed out by V. Ginzburg (cf., Mathematical Appendix), it is possible to decompose a field into a photon field (transverse wave) plus a Coulomb field that is due to a continuous charge distribution by the use of a gauge potential.

Obviously, singling out the transverse wave would be an essentially non-Lorentz invariant process.

Following the example of Ginzburg, we look for analogous gauge potentials in our own theory. It is necessary that the following relativistically non-invariant gauge ($\alpha = 1, 2, 3$) must be satisfied:

$$\frac{\partial U''_\alpha}{\partial x^\alpha} + i\mu U''_5 = 0\,. \tag{4.42}$$

Equations (4.29′) will then take the form:

$$\left.\begin{aligned}
(\Box - \mu^2)U''_\alpha &= -q_\alpha + \frac{\partial^2 U''_0}{\partial x^\alpha \partial x^0},\\
(\Delta - \mu^2)U''_0 &= -q_0,\\
(\Box - \mu^2)U''_5 &= -q_5 + i\mu\frac{\partial U''_0}{\partial x^0},
\end{aligned}\right\} \tag{4.43}$$

from which, it will follow that:

$$U_0(r, x^0) = \frac{1}{4\pi}\int \frac{q_0(r', x^0)}{|r' - r|} e^{-\mu|r'-r|} dv'\,. \tag{4.44}$$

We transform the expression for $\bar{\Theta}_{00}$ by means of formula (4.35), taking into account the gauge condition (4.42). If we discard the double prime notation then we will have:

$$\begin{aligned}
\bar{\Theta}_{00} = &\left\{\frac{1}{2}U^*_{\alpha\beta}U_{\alpha\beta} + \frac{\partial U^*_\alpha}{\partial x^0}\frac{\partial U^*_\alpha}{\partial x^0} + \frac{\partial U^*_5}{\partial x^0}\frac{\partial U^*_5}{\partial x^0} + \frac{\partial U^*_5}{\partial x^\alpha}\frac{\partial U^*_5}{\partial x^\alpha} + \mu^2 U^*_\alpha U_\alpha + 2\mu^2 U_5 U^*_5\right\} + \\
&+ \frac{1}{2}\left\{U^*_0(\mu^2 - \Delta)U_0 + U_0(\mu^2 - \Delta)U^*_0\right\} + \\
&+ \frac{\partial}{\partial x^5}\left\{i\mu U^*_\beta U_5 + U_0\frac{\partial U^*_\beta}{\partial x^0} + \frac{1}{2}U^*_0\frac{\partial U^*_0}{\partial x^\beta} - i\mu U_\beta U^*_5 + U^*_0\frac{\partial U^*_\beta}{\partial x^0} + \frac{1}{2}U^*_0\frac{\partial U^*_0}{\partial x^\beta}\right\}.
\end{aligned} \tag{4.45}$$

Integrating this over the volume will give:

$$G_0 = G^1_0 + \frac{1}{4\pi}\int \frac{q_0(r', x^0)\, q''_0(r'', x^0)}{|r' - r|}\, e^{-\mu|r'-r''|}\, dV'\, dV''\,. \tag{4.46}$$

The first term, as we shall presently see, is indeed the energy of a meson wave-field, while the second term is the interaction energy of a continuous charge distribution of density $q_0(r, x^0)$ that interacts according to Jacobi's Law.

If we consider completely wave-like fields without sources then the second term in (4.46) will equal zero, and, according to (4.44), we can assume that $U_0 = 0$.

Doing this and performing analogous transformations in formulas (4.36), as with the conclusion (4.45), will give for the meson wave-field:

$$\left.\begin{aligned}
\bar{\Theta}_{\alpha 0} &= \left\{\frac{\partial U_\beta}{\partial x^0}\frac{\partial U^*_\beta}{\partial x^\alpha}+\frac{\partial U_5}{\partial x^0}\frac{\partial U^*_5}{\partial x^\alpha}+\frac{\partial U^*_\beta}{\partial x^0}\frac{\partial U_\beta}{\partial x^\alpha}+\frac{\partial U^*_\beta}{\partial x^0}\frac{\partial U_5}{\partial x^\alpha}\right\}\\
&\quad+\frac{\partial}{\partial x^\beta}\left(U^*_\alpha\frac{\partial U_\beta}{\partial x^0}+U^*_\beta\frac{\partial U_\alpha}{\partial x^0}\right),\\
\bar{\Theta}_{00} &= \left\{\frac{1}{2}U^*_{\alpha\beta}U_{\alpha\beta}+\frac{\partial U^*_\alpha}{\partial x^0}\frac{\partial U_\alpha}{\partial x^0}+\frac{\partial U^*_5}{\partial x^0}\frac{\partial U_5}{\partial x^0}+\frac{\partial U^*_5}{\partial x^\alpha}\frac{\partial U_5}{\partial x^\alpha}+\mu^2U^*_\alpha U_\alpha+2\mu^2U^*_5U_5\right\}\\
&\quad+i\mu\frac{\partial}{\partial x^\beta}\left(U^*_5U_5-U_\beta U^*_5\right),\\
\bar{\Theta}_{50} &= i\mu\left(U^*_\beta\frac{\partial U_\beta}{\partial x^0}+U^*_5\frac{\partial U_5}{\partial x^0}-U_\beta\frac{\partial U^*_\beta}{\partial x^0}-U_5\frac{\partial U^*_5}{\partial x^0}\right)+\frac{\partial}{\partial x^\beta}\left(U^*_5\frac{\partial U_\beta}{\partial x^0}+U_5\frac{\partial U^*_\beta}{\partial x^0}\right).
\end{aligned}\right\} \tag{4.47}$$

We now calculate the expression for the 5-vector:

$$G^1_\lambda = \int \Theta_{\lambda 0}\, dV\,. \tag{4.48}$$

We can decompose these potentials U_α, U_5 into plane waves ($k_0^2 = k^2 + \mu^2$):

$$U = (2V)^{-\frac{1}{2}}\sum_{(k)}\{\mathbf{U}_+(k)\exp[i(\mathbf{kr}-k_0x^0)]+\mathbf{U}_-(k)\exp[-i(\mathbf{kr}-k_0x^0)]\}\,. \tag{4.49}$$

In that case, from condition (4.42), we will have:

$$U_5 = \frac{(2V)^{-\frac{1}{2}}}{\mu}\sum_{(k)}\left\{-(\mathbf{kU}_+)\exp[i(\mathbf{kr}-k_0x^0)]+(\mathbf{kU}_-)\exp[-i(\mathbf{kr}-k_0x^0)]\right\} \tag{4.49′}$$

Substituting this expression in (4.48) will give:

$$\left.\begin{aligned}
&\mathit{energy}: \quad G_0 = k\sum_{(k)}(N_+(k)+N_-(k),),\\
&\mathit{impulse}: \quad \mathbf{G} = \mathbf{k}\sum_{(k)}(N_+(k)+N_-(k),),\\
&\mathit{charge}: \quad G_5 = \mu\sum_{(k)}(N_+(k)-N_-(k),),
\end{aligned}\right\} \tag{4.50}$$

in which, for the sake of brevity, we have introduced the notation:

$$N(k) = k_0(\mathbf{U}^*\mathbf{U}) + \frac{k_0}{\mu^2}(\mathbf{kU}^*)(\mathbf{kU}) , \tag{4.51}$$

For the sake of singling out longitudinal and transverse components, we transform the bilinear formula (4.51) into its principal axis form:

$$\mathbf{U} = \frac{1}{\sqrt{k_0}}\left[\mathbf{e}_1 V_1 + \mathbf{e}_2 V_2 + \frac{\mu}{k_0}\mathbf{e}_3 V_3\right], \tag{4.52}$$

in which $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are three mutually-perpendicular unit vectors, such that $\mathbf{e}_3 \parallel \mathbf{k}$, so the formulas for N_+ and N_- will take on a normal form:

$$N_+ = (\mathbf{V}_+^*, \mathbf{V}_+) , \qquad N_- = (\mathbf{V}_-^*, \mathbf{V}_-) . \tag{4.53}$$

We conclude from formulas (4.52) that the longitudinal component of the force will decrease (μ / $k_0 \to 0$) with increasing speed of a meson, and will approach a transverse one.

If the same calculation is done for the case of the Proca gauge then, instead of formulas (4.51) and (4.52), one will have the following formulas (cf., Pauli [**13**], pp. 36):

$$N = k_0 \, (\mathbf{U}^* \, \mathbf{U}) - \frac{1}{k_0}(\mathbf{k} \, \mathbf{U}^*)(\mathbf{k} \, \mathbf{U}), \tag{4.51′}$$

$$U = \frac{1}{\sqrt{k_0}}\left[\mathbf{e}_1 V_1 + \mathbf{e}_2 V_2 + \frac{k_0}{k}\mathbf{e}_3 V_3\right]. \tag{4.52′}$$

We conclude from formula (4.52′) that the longitudinal component of force does not vanish (k_0 / $k \to 1$) with increasing speed of a meson, but tends towards the largest transverse value, which is physically unsatisfactory.

3. *Electrodynamics and vector meson dynamics.* We have seen that 5-optics combines electrodynamics and vector meson dynamics into a unified five-dimensional Maxwellian theory. Physically, this means that we have found it necessary to consider not only the usual photons, which relate to $Z = 0$, but also “massive photons” – i.e., vector mesons – in the electromagnetic theory of light in the presence of short wave lengths ($\lambda < h / mc$).

In order to understand this, it would be appropriate to recall the acoustic model that was discussed in § 23.

The question arises: In which cases does one use the laws of classical electrodynamics, and in which cases does it become necessary to go to the five-dimensional Maxwell equations?

If we single out the action coordinate and average over the action coordinate in equations (4.22) and (4.27) then we will get:

$$\frac{\partial \bar{W}_{mn}}{\partial x^n} = \bar{Q}_m, \qquad \frac{\partial \bar{W}_{mn}}{\partial x^k} + \frac{\partial \bar{W}_{nk}}{\partial x^m} + \frac{\partial \bar{W}_{km}}{\partial x^n} = 0, \tag{4.54}$$

$$\frac{\partial \bar{W}_{5n}}{\partial x^n} = \bar{Q}_5, \qquad \frac{\partial \bar{W}_{n5}}{\partial x^m} - \frac{\partial \bar{W}_{m5}}{\partial x^n} = 0. \tag{4.55}$$

If we identify the 4-vector $\bar{W}_k = U_k\,(0 \mid x^1, x^2, x^3, x^4)$ with the electromagnetic potential then the system of equations (4.54) will be the system of Maxwell equations precisely. If one identifies the 4-scalar $W_5 = U_5\,(0 \mid x^1, x^2, x^3, x^4)$ with the χ-potential then the system of equations (4.55) will be the system for the scalar χ-field.

In the classical approximation, when we neglect all of the Fourier components, except for $Z = 0$ (i.e., we neglect the dependence of the field components on the action coordinate) we will get a simple superposition of the electromagnetic field with the χ-field, just as in static electrodynamics we get a simple superposition of the magnetic and the electric fields.

If we take the higher Fourier components into account (i.e., field components that depend periodically upon the action coordinate) then that will bring about the "induction of action," and both fields will be reciprocally related to each other. In the corpuscular picture, this "induction of action" will correspond to the appearance of "massive photons" – i.e., vector mesons.

Thus, while we are dealing with electromagnetic waves with $\lambda \gg h / mc$, it will be legitimate for us to use the classical (i.e., averaged over the action coordinate) electromagnetic theory of light.

In the presence of short wave lengths ($\lambda \ll h / mc$), one should have to use the exact five-dimensional Maxwell theory.

We shall now consider the interaction of two continuously-distributed charges.

If we generalize formula (4.46) to the case of the Fourier expansion of the total charge density:

$$Q_0\,(x^1, x^2, x^3, x^4, x^5) = \sum_{Z=-\infty}^{+\infty} \exp\,(iZ\mu\, x^5)\; q_0\,(Z \mid x^1, x^2, x^3, x^4) \tag{4.56}$$

then we will get:

$$\frac{1}{8\pi}\int \frac{q_0(0\,|\,r', x^0)\, q_0(0\,|\,r'', x^0)}{|\,r' - r''\,|}\, dV' dV'' + \frac{1}{4\pi}\sum_{Z=1}^{\infty} \int \frac{q_0(Z\,|\,r', x^0)\, q_0(Z\,|\,r'', x^0)}{|\,r' - r''\,|}\, e^{-Z\mu |r' - r''|}\, dV' dV'' \tag{4.46′}$$

for the interaction energy.

We see that in the exact theory, in addition to the classical Coulomb interaction, interaction forces of Yukawa type will appear, with a progressively-decreasing radius of action. They can be ignored if the charge is located at a distance that is greater than $1 / \mu$.

In 5-optics, the electromagnetic field and the χ field are the components of the $G_{\mu\nu}$ field that defines the metric relationships in configuration 5-space.

Therefore, the five-dimensional Maxwell theory that is outlined here is an approximation that is legitimate only in the case where one can neglect gravitational phenomena. In the future, it should be included as part of a unified quantum theory of the $G_{\mu\nu}$ field.

§ 26. Pseudo-vector mesons.

Let a real field in 5-space be given a Lagrangian function:

$$L = \frac{1}{12} W_{\lambda\mu\nu}\, W_{\lambda\mu\nu}\,, \tag{4.57}$$

where the components of the field $W_{\lambda\mu\nu}$ define a third-rank 5-multi-vector.

The components of the field $W_{\lambda\mu\nu}$ can be expressed in terms of the potential fields $W_{\lambda\mu} = - W_{\lambda\mu}$ by the formula:

$$W_{\lambda\mu\nu} = \frac{\partial W_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial W_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial W_{\nu\lambda}}{\partial x^{\mu}}. \tag{4.58}$$

The potentials $W_{\lambda\mu}$ are defined up to a gauge:

$$W'_{\lambda\mu} = W_{\lambda\mu} + \frac{\partial F_{\mu}}{\partial x^{\lambda}} - \frac{\partial F_{\lambda}}{\partial x^{\mu}}\ . \tag{4.59}$$

If we vary the potentials for $W_{\lambda\mu\nu}$ in (4.57) then we will get the first group of field equations:

$$\frac{\partial W_{\lambda\mu\nu}}{\partial x^{\nu}} = 0. \tag{4.60}$$

The second group of field equations can be written in the form:

$$\frac{\partial W_{\lambda\mu\nu}}{\partial x^{\sigma}} - \frac{\partial W_{\mu\nu\sigma}}{\partial x^{\lambda}} + \frac{\partial W_{\nu\sigma\lambda}}{\partial x^{\mu}} - \frac{\partial W_{\sigma\lambda\mu}}{\partial x^{\nu}} = 0, \tag{4.61}$$

which is equivalent to (4.58).

Comparing (4.58) with (4.60) will give a system of wave equations for the potentials $W_{\lambda\mu}$:

$$\frac{\partial^2 W_{\lambda\mu}}{\partial x^{\nu}\, \partial x^{\nu}} + \frac{\partial^2 W_{\mu\nu}}{\partial x^{\lambda}\, \partial x^{\nu}} + \frac{\partial^2 W_{\nu\lambda}}{\partial x^{\mu}\, \partial x^{\nu}} = 0. \tag{4.62}$$

Now, use this opportunity to gauge the potentials, and require that they satisfy the following Lorentz-invariant condition:

$$\frac{\partial W_{\lambda\mu}}{\partial x^{\mu}} = \frac{\partial W_{\lambda 5}}{\partial x^{5}}. \tag{4.63}$$

In this case, equations (4.62) will take the form:

$$\frac{\partial^2 W_{\lambda\mu}}{\partial x^{\nu}\,\partial x^{\nu}} + \frac{\partial^2 W_{\mu 5}}{\partial x^{\lambda}\,\partial x^{5}} - \frac{\partial^2 W_{5\lambda}}{\partial x^{\mu}\,\partial x^{5}} = 0, \tag{4.64}$$

or, if we single out the action coordinate:

$$\begin{gathered} \left(\Box + \frac{\partial^2}{\partial x^5\,\partial x^5}\right) W_{ik} + \frac{\partial}{\partial x^5}\left(\frac{\partial W_{k5}}{\partial x^i} - \frac{\partial W_{i5}}{\partial x^k}\right) = 0, \\ W_{5kl} = \frac{\partial W_{kl}}{\partial x^5}. \end{gathered} \tag{4.65}$$

The field equations will then take the form:

$$\left.\begin{aligned} \frac{\partial W_{ikn}}{\partial x^n} + \frac{\partial W_{ik5}}{\partial x^5} \equiv \frac{\partial W_{ikn}}{\partial x^n} + \frac{\partial^2 W_{ik}}{\partial x^5\,\partial x^5} = 0, \\ \frac{\partial W_{ikn}}{\partial x^n} \equiv \frac{\partial}{\partial x^5}\left(\frac{\partial W_{kn}}{\partial x^n}\right) = 0, \end{aligned}\right\} \tag{4.60$'$}$$

$$\frac{\partial W_{ikl}}{\partial x^n} - \frac{\partial W_{kln}}{\partial x^i} + \frac{\partial W_{lni}}{\partial x^k} - \frac{\partial W_{nik}}{\partial x^l} = 0,$$

$$\frac{\partial W_{ikl}}{\partial x^5} - \frac{\partial W_{kl5}}{\partial x^i} + \frac{\partial W_{l5i}}{\partial x^k} - \frac{\partial W_{5ik}}{\partial x^l} \equiv \frac{\partial}{\partial x^5}\left(W_{ikl} - \frac{\partial W_{kl}}{\partial x^i} - \frac{\partial W_{li}}{\partial x^k} - \frac{\partial W_{ik}}{\partial x^l}\right) = 0. \tag{4.61$'$}$$

In order to clarify the physical meaning of this prescription for obtaining five-dimensional field equations, it is necessary to rewrite them in terms of Fourier components. We will get (with the condition that $Z = \pm 1$):

$$\left.\begin{aligned} \frac{\partial U_{ikn}}{\partial x^n} - \mu^2 U_{ik} = 0, \\ \frac{\partial U_{kn}}{\partial x^n} = 0, \end{aligned}\right\} \tag{4.60$''$}$$

$$\left.\begin{aligned}\frac{\partial U_{ikn}}{\partial x^n}-\frac{\partial U_{kln}}{\partial x^i}+\frac{\partial U_{lni}}{\partial x^k}-\frac{\partial U_{nik}}{\partial x^l}=0,\\ U_{ikl}=\frac{\partial U_{kl}}{\partial x^i}+\frac{\partial U_{li}}{\partial x^k}+\frac{\partial U_{ik}}{\partial x^l}.\end{aligned}\right\} \tag{4.61″}$$

If you make the change of notation $\mu^2\ U_{ik} \to U_{ik}$ in the system of equations (4.60″) and (4.61″) then that system will coincide with the system of equations that describes pseudo-vector mesons (cf. [**13**], pp. 34). Thus, we have shown that real fields $W_{\lambda\mu\nu}$ in 5-space, with a suitable gauge (4.63), will describe the entire family of pseudo-vector mesons of mass $|\ Z\ |\ m$ and charge Ze, where Z is a positive or negative integer, including zero. The case for which $Z = 0$ deserves special consideration (cf., § 29).

We shall now generalize the equations for the fields $W_{\lambda\mu\nu}$ to the case in which field sources are present in space, and write them in the form:

$$\frac{\partial W_{\lambda\mu\nu}}{\partial x^\nu}=Q_{\lambda\mu}\,,\qquad \frac{\partial W_{\lambda\mu\nu}}{\partial x^\sigma}-\frac{\partial W_{\mu\nu\sigma}}{\partial x^\lambda}+\frac{\partial W_{\nu\sigma\lambda}}{\partial x^\mu}-\frac{\partial W_{\sigma\lambda\mu}}{\partial x^\nu}=0, \tag{4.65}$$

in which:

$$Q_{\lambda\mu}=q_{\lambda\mu}\exp(i\mu\, x^5)+\overset{*}{q}_{\lambda\mu}\exp(-\,i\mu\, x^5), \tag{4.66}$$

is the source 5-tensor of the fields $W_{\lambda\mu\nu}$.

In order to focus on the use of free meson fields, we shall use the Ginzburg gauge and require the following relativistically non-invariant condition on the potentials:

$$\frac{\partial W_{\lambda\mu}}{\partial x^\mu}=\frac{\partial W_{\lambda 4}}{\partial x^4}. \tag{4.67}$$

With the gauge (4.67), the equations of the potentials will take on the form:

$$\frac{\partial^2 W_{\lambda\mu}}{\partial x^\nu\,\partial x^\nu}+\frac{\partial^2 W_{\mu 4}}{\partial x^4\,\partial x^\lambda}+\frac{\partial^2 W_{4\lambda}}{\partial x^\mu\,\partial x^4}=Q_{\lambda\mu}, \tag{4.68}$$

or, when written down in three-dimensional form and in terms Fourier components (α, β = 1, 2, 3):

$$\left.\begin{aligned}(\Box-\mu^2)U_{\alpha\beta}+\frac{\partial}{\partial x^4}\left(\frac{\partial U_{\beta 4}}{\partial x^\alpha}-\frac{\partial U_{\alpha 4}}{\partial x^\beta}\right)=q_{\alpha\beta},\\ (\Box-\mu^2)U_{\alpha 5}+\frac{\partial}{\partial x^4}\left(\frac{\partial U_{54}}{\partial x^\alpha}-i\mu U_{\alpha 4}\right)=q_{\alpha\beta},\\ (\Delta-\mu^2)U_{\alpha 4}=q_{\alpha 4},\\ (\Delta-\mu^2)U_{54}=q_{54}\,;\end{aligned}\right\} \tag{4.69}$$

We will then get:

$$\left.\begin{aligned} U_{\alpha 4} &= -\frac{1}{4\pi}\int \frac{q_{\alpha 4}(r', x^4)}{|r'-r|} e^{-\mu|r-r'|}\, dV', \\ U_{54} &= -\frac{1}{4\pi}\int \frac{q_{54}(r', x^4)}{|r'-r|} e^{-\mu|r-r'|}\, dV'. \end{aligned}\right\} \tag{4.70}$$

If one writes (4.57) in covariant form:

$$L = \frac{1}{12}\sqrt{|G|}\, W_{\sigma\tau\lambda}\, W^{\sigma\tau\lambda} \tag{4.57'}$$

and varies the metric potentials $G^{\mu\nu}$ then one will get the usual expression for the symmetrized 5-tensor of energy-impulse-charge:

$$\Theta_{\lambda\mu} = \tfrac{1}{2} W_{\lambda\sigma\tau}\, W_{\mu\sigma\tau} - \delta_{\lambda\mu}\, L\ . \tag{4.71}$$

Using the time coordinate $x^0 = -\,i\,x^4$, we calculate ($\alpha, \beta, \gamma = 1, 2, 3$):

The energy density:

$$\bar{\Theta}_{00} = \tfrac{1}{2}\left(U^*_{0\alpha\beta}U_{0\alpha\beta} + 2U^*_{05\alpha}U_{05\alpha} + \tfrac{1}{3}U^*_{\alpha\beta\gamma}U_{\alpha\beta\gamma} + U^*_{\alpha\beta 5}U_{\alpha\beta 5}\right), \tag{4.72}$$

The impulse density:

$$\bar{\Theta}_{\alpha 0} = \tfrac{1}{2}\left(U^*_{\alpha\beta\gamma}U_{0\beta\gamma} + 2U^*_{\alpha 5\gamma}U_{05\gamma} + \text{c. c.}\right), \tag{4.72'}$$

The charge density:

$$\bar{\Theta}_{50} = \tfrac{1}{2}\left(U^*_{5\alpha\beta}U_{0\alpha\beta} + \text{c. c.}\right).$$

If we take the gauge condition (4.67) into account:

$$\left.\begin{aligned} \frac{\partial U_{ik}}{\partial x^k} + i\mu U_{i5} &= \frac{\partial U_{i4}}{\partial x^4}, \\ \frac{\partial U_{5k}}{\partial x^k} &= \frac{\partial U_{54}}{\partial x^4} \end{aligned}\right\} \tag{4.67'}$$

then we can integrate by parts and convert the expression (4.72) into the form:

$$\bar{\Theta}_{00} = \frac{1}{2}\left(\frac{\partial U^*_{\alpha\beta}}{\partial x^0}\frac{\partial U_{\alpha\beta}}{\partial x^0} + 2\frac{\partial U^*_{\alpha 5}}{\partial x^0}\frac{\partial U_{\alpha 5}}{\partial x^0} + \tfrac{1}{3}U^*_{\alpha\beta\gamma}U_{\alpha\beta\gamma} + U^*_{\alpha\beta 5}U_{\alpha\beta 5}\right)$$

$$+\ \frac{1}{4}\left\{U_{0\alpha}^{*}(\mu^2-\Delta)U_{0\alpha}+U_{05}^{*}(\mu^2-\Delta)U_{05}+\text{c. c.}\right\}$$
$$+\ \text{spatial divergence}, \tag{4.72$'$a}$$

$$\bar{\Theta}_{\alpha 0} = \left(-\frac{1}{2}\frac{\partial U_{\beta\gamma}^{*}}{\partial x^{\alpha}}\frac{\partial U_{\beta\gamma}}{\partial x^{0}}-\frac{\partial U_{5\beta}^{*}}{\partial x^{\alpha}}\frac{\partial U_{5\beta}}{\partial x^{0}}+\text{c. c.}\right)$$
$$+\ \frac{1}{4}\left\{U_{\alpha\beta}^{*}(\mu^2-\Delta)U_{0\beta}+U_{\alpha 5}^{*}(\mu^2-\Delta)U_{05}+\text{c. c.}\right\}$$
$$+\ \text{spatial divergence}, \tag{4.72$'$b}$$

$$\bar{\Theta}_{50} = i\mu\left\{\frac{1}{2}\left(U_{\alpha\beta}^{*}\frac{\partial U_{\alpha\beta}}{\partial x^{0}}-U_{\alpha\beta}\frac{\partial U_{\alpha\beta}^{*}}{\partial x^{0}}\right)+\left(U_{\alpha 5}^{*}\frac{\partial U_{\alpha 5}}{\partial x^{0}}-U_{\alpha 5}\frac{\partial U_{\alpha 5}^{*}}{\partial x^{0}}\right)\right\}$$
$$+\ \frac{1}{4}\left\{U_{5\alpha}^{*}(\mu^2-\Delta)U_{0\alpha}+\text{c. c.}\right\}$$
$$+\ \text{spatial divergence}. \tag{4.72$'$c}$$

Integrating over volume and taking (4.70) and (4.69) into account will give:

$$\int\bar{\Theta}_{00}\,dV = \frac{1}{2}\int\left\{\frac{\partial U_{\alpha\beta}^{*}}{\partial x^{0}}\frac{\partial U_{\alpha\beta}}{\partial x^{0}}+2\frac{\partial U_{\alpha 5}^{*}}{\partial x^{0}}\frac{\partial U_{\alpha 5}}{\partial x^{0}}+\frac{1}{3}U_{\alpha\beta\gamma}^{*}U_{\alpha\beta\gamma}+U_{\alpha\beta 5}^{*}U_{\alpha\beta 5}\right\}dV$$
$$+\ \frac{1}{4\pi}\int\frac{q_{0\alpha}^{*}(r',x^0)q_{0\alpha}(r'',x^0)+q_{05}^{*}(r',x^0)q_{05}(r'',x^0)}{|r'-r''|}e^{-\mu(r'-r'')}dV'dV''. \tag{4.73}$$

The second term gives the interaction energy of continuously-distributed sources of the $W_{\lambda\mu\nu}$ field that interact according to the Yukawa law. It will be zero if the field vanishes in all of space; i.e., $q_{\mu\nu} = 0$. In that case, it will be the first term that gives the energy of the meson wave field.

