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THE COMBESCURE TRANSFORMATION AND OTHER ANALOGOUS ONES FOR TWISTED CURVES.

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The illustrious Prof. BIANCHI (*) has given the name of *COMBESCURE transformation* to a particular correspondence between the points of two curves, and he made some noteworthy applications to the search for the most general curve with constant flexion, to spherical helices, to the orthogonal trajectories of a simple infinitude of spheres, and to BERTRAND curves.

Now, that transformation and other analogous ones presents themselves spontaneously in the study of curves by the *intrinsic* route.

Let ρ and r be the radii of first and second curvature, resp., of a curve C at a point M, let s be the length of the arc that has M for its endpoint, when computed by starting from a fixed origin on C, let (x, y, z) be the Cartesian coordinates of a point M_1 that moves along M with respect to the fundamental trihedron of C at M, namely, with respect to the tangent, the binormal, and the principal normal of C at M. Now let $(x + \delta x, y + \delta y, z + \delta z)$ be the coordinates of the point M_1' to which M_1 moves when M passes to the infinitely-close point M' while moving along C. One will then have the fundamental formulas (**):

$$(1) \qquad \frac{\delta x}{ds} = \frac{dx}{ds} - \frac{z}{\rho} + 1, \qquad \frac{\delta y}{ds} = \frac{dy}{ds} - \frac{z}{r}, \qquad \frac{\delta z}{ds} = \frac{dz}{ds} + \frac{x}{\rho} + \frac{y}{r}.$$

If α, β, γ are the direction cosines of a direction that moves with M then one will have analogous formulas:

$$(1) \qquad \frac{\delta \alpha}{ds} = \frac{d\alpha}{ds} - \frac{\gamma}{\rho}, \qquad \frac{\delta \beta}{ds} = \frac{d\gamma}{ds} - \frac{z}{r}, \qquad \frac{\delta \gamma}{ds} = \frac{d\gamma}{ds} + \frac{\alpha}{\rho} + \frac{\beta}{r}.$$

(*) Cf., *Lezioni di Geometria differenziale*, 2nd ed., vol. I, pp. 40, *et seq.*
(**) Cf., E. CESÀRO, *Lezioni di Geometria intrinsica*, pp. 124.

Set $\delta x = \delta y = \delta z = 0$ in (1), so one will have the relations:

$$\frac{dx}{ds} = \frac{z}{\rho} - 1, \quad \frac{dy}{ds} = \frac{z}{r}, \qquad \frac{dz}{ds} = -\frac{x}{\rho} - \frac{y}{r},$$

which are *necessary and sufficient for the immobility of the point* M_1 .

The idea of annulling only two of the quantities δx, δy, δz suggests itself spontaneously.

For example, set $\delta y = \delta z = 0$, so the moving point M_1 will describe a curve C_1 that relates to C in such a way that the corresponding points M and M_1 will have parallel tangents, and C_1 will then be a curve that one can deduce from C by a COMBESCURE *transformation.*

Obviously, the binormals and the principal normals will also be parallel, and the infinitesimal angles of contingency and torsion ε and η will also be equal; it will then follow that:

$$\frac{1}{\rho_1} = \frac{\varepsilon}{\delta s_1} = \frac{\varepsilon}{ds} \cdot \frac{ds}{\delta s_1} = \frac{1}{\rho} \cdot \frac{ds}{\delta s_1}, \qquad \frac{1}{r_1} = \frac{\eta}{\delta s_1} = \frac{\eta}{ds} \cdot \frac{ds}{\delta s_1} = \frac{1}{r} \cdot \frac{ds}{\delta s_1},$$

in which one gives the index 1 to the quantities that refer to C_1, or:

$$\rho_1 = r\, f(s), \qquad r_1 = r\, f(s),$$

in which one sets:

$$\frac{\delta s_1}{ds} = f(s).$$

It then follows that *a COMBESCURE transformation leaves the ratio of the curvatures unaltered.*

One cannot deduce that the rectifying lines at C and C_1 are also parallel, because the inclination θ of the rectifying line at C to the tangent is defined by the formula (*):

$$\tan\theta = -\frac{\tau}{\rho}.$$

Recall that:

(3) $$\frac{dy}{ds} = \frac{z}{r}, \qquad \frac{dz}{ds} = -\frac{x}{\rho} - \frac{y}{r}$$

are *the necessary and sufficient conditions for the point* M_1 (x, y, z) *to describe a curve* C_1 *that is the* COMBESCURE *transform of C. The tangent, binormal, principal normal, and rectifying line will then be parallel to the analogous lines of C and will have the intrinsic equations:*

(4) $$\rho_1 = \rho\, f(s), \qquad r_1 = r\, f(s), \qquad \delta s_1 = f(s)\, ds,$$

(*) Cf., E. CESÀRO, *loc. cit.*, pp. 137.

in which:

(5) $$f(s) = \frac{\delta x}{ds} = \frac{dx}{ds} - \frac{z}{\rho} + 1.$$

Now set $\delta x = \delta z = 0$ in (1) instead. The moving point M_1 then describes a curve C_1 that relates to C in such a way that the tangent to C_1 at M_1 will be parallel to the binormal of C at the corresponding point M. We say that C_1 is deduced from C by a *transformation B* or that C_1 *is a transform of C by B.*

One can deduce the properties of the transform C_1 directly by means of (1) and (2), but one can also deduce them immediately when one knows a remarkable theorem of BIANCHI (*):

For any curve C, there exists another one C_1 (which is defined up to a translation) that corresponds to C in arc length by equivalence. The two curvatures and the directions of the tangents and binormals will be permuted by the transformation. More precisely: *If one chooses the positive sense of the tangent to C_1 to be that of the binormal to C then the positive sense of the binormal to C_1 will be that of the tangent to C, while the principal normals will be parallel, but have the opposite positive senses; one will then have:*

$$\rho_1 = -r, \qquad r_1 = -\rho.$$

Let B_0 denote the particular, but fundamental, transformation B, while noting that all of the transforms of a curve by B can be deduced from each other by COMBESCURE transformation, so one will have: *The most general transformation B is the product of B_0 and the most general COMBESCURE transformation (and vice versa).*

Symbolically, if T is a COMBESCURE transformation then one will have:

$$B = B_0 \cdot T = T\, B_0\,.$$

One recalls from the foregoing that:

(6) $$\frac{dx}{ds} = \frac{z}{\rho} - 1, \qquad \frac{dz}{ds} = -\frac{x}{\rho} - \frac{y}{r}$$

are the necessary and sufficient conditions for the point M_1 (x, y, z) to describe a curve C_1 that is the transform of C by B. The tangent and binormal to C_1 will be parallel to the binormal and tangent to C, resp. (and with equal positive senses), and the principal normal will be parallel to the principal normal to C (but with opposite positive senses), and one will have the intrinsic equations:

(7) $$\rho_1 = -r\, f(s), \quad r_1 = -\rho\, f(s), \quad \delta s_1 = f(s)\, ds,$$

in which:

(*) *Loc. cit.*, vol. I, pp. 53.

$$f(s) = \frac{\delta y}{ds} = \frac{dy}{ds} - \frac{z}{r} \quad . \tag{8}$$

Finally, set $\delta x = \delta y = 0$ in (1). The moving point M_1 (x, y, z) will then describe a curve C_1 that relates to C in such a way that the tangent at M_1 will be parallel to the principal normal to C at the corresponding point M. We say that C_1 is deduced from C by a *transformation N* or that C_1 *is a transform of C by N.*