Expanding the potentials in plane-waves will give, in the case of wave fields, and while taking (4.70) and (4.67′) into consideration:

$$\begin{aligned} U_{\alpha\beta} &= (2V)^{-1/3}\sum_{(k)}\left\{U_{\alpha\beta}^{+}(k)\exp[i(\mathbf{kr}-k_0x^0)]+U_{\alpha\beta}^{-*}(k)\exp[i(-\mathbf{kr}+k_0x^0)]\right\}, \\ -\,U_{\alpha 5} &= (2V)^{-1/3}\sum_{(k)}\left\{U_{\alpha\beta}^{+}k_{\beta}\exp[i(\mathbf{kr}-k_0x^0)]+U_{\alpha\beta}^{-*}k_{\beta}\exp[i(-\mathbf{kr}+k_0x^0)]\right\}, \end{aligned} \tag{4.74}$$

$$U_{\alpha 4} = 0, \qquad U_{54} = 0.$$

Substituting these expressions in (4.72′) and integrating over volume will give:

$$\int \bar{\Theta}_{00}\, dV = k_0 \sum_{(k)} (N_+(k) + N_-(k)), \tag{4.75a}$$

$$\int \bar{\Theta}_{0\alpha}\, dV = \sum_{(k)} k_\alpha (N_+(k) + N_-(k)), \tag{4.75b}$$

$$\int \bar{\Theta}_{05}\, dV = \mu \sum_{(k)} (N_+(k) - N_-(k)), \tag{4.75c}$$

in which, to abbreviate, we have set:

$$N = k_0 \left\{ \frac{1}{2} U^*_{\alpha\beta} U_{\alpha\beta} + \frac{1}{\mu^2} (k_\alpha U_{\alpha\beta})(k_\gamma U_{\gamma\beta}) \right\}. \tag{4.76}$$

In order to emphasize the transverse and longitudinal components, we put formula (4.76) into normal form. In order to do that, we introduce the pseudo-vector $\tilde{U}_\alpha = \frac{1}{2}\, \varepsilon_{\beta\gamma\lambda}\, U_{\beta\gamma}$ and get:

$$\tilde{\mathbf{U}} = \frac{1}{\sqrt{k_0}} \left\{ \frac{\mu_0}{k_0} (\tilde{V}_1 \mathbf{e}_1 + \tilde{V}_2 \mathbf{e}_2) + \tilde{V}_3 \mathbf{e}_3 \right\}, \tag{4.77}$$

in which, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are three mutually-perpendicular unit vectors, with $\mathbf{e}_3 \parallel \mathbf{k}$, which gives:

$$N = (\tilde{V}^* V). \tag{4.78}$$

With increasing speed of the meson with longitudinal components $\tilde{V}_1 = V_{23}$, $\tilde{V}_2 = V_{31}$, due to the fact that $\mu / k_0 \to 0$, they will gradually disappear, and the field will gradually approach a transverse one. Note that if all of the calculations are repeated in the case of normal gauge then it will emerge that for mesons at higher speeds the longitudinal components will tend to have the same magnitude as the transverse ones, which is physically unacceptable.

§ 27. Pseudo-scalar mesons.

Consider a 5-field that is described by the following group of equations:

$$\frac{\partial W_{\lambda\mu\nu\sigma}}{\partial x^\sigma} = Q_{\lambda\mu\nu}, \tag{4.78a}$$

$$\frac{\partial W_{\lambda\mu\nu\sigma}}{\partial x^\tau} + \frac{\partial W_{\mu\nu\sigma\tau}}{\partial x^\lambda} + \frac{\partial W_{\nu\sigma\tau\lambda}}{\partial x^\mu} + \frac{\partial W_{\sigma\tau\lambda\mu}}{\partial x^\nu} + \frac{\partial W_{\tau\lambda\mu\nu}}{\partial x^\sigma} = 0, \tag{4.78b}$$

where the components of the field $W_{\lambda\mu\nu\sigma}$ define a fifth-rank 5-multi-vector. The components of the 5-tensor $Q_{\lambda\mu\nu}$ of the field source define a fourth-rank 5-multi-vector. The components of the field $W_{\lambda\mu\nu\sigma}$ are expressed in terms of the potentials $W_{\lambda\mu\nu}$, which are the generators of a third-rank 5-multi-vector, by the formula:

$$W_{\lambda\mu\nu\sigma} = \frac{\partial W_{\lambda\mu\nu}}{\partial x^{\sigma}} - \frac{\partial W_{\mu\nu\sigma}}{\partial x^{\lambda}} + \frac{\partial W_{\nu\sigma\lambda}}{\partial x^{\mu}} - \frac{\partial W_{\sigma\lambda\mu}}{\partial x^{\nu}}. \tag{4.79}$$

The potentials $W_{\lambda\mu\nu}$ are defined up to a gauge:

$$\left.\begin{aligned} W'_{\lambda\mu\nu} &= W_{\lambda\mu\nu} + \frac{\partial F_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial F_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial F_{\nu\lambda}}{\partial x^{\mu}}, \\ F_{\lambda\mu} &= -F_{\mu\lambda}. \end{aligned}\right\} \tag{4.80}$$

If we substitute (4.79) into (4.78) then it will satisfy (4.78b) identically, and we will get the following system of equations for the potentials:

$$\frac{\partial^2 W_{\lambda\mu\nu}}{\partial x^{\sigma}\partial x^{\sigma}} - \frac{\partial^2 W_{\mu\nu\sigma}}{\partial x^{\sigma}\partial x^{\lambda}} + \frac{\partial^2 W_{\nu\sigma\lambda}}{\partial x^{\sigma}\partial x^{\mu}} - \frac{\partial^2 W_{\sigma\lambda\mu}}{\partial x^{\sigma}\partial x^{\nu}} = Q_{\lambda\mu\nu}. \tag{4.81}$$

We now take the opportunity to gauge the potentials, and require that they satisfy the following Lorentz-invariant potential normalization:

$$\frac{\partial W_{\lambda\mu\nu}}{\partial v^{\nu}} = \frac{\partial W_{\lambda\mu 5}}{\partial v^{5}}. \tag{4.82}$$

The system of equations (4.81) can then be put into the form:

$$\left\{\Box - \frac{\partial^2}{\partial x^5 \partial x^5}\right\} W_{ikl} - \frac{\partial}{\partial x^5}\left\{\frac{\partial W_{kl5}}{\partial x^{i}} + \frac{\partial W_{li5}}{\partial x^{k}} + \frac{\partial W_{ik5}}{\partial x^{l}}\right\} = Q_{ikl}, \tag{4.83}$$

$$\Box W_{5kl} = Q_{5\mathrm{kl}}. \tag{4.79}$$

If this field vanishes – i.e., $Q_{\lambda\mu\nu} = 0$ – throughout space then we can put $W_{5\mathrm{kl}} = 0$, and formula (4.79) will take the form:

$$\left.\begin{aligned} W_{iklm} &= \frac{\partial W_{ikl}}{\partial x^{m}} - \frac{\partial W_{klm}}{\partial x^{i}} + \frac{\partial W_{lmi}}{\partial x^{k}} - \frac{\partial W_{mik}}{\partial x^{l}}, \\ W_{klm5} &= \frac{\partial W_{klm}}{\partial x^{5}}, \end{aligned}\right\} \tag{4.79′}$$

while the field equations (4.78) will take the form:

$$\left.\begin{aligned} &\frac{\partial W_{iklm}}{\partial x^m}-\frac{\partial^2 W_{ikl}}{\partial x^5\,\partial x^5}=0,\\ &\frac{\partial}{\partial x^5}\left(\frac{\partial W_{klm}}{\partial x^m}\right)=0, \end{aligned}\right\} \tag{4.78″a}$$

$$\left.\begin{aligned} &\frac{\partial U_{iklm}}{\partial x^n}+\frac{\partial U_{klmn}}{\partial x^i}+\frac{\partial U_{lmni}}{\partial x^k}+\frac{\partial U_{mnik}}{\partial x^l}+\frac{\partial U_{lmni}}{\partial x^m}=0,\\ &U_{iklm}=\frac{\partial U_{ikl}}{\partial x^m}-\frac{\partial U_{klm}}{\partial x^l}+\frac{\partial U_{lmi}}{\partial x^i}-\frac{\partial U_{mik}}{\partial x^l}. \end{aligned}\right\} \tag{4.78″b}$$

If we make the replacement $\mu^2\ U_{ikl} \to U_{ikl}$ in our notation then the system of equations (4.78″) will coincide with the system of equations that describes pseudo-scalar mesons ([**13**], pp. 22).

Thus, we have shown that real fields $W_{\lambda\mu\nu\sigma}$ in 5-space, with a suitable gauge (4.82), describe the entire family of pseudo-scalar mesons of mass $|Z|\,m$ and charge Ze, where Z is a positive or negative integer, including zero. The case of $Z = 0$ deserves special attention (cf., § 29).

Passing to the general case of the fields $W_{\lambda\mu\nu\sigma}$ that have source fields in space, we use the Ginzburg gauge, and require that we impose the following relativistically non-invariant formula on the potentials:

$$\frac{\partial W_{\lambda\mu\nu}}{\partial x^\nu}=\frac{\partial W_{\lambda\mu 4}}{\partial x^4}. \tag{4.84}$$

The system of equations (4.81) will take following form in Fourier components:

$$(\Box-\mu^2)\,U_{\alpha\beta\gamma}-\frac{\partial}{\partial x^4}\left\{\frac{\partial U_{\alpha\beta 4}}{\partial x^\gamma}+\frac{\partial U_{\beta\gamma 4}}{\partial x^\alpha}+\frac{\partial U_{\gamma\alpha 4}}{\partial x^\beta}\right\}=q_{\alpha\beta\gamma},$$

$$(\Box-\mu^2)\,U_{\alpha\beta 5}-\frac{\partial}{\partial x^4}\left\{i\mu U_{\alpha\beta 4}+\frac{\partial U_{\beta 54}}{\partial x^\alpha}+\frac{\partial U_{5\alpha 4}}{\partial x^\beta}\right\}=q_{\alpha\beta 5},$$

$$\left.\begin{aligned} &(\Delta-\mu^2)U_{\alpha\beta 4}=q_{\alpha\beta 4,}\\ &(\Delta-\mu^2)U_{\alpha 54}=q_{\alpha 54,} \end{aligned}\right\} \tag{4.81′}$$

so:

$$\left.\begin{aligned} &U_{\alpha\beta 4}=-\frac{1}{4\pi}\int\frac{q_{\alpha\beta 4}(r',x^0)}{|r'-r|}e^{-\mu|r'-r|}dV',\\ &U_{\alpha 54}=-\frac{1}{4\pi}\int\frac{q_{\alpha 54}(r',x^0)}{|r'-r|}e^{-\mu|r'-r|}dV'. \end{aligned}\right\} \tag{4.85}$$

The expression for the symmetric 5-tensor of energy-impulse-charge $\Theta_{\lambda\mu}$ is obtained by the general rules from the Lagrangian function:

$$L = \frac{1}{48}\sqrt{|G|}\, W_{\lambda\mu\nu\sigma}\ W^{\lambda\mu\nu\sigma} \tag{4.86}$$

in the form:

$$\Theta_{\lambda\mu} = \frac{1}{6} W_{\lambda\sigma\tau\rho}\ W_{\mu\sigma\tau\rho} - \frac{1}{48} W_{\sigma\lambda\tau\mu}\ W^{\sigma\tau\rho\nu}. \tag{4.87}$$

Using the time coordinate $x^0 = -\, i\, x^4$, we compute:

Energy density:

$$\bar{\Theta}_{00} = \frac{1}{6}\left\{U^*_{0\alpha\beta\lambda} U_{0\alpha\beta\gamma} + U^*_{5\alpha\beta\lambda} U_{5\alpha\beta\gamma} + 3U^*_{50\alpha\beta} U_{50\alpha\beta}\right\}, \tag{4.88a}$$

Impulse density:

$$\bar{\Theta}_{\alpha 0} = \frac{1}{2}\{U^*_{\alpha\beta\gamma 5} U_{0\beta\gamma 5} + \text{c. c.}\}, \tag{4.88b}$$

Charge density:

$$\bar{\Theta}_{50} = \frac{1}{6}\{U^*_{0\alpha\beta\gamma} U_{5\alpha\beta\gamma} + \text{c. c.}\}. \tag{4.88c}$$

If we integrate by parts, while using the gauge condition (4.84), then we can transform these expressions as follows:

$$\begin{aligned}\bar{\Theta}_{00} = {} & \frac{1}{6}\left\{\frac{\partial U^*_{\alpha\beta\lambda}}{\partial x^0}\frac{\partial U_{\alpha\beta\gamma}}{\partial x^0} + 3\frac{\partial U^*_{5\alpha\beta}}{\partial x^0}\frac{\partial U_{5\alpha\beta}}{\partial x^0} + U^*_{5\alpha\beta\gamma} U_{5\alpha\beta\gamma}\right\} \\ & + \frac{1}{4}\{U^*_{0\alpha\beta}(\mu^2 - \Delta)U_{0\alpha\beta} + U^*_{05\alpha}(\mu^2 - \Delta)U_{05\alpha} + \text{c. c.}\} \\ & + \text{spatial divergence,}\end{aligned} \tag{4.89a}$$

$$\begin{aligned}\bar{\Theta}_{0\alpha} = {} & \frac{1}{6}\left\{\frac{\partial U^*_{\beta\lambda\delta}}{\partial x^\alpha}\frac{\partial U_{\beta\gamma\delta}}{\partial x^0} + 3\frac{\partial U^*_{5\beta\gamma}}{\partial x^\alpha}\frac{\partial U_{5\beta\gamma}}{\partial x^0} + \text{c. c.}\right\} \\ & + \frac{1}{4}\{U^*_{\alpha\beta\gamma}(\mu^2 - \Delta)U_{0\beta\gamma} + 2U^*_{\alpha 5\beta}(\mu^2 - \Delta)U_{05\beta} + \text{c. c.}\} \\ & + \text{spatial divergence,}\end{aligned} \tag{4.89b}$$

$$\begin{aligned}\bar{\Theta}_{05} = {} & \frac{i\mu}{6}\left\{\left(U^*_{\alpha\beta\lambda}\frac{\partial U_{\alpha\beta\gamma}}{\partial x^0} - U_{\alpha\beta\gamma}\frac{\partial U^*_{\alpha\beta\lambda}}{\partial x^0}\right) + 3\left(U^*_{5\alpha\beta}\frac{\partial U_{5\alpha\beta}}{\partial x^0} - U_{5\alpha\beta}\frac{\partial U^*_{5\alpha\beta}}{\partial x^0}\right)\right\} \\ & + \frac{1}{4}\{U^*_{5\alpha\beta}(\mu^2 - \Delta)U_{0\alpha\beta} + \text{c. c.}\} \\ & + \text{spatial divergence}\ .\end{aligned} \tag{4.89c}$$

Integrating over space and taking into account (4.81′) and (4.85) will give:

$$\int \bar{\Theta}_{00}\, dV = \frac{1}{6}\int \left\{ \frac{\partial U^*_{\alpha\beta\lambda}}{\partial x^0}\frac{\partial U_{\alpha\beta\gamma}}{\partial x^0} + 3\frac{\partial U^*_{5\alpha\beta}}{\partial x^0}\frac{\partial U_{5\alpha\beta}}{\partial x^0} + U^*_{5\alpha\beta\gamma}U_{5\alpha\beta\gamma} \right\} dV$$

$$+ \frac{1}{4\pi}\int \frac{1}{2}\frac{q^*_{0\alpha\beta}(r', x^0)\, q_{0\alpha\beta}(r'', x^0) + 2q^*_{05\alpha}(r', x^0)\, q_{05\alpha}(r'', x^0)}{|r' - r''|} e^{-\mu|r'-r''|}\, dV' dV'' . \quad (4.90)$$

The second term gives the interaction energy for continuously-distributed sources of meson fields that interact by the Yukawa law. If the field vanishes then it will be zero.

If we expand the potentials $U_{\alpha\beta\gamma}$ in plane-waves, as in the preceding section, then that will give, in the case of vanishing fields:

$$\int \bar{\Theta}_{00}\, dV = k_0 \sum_{(k)} (N_+(k) + N_-(k)) , \quad (4.91a)$$

$$\int \bar{\Theta}_{0\alpha}\, dV = \sum_{(k)} k_\alpha (N_+(k) + N_-(k)) , \quad (4.91b)$$

$$\int \bar{\Theta}_{05}\, dV = \mu \sum_{(k)} (N_+(k) - N_-(k)) , \quad (4.91c)$$

in which, in order to make the indicated reductions, one must set:

$$N(k) = k_0 \left\{ \frac{1}{6} U^*_{\alpha\beta\gamma} U_{\alpha\beta\lambda} + \frac{1}{2\mu^2}(k_\alpha U^*_{\alpha\beta\gamma})(k_\delta U_{\delta\beta\lambda}) \right\} . \quad (4.92)$$

Introducing the pseudo-scalar $\tilde{U} = \frac{1}{6}\, \varepsilon_{\alpha\beta\gamma} U_{\alpha\beta\gamma}$ will give:

$$N = k_0 \frac{k_0^2}{\mu^2} \tilde{U}^* U . \quad (4.93)$$

As the speed of the meson increases, since $k_0 / \mu \to \infty$, one will get $\tilde{U} = U_{123} \to 0$, as it should be, since it is the only potential component $U_{\alpha\beta\gamma}$ that contain the index 3, and should be considered to be longitudinal.

§ 28. Particles of spin 2 (metrons)

The extension of the theory of weak gravitational fields to the general case of weak metric fields in 5-space leads to the theory of particles of spin two, which we proposed to call *fundamental* in our previous communications. It seems to us that is would be better to call weak quantum metric fields *metrons,* and that is the term that we will use in what follows. The theory of metrons is essential to 5-optics, because they provide an exchange of energy, impulse, and charge between all of the elementary particles that are present.

For weak metric 5-fields, we can set:

$$G_{\mu\nu} = \delta_{\mu\nu} + H_{\mu\nu}\,, \qquad G^{\mu\nu} = \delta_{\mu\nu} - H_{\mu\nu}\,, \tag{4.94}$$

in which $H_{\mu\nu}$ are small quantities, and we are neglecting quadratic terms.

Insofar as:

$$G^{\mu\nu} = \begin{pmatrix} \tilde{g}^{ik} & -\tilde{g}^{ki} g_i \\ -\tilde{g}^{ki} g_k & \dfrac{1}{1+\chi} + \tilde{g}^{ik} g_i g_k \end{pmatrix}, \tag{4.95}$$

we find the physical meaning of the quantities $H_{\mu\nu}$, namely:

$$H_{ik} = h_{ik}\,, \qquad H_{5k} = g_k\,, \qquad H_{55} = \chi, \qquad H = h + \chi, \tag{4.96}$$

in which h_{ik} appears in the theory of gravitational fields as the components of the true weak gravitational field.

We shall confine ourselves here to the consideration of vanishing fields without sources. For weak fields, the Einstein equations $P_{\lambda\mu} - \frac{1}{2} G_{\lambda\mu} P = 0$ take the form (the first group of field equations):

$$\frac{\partial G_{\sigma,\lambda\mu}}{\partial x^{\sigma}} - \frac{\partial G_{\sigma,\lambda\sigma}}{\partial x^{\mu}} - \frac{1}{2}\delta_{\lambda\mu}\left(\frac{\partial G_{\sigma,\tau\tau}}{\partial x^{\sigma}} - \frac{\partial G_{\sigma,\tau\sigma}}{\partial x^{\tau}}\right) = 0. \tag{4.97}$$

The components of the field $G_{\sigma,\lambda\mu}$ can be expressed in terms of the potential field $H_{\mu\nu}$ by the formula:

$$G_{\sigma,\lambda\mu} = \frac{1}{2}\left(\frac{\partial H_{\sigma\lambda}}{\partial x^{\mu}} - \frac{\partial H_{\lambda\mu}}{\partial x^{\sigma}} + \frac{\partial H_{\mu\sigma}}{\partial x^{\lambda}}\right). \tag{4.98}$$

We can write down the second group of equations in the form:

$$\frac{\partial}{\partial x^{\nu}}\{G_{\sigma,\lambda\mu} + G_{\lambda,\sigma\mu}\} - \frac{\partial}{\partial x^{\mu}}\{G_{\sigma,\lambda\nu} + G_{\lambda,\sigma\nu}\} = 0, \tag{4.99}$$

which is equivalent to (4.98).

The potential field $H_{\mu\nu}$ is determined up to a gauge:

$$H'_{\mu\nu} = H_{\mu\nu} + \frac{\partial F_{\mu}}{\partial x^{\nu}} + \frac{\partial F_{\nu}}{\partial x^{\mu}}\,. \tag{4.100}$$

Substituting (4.98) in (4.97) will give the following system of wave equations for the potentials $H_{\mu\nu}$:

$$\frac{\partial^2 H_{\sigma\lambda}}{\partial x^\tau \partial x^\tau} - \frac{\partial}{\partial x^\tau}\left(\frac{\partial H_{\tau\sigma}}{\partial x^\lambda} + \frac{\partial H_{\tau\lambda}}{\partial x^\sigma}\right) + \frac{\partial^2 H}{\partial x^\sigma \partial x^\lambda} + \delta_{\sigma\lambda}\left(\frac{\partial^2 H_{\mu\nu}}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 H}{\partial x^\nu \partial x^\nu}\right) = 0. \qquad (4.101)$$

In order to discover the physical meaning of these five-dimensional equations, they need to be rewritten in terms of Fourier components.

Let:

$$H_{\mu\nu} = A_{\mu\nu} \exp(i\mu x^5) + A^*_{\mu\nu} \exp(-i\mu x^5). \qquad (4.102)$$

The wave equations (4.101) will then take the form:

$$(\Box - \mu^2)A_{ik} - \frac{\partial}{\partial x^l}\left(\frac{\partial A_{li}}{\partial x^k} + \frac{\partial A_{lk}}{\partial x^i}\right) - i\mu\left(\frac{\partial A_{5i}}{\partial x^k} + \frac{\partial A_{5k}}{\partial x^i}\right) + \frac{\partial^2 A_{nn}}{\partial x^i \partial x^k}$$
$$+ \delta_{ik}\left(\frac{\partial^2 A_{ln}}{\partial x^l \partial x^n} + 2\frac{\partial A_{5l}}{\partial x^l} i\mu - \mu^2 A_{55} - (\Box - \mu^2)A_{nn}\right) = 0. \qquad (4.103)$$

$$\Box A_{k5} - \frac{\partial}{\partial x^k}\left[\frac{\partial A_{l5}}{\partial x^l} + (A_{55} - A_{nn})i\mu\right] = 0, \qquad (4.103')$$

$$\frac{\partial^2 A_{ik}}{\partial x^i \partial x^k} + \Box(A_{55} - A_{nn}) = 0. \qquad (4.103'')$$

We now impose the following Lorentz-invariant conditions on the potentials:

$$A_{15} = A_{25} = A_{35} = A_{45} = A_{55} = 0. \qquad (4.104)$$

Equations (4.103) will then take the form:

$$(\Box - \mu^2)A_{ik} - \frac{\partial}{\partial x^l}\left(\frac{\partial A_{li}}{\partial x^k} + \frac{\partial A_{lk}}{\partial x^i}\right) + \frac{\partial^2 A_{nn}}{\partial x^i \partial x^k} + \delta_{ik}\left(\frac{\partial^2 A_{ln}}{\partial x^l \partial x^n} + (\Box - \mu^2)A_{nn}\right)A_{ik} = 0. \quad (4.105')$$

$$\mu\frac{\partial A_{nn}}{\partial x^k} = 0, \qquad (4.105'')$$

$$\frac{\partial^2 A_{ik}}{\partial x^i \partial x^k} - \Box A_{nn} = 0. \qquad (4.105''')$$

Equations (4.105′) are exactly the same as the Fierz-Pauli system of equations, which describe particles of spin two (cf., [**14**], pp. 242, *et seq.*) It is also shown there that when $\mu \neq 0$ in equations (4.105′), it will follow as a consequence that:

$$\frac{\partial A_{ik}}{\partial x^k} = 0, \qquad A_{nn} = 0, \tag{4.106}$$

and therefore equations (4.105″) and (4.105‴), which are absent from Fierz-Pauli, will be satisfied identically, and equation (4.105′) will take the form:

$$(\Box - \mu^2)A_{ik} = 0. \tag{4.107}$$

Thus, we have shown that weak metric fields in 5-space, with suitably-normalized potentials (4.104), will describe the entire family of Fierz-Pauli particles of mass $|Z|m$ and charge Ze, where Z is a positive or negative whole number. The case of $Z = 0$ needs special consideration (cf., § 29 below).

Now, use the Ginzburg gauge and require that the following relativistically non-invariant condition should be imposed upon the potentials:

$$A_{41} = A_{42} = A_{43} = A_{44} = A_{45} = 0. \tag{4.108}$$

In what follows up to the end of this section, Latin characters m, n, p, q will take the values 1, 2, 3, 5. The wave equations will now take on the following form:

$$(\Box - \mu^2)A_{mn} - \frac{\partial}{\partial x^p}\left(\frac{\partial A_{pm}}{\partial x^n} + \frac{\partial A_{pn}}{\partial x^m}\right) + \frac{\partial^2 A_{pp}}{\partial x^m \partial x^n} + \delta_{ik}\left(\frac{\partial^2 A_{pq}}{\partial x^p \partial x^q} + (\Box - \mu^2)A_{pp}\right) = 0. \tag{4.109′}$$

$$\frac{\partial}{\partial x^4}\frac{\partial A_{pp}}{\partial x^k} = 0, \tag{4.109″}$$

$$\frac{\partial^2 A_{pq}}{\partial x^p\, \partial x^q} - \Box\, A_{pp} = 0. \tag{4.109‴}$$

If the operators:

$$\frac{\partial}{\partial x^n}, \qquad \frac{\partial^2}{\partial x^m\, \partial x^n}, \qquad \delta_{mn}$$

are applied to equations (4.109′) on the left then:

$$\frac{\partial^2}{\partial x^4\, \partial x^4}\left(\frac{\partial A_{mn}}{\partial x^n} - \frac{\partial A_{pp}}{\partial x^m}\right) = 0, \tag{4.110′}$$

$$\frac{\partial^2}{\partial x^4\, \partial x^4}\left(\frac{\partial^2 A_{mn}}{\partial x^m\, \partial x^n} - (\Delta - \mu^2)A_{pp}\right) = 0, \tag{4.110″}$$

$$2\frac{\partial^2 A_{mn}}{\partial x^m \partial x^n} - \left[2(\Delta - \mu^2) + 3\frac{\partial^2}{\partial x^4 \partial x^4}\right] A_{pp} = 0. \qquad (4.110''')$$

If we exclude static fields from consideration then A_{pp} should satisfy (4.110″) and (4.110‴), and thus $\frac{\partial A_{mn}}{\partial x^n} = 0$ from (4.110′), and equation (4.110′) will take the form:

$$(\Box - \mu^2) A_{mn} = 0. \qquad (4.111)$$

The Ginzburg gauge is a generalization of the usual gauge on the components of the weak gravitational field that leads one to choose the transverse gravitational field (i.e., the *graviton*) to the general case of the weak metric field on 5-space. It is well-known that the choice of the graviton as the wave field in the theory of gravity is achieved when one sets (cf., e.g., [**12**], pp. 338):

$$h_{14} = h_{24} = h_{34} = h_{44} = h = 0. \qquad (4.112)$$

In conclusion, we shall write down the wave equations for metrons. Taking (4.96) under consideration and denoting the Fourier component of f by $\hat{f}$ will give:

1. The Fierz-Pauli gauge [equations (4.106) and (4.107)]:

$$\left.\begin{aligned} (\Box - \mu^2)\,\hat{h}_{ik} &= 0, \\ \frac{\partial \hat{h}_{ik}}{\partial x^k} &= 0, \\ \hat{h}_{ii} &= 0. \end{aligned}\right\} \qquad (4.113)$$

2. Ginzburg gauge (α, β = 1, 2, 3):

$$\left.\begin{aligned} (\Box - \mu^2)\,\hat{h}_{\alpha\beta} &= 0, & \frac{\partial \hat{h}_{\alpha\beta}}{\partial x^\beta} + i\mu\hat{g}_\alpha &= 0, \\ (\Box - \mu^2)\,\hat{g}_\alpha &= 0, & \hat{h}_{\alpha\alpha} + \hat{\chi} &= 0, \\ (\Box - \mu^2)\,\hat{\chi} &= 0, & \frac{\partial \hat{g}_\alpha}{\partial x^\varepsilon} + i\mu\hat{\chi} &= 0. \end{aligned}\right\} \qquad (4.114)$$

In both cases, the ten quantities $(\hat{h}_{ik})$ and $(\hat{h}_{\alpha\beta}, \hat{g}_\alpha, \hat{\chi})$ will be subjected to five additional conditions so that the number of independent components of the wave function of the metron will amount to five, which will correspond to a spin-two metron.

If we consider flat, harmonic waves then it will be easy to see that in the case of (4.114), as the speed of the metron increases, the longitudinal component of the field will gradually disappear, while in the case of (4.113), it will tend to the largest transverse component, which is physically unsatisfactory.

§ 29. Mesons and metrons in the zero charge state

According to 5-optics, mesons and metrons can be in a state of zero mass and charge. In the expansion of the wave function into a Fourier series and in the case for which only one component is present, which will correspond to $Z = 0$, one will have:

$$W(x_1, x_2, x_3, x_4, x_5) = U\,(0 \mid x_1, x_2, x_3, x_4) = \overline{W}\,.$$

The state with $Z = 0$ thus describes classical interaction fields in 4-space. We get the field equations by averaging the corresponding equations of 5-optics over the action coordinate.

1) Scalar mesons:

$$\frac{\partial \overline{W}_i}{\partial x^i} = 0, \qquad \frac{\partial \overline{W}_i}{\partial x^k} - \frac{\partial \overline{W}_k}{\partial x^i} = 0, \qquad \frac{\partial \overline{W}_5}{\partial x^i} = 0. \tag{4.116}$$

Introduce the scalar U: $\overline{W}_i = \partial U / \partial x^i$, $\Box U = 0$, $\overline{W}_5 = \text{const}$.

Thus, scalar mesons in the $Z = 0$ state are described a superposition of the classical scalar field U and the constant field $\overline{W}_5 = \text{const}$.

2) Vector mesons:

$$\left.\begin{aligned} &\frac{\partial \overline{W}_{ik}}{\partial x^k} = 0, \qquad \frac{\partial \overline{W}_{ik}}{\partial x^l} + \frac{\partial \overline{W}_{kl}}{\partial x^i} + \frac{\partial \overline{W}_{li}}{\partial x^k} = 0, \\ &\frac{\partial \overline{W}_{5k}}{\partial x^k} = 0, \qquad\qquad \frac{\partial \overline{W}_{5k}}{\partial x^i} - \frac{\partial \overline{W}_{5i}}{\partial x^k} = 0. \end{aligned}\right\} \tag{4.117}$$

Introduce the scalar U: $\overline{W}_{5k} = \partial U / \partial x^k$.

Thus, vector mesons in the $Z = 0$ state are described by a superposition of two fields: A Maxwellian electromagnetic field and a classical scalar field U.

3) Pseudo-vector mesons:

$$\left.\begin{aligned} &\frac{\partial \overline{W}_{ikn}}{\partial x^n} = 0, \qquad \frac{\partial \overline{W}_{ikl}}{\partial x^n} - \frac{\partial \overline{W}_{kln}}{\partial x^i} + \frac{\partial \overline{W}_{lni}}{\partial x^k} - \frac{\partial \overline{W}_{nik}}{\partial x^l} = 0, \\ &\frac{\partial \overline{W}_{5kn}}{\partial x^{kn}} = 0, \qquad\qquad \frac{\partial \overline{W}_{5kl}}{\partial x^n} + \frac{\partial \overline{W}_{5ln}}{\partial x^k} + \frac{\partial \overline{W}_{5nk}}{\partial x^l} = 0. \end{aligned}\right\} \tag{4.118}$$

Introduce the 4-pseudo-vector $\tilde{U}_i = \frac{1}{6}\varepsilon_{ikln}\overline{W}_{kln}$ and the 4-bivector $F_{kl} = \overline{W}_{5kl}$ and rewrite the equations in the form:

$$\left.\begin{array}{ll}\dfrac{\partial\tilde{U}_i}{\partial x^l}-\dfrac{\partial\tilde{U}_l}{\partial x^i}=0, & \dfrac{\partial\tilde{U}_i}{\partial x^i}=0,\\ \dfrac{\partial F_{kn}}{\partial x^n}=0, & \dfrac{\partial F_{kl}}{\partial x^n}+\dfrac{\partial F_{ln}}{\partial x^k}+\dfrac{\partial F_{nk}}{\partial x^i}=0.\end{array}\right\} \tag{4.118′}$$

Introduce the 4-pseudo-scalar $\tilde{U}$: $\tilde{U}_i = \partial\tilde{U}/\partial x^i$. The pseudo-vector meson in the Z = 0 state will then be described by the superposition of two fields: A Maxwellian electromagnetic field and pseudo-scalar field $\tilde{U}$.

4) Pseudo-scalar mesons:

$$\left.\begin{array}{r}\dfrac{\partial\bar{W}_{5kln}}{\partial x^n}=0,\\ \dfrac{\partial\bar{W}_{5kln}}{\partial x^m}-\dfrac{\partial\bar{W}_{5lnm}}{\partial x^k}+\dfrac{\partial\bar{W}_{5nmk}}{\partial x^l}-\dfrac{\partial\bar{W}_{5mkl}}{\partial x^n}=0,\\ \dfrac{\partial\bar{W}_{ikln}}{\partial x^n}=0.\end{array}\right\} \tag{4.119}$$

Introduce the 4-pseudo-vector $\tilde{U}_i = \frac{1}{6}\varepsilon_{ikln}\bar{W}_{5kln}$ and rewrite the equations in form:

$$\frac{\partial\tilde{U}_i}{\partial x^k}-\frac{\partial\tilde{U}_k}{\partial x^i}=0, \qquad \frac{\partial\tilde{U}_i}{\partial x^i}=0. \tag{4.119′}$$

Introduce the 4-pseudo-scalar $\tilde{U}_i = \partial\tilde{U}/\partial x^i$. Pseudo-scalar mesons in the Z = 0 state can then be described as the superposition of two fields: The pseudo-scalar field $\tilde{U}$ and the constant field $\bar{W}_{1234}$ = const.

5) Metrons.

The last of equations (4.101) for the action coordinate gives:

$$\left.\begin{array}{r}\Box\bar{H}_{ik}-\dfrac{\partial}{\partial x^i}\left(\dfrac{\partial\bar{H}_{li}}{\partial x^k}+\dfrac{\partial\bar{H}_{lk}}{\partial x^i}\right)+\dfrac{\partial^2\bar{H}_{nn}}{\partial x^i\,\partial x^k}+\delta_{ik}\left(\dfrac{\partial^2\bar{H}_{in}}{\partial x^i\,\partial x^n}-\Box\bar{H}_{nn}\right)=0,\\ \Box\bar{H}_{5k}-\dfrac{\partial}{\partial x^k}\left(\dfrac{\partial\bar{H}_{5l}}{\partial x^l}\right)=0,\\ \Box\bar{H}_{55}+\dfrac{\partial^2\bar{H}_{ik}}{\partial x^i\,\partial x^k}-\Box\bar{H}_{nn}=0.\end{array}\right\} \tag{4.120}$$

Due to (4.96), we conclude that metrons in the $Z = 0$ state can be described as the superposition of three fields: A weak gravitational field, an electromagnetic field, and the χ-field.

Modern physics distinguishes complex ψ-fields that describe the behavior of charged particles with non-zero rest mass from real classical fields that describe the behavior of real particles of zero rest mass. Such a sharp distinction is mainly due to the fact that in the view of contemporary physics, ψ-fields are localized relative to configuration space in the same way that classical fields are localized in the universal 4-space of the theory of relativity.

Naturally, in 5-optics, which radically refuses to be subordinate to universal space, the main distinction between complex ψ-fields and real classical fields vanishes for them and other particles that can be suitably localized in configuration space. In the mathematical apparatus of 5-optics, real classical fields are represented by the zero term in a Fourier series expansion, and complex ψ-fields are represented by appropriate higher-order terms. That will exhaust all of the differences between them.

§ 30. Complex spinor fields (electrons, positrons, neutrinos)

1. *Field equations.* In 5-spaces with pseudo-Euclidian metrics, 5-spinors arise as four-component complex quantities that define four-rowed representations of the five-dimensional group of rotations. If we restrict ourselves to Lorentz transformations (x^5 = invar.) then the 5-spinor will split into two 4-spinors whose transformation properties are well-known in Dirac theory.

We decompose each of the four components of the 5-spinor in a Fourier series:

$$W_\sigma = \sum U_\sigma (Z \mid x^1, x^2, x^3, x^4) \exp (i\, Z\mu\, x^5). \tag{4.121}$$

For the sake of simplicity in subsequent formulas, we shall assume that the Fourier series in (4.121) is represented by just one component that corresponds to $Z = + 1$:

$$W_\sigma = U_\sigma \exp (i\mu\, x^5). \tag{4.121a}$$

The generalization of this is not difficult.

Since a spinor is an essentially complex quantity, in contrast to tensor fields, one will have:

$$U_\sigma (- Z \mid x^1, x^2, x^3, x^4) \neq U_\sigma^* (Z \mid x^1, x^2, x^3, x^4). \tag{4.122}$$

This means that in the case of spinor fields, the complex-conjugates of the Fourier components do not relate to the charge-conjugate particles, as was the case for tensor fields.

From now on, we shall employ matrix notation and denote 5-spinors by one symbol W, without choosing a component.

The simplest equation for spinor fields that we shall consider will take the form:

$$\left\{\mu(1)\frac{\partial}{\partial x^1}+\mu(2)\frac{\partial}{\partial x^2}+\mu(3)\frac{\partial}{\partial x^3}+\mu(4)\frac{\partial}{\partial x^4}+\mu(5)\frac{\partial}{\partial x^5}\right\}W = 0, \tag{4.123}$$

in which W is a system of five four-rowed matrices that satisfy the following condition:

$$\mu(\alpha)\,\mu(\beta) + \mu(\beta)\,\mu(\alpha) = 2\delta(\alpha, \beta). \tag{4.124}$$

It will be convenient for us to choose the $\mu(\alpha)$ matrices to be the following system of matrices:

$$\left.\begin{aligned} \mu(1) &= -i\,\gamma^2\gamma^3\gamma^4 = i\beta\alpha^3\alpha^2, \\ \mu(2) &= \ \ i\,\gamma^1\gamma^3\gamma^4 = i\beta\alpha^1\alpha^3, \\ \mu(3) &= -i\,\gamma^1\gamma^2\gamma^4 = i\beta\alpha^2\alpha^1, \\ \mu(4) &= \ \ i\,\gamma^1\gamma^2\gamma^3 = \beta\alpha^1\alpha^2\alpha^3, \\ \mu(5) &= \gamma^1\gamma^2\gamma^3\gamma^4 = i\,\alpha^1\alpha^2\alpha^3, \end{aligned}\right\} \tag{4.125}$$

in which $\{\gamma^1, \gamma^2, \gamma^3, \gamma^4\}$ and $\{\alpha^1, \alpha^2, \alpha^3, \alpha^4\}$ are two systems of matrices that figure in Dirac's theory. (4.125) will then give:

$$\left.\begin{aligned} \gamma^1 &= i\mu(1)\mu(5), & \alpha^1 &= i\mu(4)\mu(1), \\ \gamma^2 &= i\mu(2)\mu(5), & \alpha^2 &= i\mu(4)\mu(2), \\ \gamma^3 &= i\mu(3)\mu(5), & \alpha^3 &= i\mu(4)\mu(3), \\ \gamma^4 &= i\mu(4)\mu(5), & \alpha^4 &= i\mu(4)\mu(4). \end{aligned}\right\} \tag{4.126}$$

We can write the field equations (4.123), while choosing one of the coordinates, which we shall denote by x^0. From (4.123), we have:

$$\frac{\partial W}{\partial x^0}+\mu(0)\mu(n)\frac{\partial W}{\partial x^n} = 0, \tag{4.127a}$$

in which the index n enumerates all of the other coordinates that were not selected.

If we single out the action coordinate $x^0 = x^5$ then we will have:

$$\gamma_k\frac{\partial W}{\partial x^k}+\mu U = 0. \tag{4.127b}$$

If we single out the time coordinate $x^0 = x^4 = ict$ then we will have:

$$\frac{1}{c}\frac{\partial W}{\partial t}+\left(\boldsymbol{\alpha}\frac{\partial W}{\partial \mathbf{x}}\right)+\beta\frac{\partial W}{\partial x^5} = 0, \tag{4.127c}$$

or, if we pass to Fourier components:

$$\frac{1}{c}\frac{\partial U}{\partial t}+\left(\boldsymbol{\alpha}\frac{\partial U}{\partial \mathbf{x}}\right)+i\beta\mu U = 0. \tag{4.127d}$$

2. *Adjoint spinor field*. We further introduce the adjoint spinor $\tilde{W}$:

$$\tilde{W} = \sum_{Z=-\infty}^{+\infty} \tilde{U}\ (Z \mid x^1, x^2, x^3, x^4) \exp(-\, iZ\mu\, x^5), \tag{4.121′}$$

which satisfies the field equation:

$$\left\{\frac{\partial \tilde{W}}{\partial x^1}\mu(1)+\frac{\partial \tilde{W}}{\partial x^2}\mu(2)+\frac{\partial \tilde{W}}{\partial x^3}\mu(3)+\frac{\partial \tilde{W}}{\partial x^4}\mu(4)+\frac{\partial \tilde{W}}{\partial x^5}\mu(5)\right\} = 0, \tag{4.123′}$$

or, if we single out the coordinate x^0, the equation:

$$\frac{\partial \tilde{W}}{\partial x^0}+\frac{\partial \tilde{W}}{\partial x^n}\mu(n)\mu(0) = 0. \tag{4.127′}$$

Introduce the spinor $\overset{(0)}{W}$, which is defined as:

$$\overset{(0)}{W} = \tilde{W}\,\mu(0), \tag{4.128}$$

and call this spinor "adjoint to the coordinate x^0." From (4.128), one should have $\tilde{W}$ = $\overset{(0)}{W}\,\mu(0)$, and if one substitutes this in (4.127′) then that will give the following equation for $\overset{(0)}{W}$:

$$\frac{\partial \overset{(0)}{W}}{\partial x^0}+\frac{\partial \overset{(0)}{W}}{\partial x^n}\mu(0)\mu(n) = 0. \tag{4.129}$$

If we single out the action coordinate $x^0 = x^5$ then we will have:

$$\frac{\partial \overset{(5)}{W}}{\partial x^5}-\frac{1}{i}\frac{\partial \overset{(5)}{W}}{\partial x^k}\gamma^k = 0, \tag{4.129a}$$

or, in Fourier components:

$$\frac{\partial \overset{(5)}{U}}{\partial x^k}\gamma^k - \mu\overset{(5)}{U} = 0. \tag{4.129b}$$

If we single out the time coordinate $x^0 = x^4 = ict$ then we will have:

$$\frac{1}{c}\frac{\partial \overset{(4)}{W}}{\partial t}+\left(\frac{\partial \overset{(4)}{W}}{\partial \mathbf{x}}\boldsymbol{\alpha}\right)+\frac{\partial \overset{(4)}{W}}{\partial x^5}\beta = 0, \tag{4.129c}$$

or, in Fourier components:

$$\frac{1}{c}\frac{\partial \overset{(4)}{U}}{\partial t}+\left(\frac{\partial \overset{(4)}{U}}{\partial \mathbf{x}}\boldsymbol{\alpha}\right)-i\mu\overset{(4)}{U}\beta = 0. \tag{4.129d}$$

We conclude from (4.127b) and (4.129b) that:

$$\overset{(4)}{W} = W^*, \qquad \overset{(4)}{U} = U^*, \tag{4.130}$$

in which W^* denotes the complex-conjugate transpose of the spinor W.

We have:

$$\tilde{W} = \overset{(5)}{W}\mu(5), \qquad \tilde{W} = W^*\mu(4), \qquad \overset{(5)}{W} = W^*\mu(4)\,\mu(5) = -\,i\,W^*\gamma^4,$$

and therefore, in accordance with the Pauli notation:

$$\overset{(5)}{W} = -\,iW^+.$$

We put the relationships between the various adjoints into the following table:

$$\left.\begin{aligned} W &= W^*\mu(4) = -W^+\mu(5),\\ W^* &= \tilde{W}\mu(4) = W^+\gamma^4,\\ W^+ &= i\tilde{W}\mu(5) = W^*\gamma^4, \end{aligned}\right\} \tag{4.131}$$

which we shall use often.

If we use the Fourier series (4.121a) then formulas (4.127b) and (4.129b) will give:

$$\gamma_k\frac{\partial U}{\partial x^k} + \mu U = 0, \qquad \frac{\partial U^+}{\partial x^k}\gamma_k - \mu U^+ = 0, \tag{4.132}$$

and formulas (4.127b) and (4.129b) will give:

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial U}{\partial t}+\left(\boldsymbol{\alpha}\frac{\partial U}{\partial \mathbf{x}}\right)+i\beta\mu U &= 0,\\ \frac{1}{c}\frac{\partial U^*}{\partial t}+\left(\frac{\partial U^*}{\partial \mathbf{x}}\boldsymbol{\alpha}\right)-i\mu U^*\beta &= 0, \end{aligned}\right\} \tag{4.133}$$

which are the two familiar Dirac equations.

We showed that complex spinor fields in 5-space describes the entire family of particles that have spinor fields with mass $|Z|m$ and charge Ze, where Z is a positive or negative integer, including zero. The cases of $Z = -1, 0, +1$ correspond to electrons neutrinos, and positrons, respectively, which should briefly explain how one and the same particle can be in different charge states.

3. *Energy-impulse-charge 5-tensor*. The field equations (4.123) and (4.123′) can be derived from the following Lagrangian function:

$$\left.\begin{aligned} L &= \tfrac{1}{2}(\tilde{W}\mu(\sigma)W_\sigma - \tilde{W}_\sigma\mu(\sigma)W), \\ W_\sigma &= \frac{\partial W}{\partial x^\sigma}, \qquad \tilde{W}_\sigma = \frac{\partial \tilde{W}}{\partial x^\sigma} \end{aligned}\right\} \tag{4.134}$$

by the formulas:

$$\frac{\partial}{\partial x^\sigma}\left(\frac{\partial L}{\partial \tilde{W}_\sigma}\right) - \frac{\partial L}{\partial \tilde{W}} = 0, \qquad \frac{\partial}{\partial x^\sigma}\left(\frac{\partial L}{\partial W_\sigma}\right) - \frac{\partial L}{\partial W} = 0. \tag{4.135}$$

Note that L will vanish for functions that satisfy the field equations. The canonical tensor of energy-impulse-charge is given by the formula:

$$T_{\alpha\beta} = \tilde{W}_\alpha \frac{\partial L}{\partial \tilde{W}_\beta} + \frac{\partial L}{\partial W_\beta} W_\sigma - \delta_{\alpha\beta} L. \tag{4.136}$$

Since we have been given that $L = 0$, if we single out the action coordinate and use (4.125) and (4.131) then:

$$\left.\begin{aligned} T_{ik} &= \tfrac{1}{2}\left(W^+\gamma^k \frac{\partial W}{\partial x^i} - \frac{\partial W^+}{\partial x^i}\gamma^k W\right), \\ T_{5k} &= \tfrac{1}{2}\left(W^+\gamma^k \frac{\partial W}{\partial x^5} - \frac{\partial W^+}{\partial x^5}\gamma^k W\right), \\ T_{55} &= \tfrac{1}{2}\left(W^+ \frac{\partial W}{\partial x^5} - \frac{\partial W^+}{\partial x^5} W\right). \end{aligned}\right\} \tag{4.137}$$

If we pass on to the formulation of things in terms of Fourier components then we will assume that just one component in the expansion is represented. We will have:

Energy-impulse 4-tensor:

$$\bar{T}_{ik} = \tfrac{1}{2}\left(U^+\gamma^k \frac{\partial U}{\partial x^i} - \frac{\partial U^+}{\partial x^i}\gamma^k U\right), \tag{4.138}$$

Current 4-vector:

$$\bar{T}_{5k} = i\mu\, U^+\gamma_k\, U, \tag{4.139}$$

4-scalar:

$$\bar{T}_{55} = \mu\, U^{+}U. \tag{4.140}$$

By virtue of the field equations, the 5-tensor $T_{\mu\nu}$ will satisfy the equation:

$$\frac{\partial T_{\mu\nu}}{\partial x^{\nu}} = 0. \tag{4.141}$$

If we single out the action coordinate then that will give:

$$\frac{\partial \bar{T}_{ik}}{\partial x^{k}} = 0, \qquad \frac{\partial \bar{T}_{5k}}{\partial x^{k}} = 0; \tag{4.141′}$$

i.e., the laws of conservation of energy and charge. Note that in our formulas the current 4-vector $\bar{T}_{5k}$ is characterized by the usual dimensional factor $mc/e\hbar$.