If one applies (2) to the direction of the tangent to C_1 – i.e., one takes $\alpha = 0$, $\beta = 0$, $\gamma = 1$ – then one will have:

$$\frac{\delta\alpha}{ds} = -\frac{1}{\rho}, \qquad \frac{\delta\beta}{ds} = -\frac{1}{r}, \qquad \frac{\delta\gamma}{ds} = 0,$$

and if one then sets:

$$\delta\sigma = \sqrt{\delta\alpha^2 + \delta\beta^2 + \delta\gamma^2} = \frac{\sqrt{\rho^2 + r^2}}{\rho\, r}\, ds$$

then:

$$\xi = \frac{\delta\alpha}{\delta\sigma} = -\frac{r}{\sqrt{\rho^2 + r^2}}, \qquad \eta = \frac{\delta\beta}{\delta\sigma} = -\frac{\rho}{\sqrt{\rho^2 + r^2}}, \qquad \zeta = \frac{\delta\gamma}{\delta\sigma} = 0$$

will be the direction cosines of the principal normal to C_1 and therefore:

$$\lambda = -\frac{\rho}{\sqrt{\rho^2 + r^2}}, \qquad \mu = \frac{r}{\sqrt{\rho^2 + r^2}}, \qquad \nu = 0$$

will be those of the binormal, and the flexion will be:

$$\frac{1}{\rho_1} = \frac{\delta\sigma}{\delta s_1} = \frac{ds}{\delta s_1} \cdot \frac{\delta\sigma}{\delta s} = \frac{ds}{\delta s_1} \cdot \frac{\sqrt{\rho^2 + r^2}}{\rho\, r} .$$

Set:

$$\tan\theta = -\frac{r}{\rho},$$

in which θ is the inclination of the rectifying line of C with respect to the tangent, and one will have:

$$\begin{aligned} &\xi = -\sin\theta, && \eta = \cos\theta, && \zeta = 0, \\ &\lambda = \cos\theta, && \mu = \sin\theta, && \nu = 0, \end{aligned}$$

so the binormal to C_1 will be parallel to the rectifying line to C.

Finally, apply (2) to the direction λ, μ, ν, so one will have:

$$\frac{\delta\lambda}{ds} = -\sin\theta\,\frac{d\theta}{ds}, \qquad \frac{\delta\mu}{ds} = \cos\theta\,\frac{d\theta}{ds}, \qquad \frac{\delta\nu}{ds} = 0,$$

and then:

$$\frac{1}{r_1} = \sqrt{\left(\frac{\delta\lambda}{\delta s_1}\right)^2 + \left(\frac{\delta\mu}{\delta s_1}\right)^2 + \left(\frac{\delta\nu}{\delta s_1}\right)^2} = \frac{ds}{\delta s_1}\cdot\frac{d\theta}{ds}.$$

Recall that:

(9) $$\frac{dx}{ds} = \frac{z}{\rho} - 1, \qquad \frac{dy}{ds} = \frac{z}{r}$$

are the necessary and sufficient conditions for the point M_1 (x, y, z) *to describe a curve* C_1 *that is the transform of* C *by* N. *The tangent and binormal to* C_1 *are parallel to the principal normal and the rectifying line to* C. *The intrinsic equations to* C_1 *are:*

(10) $$\frac{f(s)}{\rho_1} = \frac{\sqrt{\rho^2 + r^2}}{\rho r}, \qquad \frac{f(s)}{r_1} = \frac{d\theta}{ds}, \qquad \delta s_1 = f(s)\, ds$$

in which:

(11) $$f(s) = \frac{\delta z}{ds} = \frac{dz}{ds} + \frac{x}{\rho} + \frac{y}{r}.$$

Any curve has an infinitude of transforms *T, B, N*, because one of the four functions x, y, z, f of s will be arbitrary in each transformation. If one specializes one of those functions conveniently then one make some applications of the preceding results, but it is clear that those applications can go on indefinitely. We shall omit the applications that Prof. BIANCHI made already and was the first to mention.