We compute the antisymmetric part of the canonical 5-tensor $T_{\alpha\beta}$; from (4.136), we will get:

$$T_{\beta\alpha} - T_{\alpha\beta}$$

$$= \tfrac{1}{2}\{\tilde{W}[\mu(\alpha)\,\delta(\beta,\rho) - \mu(\beta)\,\delta(\alpha,\rho)]W_{\sigma} + \tilde{W}_{\sigma}[\delta(\alpha,\rho)\,\mu(\beta) - \delta(\beta,\rho)\mu(\alpha)]W\}. \tag{4.142}$$

Using (4.124), we deduce that:

$$\begin{aligned} T_{\beta\alpha} - T_{\alpha\beta} &= \tfrac{1}{4}\tilde{W}[\mu(\alpha)\mu(\beta) - \mu(\beta)]\mu(\alpha)\, W_{\sigma} \\ &+ \tfrac{1}{4}\tilde{W}_{\rho}\, \mu(\rho)[\mu(\alpha)\mu(\beta) - \mu(\beta)\mu(\alpha)]W \\ &+ \tfrac{1}{4}\tilde{W}[\mu(\alpha)\mu(\rho)\mu(\beta) - \mu(\beta)\mu(\rho)\mu(\alpha)]W_{\rho} \\ &+ \tfrac{1}{4}\tilde{W}_{\rho}[\mu(\alpha)\mu(\rho)\mu(\beta) - \mu(\beta)\mu(\rho)\mu(\alpha)]W. \end{aligned} \tag{4.142a}$$

The first two terms in (4.142a) will vanish by virtue of the field equations. If we introduce the notation:

$$K(\alpha, \rho, \beta) = \tfrac{1}{4}\tilde{W}[\mu(\alpha)\mu(\rho)\mu(\beta) - \mu(\beta)\mu(\rho)\mu(\alpha)]W, \tag{4.143}$$

in which $K(\alpha, \rho, \beta)$ is the antisymmetrization of all three indices of the third-rank tensor, then the expression (4.142a) can be written in the form:

$$T_{\beta\alpha} - T_{\alpha\beta} = \frac{1}{2}\frac{\partial}{\partial x^{\rho}} K(\alpha, \rho, \beta), \tag{4.142b}$$

so the symmetric part of the tensor $T_{\alpha\beta}$, namely:

$$\theta_{\alpha\beta} = \tfrac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) = T_{\alpha\beta} + \tfrac{1}{2}(T_{\beta\alpha} - T_{\alpha\beta})$$

$$= \frac{1}{2}\left(\tilde{W}\mu(\beta)\frac{\partial W}{\partial x^{\sigma}} - \frac{\partial \tilde{W}}{\partial x^{\sigma}}\mu(\beta)W\right) - \frac{1}{4}\frac{\partial}{\partial x^{\sigma}}K(\alpha,\rho,\beta), \tag{4.143a}$$

will satisfy the same equation (4.141) and we can consider it to be the symmetric energy-impulse-charge 5-tensor.

If we single out the action coordinates and use (4.125) and (4.131) then that will give:

$$\left.\begin{aligned}
\theta_{ik} &= \frac{1}{2}\left(W^{+}\gamma^{k}\frac{\partial W}{\partial x^{i}} - \frac{\partial W^{+}}{\partial x^{i}}\gamma^{k}W\right) - \frac{1}{4}\frac{\partial}{\partial x^{\rho}}K(\rho,i,k),\\
\theta_{5k} &= \frac{1}{2}\left(W^{+}\gamma^{k}\frac{\partial W}{\partial x^{5}} - \frac{\partial W^{+}}{\partial x^{5}}\gamma^{k}W\right) - \frac{1}{4}\frac{\partial}{\partial x^{n}}K(n,5,k),\\
\theta_{k5} &= \frac{-i}{2}\left(W^{+}\frac{\partial W}{\partial x^{5}} - \frac{\partial W^{+}}{\partial x^{5}}W\right) + \frac{1}{4}\frac{\partial}{\partial x^{n}}K(n,5,k),\\
\theta_{55} &= \frac{-i}{2}\left(W^{+}\frac{\partial W}{\partial x^{5}} - \frac{\partial W^{+}}{\partial x^{5}}W\right).
\end{aligned}\right\} \tag{4.144}$$

In the formulation of the Fourier series, one will have, assuming that the expansion is represented by just one component $Z = +\,1$:

$$\bar{\theta}_{ik} = \frac{1}{2}\left(U^{+}\gamma^{k}\frac{\partial U}{\partial x^{i}} - \frac{\partial U^{+}}{\partial x^{i}}\gamma^{k}U\right) - \frac{1}{4}\frac{\partial}{\partial x^{n}}\bar{K}(n,i,k), \tag{4.145}$$

$$\bar{\theta}_{5k} = i\mu U^{+}\gamma^{k}U + \frac{i}{4}\frac{\partial \bar{M}(k,n)}{\partial x^{n}}, \tag{4.146}$$

$$\bar{\theta}_{k5} = -\frac{i}{2}\left(U^{+}\frac{\partial U}{\partial x^{i}} - \frac{\partial U^{+}}{\partial x^{i}}U\right) - \frac{1}{4}\frac{\partial \bar{M}(k,n)}{\partial x^{n}}, \tag{4.147}$$

$$\bar{\theta}_{55} = \mu\, U^{+}U, \tag{4.148}$$

in which we have introduced the notations:

$$\left.\begin{aligned}
\bar{K}(n,i,k) &= \tfrac{1}{2}U^{+}(\gamma^{n}\gamma^{i}\gamma^{k} - \gamma^{k}\gamma^{i}\gamma^{n})U,\\
\bar{M}(k,n) &= \tfrac{1}{2}U^{+}(\gamma^{k}\gamma^{n} - \gamma^{n}\gamma^{k})U,
\end{aligned}\right\} \tag{4.149}$$

and from formula (4.145), the 4-tensor $\bar{\theta}_{ik}$ will be the symmetric energy-impulse 4-tensor of Dirac's theory precisely.

4. *Current vector*. The essentially new fact, in comparison to Dirac's theory, when one is considering the current 4-vector [viz., formula (4.146)] is the appearance of the extra term $\frac{i}{4}\frac{\partial M(k,n)}{\partial x^n}$, which is missing from Dirac's theory, and corresponds to a further polarization current.

The symmetry condition $\bar{\theta}_{5k} = \bar{\theta}_{k5}$, which can be written in the form:

$$i\mu\, U^+\gamma^k U = -\frac{i}{2}\left(U^+\frac{\partial U}{\partial x^k} - U\frac{\partial U^+}{\partial x^k}\right) - \frac{i}{2}\frac{\partial M(k,n)}{\partial x^n}, \tag{4.150}$$

is known as the Gordon identity in Dirac's theory, and is derived as follows: Substitute μU and μU^+ from equation (4.132) in the left-hand side of equation (4.150):

$$\mu U = -\gamma_n\frac{\partial U}{\partial x^n}, \qquad \mu U^+ = \gamma_n\frac{\partial U^+}{\partial x^n};$$

that will give:

$$i\mu\, U^+\gamma^k U = -\frac{i}{2}\left(U^+\gamma^k\gamma^n\frac{\partial U}{\partial x^n} - \frac{\partial U^+}{\partial x^n}\gamma^n\gamma^k U\right). \tag{4.151}$$

If we separate the terms with $k = n$ from the ones with $k \neq n$ in the summation over n in (4.151) then that will give:

$$i\mu\, U^+\gamma^k U = -\frac{i}{2}\left(U^+\frac{\partial U}{\partial x^k} - \frac{\partial U^+}{\partial x^k}U\right) - \frac{i}{4}\frac{\partial}{\partial x^n}U^+\left(\gamma^k\gamma^n - \gamma^n\gamma^k\right)U, \tag{4.152}$$

which agrees with (4.150).

According to Pauli's findings ([**13**], pp. 51), the appearance of the extra term $\frac{i}{4}\frac{\partial M(k,n)}{\partial x^n}$ in the expression for the 4-current makes one expect to see a term $-\frac{i}{8} f_{mn}\, \gamma_m\, \gamma_m\, U$ in the Dirac equation in the presence of an electromagnetic field, which would correspond to an extra magnetic moment for the particle. We shall see that this is true in Chapter VI, § 41.

Appendix: Ginzburg's electromagnetic gauge potential [11]

The equations for the electromagnetic field potentials take the form:

$$\left.\begin{aligned} \Box\varphi + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\partial A_k}{\partial x^k} + \frac{1}{c}\frac{\partial\varphi}{\partial t}\right) &= -4\pi\rho, \\ \Box A_i - \frac{\partial}{\partial x^i}\left(\frac{\partial A_k}{\partial x^k} + \frac{1}{c}\frac{\partial\varphi}{\partial t}\right) &= -\frac{4\pi}{c}\, j_i . \end{aligned}\right\} \tag{4.153}$$

One usually assumes that the potentials satisfy the Lorentz condition div **A** + $\frac{1}{c}\frac{\partial\varphi}{\partial t}$ = 0 and the field equations will then take the form:

$$\Box\varphi = -4\pi\rho, \qquad \Box A_i = -\frac{4\pi}{c}\, j_i\,. \tag{4.153′}$$

In Ginzburg's paper, the following condition was imposed upon the potential:

$$\frac{\partial A_i}{\partial x^i} = 0. \tag{4.154}$$

With that condition, the field equations (4.153) will take the form:

$$\Delta\varphi = -4\pi\rho, \qquad \varphi(r,t) = \int \frac{\rho(r',t)}{R}\,dV', \qquad R = |\,r - r'\,|, \tag{4.155}$$

$$c\,\Delta A_i = -4\pi j_i + \frac{\partial}{\partial x^i}\left(\frac{\partial\varphi}{\partial t}\right) + \frac{1}{c}\frac{\partial^2 A_i}{\partial t^2}\,. \tag{4.156}$$

We will solve equation (4.156) by the method of successive approximations:

$$A_i = \tilde{A}_i + A_i^{(0)} + \frac{1}{c^2}A_i^{(1)} + \frac{1}{c^4}A_i^{(2)} + \ldots, \tag{4.157}$$

in which $\tilde{A}_i$, which satisfies the condition $\partial\tilde{A}_i/\partial x^i = 0$, is a solution of the homogeneous wave equation $\Box\tilde{A}_i = 0$ (i.e., a light wave) and $A_i^{(0)}$, $A_i^{(k)}$ are solutions of the equations:

$$\Delta A_i^{(0)} = -4\pi j_i + \frac{\partial}{\partial x^i}\left(\frac{\partial\varphi}{\partial t}\right), \tag{4.156′}$$

$$\Delta A_i^{(k)} = \frac{\partial^2 A_i^{(k-1)}}{\partial t^2}, \qquad k = 1, 2, 3, \ldots \tag{4.156″}$$

In view of formula (with $k \neq 1$):

$$\Delta R^k = k\,(k+1)\,R^{k-2}, \tag{4.158}$$

we find that:

$$c\,A_i^{(0)}(r,t) = \int \frac{j_i(r',t)}{R}\,dv' + \frac{1}{2!}\frac{\partial^2}{\partial x^i\,\partial t}\int \rho(r',t)\,R\,dv', \tag{4.159}$$

$$c\,A_i^{(k)}(r,t) \;=\; \frac{1}{(2k)!}\int \frac{\partial^{2k} j_i(r',t)}{\partial t^{2k}} R^{(2k-1)} dv' + \frac{1}{(2k+2)!}\frac{\partial^2}{\partial x^i\,\partial t}\int \frac{\partial^{2k}\rho(r',t)}{\partial t^{2k}} R^{2k+1}\,dv' . \quad (4.159')$$

If we take the continuity equation:

$$\frac{\partial j_k}{\partial x^k} + \frac{\partial \rho}{\partial t} = 0$$

into account, along with the formulas:

$$\frac{\partial^2 R^k}{\partial x^i\,\partial x^s} = k\,[\delta_{is} + n_i\,n_s\,(k-2)]\,R^{k-2}, \qquad \mathbf{n} = \frac{\mathbf{R}}{R}, \qquad (4.160)$$

and integrate (4.159) and (4.159′) by parts then that will give:

$$\begin{aligned} c\,A_i^{(0)}(r,t) &= \int \frac{j_i(r',t)}{R} dV - \frac{1}{2!}\int j_s(r',t)\frac{\partial^2 R}{\partial x^i\,\partial t} dV' \\ &= \frac{1}{2!}\int \frac{j_s(r',t)(\delta_{si} + n_s n_i)}{R} dV' , \end{aligned} \qquad (4.161)$$

$$c\,A_i^{(k)}(r,t) =$$

$$\frac{1}{(2k)!\,2(k+1)}\int j_s(r',t)\frac{\partial^{2k} j_s(r',t)}{\partial t^{2k}} R^{2k-1}[(2k+1)\delta_{is} - (2k-1)\,n_i n_k\,]\,dV' . \quad (4.161')$$

The energy of the electromagnetic field is given by:

$$W = \frac{1}{8\pi}\int \left\{ \frac{1}{2}\left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}\right)^2 + \left(\frac{\partial \varphi}{\partial x^i} + \frac{1}{c}\frac{\partial A_i}{\partial t}\right)^2 \right\} dV , \qquad (4.162)$$

which, by virtue of (4.154), and after integrating by parts, will be converted into the form:

$$W = \frac{1}{8\pi}\int \left\{ \frac{1}{2}\left(\frac{\partial A_i}{\partial x^k}\right)^2 + \frac{1}{c^2}\left(\frac{\partial A_i}{\partial t}\right)^2 + \left(\frac{\partial \varphi}{\partial x^i}\right)^2 \right\} dV . \qquad (4.163)$$

We shall now assume that the speeds of all charges in the system are small in comparison to the speed of light. We can then neglect the magnitudes of all terms in (4.157), beginning with the terms $A_i^{(1)}$. We then calculate the expression for energy W in this approximation. If we substitute $A_i = \tilde{A}_i + A_i^{(0)}$ in (4.162′) then that will give:

$$W = \frac{1}{8\pi}\int \left\{ \left(\frac{\partial \tilde{A}_i}{\partial x^k}\right)^2 + \frac{1}{c^2}\left(\frac{\partial \tilde{A}_i}{\partial t}\right)^2 \right\} dV + \frac{1}{4\pi}\int \left\{ \frac{1}{c^2}\frac{\partial \tilde{A}_i}{\partial t}\frac{\partial \tilde{A}_i^{(0)}}{\partial t} - \tilde{A}_i\,\Delta \tilde{A}_i^{(0)} \right\} dV$$

$$+\ \frac{1}{8\pi}\int \left\{ \frac{1}{c^2}\left(\frac{\partial A_i^{(0)}}{\partial t}\right)^2 - A_i^{(0)}\Delta A_i^{(0)} - \varphi\,\Delta\varphi \right\} dV\,. \qquad (4.163)$$

With this approximation, the first term in the second and third integrals can be neglected in comparison to the ones that follow. Calculation will give:

$$-\frac{1}{8\pi}\int \varphi\Delta\varphi\, dV \quad = \frac{1}{2}\int \frac{\rho(r',t)\rho(r,t)}{R}\, dV\, dV',$$

$$-\frac{1}{8\pi}\int A_i^{(0)}\,\Delta A_i^{(0)}\, dV \quad = \frac{1}{2c^2}\int \frac{j_i(r',t)\, j_s(r,t)}{R}(\delta_{is}+n_i n_s)\, dV\, dV',$$

$$-\frac{1}{4\pi}\int \tilde{A}_i\,\Delta A_i^{(0)}\, dV \quad = \frac{1}{c}\int \tilde{A}_i\; j_i\, dV\,.$$

When one is computing the last two integrals, one must take (4.154) into account. We will finally have:

$$W = \frac{1}{8\pi}\int (\tilde{E}^2+\tilde{H}^2)\, dV + \frac{1}{c}\int \tilde{A}_i\; j_i\, dV + \frac{1}{2}\int \frac{\rho(r',t)\,\rho(r,t)}{R}\, dV\, dV'$$
$$+\ \frac{1}{2c^2}\int \frac{j_i(r',t)\; j_s(r,t)}{R}(\delta_{is}+n_i\, n_s)\, dV\, dV'\,. \qquad (4.164)$$

The first term is the energy of the transverse light waves, the second term is the energy of the interaction of the light waves with a continuously-distributed system of currents, and the last two terms given the instantaneous interaction energy of the system of continuously-distributed charges and currents. We see that the proposed Ginzburg gauge (4.154) not only singles out the energy of the photon field and the Coulomb interaction of the charges, but it also takes into account the instantaneous interaction energy of the currents in the same approximation (cf., e.g., [**12**], § 65, which was a situation that the author pointed out in the paper [**11**]).

CHAPTER V

TENSOR ANALYSIS AND METRIC *n*-BEINS (†)

INTRODUCTION

Before proceeding on to wave-like 5-optics in Riemannian 5-spaces, in this chapter, we shall construct the formal mathematical machinery that is necessary for that purpose.

It is well-known that it is impossible to introduce the concept of a spinor into Riemannian space without abandoning the classical techniques in the study of Riemannian geometry.

As early as 1929, V. A. Fock [**10**] showed that one could overcome the difficulties that were associated with introducing the metric by introducing *n*-bein coefficients, instead of the conventional Gauss coefficients.

We shall show that metric *n*-beins can be constructed upon the basis of tensor analysis, which has been used successfully for the generally covariant formulation of not only the tensor equations of mathematical physics, but also the ones that involved spin.

The advantage of the approach to tensor analysis that will be presented is that it consistently allows one to obtain Hilbert invariant integrals, field equations, and conservation laws for both tensor and spinor fields.

In what follows, we shall consider the general of *n*-dimensional spaces.

§ 31. Metric tensor *n*-bein.

In conventional tensor analysis, one introduces a metric in Riemannian space by means of Gauss matrices $\| g_{ik} \|$:

$$ds^2 = g_{ik}\, dx^i\, dx^k. \tag{5.1}$$

We can also introduce a metric by means of asymmetric quadratic matrices $\| \Omega_i(\alpha) \|$ using the formula:

$$ds^2 = \sum_{(\alpha)} \Omega_i(\alpha)\, \Omega_k(\alpha)\, dx^i dx^k\ . \tag{5.2}$$

If the matrix $\| \Omega_i(\alpha) \|$ is diagonal then its elements will be the *n-bein* coefficients, so we will then call it the *symmetric n-bein matrix.*

From (5.1) and (5.2), we should have:

$$\sum_{(\alpha)} \Omega_\sigma(\alpha)\, \Omega_\tau(\alpha) = g_{\sigma\tau}, \qquad \| \Omega_i(\alpha) \|^2 = \| g_{\sigma\tau} \|. \tag{5.3}$$

(†) Translator's note: The Russian word "ламэ" did not seem to have an obvious translation (although it might have been a translation of a non-Russian name, such as Lamé), so I used equation (5.2) and a remark below as a justification for saying that he is introducing *n*-beins.

A metric will persist when the elements of the *n*-bein matrices are subjected to an orthogonal transformation:

$$\Omega'_\sigma(\alpha) = \sum_{(\beta)} L(\alpha,\beta)\,\Omega_\sigma(\beta)\,, \tag{5.4}$$

in which $L(\alpha, \beta \,|\, x)$ are orthogonal matrices that vary from point to point:

$$\sum_{(\alpha)} L(\alpha,\beta \,|\, x) L(\alpha,\gamma \,|\, x) \;=\; \delta(\beta,\,\gamma). \tag{5.5}$$

Indeed, we have:

$$g'_{\sigma\tau} \;=\; \sum_{(\alpha)} \Omega'_\sigma(\alpha)\,\Omega'_\tau(\alpha) \;=\; \sum_{(\alpha)}\sum_{(\beta)}\sum_{(\gamma)} \Omega_\sigma(\beta)\,L(\alpha,\beta)\,\Omega_\tau(\gamma)\,L(\alpha,\gamma) = \sum_{(\alpha)} \Omega_\sigma(\alpha)\,\Omega_\tau(\alpha)$$
$$= g_{\sigma\tau}\,. \tag{5.6}$$

Since orthogonal transformations are determined by means of $n\,(n - 1) / 2$ parameters, *n*-bein matrices can always be reduced to the normal triangular form:

$$\begin{pmatrix} \Omega_1(1) & \Omega_1(2) & \cdots & \Omega_1(n) \\ 0 & \Omega_2(2) & \cdots & \Omega_2(n) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \Omega_n(n) \end{pmatrix}, \tag{5.7}$$

so they, as well as the symmetric Gauss matrices $\|\, g_{\sigma\tau} \|$, will be determined by only $n\,(n + 1) / 2$ elements. From (5.3), we have:

$$\Delta = |\,\mathrm{Det}\,(\Omega_i(\alpha))\,| = +\sqrt{\mathrm{Det}(g_{ik})} \;\neq\; 0. \tag{5.8}$$

If we let $\Omega^i(\alpha)$ denote the elements of the inverse matrix $\Omega_i \,\|\, (\alpha) \,\|^{-1}$ then we will have:

$$\Omega^\sigma(\alpha)\,\Omega_\tau(\alpha) = \delta(\alpha,\,\beta), \qquad \sum_{(\alpha)} \Omega^\sigma(\alpha)\,\Omega_\tau(\alpha) \;=\; \delta^\sigma_\tau\,. \tag{5.9}$$

Using the *n*-bein matrices, we can define the following system of quantities from the components of the contravariant vector A_i and the covariant vector B_i :

$$A(\alpha) = \Omega_i(\alpha)\,A^i, \qquad B(\alpha) = \Omega^i(\alpha)\,A_i\,, \tag{5.10}$$

which are invariant under the general coordinate transformation, as well as the transformations in the formula:

$$A'(\alpha) = \sum_{(\beta)} L(\alpha,\beta)\,A(\beta)\,, \tag{5.11}$$

which are orthogonal transformations of the elements of the n-bein matrices.

We shall call the system of quantities $A(\alpha)$ and $B(\alpha)$ the *invariant components* of the vectors A^i and B_i .

The raising and lowering of indices is achieved in two steps with the aid of intermediate indices:

$$A(\alpha) = \Omega_i(\alpha)\, A^i, \qquad A_k = \sum_{(\alpha)} \Omega_k(\alpha)\, A(\alpha)\,. \tag{5.12}$$

By using the metric n-bein as a basis, we can construct tensor analysis as theory of two simultaneous groups of transformations:

A) The group of general transformations of all coordinates:

$$x'^i = x^i + f^{\,i}(x^1, \ldots, x^n). \tag{5.13a}$$

B) The group of orthogonal transformations of the elements of the n-bein matrices:

$$\Omega'_\sigma(\alpha) = \sum_{(\alpha)} L(\alpha,\beta)\,\Omega_\sigma(\beta)\,. \tag{5.13b}$$

We agree to omit the summation sign whenever two identical indices are being summed over:

$$\sum_{(\alpha)} A(\alpha)\,B(\alpha) \equiv A(\alpha)\,B(\alpha), \quad \Omega'_\sigma(\alpha) = \sum_{(\alpha)} L(\alpha,\beta)\,\Omega_\sigma(\beta) \equiv L(\alpha,\beta)\,\Omega_\sigma(\alpha)\,.$$

In general, tensors will have indices of all three kinds:

$$T_\sigma(\alpha), \qquad T^\tau_\sigma(\alpha,\beta)\,,$$

and will transform under the groups of transformations *A* and *B* according to the formula:

$$\bar{T}^{\sigma\tau}_\lambda(\xi,\eta) = T^{kl}_c(\alpha,\beta)\,\frac{\partial x^i}{\partial \bar{x}^\lambda}\,\frac{\partial \bar{x}^\sigma}{\partial x^k}\cdot\frac{\partial \bar{x}^\tau}{\partial x^i}\; L(\xi,\alpha)\,L(\eta,\beta). \tag{5.14}$$

By virtue of the fact that $\Omega_k(\alpha) = g_{ki}\,\Omega^i(\alpha)$, the elements of the n-bein matrices should be regarded as the covariant-invariant components of the metric tensor g_{ik}, which transform under the groups of transformations *A* and *B* according to the formula:

$$\bar{\Omega}_i(\alpha) = \Omega_\sigma(\beta)\,\frac{\partial x^\sigma}{\partial \bar{x}^i}\,L(\alpha,\beta). \tag{5.14′}$$

We will therefore call the quantities $\Omega_i(\alpha)$ the *components* of the metric tensor n-bein.

Remark. Note that in the usual presentation, the quantities $\Omega_i(\alpha)$ are understood to mean the covariant components of vectors that define a local rectangular *repère* (i.e., an *n*-hedron) at each point of space, while for us they will define a covariant-invariant tensor. The advantage of our notation will become clear in the individual cases.

§ 32. Covariant differentiation of tensors

In order to introduce covariant differentiation, it is necessary to define the concept of the parallel displacement of vectors. The invariant components of the vector $A(a)$ at two infinitely-close points (x) and $(x + dx)$ must be coupled by a linear transformation:

$$A(\alpha \mid x + dx) = \{\delta(\alpha, \beta) + \Delta_\sigma(\alpha, \beta)\, dx^\sigma\}\, A(\alpha \mid x). \tag{5.15}$$

Since the square of a vector $A(\alpha)\, A(\alpha) \equiv A_i\, A^i$, as a scalar, should remain unchanged under the parallel displacement of vectors, we conclude that in the case where the vector $A(\alpha \mid x + dx)$ is obtained from the vector $A(\alpha \mid x)$ by means of parallel displacement by using (5.15), the corresponding linear transformation must be an infinitesimal orthogonal transformation; i.e., the quantities $\Delta_\sigma(\alpha, \beta)$ must be antisymmetric in the invariant indices α, β:

$$\Delta_\sigma(\alpha, \beta) = -\Delta_\sigma(\beta, \alpha). \tag{5.16}$$

They are called the *rotation coefficients* or the *Ricci symbols*.