I. For $x = 0$, (6) will give $z = \rho$, $y = -\rho\, d\rho / ds$, so M_1 (x, y, z) will be a point on the axis of the osculating circle to C, and that axis will be tangent to C_1 at M_1. Therefore, if one keeps (7) and (8) in mind then one will have:

The axes of the osculating circles to a curve C will define a developable surface (polar developable to C) that is the envelope of its normal planes. The edge of regression C_1 (*viz., the locus of centers of the osculating spheres to C) is a transform of C by B and has* (7) *for its intrinsic equation, or:*

$$-f(s) = \frac{\rho}{r} + \frac{d}{ds}\left(r\frac{d\rho}{ds}\right).$$

ρ is constant if and only if $y = 0$. In that case, M_1 will have the coordinates $(0, 0, \rho)$, and C_1 is the locus of centers of the osculating circles to C; hence:

$$f(s) = -\frac{\rho}{r}, \qquad \rho_1 = \rho, \qquad r\, r_1 = \rho^2.$$

One will then have the following theorem of BOUQUET that characterizes the twisted circles:

The locus of centers of curvature of a twisted circle C is also a twisted circle C_1. C and C_1 are the loci of the centers of the osculating circles to each other and can be deduced from each other by a transformation B. They have equal flexion, and the product of their torsions is equal to the square of their common flexion.

$f(s) = 0$ iff $\rho_1 = r_1 = 0$, and C_1 will then reduce to a point; hence:

$$(M\,M_1)^2 = x^2 + y^2 + z^2 = \rho^2 + \left(r\frac{d\rho}{ds}\right)^2,$$

but if $f(s) = 0$ then $\rho^2 + \left(r\frac{d\rho}{ds}\right)^2$ will be constant, so C will be a spherical curve. One will then have the known theorem: $\frac{\rho}{r}+\frac{d}{ds}\left(r\frac{d\rho}{ds}\right) = 0$ *is the necessary and sufficient condition for a curve to be traced on a sphere of radius* $\sqrt{\rho^2+\left(r\frac{d\rho}{ds}\right)^2}$.

II. More generally, set $x = c$ (constant). From (6), one will then have:

$$y = -\,r\frac{d\rho}{ds} - c\frac{r}{\rho}, \qquad z = \rho,$$

and the preceding theorem will generalize to the following one:

In the plane that goes through the axis of the osculating circle to a curve and is parallel to the rectifying plane, the line that is parallel to that axis and rigidly linked to it will generate a developable surface whose edge of regression C_1 is the transform of the curve by B and has (7) *for its intrinsic equation, in which:*

$$-f(s) = \frac{\rho}{r}+\frac{d}{ds}\left(r\frac{d\rho}{ds}\right)+c\frac{d}{ds}\left(\frac{r}{\rho}\right).$$

Those curves C_1 are all transforms by T of the edge of regression to the polar developable to C. They will be congruent to that edge of regression only when C is a cylindrical helix, and one will then have $\frac{d}{ds}\left(\frac{r}{\rho}\right) = 0$.

III. (6) are not satisfied for $x = z = 0$, which will be when *the binormals to a curve do not define a developable surface.*

IV. We would like to invert problem I. That is, *we would like to find all of the curves that have a given curve C as the locus of centers of the osculating sphere;* it would then be sufficient to set $y = 0$. (6) will give:

$$\frac{dx}{ds} = \frac{z}{\rho} - 1, \qquad \frac{dz}{ds} = -\frac{x}{\rho},$$

and when one eliminates z from this:

$$\rho \frac{d^2 x}{ds^2} + \frac{d\rho}{ds} \cdot \frac{dx}{ds} + \frac{x}{\rho} + \frac{d\rho}{ds} = 0,$$

or:

$$\frac{d^2 x}{d\varphi^2} + x + \frac{d\rho}{d\varphi} = 0,$$

in which one has set:

$$\varphi = \int_0^s \frac{ds}{\rho};$$

when one integrates this, it will follow that:

$$(12) \qquad \begin{cases} x = c \sin\varphi + c' \cos\varphi - \sin\varphi \int \sin\varphi \, ds - \cos\varphi \int \cos\varphi \, ds, \\ z = c \cos\varphi - c' \sin\varphi - \cos\varphi \int \sin\varphi \, ds + \sin\varphi \int \cos\varphi \, ds, \end{cases}$$

in which c and c' are arbitrary constants.