Therefore, when the vector $A(\alpha)$ is parallel displaced, it will take on an increment:

$$\delta A(\alpha) = \Delta_\sigma(\alpha, \beta)\, dx^\sigma A(\beta). \tag{5.17}$$

Under an arbitrary displacement of the vector $A(\alpha)$, it will take on an increment:

$$dA(\alpha) = \frac{\partial A(\alpha)}{\partial x^\sigma}\, dx^\sigma. \tag{5.18}$$

The difference $dA(\alpha) - \delta A(\alpha)$ of these vectors is the *absolute increment* in the vector $A(\alpha)$:

$$dA(\alpha) - \delta A(\alpha) = \left\{\frac{\partial A(\alpha)}{\partial x^\sigma} - \Delta_\sigma(\alpha, \beta)\, A(\beta)\right\} dx^\sigma, \tag{5.19}$$

and therefore the expression in curly brackets is its *covariant derivative*:

$$\nabla_\sigma A(\alpha) = \frac{\partial A(\alpha)}{\partial x^\sigma} - \Delta_\sigma(\alpha, \beta)\, A(\beta)\,. \tag{5.20}$$

The requirement that:

$$\left.\begin{aligned} \nabla_\sigma A^\tau &= \nabla_\sigma[\Omega^\tau(\alpha) A(\alpha)] = \Omega^\tau(\alpha) \nabla_\sigma A(\alpha), \\ \nabla_\sigma A_\tau &= \nabla_\sigma[\Omega_\tau(\alpha) A(\alpha)] = \Omega_\tau(\alpha) \nabla_\sigma A(\alpha) \end{aligned}\right\} \tag{5.21}$$

stems from the requirement that:

$$\nabla_\sigma \Omega^\tau(\alpha) = 0, \qquad \nabla_\sigma \Omega_\tau(\alpha) = 0. \tag{5.22}$$

If one multiplies (5.20) by $\Omega_\tau(\alpha)$ and $\Omega^\tau(\alpha)$ then one will get, after an obvious conversion:

$$\left.\begin{aligned} \nabla_\sigma A_\tau &= \frac{\partial A_\tau}{\partial x^\sigma} - \left\{\frac{\partial \Omega_\tau(\beta)}{\partial x^\sigma} + \Delta_\sigma(\alpha,\beta)\Omega_\tau(\alpha)\right\}\Omega^\lambda(\beta)A_\lambda, \\ \nabla_\sigma A^\tau &= \frac{\partial A^\tau}{\partial x^\sigma} - \left\{\frac{\partial \Omega_\lambda(\beta)}{\partial x^\sigma} + \Delta_\sigma(\alpha,\beta)\Omega_\lambda(\alpha)\right\}\Omega^\tau(\beta)A^\lambda. \end{aligned}\right\} \tag{5.23}$$

We introduce the abbreviations:

$$\Gamma^\lambda_{\sigma\tau} = \Omega^\lambda(\beta)\frac{\partial \Omega_\tau(\beta)}{\partial x^\sigma} + \Delta_{\sigma,\tau}^{\cdot\,\cdot\,\lambda}, \tag{5.24}$$

$$\Gamma_{\lambda,\,\sigma\tau} = \Omega_\lambda(\beta)\frac{\partial \Omega_\tau(\beta)}{\partial x^\sigma} + \Delta_{\sigma,\tau\lambda}, \tag{5.24a}$$

and rewrite (5.23) in the usual form:

$$\nabla_\sigma A_\tau = \frac{\partial A_\tau}{\partial x^\sigma} - \Gamma^\lambda_{\sigma\tau}A_\lambda, \qquad \nabla_\sigma A^\tau = \frac{\partial A^\tau}{\partial x^\sigma} + \Gamma^\tau_{\sigma\lambda}A^\lambda. \tag{5.23a}$$

We now show that the symbols in formulas (5.24) and (5.24a) coincide with the usual Christoffel symbols. In order to do that, switch the indices τ, λ in (5.24a), and if one takes (5.16) into account then that will give:

$$\Gamma_{\lambda,\,\sigma\tau} + \Gamma_{\tau,\,\sigma\lambda} = \Omega_\lambda(\beta)\frac{\partial \Omega_\tau(\beta)}{\partial x^\sigma} + \Omega_\tau(\beta)\frac{\partial \Omega_\lambda(\beta)}{\partial x^\sigma} = \frac{\partial g_{\lambda\tau}}{\partial x^\sigma}, \tag{5.25}$$

which will make:

$$\Gamma_{\lambda,\,\sigma\tau} = \frac{1}{2}\left(\frac{\partial g_{\lambda\sigma}}{\partial x^\tau} - \frac{\partial g_{\sigma\tau}}{\partial x^\lambda} + \frac{\partial g_{\tau\lambda}}{\partial x^\sigma}\right), \tag{5.25′}$$

and the assertion will be proved.

One arrives at the rules for calculating covariant derivatives by starting with the formula:

$$\nabla_\sigma A^\tau(\alpha) = \frac{\partial A^\tau(\alpha)}{\partial x^\sigma} + \Gamma^\tau_{\sigma\lambda}A^\lambda(\alpha) - \Delta_\sigma(\alpha,\beta)\,A^\tau(\beta). \tag{5.26}$$

One generalizes this to the following rule for the covariant differentiation of a tensor of any rank:

In order to get the covariant derivative of the tensor $A^{\cdot\cdot}_{\cdot\cdot}$ with respect to x^σ, one should add a term $-\Delta_\sigma(\alpha, \beta)\, A^\tau(\ldots, \beta, \ldots)$ for each invariant index α in $A(\ldots, \alpha, \ldots)$.

Using that rule, we can reveal the true meaning of the terms in (5.22):

$$\nabla_\sigma \Omega^\tau(\alpha) = \frac{\partial \Omega^\tau(\alpha)}{\partial x^\sigma} + \Gamma^\tau_{\sigma\lambda}\Omega^\lambda(\alpha) - \Delta_\sigma(\alpha, \beta)\, \Omega^\tau(\beta) = 0, \tag{5.27}$$

$$\nabla_\sigma \Omega_\tau(\alpha) = \frac{\partial \Omega_\tau(\alpha)}{\partial x^\sigma} - \Gamma^\lambda_{\sigma\tau}\Omega_\lambda(\alpha) - \Delta_\sigma(\alpha, \beta)\, \Omega_\tau(\beta) = 0. \tag{5.27a}$$

Remark. Note that in the usual presentation of the quantities $\Omega^\tau(\alpha)$, they are not regarded as the covariant-invariant components of a tensor, but as the components of a system of *n* contravariant vectors. Therefore, in the usual presentation of the covariant derivative of $\Omega^\tau(\alpha)$ [which is denoted by $\Omega^\tau(\alpha)_{;\,\sigma}$], we understand that the quantities:

$$\Omega^\tau(\alpha)_{;\,\sigma} = \frac{\partial \Omega^\tau(\alpha)}{\partial x^\sigma} + \Gamma^\tau_{\sigma\lambda}\Omega^\lambda(\alpha) \tag{5.28}$$

are not equal to zero, but according to (5.27), they should be equal to:

$$\Omega^\tau(\alpha)_{;\,\sigma} = \Delta_\sigma(\alpha, \beta)\, \Omega^\tau(\beta)\,. \tag{5.29}$$

§ 33. The connection between the Ricci symbols and the metric tensor *n*-bein

Permuting the indices τ, σ in (5.24a) will give:

$$\Delta_{\sigma,\,\tau\lambda} - \Delta_{\tau,\,\sigma\lambda} = \Omega_\lambda(\alpha) \left\{ \frac{\partial \Omega_\sigma(\alpha)}{\partial x^\tau} - \frac{\partial \Omega_\tau(\alpha)}{\partial x^\sigma} \right\}. \tag{5.30}$$

If we solve this equation for $\Delta_{\sigma,\,\tau\lambda}$ then we will find that:

$$\Delta_{\sigma,\,\tau\lambda} = \frac{1}{2}\left\{ \Omega_\sigma(\alpha)\left(\frac{\partial \Omega_\lambda(\alpha)}{\partial x^\tau} - \frac{\partial \Omega_\tau(\alpha)}{\partial x^\lambda} \right) + \Omega_\lambda(\alpha)\left(\frac{\partial \Omega_\sigma(\alpha)}{\partial x^\tau} - \frac{\partial \Omega_\tau(\alpha)}{\partial x^\sigma} \right) \right.$$
$$\left. + \ \Omega_\tau(\alpha)\left(\frac{\partial \Omega_\lambda(\alpha)}{\partial x^\varsigma} - \frac{\partial \Omega_\sigma(\alpha)}{\partial x^\lambda} \right) \right\}. \tag{5.31}$$

Multiplying (5.30) by $\Omega^\sigma(\alpha)\, \Omega^\tau(\beta)\, \Omega^\lambda(\gamma)$ will give:

$$\Delta(\alpha, \beta, \gamma) - \Delta(\beta, \alpha, \gamma) = \frac{\partial \Omega_\sigma(\gamma)}{\partial x^\tau}\, [\Omega^\sigma(\alpha)\, \Omega^\tau(\beta) - \Omega^\tau(\alpha)\, \Omega^\sigma(\beta)]. \tag{5.30a}$$

If we solve this equation for $\Delta(\alpha, \beta, \gamma)$ then that will give:

$$\Delta(\alpha, \beta, \gamma) = \tfrac{1}{2}\left\{\frac{\partial\Omega_\sigma(\alpha)}{\partial x^\lambda}[\Omega^\lambda(\beta)\Omega^\sigma(\gamma)-\Omega^\lambda(\gamma)\Omega^\sigma(\beta)] + \frac{\partial\Omega_\sigma(\beta)}{\partial x^\lambda}[\Omega^\lambda(\alpha)\Omega^\sigma(\gamma)-\Omega^\lambda(\gamma)\Omega^\sigma(\alpha)] + \frac{\partial\Omega_\sigma(\gamma)}{\partial x^\lambda}[\Omega^\lambda(\beta)\Omega^\sigma(\alpha)-\Omega^\lambda(\alpha)\Omega^\sigma(\beta)]\right\} .\tag{5.31a}$$

Formulas (5.31) and (5.31a) express the Ricci symbols in terms of the components of the metric tensor *n*-bein.

Contracting over the indices σ, τ in (5.27) and taking into account the fact that $\Gamma^\sigma_{\sigma\lambda} = \frac{1}{\Lambda}\frac{\partial\Lambda}{\partial x^\lambda}$ will give the formula:

$$\frac{\partial(\Lambda\Omega^\sigma(\alpha))}{\Lambda\,\partial x^\lambda} = \Delta(\beta, \alpha, \beta), \tag{5.32}$$

§ 34. Invariant differentiation of tensors

Introduce two operators, which are defined as:

$$D(\tau) = \Omega^\sigma(\tau)\frac{\partial}{\partial x^\sigma}, \tag{5.33}$$

$$\nabla(\tau) = \Omega^\sigma(\tau)\,\nabla_\sigma . \tag{5.33a}$$

Multiplying the expression (5.20) by $\Omega^\sigma(\alpha)$ will give:

$$\nabla(\tau)\,A(\alpha) = D(\tau)\,A(\alpha) - \Delta(\tau, \alpha, \beta)\,A(\beta). \tag{5.34}$$

It is convenient to call this expression the "invariant derivative" of the vector $A(\alpha)$. If we calculate the invariant derivatives of the products $A(\alpha)\,B(\beta)$ and $A(\alpha)\,B(\beta)\,C(\gamma)$ using the formula (5.34) then we will come to the formulas for the invariant differentiation of second and third-rank tensors:

$$\nabla(\tau)\,A(\alpha, \beta) = D(\tau)\,A(\alpha, \beta) - \Delta(\tau, \alpha, \varepsilon)\,A(\varepsilon, \beta) - \Delta(\tau, \beta, \varepsilon)\,A(\alpha, \varepsilon), \tag{5.35a}$$

$$\nabla(\tau)\,A(\alpha, \beta, \gamma) = D(\tau)\,A(\alpha, \beta, \gamma) - \Delta(\tau, \alpha, \varepsilon)\,A(\varepsilon, \beta, \gamma) - \Delta(\tau, \beta, \varepsilon)\,A(\alpha, \varepsilon, \gamma) - \Delta(\tau, \gamma, \varepsilon)\,A(\alpha, \beta, \varepsilon), \tag{5.35a}$$

which can be easily generalized to tensors of any rank. Contracting the expressions (5.34) and (5.35) will give corresponding formulas for the divergences:

$$\nabla(\tau)\,A(\tau) = D(\tau)\,A(\tau) - \Delta(\tau, \tau, \beta)\,A(\beta). \tag{5.36$'$}$$

$$\nabla(\tau)\, A(\alpha, \tau) = D(\tau)\, A(\alpha, \tau) - \Delta(\tau, \alpha, \varepsilon)\, A(\varepsilon, \tau) - \Delta(\tau, \tau, \varepsilon)\, A(\alpha, \varepsilon), \tag{5.36″}$$

$$\nabla(\tau)\, A(\alpha, \beta, \tau) = D(\tau)\, A(\alpha, \beta, \tau) - \Delta(\tau, \alpha, \varepsilon)\, A(\varepsilon, \beta, \tau) - \Delta(\tau, \beta, \varepsilon)\, A(\alpha, \varepsilon, \tau) - \Delta(\tau, \tau, \varepsilon)\, A(\alpha, \beta, \varepsilon). \tag{5.36‴}$$

If we substitute $A(\tau) = D(\tau)\,\Phi$, in particular, in formula (5.36′) then we will find the expression for the Laplacian of the scalar Φ:

$$\{D(\tau)\, D(\tau) - \Delta(\tau, \tau, \varepsilon)\, D(\varepsilon)\}\Phi. \tag{5.37}$$

In order to check formula (5.37), we start with the fact that we have:

$$\frac{\partial \Lambda g^{\sigma\tau} \dfrac{\partial\Phi}{\partial x^\tau}}{\Lambda \partial x^\sigma} = \frac{\partial \Lambda \Omega^\sigma(\alpha)\Omega^\tau(\alpha) \dfrac{\partial\Phi}{\partial x^\tau}}{\Lambda \partial x^\sigma} = \frac{\partial \Lambda \Omega^\sigma(\tau)}{\Lambda \partial x^\sigma} D(\tau)\,\Phi + D(\tau)\, D(\tau)\,\Phi.$$

Using formula (5.32) will give (5.37).

By definition, the antisymmetric tensor $A(\alpha, \beta) = \nabla(\alpha)\, A(\beta) - \nabla(\beta)\, A(\alpha)$ is called the *rotor* of the vector $A(\alpha)$.

We find from (5.34) that:

$$A(\alpha, \beta) = D(\alpha)\, A(\beta) - D(\beta)\, A(\alpha) - \{\Delta(\alpha, \beta, \tau) - \Delta(\beta, \alpha, \tau)\}\, A(\tau). \tag{5.38}$$

The third-rank tensor $A(\alpha, \beta, \gamma) = \nabla(\alpha)\, A(\beta, \gamma) + \nabla(\beta)\, A(\gamma, \alpha) + \nabla(\gamma)\, A(\alpha, \beta)$ is called the rotor of the antisymmetric tensor $A(\alpha, \beta) = -A(\beta, \alpha)$. If we substitute the expression (5.35a) then we will find that:

$$\begin{aligned} A(\alpha, \beta, \gamma) = {} & \nabla(\alpha)\, A(\beta, \gamma) + \nabla(\beta)\, A(\gamma, \alpha) + \nabla(\gamma)\, A(\alpha, \beta) \\ & + A(\alpha, \varepsilon)\, \Delta(\beta, \gamma, \varepsilon) - \Delta(\gamma, \tau, \varepsilon) \\ & + A(\beta, \varepsilon)\, \Delta(\gamma, \alpha, \varepsilon) - \Delta(\alpha, \gamma, \varepsilon) \\ & + A(\gamma, \varepsilon)\, \Delta(\alpha, \beta, \varepsilon) - \Delta(\beta, \alpha, \varepsilon). \end{aligned} \tag{5.39}$$

Formulas (5.38) and (5.39) are easily checked by direct calculation:

$$A(\alpha, \beta) = \Omega^\sigma(\alpha)\, \Omega^\tau(\beta) \left(\frac{\partial A_\tau}{\partial x^\sigma} - \frac{\partial A_\sigma}{\partial x^\tau} \right),$$

$$A(\alpha, \beta, \gamma) = \Omega^\sigma(\alpha)\, \Omega^\tau(\beta)\, \Omega^\lambda(\gamma) \left(\frac{\partial A_{\sigma\tau}}{\partial x^\lambda} + \frac{\partial A_{\tau\lambda}}{\partial x^\tau} + \frac{\partial A_{\lambda\sigma}}{\partial x^\tau} \right).$$

In conclusion, we shall find the commutation relations of the operators $D(\alpha)$. We have:

$$D(\alpha)\,D(\beta) - D(\beta)\,D(\alpha) = \left(\Omega^{\sigma}(\alpha)\frac{\partial\Omega^{\tau}(\beta)}{\partial x^{\sigma}} - \Omega^{\sigma}(\beta)\frac{\partial\Omega^{\tau}(\alpha)}{\partial x^{\sigma}}\right)\frac{\partial}{\partial x^{\tau}}\,. \tag{5.40}$$

If we eliminate the derivatives $\frac{\partial\Omega^{\tau}(\beta)}{\partial x^{\sigma}}$, $\frac{\partial\Omega^{\tau}(\beta)}{\partial x^{\sigma}}$ by means of (5.27a) then we will find that:

$$D(\alpha)\,D(\beta) - D(\beta)\,D(\alpha) = [\Delta(\alpha,\beta,\varepsilon) - \Delta(\beta,\alpha,\varepsilon)]\,D(\varepsilon). \tag{5.41}$$

§ 35. Riemann tensor

We shall now calculate the change in a vector under parallel displacement around a closed circuit.

From formula (5.17), we have (applying Stokes's theorem):

$$\begin{aligned}\oint \delta A &= \frac{1}{2}\iint \left\{\frac{\partial\Delta_{\tau}(\alpha,\beta)A(\beta)}{\partial x^{\sigma}} - \frac{\partial\Delta_{\sigma}(\alpha,\beta)A(\beta)}{\partial x^{\tau}}\right\} A(\beta)\,df^{\sigma\tau} \\ &= \frac{1}{2}\iint \left\{\frac{\partial\Delta_{\tau}(\alpha,\beta)}{\partial x^{\sigma}} - \frac{\partial\Delta_{\sigma}(\alpha,\beta)}{\partial x^{\tau}} + \Delta_{\tau}(\alpha,\varepsilon)\Delta_{\sigma}(\varepsilon,\beta) - \Delta_{\sigma}(\alpha,\varepsilon)\Delta_{\tau}(\varepsilon,\beta)\right\} A(\beta)\,df^{\sigma\tau}\,.\end{aligned} \tag{5.42}$$

and in the curly brackets, we have the *covariant-invariant Riemann tensor*:

$$R_{\sigma\tau}(\alpha,\beta) = \frac{\partial\Delta_{\tau}(\alpha,\beta)}{\partial x^{\sigma}} - \frac{\partial\Delta_{\sigma}(\alpha,\beta)}{\partial x^{\tau}} + \Delta_{\tau}(\alpha,\varepsilon)\Delta_{\sigma}(\varepsilon,\beta) - \Delta_{\sigma}(\alpha,\varepsilon)\Delta_{\tau}(\varepsilon,\beta)\,. \tag{5.43}$$

In order to obtain the ordinary Riemann tensor $R_{\sigma\tau\lambda}^{\cdot\cdot\cdot\mu}$ multiply (5.43) by $\Omega_{\lambda}(\alpha)\,\Omega^{\mu}(\beta)$, which will give:

$$\begin{aligned}&\frac{\partial\Delta_{\tau\lambda\cdot}^{\cdot\cdot\mu}}{\partial x^{\sigma}} - \frac{\partial\Delta_{\sigma\lambda\cdot}^{\cdot\cdot\mu}}{\partial x^{\tau}} \\ &\quad + \Delta_{\tau\lambda\cdot}^{\cdot\cdot\varepsilon}\Delta_{\tau\varepsilon\cdot}^{\cdot\cdot\mu} - \Delta_{\sigma\lambda\cdot}^{\cdot\cdot\varepsilon}\Delta_{\tau\varepsilon\cdot}^{\cdot\cdot\mu} - \Delta_{\tau}(\alpha,\beta)\frac{\partial}{\partial x^{\sigma}}[\Omega_{\lambda}(\alpha)\Omega^{\mu}(\beta)] - \Delta_{\sigma}(\alpha,\beta)\frac{\partial}{\partial x^{\tau}}[\Omega_{\lambda}(\alpha)\Omega^{\mu}(\beta)]\,.\end{aligned}$$

Substituting the expression for $\Delta_{\tau\lambda\cdot}^{\cdot\cdot\mu}$ in formula (5.24) and eliminating the derivatives of $\Omega_{\lambda}(\alpha)$ and $\Omega^{\mu}(\beta)$ using (5.27) will give:

$$R_{\sigma\tau}(\alpha,\beta)\,\Omega_{\lambda}(\alpha)\,\Omega^{\mu}(\beta) = \frac{\partial\Gamma_{\tau\lambda}^{\mu}}{\partial x^{\sigma}} - \frac{\partial\Gamma_{\sigma\lambda}^{\mu}}{\partial x^{\tau}} + \Gamma_{\tau\lambda}^{\varepsilon}\Gamma_{\sigma\varepsilon}^{\mu} - \Gamma_{\sigma\lambda}^{\varepsilon}\Gamma_{\tau\varepsilon}^{\mu} = R_{\sigma\tau\lambda}^{\cdot\cdot\cdot\mu}\,. \tag{5.44}$$

In order to obtain the invariant Riemann tensor $R(\alpha,\beta,\mu,\nu)$, multiply (5.43) by $\Omega^{\sigma}(\mu)\,\Omega^{\tau}(\nu)$. That will give:

$$\Omega^{\sigma}(\mu)\left\{\frac{\partial\Delta(\nu,\alpha,\beta)}{\partial x^{\sigma}}-\frac{\partial\Omega^{\tau}(\alpha,\beta)}{\partial x^{\sigma}}\Delta_{\tau}(\alpha,\beta)\right\}-\Omega^{\tau}(\nu)\left\{\frac{\partial\Delta(\mu,\alpha,\beta)}{\partial x^{\tau}}-\frac{\partial\Omega^{\sigma}(\alpha,\beta)}{\partial x^{\tau}}\Delta_{\sigma}(\alpha,\beta)\right\}$$
$$+\,\Delta(\nu,\alpha,\varepsilon)\,\Delta(\mu,\varepsilon,\beta)-\Delta(\mu,\alpha,\varepsilon)\,\Delta(\nu,\varepsilon,\beta).$$

Eliminating the derivatives $\frac{\partial\Omega^{\tau}(\nu)}{\partial x^{\sigma}}$, $\frac{\partial\Omega^{\sigma}(\mu)}{\partial x^{\tau}}$ with the help of (5.27) will result in the formula:

$$R_{\sigma\tau}(\alpha,\beta)\,\Omega_{\lambda}(\alpha)\,\Omega^{\mu}(\beta)=R(\alpha,\beta,\mu,\nu)$$

$$\begin{aligned}&=D(\mu)\,\Delta(\nu,\alpha,\beta)-D(\nu)\,\Delta(\mu,\alpha,\beta)\\&\quad+\Delta(\mu,\varepsilon,\beta)\,\Delta(\nu,\alpha,\varepsilon)-\Delta(\nu,\varepsilon,\beta)\,\Delta(\mu,\alpha,\varepsilon)\\&\quad+\Delta(\mu,\varepsilon,\beta)\,\Delta(\mu,\alpha,\beta)-\Delta(\nu,\varepsilon,\mu)\,\Delta(\varepsilon,\alpha,\beta).\end{aligned}\tag{5.45}$$

If we contract over the indices ν, α and renumber the indices then we will get the invariant Ricci tensor:

$$R(\alpha,\beta)$$
$$=D(\alpha)\Delta(\sigma,\sigma,\beta)-D(\sigma)\Delta(\alpha,\sigma,\beta)+\Delta(\alpha,\sigma,\beta)\Delta(\tau,\tau,\sigma)-\Delta(\tau,\sigma,\alpha)\Delta(\sigma,\tau,\beta)\,.\tag{5.46}$$

Contracting over the indices α, β will give the scalar:

$$R=2\,D(\sigma)\,\Delta(\sigma,\sigma,\beta)-\Delta(\sigma,\sigma,\varepsilon)\,\Delta(\tau,\tau,\varepsilon)-\Delta(\sigma,\tau,\varepsilon)\,\Delta(\tau,\varepsilon,\sigma)\,.\tag{5.47}$$

§ 36. Spinors in Riemannian spaces

We shall now show that the case that was treated by V. A. Fock [**10**], which provides the framework for tensor analysis, will also allow us to introduce spinors into Riemannian space.

We construct a system of Hermitian matrices $\mu(\alpha)$ that satisfy the following conditions:

$$\mu(\alpha)\,\mu(\beta)+\mu(\beta)\,\mu(\alpha)=2\delta(\alpha,\beta).\tag{5.48}$$

In spinor analysis, it is proved that if $n = 2\nu$ or $n = 2\nu + 1$ then these matrices $\mu(\alpha)$ will have $s = 2^{\nu}$ rows.

Two complex *s*-component quantities:

$$W=(W_1,W_2,\ldots,W_s),\qquad \tilde{W}=(\tilde{W}_1,\tilde{W}_2,\ldots,\tilde{W}_s),\tag{5.49}$$

are called *conjugate spinors* when the *n* Hermitian forms $\tilde{W}\,\mu(\alpha)\,W$ form a vector with invariant components.

The components of the spinor remain invariant under the transformations of group *A*:

$$\tilde{W}' \mu(\alpha) W' = \tilde{W} \mu(\alpha) W. \tag{5.50}$$

Under the transformations of the group B:

$$\tilde{W}' \mu(\alpha) W' = \sum_{(\beta)} L(\alpha, \beta)(\tilde{W} \mu(\beta) W), \tag{5.51}$$

the components of the spinor are converted into the other ones by s-rowed representations of the group of orthogonal transformations:

$$W' = S\,W, \qquad \tilde{W}' = \tilde{W} S^{-1}, \tag{5.52}$$

in which S is an s-rowed matrix that changes from point to point and is related to the values of the matrix $\| L(\alpha, \beta) \|$ by:

$$S^{-1} \mu(\alpha) S = \sum_{(\beta)} L(\alpha, \beta) \mu(\beta), \tag{5.53}$$

which follows from formula (5.51).

In order to deduce the formula for the covariant differentiation of spinors, we shall define the concept of the parallel displacement of spinors in the case that was treated by Fock.

The components of a spinor at two infinitely-close points (x) and $(x + dx)$ should be associated with infinitesimal linear transformations:

$$\left.\begin{aligned} W(x+dx) &= \{I + B_\sigma dx^\sigma\} W(x), \\ \tilde{W}(x+dx) &= \tilde{W}(x) \{I - B_\sigma dx^\sigma\}, \end{aligned}\right\} \tag{5.54}$$

in which B_σ are some n-rowed matrices and I is the s-rowed identity matrix. From the transformations of (5.54) that determine parallel displacement, it is necessary to construct the vector $\tilde{W} \mu(\alpha) W$ that undergoes parallel displacement from the spinors $\tilde{W}$ and W; i.e., in accordance with formulas (5.15) and (5.54), one must have:

$$\begin{aligned} &\tilde{W}(x) \{I - B_\sigma dx^\sigma\} \mu(\alpha) \{I + B_\sigma dx^\sigma\} W(x) \\ &\qquad = \sum_{(\beta)} \tilde{W} \mu(\beta) W \{\delta(\alpha, \beta) + \Delta_\sigma(\alpha, \beta)\, dx^\sigma\}, \end{aligned} \tag{5.55}$$

from which, we will get the equation for the Fock matrices B_σ:

$$\mu(\alpha) B_\sigma - B_\sigma \mu(\alpha) = \sum_{(\beta)} \Delta_\sigma(\alpha, \beta) \mu(\beta). \tag{5.56}$$

The general solution to these equations is, as is easily verified:

$$B_\sigma = \tfrac{1}{4}\Delta_\sigma(\alpha, \beta)\,\mu(\alpha)\,\mu(\beta) + i\,f_\sigma I, \tag{5.57}$$

in which f_σ are arbitrary functions and I is the identity matrix.

Therefore, we have the following formulas for the covariant derivatives of an arbitrary spinor:

$$\nabla_\sigma W = \frac{\partial W}{\partial x^\sigma} - B_\sigma W, \qquad \nabla_\sigma \tilde{W} = \frac{\partial \tilde{W}}{\partial x^\sigma} + \tilde{W} B_\sigma, \tag{5.58}$$

and for the invariant derivatives:

$$\nabla(\alpha)\,W = D(\alpha)\,W - B(\alpha)\,W, \qquad \nabla(\alpha)\tilde{W} = D(\alpha)\tilde{W} + \tilde{W}\,B(\alpha). \tag{5.59}$$

§ 37. Application to 5-optics

In 5-optics, we are dealing with a five-dimensional Riemannian space whose metric tensor takes the form:

$$G_{\mu\nu} = \begin{pmatrix} g_{ik} + (1+\chi)g_i g_k & (1+\chi)g_k \\ (1+\chi)g_i & (1+\chi) \end{pmatrix},$$

$$G^{\mu\nu} = \begin{pmatrix} \tilde{g}^{ik} & -\tilde{g}^{ik}g_k \\ -\tilde{g}^{ik}g_i & \dfrac{1}{1+\chi} + \tilde{g}^{ik}g_i g_k \end{pmatrix}. \tag{5.60}$$

It is very easy to check that the components of the *n*-bein matrices will then take the form:

$$\Omega_\sigma(\alpha) = \begin{pmatrix} \omega_i(n) & \sqrt{1+\chi}\,g_i \\ 0 & \sqrt{1+\chi} \end{pmatrix},$$

$$\Omega^\sigma(\alpha) = \begin{pmatrix} \omega^i(n) & 0 \\ -\omega^i(n)g_i & \dfrac{1}{\sqrt{1+\chi}} \end{pmatrix}, \tag{5.61}$$

in which $\|\ \omega_i(n)\ \|$ is the four-dimensional *n*-bein matrix of components of the 4-tensor $\tilde{g}_{ik}$.

For example, as is easily verified, the *n*-bein matrix of the components of the Schwarzschild field [formula (3.22)] will take the form:

$$\Omega_\sigma(\alpha) = \left\{\begin{matrix} \delta_i(k)+\left(e^{\nu/2}-1\right)n_i n(k) & 0 & 0 \\ 0 & e^{\mu/2} & e^{\lambda/2}g \\ 0 & 0 & e^{\lambda/2} \end{matrix}\right\},$$

$$\Omega^\sigma(\alpha) = \left\{\begin{matrix} \delta_i(k)+\left(e^{-\nu/2}-1\right)n_i n(k) & 0 & 0 \\ 0 & e^{-\mu/2} & 0 \\ 0 & -e^{-\lambda/2}g & e^{-\lambda/2} \end{matrix}\right\}. \tag{5.61′}$$

The operators $D(\alpha)$ in formula (5.33) take the form:

$$D(n) = \omega^i(n)\left(\frac{\partial}{\partial x^i} - g_i\frac{\partial}{\partial x^5}\right), \qquad D(5) = \frac{1}{\sqrt{1+\chi}}\frac{\partial}{\partial x^5}. \tag{5.62}$$

Of particular importance is the special case in which $G_{\mu\nu}$ are purely electromagnetic fields and their dependence upon the fifth action coordinate can be neglected. This is the only special case that is considered in modern quantum mechanics.

The following conditions emerge from the general case:

$$\left.\begin{array}{ll} \text{1. No gravitational field} & g_{ik} = \delta_{ik}. \\ \text{2. No } \chi-\text{field} & \chi = 0. \\ \text{3. Cylindricality condition} & \dfrac{\partial g_i}{\partial x^5} = 0. \\ \text{4. Harmonicity condition} & \dfrac{\partial g_i}{\partial x^i} = 0. \end{array}\right\} \tag{5.63}$$

Metric tensors:

$$G_{\mu\nu} = \begin{pmatrix} \delta_{ik}+g_i g_k & g_k \\ g_i & 1 \end{pmatrix}, \qquad G^{\mu\nu} = \begin{pmatrix} \delta_{ik} & -g_k \\ -g_i & 1+g_i g^i \end{pmatrix}. \tag{5.64}$$

Metric tensor n-beins:

$$\Omega_\sigma(\alpha) = \begin{pmatrix} \delta_{ik} & g_i \\ 0 & 1 \end{pmatrix}, \quad \Omega^\sigma(\alpha) = \begin{pmatrix} \delta_{ik} & 0 \\ -g_k & 1 \end{pmatrix}. \tag{5.65}$$

From (5.62), the $D(\alpha)$ operators give:

$$D(n) = \frac{\partial}{\partial x^n} - g_n\frac{\partial}{\partial x^5}, \qquad D(5) = \frac{\partial}{\partial x^5}, \tag{5.66}$$

which imply the commutation relations:

$$D(n)\,D(m) - D(m)\,D(n) = f(m, n)\,D(5),$$
$$D(5)\,D(n) - D(n)\,D(5) = 0. \tag{5.67}$$

It is very important that by virtue of (5.65), one will have:

$$A^i = \Omega_i\,(\alpha)\,A(\alpha) = A(i), \qquad A_5 = \Omega_5\,(\alpha)\,A(\alpha) = A(5)\,; \tag{5.68}$$

i.e., the gradient-invariant components of any 5-vector $A(\alpha)$ will coincide with its invariant components, and therefore:

$$\{A^i, A_5\} = \{A(i), A(5)\}. \tag{5.69}$$

Hence, using the symbol of invariant differentiation will give a gradient-invariant expression in this particular case.

Calculating the Ricci symbols from (5.30′) will give:

$$\left.\begin{array}{ll} \Delta(i, kl) = 0, & \Delta(5, ik) = \tfrac{1}{2}\,f(i, k), \\ \Delta(5, 5l) = 0, & \Delta(i, k5) = \tfrac{1}{2}\,f(k, i), \end{array}\right\} \tag{5.70}$$

$$\Delta(\beta, \beta\alpha) = 0.$$

Using the general formulas of § 34, and singling out the action coordinate, we calculate:

1. (Invariant) divergence of a vector:

$$D(n)\,A(n) + D(5)\,A(5). \tag{5.71}$$

2. Invariant divergence of a symmetric tensor:

$$\left.\begin{array}{l} D(n)\,Q(k, n) + D(5)\,Q(k, 5) + f(k, n)\,Q(5, n), \\ D(n)\,Q(5, n) + D(5)\,Q(5, 5). \end{array}\right\} \tag{5.72}$$

3. Invariant divergence of an antisymmetric tensor:

$$\left.\begin{array}{l} D(n)\,W(k, n) + D(5)\,W(k, 5), \\ D(n)\,W(5, n) + \tfrac{1}{2}\,f(k, n)\,W(k, n). \end{array}\right\} \tag{5.73}$$

4. Invariant divergence of an antisymmetric tensor of rank three:

$$\left.\begin{array}{l} D(n)\,K(n,i,k)+D(5)\,K(5,i,k), \\ D(n)\,K(n,5,k)-\tfrac{1}{2}\,f(l,n)\,K(i,n,k). \end{array}\right\} \tag{5.74}$$

5. Rotor of a vector:

$$\left.\begin{array}{l} D(n)\,A(k)-D(k)\,A(i)+f(i,k)A(5), \\ D(5)\,A(k)-D(k)\,A(5). \end{array}\right\} \tag{5.75}$$

6. Rotor of an antisymmetric tensor:

$$D(l)\,A(i,k)+D(i)\,A(k,l)+D(k)\,A(l,i)+f(i,k). \tag{5.76}$$

7. Laplacian of a scalar:

$$\{D(n)\,D(n)+D(5)\,D(5)\}\,\Phi. \tag{5.77}$$

Using the formulas of § 36, we compute:

$$\left.\begin{array}{c} B^i=B(i)=\tfrac{1}{4}\,f(l,k)\mu(5)\mu(i)\mu(k), \\[2ex] B_5=B(5)=\tfrac{1}{8}\,f(i,k)\mu(i)\mu(k), \\[2ex] \mu(\alpha)B(\alpha)+B(\alpha)\mu(\alpha)=-\tfrac{1}{8}\,f(i,k)\mu(5)\mu(l)\mu(k) \\[2ex] \tilde{W}[B(k)\mu(i)+\mu(i)B(k)]W=\tfrac{1}{2}\,f(k,n)K(5,n,i), \\[2ex] \tilde{W}[B(5)\mu(k)+\mu(k)B(5)]W=\tfrac{1}{4}\,f(m,n)K(m,n,k), \\[2ex] \tilde{W}[B(k)\mu(5)+\mu(5)B(k)]W=0, \\[2ex] \tilde{W}[B(5)\mu(5)+\mu(5)B(5)]W=\tfrac{1}{4}\,f(m,n)K(m,n,5), \end{array}\right\} \tag{5.78}$$

in which $K(\alpha, \beta, \gamma)$ is the antisymmetric tensor of rank three:

$$K(\alpha,\beta,\gamma)=\tfrac{1}{2}\tilde{W}\,[\mu(\alpha)\,\mu(\beta)\,\mu(\gamma)-\mu(\gamma)\,\mu(\beta)\,\mu(\alpha)]\,W. \tag{5.79}$$

§ 38. Hilbert's invariant integral.

Consider the integral:

$$\left.\begin{aligned} J &= \int \Lambda\, L\left(\Omega^\sigma(\alpha); \frac{\partial \Omega^\sigma(\alpha)}{\partial x^\sigma}\right) dx, \\ \Lambda &= \mathrm{Det}[\Omega_\sigma(\alpha)]; \qquad dx = dx_1 dx_2 \dots dx_n, \end{aligned}\right\} \tag{5.80}$$

which will remain invariant under transformations of the groups A and B.

We have three types of variational quantities $\Omega^\sigma(\alpha)$.

1. Arbitrary variations: $\delta\Omega^\sigma(\alpha)$.

2. Variations that are generated by the infinitesimal transformations of the group A: $\delta_1\Omega^\sigma(\alpha)$.

3. Variations that are generated by the infinitesimal transformations of the group B: $\delta_2\Omega^\sigma(\alpha)$.

It is obvious that $\delta J \neq 0$, but one will always have $\delta_1 J = \delta_2 J = 0$.

I. The group of transformations A.

One has (ε is an arbitrary parameter):

$$\left.\begin{aligned} x'^\sigma &= x^\sigma + \varepsilon f^\sigma(x), \\ dx'^\sigma &= dx^\sigma + \varepsilon \left(\frac{\partial f^\sigma}{\partial x^\tau}\right) dx^\tau, \\ \Omega'^\sigma(x^\sigma + \varepsilon f^\sigma(x) \mid \alpha) &= \Omega^\sigma(x \mid \alpha) + \varepsilon\, \Omega^\tau(x \mid \alpha) \frac{\partial f^\sigma}{\partial v^\tau}, \end{aligned}\right\} \tag{5.81}$$

in which one will have, up to terms that are linear in ε:

$$\Omega'^\sigma(\alpha) = \Omega^\sigma(\alpha) + \varepsilon \left(\Omega^\tau(\alpha) \frac{\partial f^\sigma}{\partial x^\tau} - f^\tau \frac{\partial \Omega^\sigma}{\partial x^\tau}\right), \tag{5.82}$$

and therefore:

$$\delta_1\Omega^\sigma(\alpha) = \varepsilon \left(\Omega^\tau(\alpha) \frac{\partial f^\sigma}{\partial x^\tau} - f^\tau \frac{\partial \Omega^\sigma}{\partial x^\tau}\right). \tag{5.83}$$

II. The group of transformations B:

We have:

$$\Omega'^\sigma(\alpha) = \{\delta(\alpha, \beta) + \varepsilon\, \delta(\alpha, \beta)\}\, \Omega^\sigma(\beta), \tag{5.84}$$

in which $\| A(\alpha, \beta) \|$ is an arbitrary antisymmetric matrix.

Therefore:

$$\delta_2\Omega^\sigma(\alpha) = \varepsilon A(\alpha, \beta)\, \Omega^\sigma(\beta). \tag{5.85}$$

We shall now compute the invariant variation of (5.80):

$$\begin{aligned}\delta J &= \int \left\{ \frac{\partial \Lambda L}{\partial \Omega^\sigma(\alpha)} - \frac{\partial}{\partial x^\tau}\left(\frac{\partial \Lambda L}{\partial\left(\dfrac{\partial \Omega^\sigma(\alpha)}{\partial x^\tau}\right)} \right) \right\} \delta\Omega^\sigma(\alpha)\, dx \\ &= \int \Lambda \theta_\sigma(\alpha)\, \delta\Omega^\sigma(\alpha)\, dx\,,\end{aligned} \tag{5.86}$$

in which we have introduced the notation:

$$\Lambda \theta_\sigma(\alpha) = \frac{\partial \Lambda L}{\partial \Omega^\sigma(\alpha)} - \frac{\partial}{\partial x^\tau}\left(\frac{\partial \Lambda L}{\partial\left(\dfrac{\partial \Omega^\sigma(\alpha)}{\partial x^\tau}\right)} \right). \tag{5.87}$$

If we now substitute $\delta\Omega^\sigma(\alpha) = \delta_2\Omega^\sigma(\alpha)$ in (5.86) from formula (5.85) then that will give:

$$0 = \varepsilon \int \Lambda \theta_\sigma(\alpha)\, A(\alpha,\beta)\, \Omega^\sigma(\beta)\, dx \ = \varepsilon \int \Lambda \theta(\alpha)\, A(\alpha,\beta)\, dx\,, \tag{5.88}$$

in which, due to the arbitrariness in $A(\alpha,\beta)$, we should have the symmetric tensor $\theta(\alpha, \beta)$:

$$\theta(\alpha, \beta) = \frac{\Omega^\sigma(\beta)}{\Lambda}\frac{\partial \Lambda L}{\partial \Omega^\sigma(\alpha)} - \frac{\Omega^\sigma(\beta)}{\Lambda}\frac{\partial}{\partial x^\tau}\left(\frac{\partial \Lambda L}{\partial\left(\dfrac{\partial \Omega^\sigma(\alpha)}{\partial x^\tau}\right)} \right). \tag{5.89}$$

If we now substitute $\delta\Omega^\sigma(\alpha) = \delta_1\Omega^\sigma(\alpha)$ in (5.86) from formula (5.83) then that will give:

$$\begin{aligned} 0 \ &= \varepsilon \int \Lambda \theta_\sigma(\alpha) \left\{ \Omega^\tau(\alpha) \frac{\partial f^\sigma}{\partial x^\tau} - f^\tau \frac{\partial \Omega^\sigma(\alpha)}{\partial x^\tau} \right\} dx \\ &= -\, \varepsilon \int \Lambda f^\tau \left\{ \frac{\partial \Omega^\tau(\alpha)}{\partial x^\tau}\theta_\sigma(\alpha) + \frac{1}{\Lambda}\frac{\partial \Lambda \Omega^\nu(\alpha)\,\theta_\tau(\alpha)}{\partial x^\nu} \right\} dx \\ &= -\, \varepsilon \int \Lambda f^\tau \left\{ \frac{1}{\Lambda}\frac{\partial \Lambda \theta^\nu_\tau}{\partial x^\nu} + \frac{\partial \Omega^\sigma(\alpha)}{\partial x^\tau}\Omega^\nu(\alpha)\,\theta_{\sigma\nu} \right\} dx\,, \end{aligned} \tag{5.90}$$

in which, due to the symmetry of the tensor $\theta_{\sigma\nu}$ and the arbitrariness in the functions $f^{\,\tau}$, one will get:

$$\frac{1}{\Lambda}\frac{\partial \Lambda \theta^{\nu}_{\tau}}{\partial x^{\nu}} + \frac{\partial \Omega^{\sigma}(\alpha)}{\partial x^{\tau}}\Omega^{\nu}(\alpha)\,\theta_{\sigma\nu} \; = \; \nabla_{\nu}\theta^{\nu}_{\tau} \; = 0. \tag{5.91}$$

Using formula (5.37″), we rewrite (5.91) in the form:

$$\nabla(\beta)\;\theta(\alpha,\beta) = D(\beta)\theta(\alpha,\beta) - \Delta(\beta,\alpha,\tau)\theta(\tau,\beta) - \Delta(\beta,\beta,\tau)\theta(\tau,\alpha) = 0. \tag{5.91′}$$

Now, consider an important special case for which $\Omega^{\sigma}(\alpha)$ and $\dfrac{\partial \Omega^{\sigma}(\alpha)}{\partial x^{\tau}}$ are included in the function L only in the combinations:

$$\left.\begin{aligned} g^{\sigma\tau} &= \Omega^{\sigma}(\alpha)\,\Omega^{\tau}(\alpha), \\ \frac{\partial g^{\sigma\tau}}{\partial x^{\lambda}} &= \frac{\partial \Omega^{\sigma}(\alpha)}{\partial x^{\lambda}}\Omega^{\tau}(\alpha) + \Omega^{\sigma}(\alpha)\frac{\partial \Omega^{\tau}(\alpha)}{\partial x^{\lambda}}. \end{aligned}\right\} \tag{5.92}$$

In this particular case, the invariant integral can be put into the form:

$$J = \int \Lambda\, L\left(g^{\sigma\tau}, \frac{\partial g^{\sigma\tau}}{\partial x^{\lambda}}\right) dx\,, \tag{5.93}$$

and its variation will take the form:

$$\begin{aligned} \delta J \;&=\; \int \left\{ \frac{\partial \Lambda L}{\partial g^{\sigma\tau}} - \frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial \Lambda L}{\partial\left(\dfrac{\partial g^{\sigma\tau}}{\partial x^{\lambda}}\right)} \right) \right\} \delta g^{\sigma\tau}\, dx \\ &= 2\int \left\{ \frac{\partial \Lambda L}{\partial g^{\sigma\tau}} - \frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial \Lambda L}{\partial\left(\dfrac{\partial g^{\sigma\tau}}{\partial x^{\lambda}}\right)} \right) \right\} \Omega^{\tau}(\alpha)\,\delta\Omega^{\sigma}(\alpha)\, dx\,. \end{aligned} \tag{5.94}$$

Comparing this with the general formula (5.86), one will find that:

$$\Lambda\theta_{\sigma}(\alpha) = \left\{ \frac{\partial \Lambda L}{\partial g^{\sigma\tau}} - \frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial \Lambda L}{\partial\left(\dfrac{\partial g^{\sigma\tau}}{\partial x^{\lambda}}\right)} \right) \right\} \Omega^{\tau}(\alpha), \tag{5.95}$$

so, upon multiplying this by $\Omega_\nu(\alpha)$, it will result that:

$$\tfrac{1}{2}\Lambda\theta_{\sigma\tau} = \frac{\partial \Lambda L}{\partial g^{\sigma\tau}} - \frac{\partial}{\partial x^\lambda}\left(\frac{\partial \Lambda L}{\partial\left(\frac{\partial g^{\sigma\tau}}{\partial x^\lambda}\right)}\right). \tag{5.96}$$

The symmetry of the tensor $\theta_{\sigma\tau}$ is obvious from its derivation. If we now substitute $\delta\Omega^\sigma(\alpha) = \delta_1\Omega^\sigma(\alpha)$ from formula (5.83) in (5.94) then that will give:

$$\begin{aligned} 0 &= \varepsilon \int \Lambda\theta_{\sigma\tau}\Omega^\tau(\alpha)\left\{\Omega^\nu(\alpha)\frac{\partial f^\sigma}{\partial x^\nu} - f^\nu \frac{\partial\Omega^\sigma(\alpha)}{\partial x^\nu}\right\} dx \\ &= -\varepsilon \int \Lambda f^\sigma\left\{\frac{1}{\Lambda}\frac{\partial\theta^\nu_\sigma}{\partial x^\nu} + \frac{1}{2}\theta_{\tau\nu}\frac{\partial g^{\tau\nu}}{\partial x^\sigma}\right\} dx, \end{aligned} \tag{5.97}$$

which should once more satisfy $\nabla_\nu\theta^\nu_\sigma = 0$.

The cases that we considered were based upon the Hilbert's findings regarding conservation laws in the theory of gravity. Following Hilbert, in the next chapter, we will give the derivation of the conservation laws for tensor fields, which will be based upon an invariant integral of the type (5.93), and the derivation of conservation laws for spinor fields, which will be based upon an invariant integral of the more general type (5.80).

In conclusion, we shall consider, as an example, the invariant integral:

$$J = \int \Lambda\, \Omega^\sigma(\alpha)\, \Omega^\tau(\alpha)\, R_{\sigma\tau}(\alpha, \beta)\, dx, \tag{5.98}$$

in which $R_{\sigma\tau}(\alpha, \beta)$ is the Riemann tensor in formula (5.32).

One has:

$$\delta J = \int \delta\Lambda\ \Delta R\, dx + 2\int \Lambda R_\sigma\, \delta\Omega^\sigma(\alpha)\, \Omega^\tau(\alpha)\, dx + \int \Lambda\Omega^\sigma(\alpha)\, \Omega^\tau(\alpha)\, \delta R_{\sigma\tau}(\alpha, \beta)\, dx, \tag{5.99}$$

in which we have introduced the notations:

$$R_\sigma(\alpha) = \Omega^\tau(\beta)\, R_{\sigma\tau}(\alpha, \beta), \qquad R = \Omega^\sigma(\alpha)\, \Omega^\tau(\alpha)\, R_{\sigma\tau}(\alpha, \beta)\,.$$

In order to calculate the last integral, we use a known trick ([**12**], § 94). Note that $\delta\Delta_\alpha(\alpha, \beta)$ is a tensor, and choose a coordinate system in which all $\dfrac{\partial\Omega^\sigma(\alpha)}{\partial x^\tau}$ are equal to zero at some point. One will have:

$$\Omega^\sigma(\alpha)\, \Omega^\tau(\beta)\, \delta R_{\sigma\tau}(\alpha, \beta) = 2\, \frac{\partial}{\partial x^\sigma}\{\Omega^\sigma(\alpha)\, \Omega^\tau(\beta)\, \delta\Delta_\tau(\alpha, \beta)\}.$$

In an arbitrary coordinate system one will have:

$$\Omega^{\sigma}(\alpha)\,\Omega^{\tau}(\beta)\,\delta R_{\sigma\tau}(\alpha,\beta) = \frac{2}{\Lambda}\frac{\partial}{\partial x^{\sigma}}\{\Lambda\,\Omega^{\sigma}(\alpha)\,\Omega^{\tau}(\beta)\,\delta\Delta_{\tau}(\alpha,\beta)\}.$$

Therefore, the last integral in (5.99) can be converted into a surface integral, and will disappear.

If one notices that:

$$\delta\Lambda = \Lambda\Omega^{\sigma}(\alpha)\,\delta\Omega_{\sigma}(\alpha) = -\,\Lambda\Omega_{\sigma}(\alpha)\,\delta\Omega^{\sigma}(\alpha) \tag{5.100}$$

then that will make:

$$\delta J = \delta\int \Delta R\,dx = 2\int \Lambda\left(R_{\sigma}(\alpha) - \tfrac{1}{2}\Omega_{\sigma}(\alpha)R\right)\delta\Omega^{\sigma}(\alpha)\,dx. \tag{5.101}$$

If we substitute $\delta_2\,\Omega^{\sigma}(\alpha)$ from formula (5.85) then we will get:

$$0 = 2\varepsilon\int \Lambda\left(R_{\sigma}(\alpha) - \tfrac{1}{2}\Omega_{\sigma}(\alpha)\,R\right)A(\alpha,\beta)\,\Omega^{\sigma}(\beta)\,dx, \tag{5.102}$$

which will imply the symmetry of the tensor $R(\alpha, \beta)$.