(12) solves the proposed problem, and therefore, the following equivalent one, as well: *Find all orthogonal trajectories to a simple infinitude of planes* (viz., the osculating planes to C). The problem that was solved before in a different way will then be solved by means of three quadratures.

The intrinsic equations of that trajectory are:

$$\rho_1 = z, \qquad r_1 = z \frac{\rho}{r}, \qquad \delta s_1 = \frac{z}{\rho} ds .$$

V. Passing to the transformation T, set $x = 0$ in (3); it will result that:

$$\frac{dy}{ds} = \frac{z}{r}, \qquad \frac{dz}{ds} = -\frac{y}{r},$$

and eliminating z will give:

$$r \frac{d^2 y}{ds^2} + \frac{dr}{ds} \cdot \frac{dy}{ds} + \frac{y}{r},$$

or

$$\frac{d^2y}{d\psi^2} + y = 0,$$

in which one has set:

$$\psi = \int_0^s \frac{ds}{r}; \tag{13}$$

integration will give:

$$\begin{cases} y = c\cos\psi + c'\sin\psi, \\ z = c'\cos\psi - c\sin\psi, \end{cases} \tag{14}$$

in which c and c' are arbitrary constants.

If one draws two normals to C through M that have inclinations ψ and $\pi / 2 + \psi$ with respect to the principal normals, resp., then it will be clear that the point M_1 $(0, y, z)$ that is defined by (14) is that point on the normal plane to C that is at distances of c and c' from those normals. Now, if one keeps in mind that (13) is the necessary and sufficient condition for one of the normals and (therefore) the other one to generate a developable surface (*) then one can assert that:

A point on the normal plane to a curve C that is invariably linked with two mutually-orthogonal normals will generate a curve C_1 that is the transform of C by T when one of the two normals (and therefore the other one) generates a developable surface.

Now observe that the curves C_1 are orthogonal trajectories to the normal planes to C, and one will find that (14) solves the following problem: *Construct the orthogonal trajectories to a simple infinitude of planes when one knows one of them.* That problem is therefore *solved by means of just one quadrature*, namely, (13), while the general problem requires three quadratures (cf., IV).

Now let:

$$(M\,M_1)^2 = x^2 + y^2 + z^2 = c^2 + c'^2.$$

It will then follow that: *The orthogonal trajectories to a simple infinitude of planes are everywhere equidistant.*

VI. Set $z = 0$, so (3) will give $y = c$ (constant) and $x = -(\rho / r)\, c$, and therefore M_1 (x, y, z) will be on the rectifying line to C at M. One will then have CESÀRO's theorem (**):

The line that is parallel to the tangent to a curve C that is contained in the rectifying plane and is rigidly linked with its fundamental trihedron will generate a developable surface.

One can add that:

The edges of regression of C_1 are transforms of C by T; their intrinsic equations are:

(*) CESÀRO, *loc. cit.*, pp. 139.
(**) *Loc. cit.*, pp. 149.

$$s_1 = s - \frac{\rho}{r}c, \quad \rho_1 = \rho\left[1 - c\frac{d}{ds}\left(\frac{\rho}{r}\right)\right], \qquad r_1 = r\left[1 - c\frac{d}{ds}\left(\frac{\rho}{r}\right)\right],$$

and have the rectifying developable in common with C.

A curve is geodetic on the rectifying developable, so C and C_1 are geodetics of the common rectifying developable, and are inclined equally from its generators; hence, $c\sqrt{1+\left(\frac{\rho}{r}\right)^2}$.