If we substitute $\delta_1\,\Omega^{\sigma}(\alpha)$ from formula (5.83) then we will get:

$$0 = 2\varepsilon\int \Lambda\left(R_{\sigma}(\alpha) - \tfrac{1}{2}\Omega_{\sigma}(\alpha)\,R\right)\left(\Omega^{\tau}(\alpha)\frac{\partial f^{\sigma}}{\partial x^{\tau}} - f^{\tau}\,\frac{\partial\Omega^{\sigma}(\alpha)}{\partial x^{\tau}}\right)dx\,. \tag{5.103}$$

If we do the same calculations that we did in the derivation of (5.90) then that will give:

$$\nabla(\tau)\left(R(\sigma,\tau) - \tfrac{1}{2}\delta(\sigma,\tau)\,R\right) = 0. \tag{5.104}$$

Our study has shown that the elementary geometric objects that define the metric in a Riemannian space are the elements of the metric *n*-bein matrices, while the components of the metric tensor g_{ik} are derived from them by quadratic constructions. As long as we are dealing with ordinary tensors, which are spin-tensors of *even rank*, we can use the usual metric. However, as soon as we turn to spin-tensors of *odd* rank, we must go back to the original metric *n*-bein.

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CHAPTER VI

WAVE-LIKE 5-OPTICS IN RIEMANNIAN SPACES

INTRODUCTION

In order to embark upon wave-like 5-optics in Riemannian spaces, we might wish to consistently demand the periodic dependence of the components $G_{\mu\nu}$ of the metric field upon the action coordinate.

However, we shall not do that in this monograph, and, as is done in modern quantum mechanics, we shall demand that the components $G_{\mu\nu}$ of the metric field should *not* depend upon the fifth action coordinate; i.e., it should satisfy the cylindricality condition. Including that dependency will have to be the subject of further study.

Obviously, the transition of elementary particles (tensors and spinors) from one charged state to another with the emission or absorption of a massive, charged quantum of metric field – viz., the metron – is a new, specifically 5-optical effect.

As we mentioned already, we shall not discuss that effect here, but confine ourselves to the special case in which the metric field $G_{\mu\nu}$ is purely electromagnetic; i.e., the only case that is considered in modern quantum mechanics.

§ 39. Derivation of some formulas from the theory of tensor and spinor fields

In the case of the tensor field $W^{(r)}$, we can derive all of the main formulas for field theory from an invariant Lagrange integral:

$$\int \Lambda\, L\left(W^{(r)}, \frac{\partial W^{(r)}}{\partial x^{\sigma}} \middle| G^{\sigma\tau}, \frac{\partial G^{\sigma\tau}}{\partial x^{\lambda}} \right) dx^1\, dx^2\, dx^3\, dx^4\, dx^5. \tag{6.1}$$

Varying the field components $W^{(r)}$ at constant $G^{\sigma\tau}$ will give the field equations in generally-covariant form:

$$\frac{\partial \Lambda L}{\partial W^{(r)}} - \frac{\partial}{\partial x^{\sigma}} \left(\frac{\partial \Lambda L}{\partial \left(\dfrac{\partial W^{(r)}}{\partial x^{\sigma}} \right)} \right) = 0. \tag{6.2}$$

Varying the metric potentials $G^{\sigma\tau}$ will give an expression for the symmetric 5-tensor $\theta_{\sigma\tau}$ of energy-impulse-charge:

$$\tfrac{1}{2}\Lambda\theta_{\sigma\tau} = \frac{\partial \Lambda L}{\partial G^{\sigma\tau}} - \frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial \Lambda L}{\partial\left(\dfrac{\partial G^{\sigma\tau}}{\partial x^{\lambda}}\right)}\right), \tag{6.3}$$

or, in view of the fact that $\dfrac{\partial \Lambda}{\partial G^{\sigma\tau}} = -\tfrac{1}{2}\Lambda\, G_{\sigma\tau}$:

$$\tfrac{1}{2}\,\theta_{\sigma\tau} = \frac{\partial L}{\partial G^{\sigma\tau}} - \frac{1}{\Lambda}\frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial \Lambda L}{\partial\left(\dfrac{\partial G^{\sigma\tau}}{\partial x^{\lambda}}\right)}\right) - \frac{1}{2}G_{\sigma\tau}L\,. \tag{6.3′}$$

It was proved in § 36 that it satisfies:

$$\nabla_{\tau}\theta_{\sigma}^{\tau} = 0. \tag{6.4}$$

In the case of complex spinor fields W, $\tilde{W}$, we can derive all of the main formulas for field theory from the invariant Lagrange integral:

$$\int \Lambda\, L\left(W, \tilde{W}, \frac{\partial W}{\partial x^{\sigma}}, \frac{\partial \tilde{W}}{\partial x^{\sigma}} \middle| \Omega^{\sigma}(\alpha), \frac{\partial \Omega^{\sigma}(\alpha)}{\partial x^{\tau}}\right) dx^1\, dx^2\, dx^3\, dx^4\, dx^5. \tag{6.5}$$

Varying the field components $\tilde{W}$ and W at constant $\Omega^{\sigma}(\alpha)$ will give the spinor field equations in the generally-covariant form:

$$\frac{\partial \Lambda L}{\partial \tilde{W}} - \frac{\partial}{\partial x^{\sigma}}\left(\frac{\partial \Lambda L}{\partial\left(\dfrac{\partial \tilde{W}}{\partial x^{\sigma}}\right)}\right) = 0,$$

$$\frac{\partial \Lambda L}{\partial W} - \frac{\partial}{\partial x^{\sigma}}\left(\frac{\partial \Lambda L}{\partial\left(\dfrac{\partial W}{\partial x^{\sigma}}\right)}\right) = 0. \tag{6.6}$$

Varying the metric potentials $\Omega^{\sigma}(\alpha)$ will give an expression for the symmetric tensor $\theta(\alpha, \beta)$ of energy-impulse-charge by means of the formula:

$$\Lambda\theta(\alpha,\beta) = \Omega^\sigma(\beta)\frac{\partial\Lambda L}{\partial\Omega^\sigma(\alpha)} - \Omega^\sigma(\beta)\frac{\partial}{\partial x^\lambda}\left(\frac{\partial\Lambda L}{\partial\left(\dfrac{\partial\Omega^\sigma(\alpha)}{\partial x^\lambda}\right)}\right), \tag{6.7}$$

or, in view of the fact that $\dfrac{\partial\Lambda L}{\partial\Omega^\sigma(\alpha)} = -\,\Lambda\,\Omega_\sigma(\beta)$:

$$\theta(\alpha,\beta) = \Omega^\sigma(\beta)\frac{\partial L}{\partial\Omega^\sigma(\alpha)} - \frac{\Omega^\sigma(\beta)}{\Lambda}\frac{\partial}{\partial x^\lambda}\left(\frac{\partial\Lambda L}{\partial\left(\dfrac{\partial\Omega^\sigma(\alpha)}{\partial x^\lambda}\right)}\right) - \delta(\alpha,\beta)L\,. \tag{6.8}$$

As was proved in § 36, the tensor $\theta(\alpha,\beta)$ is symmetric and satisfies the equation:

$$\nabla(\beta)\;\theta(\alpha,\beta) = 0. \tag{6.9}$$

For some applications, it is interesting to consider invariant Lagrange integrals that are more general than (6.5):

$$\int \Lambda\, L\left(\cdots \,|\, \Omega^\sigma(\alpha), \Omega_\tau(\alpha), \frac{\partial\Omega^\sigma(\alpha)}{\partial x^\tau}\right) dx^1\, dx^2\, dx^3\, dx^4\, dx^5, \tag{6.5}$$

in which all $\Omega_\tau(\gamma)$ should be considered to be functions of $\Omega^\tau(\alpha)$. We have:

$$\frac{\partial L}{\partial\Omega^\sigma(\alpha)} = \left(\frac{\partial L}{\partial\Omega^\sigma(\alpha)}\right) + \frac{\partial L}{\partial\Omega_\tau(\gamma)}\cdot\frac{\partial\Omega_\tau(\gamma)}{\partial\Omega^\sigma(\alpha)}, \tag{6.10}$$

in which the parentheses indicate that the derivative is taken at constant $\Omega_\tau(\gamma)$.

However, since $\Omega^\sigma(\alpha)\;\delta\Omega_\tau(\alpha) = -\,\Omega_\tau(\alpha)\;\delta\Omega^\sigma(\alpha)$, it should result from multiplying by $\Omega_\sigma(\alpha)$ that:

$$\begin{gathered}\delta\Omega_\tau(\alpha) = -\,\Omega_\tau(\alpha)\,\Omega_\sigma(\gamma)\;\delta\Omega^\sigma(\alpha), \\ \frac{\partial\Omega_\tau(\gamma)}{\partial\Omega^\sigma(\alpha)} = -\,\Omega_\tau(\alpha)\,\Omega_\sigma(\gamma)\,.\end{gathered} \tag{6.11}$$

Substituting (6.11) in (6.10) will give:

$$\frac{\partial L}{\partial\Omega^\sigma(\alpha)} = \left(\frac{\partial L}{\partial\Omega^\sigma(\alpha)}\right) - \frac{\partial L}{\partial\Omega_\tau(\gamma)}\Omega_\tau(\alpha)\Omega_\sigma(\gamma)\,. \tag{6.12}$$

Substituting (6.12) in (6.8) will give:

$$\theta(\alpha, \beta)$$

$$= \Omega^\sigma(\beta)\left(\frac{\partial L}{\partial \Omega^\sigma(\alpha)}\right) - \Omega_\tau(\alpha)\frac{\partial L}{\partial \Omega_\tau(\beta)} - \frac{\Omega^\sigma(\beta)}{\Lambda}\frac{\partial}{\partial x^\lambda}\left(\frac{\partial \Lambda L}{\partial\left(\dfrac{\partial \Omega^\sigma(\alpha)}{\partial x^\lambda}\right)}\right) - \delta(\alpha, \beta)\, L. \qquad (6.8')$$

We shall now reveal the physical meaning of equations (6.4) and (6.9), which, from formula (5.36″), can be written in the form:

$$D(\tau)\ \theta(\alpha, \beta) - \Delta(\tau, \alpha, \varepsilon)\ \theta(\varepsilon, \tau) - \Delta(\tau, \tau, \varepsilon)\ \theta(\alpha, \varepsilon) = 0. \qquad (6.13)$$

In order to do that, we shall consider the special case in which the external field $G_{\mu\nu}$ is purely-electromagnetic and its dependence upon x^5 can be neglected.

From the general formulas (5.72), we can rewrite equation (6.13) for this special case in the form:

$$\left.\begin{aligned} D(n)\,\theta(k,n) + D(5)\,\theta(k,5) + f(k,n)\,\theta(5,n) &= 0, \\ D(n)\,\theta(5,n) + D(5)\,\theta(5,5) \qquad\qquad\quad &= 0. \end{aligned}\right\} \qquad (6.14)$$

Averaging over the action coordinate will give:

$$\frac{\partial \bar\theta(k,n)}{\partial x^n} + f(k,n)\,\bar\theta(5,n) = 0, \qquad (6.15)$$

$$\frac{\partial \bar\theta(5,n)}{\partial x^n} = 0. \qquad (6.16)$$

Equation (6.15) expresses the conservation of energy and impulse in the presence of an external electromagnetic field that does not depend upon x^5, while equation (6.16) expresses the law of conservation for the electric current. Hence, equation (6.13) expresses the law of conservation of energy, impulse, and charge in the case of a general field $G_{\mu\nu}$.

§ 40. Real tensor fields in Riemannian 5-spaces

In Chapter IV, we established the complete system of equations for meson fields in the presence of source fields. When written in generally-covariant form, they will take the following forms:

1. Scalar mesons:

$$\frac{\partial \Lambda W^\lambda}{\partial x^\lambda} = \Lambda Q, \qquad \frac{\partial W_\lambda}{\partial x^\mu} - \frac{\partial W_\mu}{\partial x^\lambda} = 0. \qquad (6.17)$$

2. Vector mesons:

$$\frac{\partial \Lambda W^{\lambda\mu}}{\partial x^{\mu}} = \Lambda Q^{\lambda}, \qquad \frac{\partial W_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial W_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial W_{\nu\lambda}}{\partial x^{\mu}} = 0. \tag{6.17}$$

3. Pseudo-vector mesons:

$$\frac{\partial \Lambda W^{\lambda\mu\nu}}{\partial x^{\nu}} = \Lambda Q^{\lambda\mu}, \qquad \frac{\partial W_{\lambda\mu\nu}}{\partial x^{\sigma}} - \frac{\partial W_{\mu\nu\sigma}}{\partial x^{\lambda}} + \frac{\partial W_{\nu\sigma\lambda}}{\partial x^{\mu}} - \frac{\partial W_{\sigma\lambda\mu}}{\partial x^{\nu}} = 0. \tag{6.17}$$

4. Pseudo-scalar mesons:

$$\frac{\partial \Lambda W^{\lambda\mu\nu\sigma}}{\partial x^{\sigma}} = \Lambda Q^{\lambda\mu\nu}, \quad \frac{\partial W_{\lambda\mu\nu\sigma}}{\partial x^{\tau}} + \frac{\partial W_{\mu\nu\sigma\tau}}{\partial x^{\lambda}} + \frac{\partial W_{\nu\sigma\tau\lambda}}{\partial x^{\mu}} + \frac{\partial W_{\sigma\tau\lambda\mu}}{\partial x^{\nu}} + \frac{\partial W_{\tau\lambda\mu\nu}}{\partial x^{\sigma}} = 0. \tag{6.17}$$

Here, $W_{\lambda\mu}$, $W_{\lambda\mu\nu}$, $W_{\lambda\mu\nu\sigma}$, $Q^{\lambda\mu}$, and $Q^{\lambda\mu\nu}$ are antisymmetric in all tensor indices. The structure of the equations is the same for all of these meson fields: The first group of equations expresses the idea that the divergence of the field strength tensor is equal to the source density field. The second group of equations expresses the idea that the rotor of the field-strength tensor is equal to zero in all four cases.

Using the rules that were set down in § 37, we can write these equations in terms of the invariant differentiation symbol.

1. Scalar mesons:

$$\left.\begin{aligned} D(i)W(i) + D(5)W(5) &= Q, \\ D(i)W(k) - D(k)W(i) + f(i,k)W(5) &= 0, \\ D(5)W(i) + D(i)W(5) &= 0. \end{aligned}\right\} \tag{6.17′}$$

2. Vector mesons:

$$\left.\begin{aligned} D(i)W(k,i) + D(5)W(k,5) &= Q(k), \\ D(i)W(5,i) - \tfrac{1}{2} f(i,k)W(k,i) &= Q(5), \\ D(i)W(k,l) + D(k)W(l,i) + D(l)W(i,k) & \\ + f(i,k)W(5,l) + f(k,l)W(5,i) + f(l,i)W(5,k) &= 0, \\ D(5)W(i,k) + D(i)W(k,5) + D(k)W(5,i) &= 0. \end{aligned}\right\} \tag{6.18′}$$

3. Pseudo-vector mesons:

$$\left.\begin{aligned} D(i)W(k,l,i) + D(5)W(k,l,5) &= Q(k,l), \\ D(i)W(5,l,i) - \tfrac{1}{2} f(i,k)W(k,l,i) &= Q(5,l), \end{aligned}\right\}$$

$$\left.\begin{aligned} D(i)W(k,l,m)-D(k)W(l,m,i)+D(l)\,W(m,i,k)-D(m)W(i,k,l)\\ f(i,k)W(5,l,m)-f(k,l)W(5,m,i)+f(l,m)W(5,i,k)-f(m,i)W(5,k,l)=0,\\ \\ D(5)W(i,k,l)-D(i)W(k,l,5)+D(k)W(l,5,i)-D(i)W(5,i,k)=0.\end{aligned}\right\} \qquad (6.19')$$

4. Pseudo-scalar mesons:

$$\left.\begin{aligned} &D(i)W(k,l,m,i)+D(5)W(k,l,m,5)=Q(k,l,m),\\ \\ &D(i)W(5,l,m,i)-\tfrac{1}{2}f(i,k)W(k,l,m,i)=Q(5,l,m),\\ \\ &D(i)W(k,l,m,n)+D(k)W(l,m,n,i)+D(n)W(i,k,l,m)+D(m)W(m,i,k,l)\\ &+D(n)W(i,k,l,m)+f(i,k)W(5,l,m,m)+f(k,l)W(5,m,n,i)+\\ &+f(l,m)W(5,n,i,k)+f(m,n)W(5,i,k,l)+f(n,i)W(5,k,l,m)=0,\\ \\ &D(5)W(k,l,m,n)+D(k)W(l,m,n,5)+D(l)W(m,n,5,k)+D(m)W(n,5,k,l)\\ &+D(n)W(5,k,l,m)=0.\end{aligned}\right\} \qquad (6.20')$$

One consequently imposes the cyclicality condition upon the field components:

$$\left.\begin{aligned} W(x^1,x^2,x^3,x^4,x^5)=U(x^1,x^2,x^3,x^4)\exp(i\mu x^5),\\ Q(x^1,x^2,x^3,x^4,x^5)=q(x^1,x^2,x^3,x^4)\exp(i\mu x^5),\end{aligned}\right\} \qquad (6.21)$$

and rewrites the equation in Fourier components.

When the D-differentiation operator is applied to the Fourier components, it will replace $\partial / \partial x^5$ with $i\mu$. One will have:

$$D(k) = \frac{\partial}{\partial x^k} - i\mu\, g_k, \qquad D(5) = i\mu\,, \qquad (6.22)$$

and the equations for the meson field will take on the following form:

1. Scalar meson:

$$\left.\begin{aligned} D(i)U(i)+i\mu U(5)=q,\\ D(i)U(k)-D(k)U(i)+f(i,k)U(5)=0,\\ i\mu U(i)-D(i)U(5)=0.\end{aligned}\right\} \qquad (6.17'')$$

2. Vector mesons:

$$\left.\begin{aligned}
&D(i)U(k,i)+i\mu\, U(k,5)=q(k),\\
&D(i)U(5,i)-\tfrac{1}{2}f(i,k)U(k,i)=q(5),\\
&D(i)U(k,l)+D(k)U(l,i)+D(l)U(i,k)\\
&+f(i,k)U(5,l)+f(k,l)U(5,i)+f(l,i)U(5,k)=0,\\
&i\mu W(i,k)+D(i)U(k,5)+D(k)U(5,i)=0.
\end{aligned}\right\}\qquad(6.18'')$$

3. Pseudo-vector mesons:

$$\left.\begin{aligned}
&D(i)U(k,l,i)+i\mu U(k,l,5)=q(k,l),\\
&D(i)U(5,l,i)-\tfrac{1}{2}f(i,k)U(k,l,i)=q(5,l),\\
&D(i)U(k,l,m)-D(k)U(l,m,i)+D(l)\,U(m,i,k)-D(m)U(i,k,l)\\
&f(i,k)U(5,l,m)-f(k,l)U(5,m,i)+f(l,m)U(5,i,k)-f(m,i)U(5,k,l)=0,\\
&i\mu U(i,k,l)-D(i)U(k,l,5)+D(k)U(l,5,i)-D(i)U(5,i,k)=0.
\end{aligned}\right\}\qquad(6.19'')$$

4. Pseudo-scalar mesons:

$$\left.\begin{aligned}
&D(i)U(k,l,m,i)+i\mu U(k,l,m,5)=q(k,l,m),\\
&D(i)U(5,l,m,i)-\tfrac{1}{2}f(i,k)U(k,l,m,i)=q(5,l,m),\\
&D(i)U(k,l,m,n)+D(k)U(l,m,n,i)+D(n)U(i,k,l,m)+D(m)U(m,i,k,l)\\
&+D(n)U(i,k,l,m)+f(i,k)U(5,l,m,m)+f(k,l)U(5,m,n,i)+\\
&+f(l,m)U(5,n,i,k)+f(m,n)U(5,i,k,l)+f(n,i)U(5,k,l,m)=0,\\
&i\mu U(k,l,m,n)+D(k)U(l,m,n,5)+D(l)U(m,n,5,k)+D(m)U(n,5,k,l)\\
&+D(n)U(5,k,l,m)=0.
\end{aligned}\right\}\qquad(6.20'')$$

In the case for which the meson field in question is purely wave-like – i.e., there are no source fields – we can regroup the equations and rename them:

1) Scalar mesons: $U(5)\to i\mu\, U$,
2) Vector mesons: $U(5,\,k)\to i\mu\, U(k)$,

3) Pseudo-vector mesons: $i\mu\, U(k, l, 5) \to U(l, k)$,
4) Pseudo-scalar mesons: $i\mu\, U(k, l, m, 5) \to U(k, l, m)$,

and we can write the systems for the four kinds of mesons in the following forms:

1. Scalar mesons:

$$\left.\begin{aligned} D(i)U(i) - \mu^2 U = 0, \\ U(i) = D(i)U, \end{aligned}\right\} \tag{A}$$

$$D(i)\, U(k) - D(k)\, U(i) + i\mu f(i, k) = 0. \tag{B}$$

2. Vector mesons:

$$\left.\begin{aligned} D(i)U(k,i) = \mu^2 U(k) = 0, \\ U(i,k) = D(i)U(k) - D(k)U(i), \end{aligned}\right\} \tag{A}$$

$$\left.\begin{aligned} D(i)U(k,l) + D(k)U(l,i) + D(l)U(i,k) \\ +i\mu\{ f(i,k)U(l) + f(k,l)U(i) + f(l,i)U(k) = 0, \\ \mu^2 D(i)U(i) + \frac{i\mu}{2} f(i,k)U(k,i) = 0. \end{aligned}\right\} \tag{B}$$

3. Pseudo-vector mesons:

$$\left.\begin{aligned} D(i)U(k,l,i) - U(k,l) = 0, \\ D(i)U(k,l) + D(k)U(l,i) + D(l)U(i,k) - \mu^2 U(i,k,l) = 0, \end{aligned}\right\} \tag{A}$$

$$\left.\begin{aligned} \mu^2\{D(m)U(i,k,l) - D(i)U(k,l,m) + D(k)U(l,m,i) - D(l)U(m,i,k)\} \\ +i\mu\{ f(i,k)U(l,m) - f(k,l)U(m,i) + f(l,m)U(i,k) - f(m,i)U(k,l)\} = 0, \\ D(i)U(l,i) - \tfrac{1}{2} i\mu\, f(i,k)U(k,l,i) = 0. \end{aligned}\right\} \tag{B}$$

4. Pseudo-scalar mesons:

$$\left.\begin{aligned} D(i)U(k,l,m,i) - U(k,l,m) = 0, \\ D(i)U(k,l,m) - D(k)U(l,m,i) + D(l)U(m,i,k) - D(m)U(i,k,l) \\ -\mu^2 U(i,k,l,m) = 0, \end{aligned}\right\} \tag{A}$$

$$\left.\begin{aligned} &\mu^2\{D(n)U(i,k,l,m)+D(l)U(k,l,m,n)+D(n)U(l,m,n,i)+D(l)U(m,n,i,k)\\ &D(m)U(n,i,k,l)+i\mu\{f(i,k)U(l,m,n)+f(k,l)U(m,n,i)+f(l,m)U(n,i,k)\\ &\qquad\qquad+f(m,n)U(i,k,l)+f(n,i)U(k,l,m)=0,\\ &\qquad\qquad D(m)U(k,l,m)-\tfrac{1}{2}i\mu\, f(i,m)U(k,l,m,i)=0. \end{aligned}\right\} \qquad (B)$$

The group of equations (A) for each kind of meson is the one that is usually given in the literature (cf., especially [**13**]) for the wave equations of mesons. If we exclude the case of $\mu = 0$ from consideration then the group of equations (B) will be a consequence of the group of equations (A). Indeed, let the operator D(.) (the dot means one of the indices 1, 2, 3, 4) act upon the second group of equations (A). Alternating them and taking (5.67) into account will give the first group of equations (A). Contracting and taking (5.67) into account will give the group of equations (B).

We shall explain this using an example from electrodynamics. Write the system of wave equations for a photon of frequency ω as:

$$\text{rot}\, H - \frac{i\omega}{c} E = 0, \qquad \text{rot}\, E + \frac{i\omega}{c} H = 0. \qquad (A)$$

If we exclude the case of $\omega = 0$ from consideration then, from (A), we will get the following system of equations as a consequence:

$$\text{div}\, H = 0, \qquad \text{div}\, E = 0. \qquad (B)$$

If we turn to the general case of electromagnetic fields with sources then both systems (A) and (B) will combine into a single, complete, system of Maxwell equations:

$$\left.\begin{aligned} &\text{rot}\, H - \frac{i\omega}{c} E = \frac{4\pi}{c} j, \qquad \text{div}\, E = 4\pi\rho,\\ &\text{rot}\, E + \frac{i\omega}{c} H = 0, \qquad \text{div}\, H = 0, \end{aligned}\right\} \qquad (C)$$

in which ω now means the operator $\frac{1}{i}\frac{\partial}{\partial t}$.

We see that 5-optics gives a rigorous justification for the rule that accounting for an external electromagnetic field is achieved by replacing the operator $\frac{\partial}{\partial x^k}$ with the operator $\frac{\partial}{\partial x^k} + \frac{ie}{\hbar c} A_\kappa$. That rule is confirmed under these limitations if one is considering only wave fields for mesons [group of equations (A)]. If we turn to the complete system of equations for the meson fields with sources then that rule will be violated since terms that contain $f(i, k)$ will appear in the group of equations (B). The situation will be different when we pass from the consideration of tensor fields to the consideration of spinor fields.