Hence: *If one starts from each point M of a geodetic C on a developable surface and transports the segment* $MM_1 = c\sqrt{1+\left(\frac{\rho}{r}\right)^2}$ *along the corresponding generator then the locus of points* M_1 *will be another geodetic that has the same inclination with respect to the generator.*

If C is a cylindrical helix – i.e., if ρ / r is constant – then one will have:

$$s_1 = s + \text{const.}, \qquad \rho_1 = \rho, \qquad r_1 = r,$$

and the following theorem will be obvious: *A cylindrical helix can be slid along a cylinder in the sense of the generators and without deformation,* and from that, one can maintain the preceding as a generalization.

VII. For $y = \pm\rho$, (3) will give:

$$z = \pm r\frac{d\rho}{ds}, \qquad x = \mp\rho\left[\frac{\rho}{r} + \frac{d}{ds}\left(r\frac{d\rho}{ds}\right)\right].$$

Now, as is easy to see, the point $P\ (y = \pm\rho, z = \pm r\, d\rho / ds)$ on the normal plane to C at M is nothing but the center of the osculating sphere (cf., I) ($y = -r\, d\rho / ds$, $z = \rho$), when rotated around M through the angle $\pm\pi / 2$ in that plane. *If the parallel to the tangent at a point M of a curve C that is drawn through the center of the osculating sphere is rotated through an angle* $\pm\pi / 2$ *around M then it will generate a developable surface when M moves along that curve.*

If C is a spherical curve then it will result that (cf., I) $x = 0$, and therefore:

The center of a sphere that is rotated through a right angle around a point M on a curve that is traced on that sphere and in the plane normal to that curve will generate a transform T of the curve when M moves along the curve.

VIII. (9) will not be satisfied when $x = y = 0$; that is, when *the principal normals to a curve do not form a developable surface.*

IX. It follows from (10) that $1 / r_1 = 0$ only when θ is constant, or when ρ / r is constant; hence:

Among the twisted curves, the cylindrical helices are characterized by the fact that all of their transforms by N are plane curves that are contained in planes that are perpendicular to the generators of the cylinder.

X. For $y = 0$, (9) gives $z = 0$, $x = c - s$, with c and arbitrary constant. M_1 (x, y, z) will then be a point on the tangent at M, and one will have $MM_1 = x = c - s$, so the locus C_1 of points M_1 is a developable of C; hence: *The developables of a curve are transforms of it by N.*

More generally, for $y = c'$ (constant), (9) will give only $z = 0$, $x = c - s$; hence:

A point of a developable on a curve that is displaced through a constant segment c' in the sense of the binormal at the corresponding point of the curve will generate a transform of the curve by N.

XI. Set $x = 0$ in (9), so one will have $y = \int \rho / r \, ds$, $z = \rho$. The curve C_1 that is described by the point M_1 (x, y, z) that lies on the polar developable to C and is an orthogonal trajectory to its generators will be perpendicular to the axis of the osculating circle since it is parallel to the principal normal to C at M. Therefore:

The principal normal to a curve that is displaced parallel to itself in the normal plane to $y = \int \rho / r \, ds$ describes a developable surface whose edge of regression lies on the polar developable to the curve and does not meet the generators at a right angle.

The point P $(0, \int \rho / r \, ds, \rho)$ on the polar axis to a curve is then such that the parallel to that axis on the principal normal and its polar axis will generate developable surfaces. However, the values $y = \int \rho / r \, ds$, $r = \rho$ also satisfy (3) and give:

$$x = -\rho\left(\frac{d\rho}{ds} + \frac{1}{r}\int \frac{\rho}{r} ds\right),$$

so the parallel to the tangent that is drawn through P will also generate a developable surface. Therefore:

The parallels to the tangents, the binormals, and the principal normals to a curve that is drawn through a point in the normal plane at distances of ρ and $\int \rho / r \, ds$, resp., in the latter two lines will generate three developable surfaces whose edges of regression will be transforms of the curve by T, B, N, resp.

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