§ 41. Complex spinor fields in Riemannian 5-spaces

We shall obtain the equations of spinor fields and the expression for the 5-tensor of energy-impulse-charge from the general formulas in § 37 by starting with the invariant Lagrange integral:

$$\tfrac{1}{2}\int \Lambda\left\{\tilde{W}\,\Omega^\sigma(\alpha)\,\mu(\alpha)\left(\frac{\partial W}{\partial x^\sigma}-B_\sigma W\right)-\left(\frac{\partial \tilde{W}}{\partial x^\sigma}+B_\sigma \tilde{W}\right)\Omega^\sigma(\alpha)\,\mu(\alpha)W\right\} \\ \times dx^1\,dx^2\,dx^3\,dx^4\,dx^5. \tag{6.23}$$

Using formula (6.33) from the Appendix, we can express the invariant Lagrange function in the form:

$$L = \tfrac{1}{2}\Omega^\sigma(\alpha)\left\{\tilde{W}\mu(\alpha)\frac{\partial W}{\partial x^\sigma}-\frac{\partial \tilde{W}}{\partial x^\sigma}\mu(\alpha)W\right\}+\tfrac{1}{4}\frac{\partial \Omega^\tau(\gamma)}{\partial x^\sigma}\Omega_\tau(\rho)\,\Omega^\sigma(\alpha)\,K(\rho,\gamma,\alpha), \tag{6.24}$$

in which $K(\rho, \gamma, \alpha)$ is the tensor that was specified in formula (5.79).

From formula (6.6), we will get the equations of spinor fields in the form:

$$\left.\begin{aligned} &\Lambda\Omega^\sigma(\alpha)\mu(\alpha)\frac{\partial W}{\partial x^\sigma}+\frac{\partial}{\partial x^\sigma}[\Lambda\Omega^\sigma(\alpha)\mu(\alpha)W]-\Lambda[\mu(\alpha)B(\alpha)+B(\alpha)\mu(\alpha)]W=0,\\ &\Lambda\frac{\partial \tilde{W}}{\partial x^\sigma}\Omega^\sigma(\alpha)\mu(\alpha)+\frac{\partial}{\partial x^\sigma}[\Lambda\tilde{W}\Omega^\sigma(\alpha)\mu(\alpha)]-\Lambda\tilde{W}[\mu(\alpha)B(\alpha)+B(\alpha)\mu(\alpha)]=0.\end{aligned}\right\} \tag{6.25}$$

By virtue of formula (6.32) in the Appendix, we can rewrite equations (6.25) in the form:

$$\Omega^\sigma(\alpha)\,\mu(\alpha)\left(\frac{\partial}{\partial x^\sigma}-B_\sigma\right)W=0, \qquad \Omega^\sigma(\alpha)\left(\frac{\partial \tilde{W}}{\partial x^\sigma}+WB_\sigma\right)\mu(\alpha)=0. \tag{6.25$'$}$$

In order to derive an expression for the symmetric 5-tensor $\theta(\alpha, \beta)$, we will use the general formula (6.5′) for the invariant integral and perform the calculations in formula (6.8′).

We will have:

$$\Omega^\sigma(\beta)\left(\frac{\partial L}{\partial \Omega^\sigma(\alpha)}\right)-\Omega_\tau(\alpha)\frac{\partial L}{\partial \Omega_\tau(\beta)}$$

$$= \tfrac{1}{2}\Omega^{\sigma}(\beta)\left\{\tilde{W}\,\mu(\alpha)\frac{\partial W}{\partial x^{\sigma}} - \frac{\partial \tilde{W}}{\partial x^{\sigma}}\mu(\alpha)W\right\}$$

$$+\ \tfrac{1}{4}\frac{\partial \Omega_{\tau}(\gamma)}{\partial x^{\sigma}}\Omega_{\tau}(\rho)\,\Omega^{\sigma}(\beta)\,K(\rho,\gamma,\alpha) - \tfrac{1}{4}\frac{\partial \Omega^{\tau}(\gamma)}{\partial x^{\sigma}}\Omega^{\sigma}(\rho)\,\Omega_{\tau}(\alpha)\,K(\gamma,\rho,\beta)\,,$$

$$\frac{\Omega^{\sigma}(\beta)}{\Lambda}\frac{\partial}{\partial x^{\tau}}\left(\frac{\partial \Lambda L}{\partial\left(\dfrac{\partial \Omega^{\sigma}(\alpha)}{\partial x^{\tau}}\right)}\right) = \frac{1}{4\Lambda}\frac{\partial}{\partial x^{\tau}}\left(\Lambda K^{\tau}(\beta,\alpha)\right) - \frac{1}{4}\frac{\partial \Omega^{\tau}(\beta)}{\partial x^{\sigma}}\Omega_{\tau}(\rho)\,\Omega^{\sigma}(\gamma)\;K(\rho,\alpha,\gamma)\,.$$

If one eliminates the expressions for the derivatives $\dfrac{\partial \Omega^{\tau}(\gamma)}{\partial x^{\sigma}}$ and $\dfrac{\partial \Lambda K^{\tau}(\alpha,\beta)}{\partial x^{\tau}}$ using the formulas:

$$\frac{\partial \Omega^{\tau}(\gamma)}{\partial x^{\sigma}} + G^{\tau}_{\sigma\lambda}\,\Omega^{\lambda}(\gamma) - \Delta_{\sigma}(\gamma,\varepsilon)\,\Omega^{\tau}(\varepsilon)\ = 0,$$

$$\frac{1}{\Lambda}\frac{\partial \Lambda K^{\tau}(\beta,\alpha)}{\partial x^{\tau}} - \Delta(\gamma,\beta,\rho)\;K(\gamma,\rho,\alpha) - \Delta(\gamma,\alpha,\rho)\;K(\gamma,\beta,\rho) = \nabla_{\tau}K^{\tau}(\beta,\alpha),$$

which arise from the formulas for covariant differentiation, and considers the antisymmetry of the tensor $K(\gamma, \beta, \rho)$ then formula (6.8′) will give:

$$\theta(\alpha,\beta) = \frac{1}{2}\Omega^{\sigma}(\beta)\left(\tilde{W}\,\mu(\alpha)\frac{\partial W}{\partial x^{\sigma}} - \frac{\partial \tilde{W}}{\partial x^{\sigma}}\mu(\alpha)W\right)$$
$$-\tfrac{1}{4}\Delta(\beta,\rho,\gamma)\;K(\alpha,\beta,\gamma) - \tfrac{1}{4}\nabla_{\tau}K^{\tau}(\beta,\alpha) - \delta(\alpha,\beta)\;L. \qquad (6.26')$$

In Minkowski space, the expression (6.26′) will go to the expression (4.143) in § 30.

We shall give an expression for $\theta(\alpha, \beta)$ in the case of a spinor field that is based upon the invariant Lagrange integral while bypassing the canonical formalism.

The Lagrange function will vanish, by virtue of the fact that the fields W and $\tilde{W}$ satisfy the field equations, and if we use the formula for invariant differentiation then we can rewrite the field equations (6.25) in the form:

$$\left.\begin{aligned} \mu(\alpha)[D(\alpha) - B(\alpha)]W &= 0, \\ \{D(\alpha)\tilde{W} + \tilde{W}\,B(\alpha)\}\mu(\alpha) &= 0, \end{aligned}\right\} \qquad (6.25'')$$

and the energy-impulse-charge tensor $\theta(\alpha, \beta)$ in the form:

$$\theta(\alpha, \beta)$$

$$= \tfrac{1}{2}\left\{\tilde{W}\mu(\alpha)[D(\beta) - B(\beta)]W - [D(\beta)\tilde{W} - \tilde{W}\,B(\beta)]\,\mu(\alpha)W\right\} - \tfrac{1}{4}\nabla(\tau)\,K(\tau,\alpha,\beta)\,. \quad (6.26'')$$

Now, consider the special case of a purely electromagnetic field that does not depend upon x^5. From formula (5.78) in § 37, one will have:

Field equations:

$$\left.\begin{aligned} \left\{\mu(k)\,D(k) + \mu(5)\,D(5) + \tfrac{1}{8}\,\mu(5)\mu(i)\mu(k)\,f(i,k)\right\}W = 0, \\ \left\{D(k)\tilde{W}\mu(k) + D(5)\tilde{W}\mu(5) - \tfrac{1}{8}\tilde{W}\mu(5)\mu(i)\mu(k)\,f(i,k)\right\} = 0. \end{aligned}\right\} \quad (6.25''')$$

Using formula (5.74) will give:

Energy-impulse 4-tensor:

$$\begin{aligned} \theta(i, k) = {} & \tfrac{1}{2}\left\{\tilde{W}\mu(i)\,D(k)W - D(k)\tilde{W}\mu(i)W\right\} \\ & - \tfrac{1}{4}\,f(k,n)\,K(5,n,i) - \tfrac{1}{4}D(n)K(n,k,i) - \tfrac{1}{4}D(5)K(5,k,i)\,. \end{aligned} \quad (6.26'''\text{a})$$

Current 4-vector:

$$\left.\begin{aligned} \theta(5,k) &= \tfrac{1}{2}\{\tilde{W}\mu(5)D(k)W - D(k)\tilde{W}\mu(5)W\} \\ &\quad + \tfrac{1}{4}D(n)\,K(n,5,k) + \tfrac{1}{8}\,f(i,n)\,K(i,n,k), \\ \theta(k,5) &= \tfrac{1}{2}\{\tilde{W}\mu(k)D(5)W - D(5)\tilde{W}\mu(k)W\} - \tfrac{1}{4}D(n)\,K(n,5,k). \end{aligned}\right\} \quad (6.26'''\text{b})$$

The symmetry condition $\theta(5, k) = \theta(k, 5)$ expresses the Gordon identity in the presence of an external field.

In the transition to the representation by Fourier components, we shall make the usual assumption that only one component of the decomposition is presented, namely, the one that corresponds to $Z = 1$; i.e., we impose the cyclicality condition upon the spinors W and $\tilde{W}$:

$$W = U \exp(i\mu\, x^5), \qquad \tilde{W} = \tilde{U} \exp(-i\mu\, x^5) \quad (6.27)$$

and introduce the notations:

$$\left.\begin{aligned} \bar{K}(i,k,l) &= \tfrac{1}{2}U^{+}[\gamma(i)\gamma(k)\gamma(l) - \gamma(l)\gamma(k)\gamma(i)], \\ \bar{M}(i,k) &= i\,\bar{K}(i,k,5) = \tfrac{1}{2}U^{+}[\gamma(i)\gamma(k) - \gamma(k)\gamma(i)]. \end{aligned}\right\} \quad (6.28)$$

We will get:

Field equations:

$$\gamma(k)\, D(k)\, U + \mu\, U - \gamma(i)\, \gamma(k)\, f(i, k)\, U = 0, \tag{6.29}$$

Energy-impulse 4-tensor:

$$\bar{\theta}(i,k) = \tfrac{1}{2}\{U^{+}\gamma(i)\, D(k)U - (D(k)U^{+})\gamma(i)\, U\} - \frac{1}{4}\frac{\partial \bar{K}(k,l,n)}{\partial x^{n}} + \frac{1}{4} f(k,l)\bar{M}(i,l)\,, \tag{6.30}$$

Current 4-vector:

$$\bar{\theta}(k,5) = i\mu\,[U^{+}\gamma(i)\, U] - \frac{1}{4}\frac{\partial \bar{M}(k,n)}{\partial x^{n}}\,. \tag{6.31}$$

We see [formula (6.29)] that in 5-optics the Dirac equation will include an extra term $-\frac{i}{8}\gamma(i)\,\gamma(k)\; f(i,\; k)U$. This extra term will result from the requirement of general covariance of the equations for the spinor field, which will imply that one must replace the ordinary derivatives $\partial W / \partial x^{\sigma}$ of the spinor with the covariant ones $\partial W / \partial x^{\sigma} - B_{\sigma} W$.

The appearance of this extra term shows that the usual rule from the *D*-formalism is violated in the case of spinor fields. However, as is well-known, the Dirac equation agrees with experiment, and that agreement would deteriorate sharply if there were an extra term. We are now confronted with a dilemma whose importance must not be understated. We should hope that further developments in this theory might overcome that dilemma. In our opinion, it would be appropriate to specify a direction in which to conduct such research.

One should not think that equation (6.29) does not imply any specific 5-optical effect (in addition to the conventional radiative correction) that would compensate almost completely for the additional term, so as to bring about agreement with experiment. This effect would be consistent with the charged (massive) state of the radiation field; i.e., virtual transitions of electrons to other charged states (including a neutrino in the transition state) with the emission or absorption of a charged (massive) quantum. The transition of an electron to a neutrino state with the emission of a massive quantum is analogous to the transition from a proton to a neutron with the emission of a meson, and is the reason for the deviation of the experimental value of the magnetic moment for the electron from the one that is calculated from the Dirac equation, which, in theory, is calculated from equation (6.29).

Here, we come to some still-unsolved problems of 5-optics, the discussion of which would lead beyond the scope of this monograph.

Although using the canonical formalism will give expressions for the energy-impulse 4-tensor T_{ik} and the 4-vector s_k , these expressions will be undetermined, in that one can add expressions of the form $\frac{\partial \psi_{ikl}}{\partial x^{l}}$ and $\frac{\partial \psi_{ik}}{\partial x^{i}}$, where ψ_{ikl} and ψ_{ik} are antisymmetric tensors:

$$T'_{ik} = T_{ik} + \frac{\partial \psi_{ikl}}{\partial x^{l}}\,, \qquad s'_{k} = s_{k} + \frac{\partial \psi_{ik}}{\partial x^{i}}\,.$$

That means that the values of the energy and charge integrals:

$$\int T_{44}\, dx^1\, dx^2\, dx^3, \qquad\qquad \int s_4\, dx^1\, dx^2\, dx^3$$

of the densities of these quantities in space will also remain undetermined. This, in turn, will imply an indefinite value for the mechanical and magnetic moments of elementary particles.

The theory of gravitation eliminates this indeterminacy with respect to the energy density and mechanical moment of an elementary particle by giving the symmetric 4-tensor energy-impulse θ_{ik} a direct physical meaning.

However, the charge density still remains undetermined, which will imply an indeterminacy in the values of the magnetic moments of elementary particles.

5-optics, as a natural outgrowth of the theory of gravitation, eliminates the indeterminacy in the charge density, and consequently in the value of the magnetic moment.

APPENDIX

1. The derivation of the formula:

$$\mu(\alpha)\,\frac{\partial \Lambda \Omega^{\sigma}(\alpha)}{\partial x^{\sigma}} = \Delta[B(\alpha)\,\mu(\alpha) - \mu(\alpha)\,B(\alpha)]\,. \tag{6.32}$$

By definition, one has:

$$\begin{aligned} &B(\alpha)\,\mu(\alpha) - \mu(\alpha)\,B(\alpha) \\ &\quad = \tfrac{1}{4}\,\Delta(\alpha, \rho, \gamma)\,(\mu(\rho)\,\mu(\gamma)\,\mu(\alpha) - \mu(\alpha)\,\mu(\rho)\,\mu(\gamma)) \\ &\quad = \tfrac{1}{2}\,\Delta(\alpha, \rho, \gamma)\,(\delta(\alpha, \gamma)\,\mu(\rho) - \delta(\alpha, \rho)\,\mu(\gamma)) \\ &\quad = \left(\tfrac{1}{2}\Delta(\alpha,\rho,\alpha) - \tfrac{1}{2}\Delta(\alpha,\alpha,\rho)\right)\mu(\rho) \\ &\quad = \Delta(\alpha, \rho, \alpha)\,\mu(\rho)\,. \end{aligned}$$

Multiplying formula (5.32) by $\mu(\alpha)$ will give:

$$\mu(\alpha)\,\frac{\partial \Lambda \Omega^{\sigma}(\alpha)}{\Lambda\,\partial x^{\sigma}} = \Delta(\beta, \alpha, \beta)\,\mu(\alpha)\,.$$

Comparing this will give (6.32).

2. Compute:

$$\begin{aligned} &\tilde{W}\,[\mu(\alpha)\,B(\beta) + B(\beta)\mu(\alpha)]\,W \\ &\quad = \tfrac{1}{4}\Delta(\beta, \rho, \gamma)\,\tilde{W}\,[(\mu(\alpha)\,\mu(\rho)\,\mu(\gamma) - \mu(\gamma)\,\mu(\rho)\,\mu(\alpha)]\,W \\ &\quad = \tfrac{1}{2}\Delta(\beta, \rho, \gamma)\,K(\alpha, \rho, \gamma)\,. \end{aligned}$$

3. Substitution gives:

$$\tilde{W}\,[\mu(\alpha)\,B(\alpha) + B(\alpha)\mu(\alpha)]\,W = \tfrac{1}{2}\Delta(\alpha, \rho, \gamma)\,K(\alpha, \rho, \gamma)\ . \qquad (6.33).$$

Multiplying formula (5.30a) by $K(\alpha, \rho, \gamma)$ will give:

$$\begin{aligned}
&\Delta(\alpha, \beta, \gamma)\,K(\alpha, \beta, \gamma)\\
&\quad = \frac{\partial\Omega^{\sigma}(\gamma)}{\partial x^{\tau}}\,\Omega^{\sigma}(\alpha)\,\Omega^{\tau}(\beta)\,K(\alpha, \beta, \gamma)\\
&\quad = -\frac{\partial\Omega^{\sigma}(\alpha)}{\partial x^{\tau}}\,\Omega^{\sigma}(\gamma)\,\Omega^{\tau}(\beta)\,K(\alpha, \beta, \gamma)\ .
\end{aligned}$$

Comparing this will give:

$$\tilde{W}\,[\mu(\alpha)\,B(\alpha) + B(\alpha)\mu(\alpha)]\,W = -\frac{\partial\Omega^{\tau}(\gamma)}{\partial x^{\sigma}}\,\Omega^{\tau}(\rho)\,\Omega^{\sigma}(\alpha)\,K(\rho, \gamma, \alpha)\ . \qquad (6.34)$$

AFTERWORD

In his lectures on basic quantum mechanics, L. I. Mandelstam raised the question of the structure of physical theories of any construction in general, and answered it as follows:

"We can say, somewhat schematically (as usual), that the construction of any physical theory consists of two complementary stages… The first stage consists of learning how to attribute values (which will mostly take the form of numbers) to certain objects in nature in a rational way. Secondly: One determines mathematical relations between these values. Without the first stage, the theory would be illusory and vacuous. Without the second stage, there would be no theory, in general. Only the two stages in combination will give a physical theory."

Developing this idea further, Mandelstam said:

"Modern theoretical physics has taken a different path from the classical one. I would not say this happened consciously, but historically this was true. It happened by itself. Nowadays, we primarily try to guess at mathematical tools, and operate upon the values, or the part of the model that is known in advance, although it is not entirely clear what they mean… Undoubtedly, so did Einstein, when he created the principle of general relativity. This is especially true in the example of the way that the Schrödinger equation was created."

How does the foregoing viewpoint relate to the case of 5-optics?

The formal structure of 5-optics was, in substance, available many years ago, and it was constructed in the work of Kaluza and O. Klein, and later on, of Einstein and Bergmann.

The construction of that formal structure proceeded in the following stages:

I. T. Kaluza (c. 1921).

1. The extra fifth dimension was introduced into the four-dimensional physical space of the theory of gravitation. The physical significance of that extra fifth dimension remained open.

2. It was discovered that the metric potential of 5-space should not depend upon the extra fifth coordinate of space. The physical meaning of that cylindricality condition remained open.

3. In order to get a one-to-one correspondence, the 10 + 4 = 14 potentials from the theories of gravitation and electrodynamics and 15 metric potentials of 5-space implied that one must pose an additional requirement; e.g., that $G_{55} = 1$. The question of the physical significance of that requirement remained open.

II. O. Klein and V. A. Fock (c. 1926).

1. The specified comparison of the 14 potentials of the theories of gravitation and electrodynamics and the 15 metric potentials of 5-space and the discovery that the trajectories of charged particles correspond to null-length geodesic lines (i.e., the geometry of light) in 5-space. In fact, this is the equivalence of the problem of the relativistic, classical mechanics of the motion of charged, material points with the geometrical optics problem of the propagation of rays in 5-space.

2. An opportunity was found to formulate the quantum-mechanical problem of the motion of a charged particle as the wave-optical problem of the propagation of a scalar field in 5-space, if the wave function in 5-space has the cyclicality condition:

$$W(x^1, x^2, x^3, x^4, x^5) = U(x^1, x^2, x^3, x^4) \exp\left(i \left(\frac{mc}{\hbar} \right) x^5 \right)$$

imposed upon it, which relates to the cylindricality condition on the metric potentials.

3. The physical meaning of the fifth coordinate, the cylindricity condition on the metric potentials, and the cyclicality condition on the wave function remained open questions. As before, the physical meaning of the requirement that $G_{55} = 1$ still remained open.

III. A. Einstein and P. Bergmann (c. 1938).

The cylindricality condition was replaced with the weaker requirement of the periodicity of the metric potentials in the fifth coordinate. The period that was adopted had microscopic values that could be set equal to zero in the first approximation. The periodicity condition once more degenerated into the cylindricality condition then.

Since the equivalence principle does not apply to the electromagnetic field, in all of these works, the metric tensor of 5-space turns out to depend upon the ratio e / m of the particle for the motion in question, while the metric tensor of 4-space of the theory of gravitation is seen to be universal.

Hence, it must be concluded that 5-space, as a five-dimensional generalization of the theory of gravitation, cannot be the universal physical space of the general theory of relativity (extended by one extra dimension), and should have a very different physical meaning.

IV. 5-optics.

1. The 5-space of 5-optics defines the configuration space of test particles whose motion is being considered (extended by one extra dimension). The metric and topological closure of that space reflects the effect of the rest of the universe on the test particle.

2. The fifth coordinate of configuration space takes on the distinct physical meaning of action. This relates to the fact that the fifth coordinate of configuration 5-space is topologically closed.

3. Instead of the cylindricality condition on the metric potentials and the cyclicality conditions on the wave function, all physical quantities must satisfy the single condition that they should be periodic in the fifth coordinate of action.

4. It is discovered that the period of the fifth coordinate has the universal value of Planck's constant, which then has a distinct physical meaning.

5. The quantum motion of material points is a problem of physical quantities that depend upon the action coordinate periodically.

6. The fact that one had to assume that $G_{55} = 1$ in the previous theories is due to the fact that the 5-eikonal equation:

$$G^{\mu\nu}\frac{\partial\Sigma}{\partial x^{\mu}}\frac{\partial\Sigma}{\partial x^{\nu}} = 0$$

formulates the problem of classical mechanics of the motion of a charged material point *homogeneously* in terms of the metric potentials $G^{\mu\nu}$. Therefore, in that problem, the physical meaning of the condition is that the metric potential in 5-space has only fourteen ratios, so the requirement that $G_{55} = 1$ will not lead to a contradiction.

7. We find a different state of affairs in the problem of determining the metric potential sources for a given field that satisfies Einstein's equation for 5-space:

$$P_{\lambda\mu} - \tfrac{1}{2}G_{\lambda\mu}P = \kappa Q_{\lambda\mu}\,,$$

which is inhomogeneous in the metric potentials.

When solving this problem, the ratio e / m that appears in the expression for the metric tensor of 5-space should be replaced with the universal value $c^2\sqrt{\dfrac{\kappa}{2\pi}}$. The value of the potential G_{55} must be determined from the field equations. To believe that $G_{55} = 1$, *a priori*, is not permitted, and will lead to wrong conclusions, for example, in the problem of the field of a charged mass point.

8. Accounting for the periodic dependence of an electromagnetic field on the fifth coordinate of action will automatically lead to appearance of short-range interaction forces of Yukawa type, in addition to the long-range forces of Coulomb type (§ 25, sec. 3).

9. In any consistent classical theory, we must assume that $h \to 0$, i.e., we must neglect the periodic dependence of physical values on the action coordinate. In any consistent quantum theory, we must take into account that periodic dependence of

physical values on the action coordinate. Therefore, from the standpoint of 5-optics, it is inconsistent to neglect the periodic dependency of the components of external fields on the action coordinate, as modern quantum mechanics does.

Accounting for this dependency should lead to the prediction and discovery of a number of specialized 5-optical effects that could be used to verify the theory experimentally.

BIBLIOGRAPHY

1. F. Klein, "Ueber neuere englische Arbeiten zur Mechanik," Ges. Abhandlungen, v. II, 601.
2. F. Klein, Zeit. Math. Phys. (1901), 375.
3. Th. Kaluza, "Zum Unitätsproblem der Physik," Sitzungsberichte d. Preuss. Akad. (1921), 966.
4. H. Mandel, "Zur Herleitung der Feldgleichungen in der allgemeinen Relativitätstheorie," Zeit. Phys. **39** (1926), 136.
5. O. Klein, "Quantentheorie und fünfdimensionale Relativitätstheorie," Zeit. Phys. **37** (1926), 895.
6. V. Fock, "Ueber die invariante Form der Wellen-and der Bewegungsgleichungen für einen geladenen Massenpunkt," Zeit. Phys. **39** (1926), 226.
7. A. Einstein and P. Bergmann, "On a generalization of Kaluza's Theory of Electricity," Ann. Math. **39** (1938), 683.
8. Yu. B. Rumer:
   – "Action as a spatial coordinate," JETP **19** (1949), 1, 3, 10; **21** (1951), 3, 12; **22** (1952), 6, 7; **23** (1952), 7; **24** (1953), 1, 3. (In Russian).
   – "Physical meaning of 5-spaces in 5-optics," JETP **24** (1953), 3. (In Russian).
9. V. A. Fock, "The motion of finite masses in the general theory of relativity," JETP **8** (1939). (In Russian).
10. V. A. Fock, "Dirac's wave equation in Riemannian geometry," Zh. R. F. Kh. O., phys. section **62** (1930). (In Russian).
11. V. L. Ginzburg, JETP **9** (1939), 981. (In Russian).
12. L. Landau and E. Lifschitz, *Theory of Fields*, Gostekhizdat, 1948. (In Russian).
13. W. Pauli, *Relativistic Theory of Elementary Particles*, I. L., 1947.
14. G. Wentzel, *Introduction to the Quantum Theory of Wave Fields*, Gostekhizdat, 1947.

14\. A. Einstein, *The Mathematical Theory of Relativity*, DNTVU, Kharkov, 1933.