"Deformazioni infinitesime delle curve inestendibili e corrispondenza per orthogonalità di elementi," Rend. Circ. Mat. Palermo **21** (1906), 229-256.

Infinitesimal deformations of inextensible curves and correspondence by orthogonality of elements

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I. Infinitesimal deformation.

1. In a preceding note (*), I addressed those curves C' that could be referred, point-by-point, to an assigned curve C in such a way that the tangent to C' at each point would be parallel to the tangent or binormal or principal normal to C at the corresponding point. The first case is called the COMBESCURE *transform of C*, which was defined by Prof. BIANCHI (**); the other ones are called the *B-transform and N-transform of C*, respectively.

Now, that transformation (along with an infinitude of other ones) enters into the study of infinitesimal deformations of curves, *when they are considered to be flexible, but inextensible filaments.*

Dr. A. PERNA (***), and then Dr. F. FOÀ (†), have already addressed the infinitesimal deformations of *flexible and extensible* curves, more generally.

Here, I shall limit myself to inextensible curves in order to construct a theory that is entirely analogous to that of infinitesimal deformations of surfaces that are considered to be flexible and inextensible membranes, which are also important.

2. First of all, recall that if x, y, z are the Cartesian coordinates of a point M with respect to the tangent, the binormal, and the principal normal, resp., to a curve C at a

(*) "Trasformazione di COMBESCURE ed altre analoghe per le curve storte" [These Rendiconti, **20** (1905), 83-92.]

(**) *Lezioni di Geometria differenziale*, 2nd ed., vol. I, pp. 40.

(***) "Sulle deformazioni infinitesime delle curve," Giornale di Matematiche di BATTAGLINI **36** (1898), 286-299.

(†) "Sulle deformazioni infinitesime delle curve" [Publisher for the R. Accademia delle Scienze Fis. e Mat. in Napoli 1901].

point M then the variations δx, δy, δz of those coordinates when M passes to the infinitely-close point to C are given by the formulas (*):

$$\frac{\delta x}{ds} = \frac{dx}{ds} - \frac{z}{\rho} + 1, \qquad \frac{\delta y}{ds} = \frac{dy}{ds} - \frac{z}{r} + 1, \qquad \frac{\delta z}{ds} = \frac{dz}{ds} + \frac{x}{\rho} + \frac{y}{r}, \tag{1}$$

in which ρ and r are the radii of first and second curvature of C at M, and s is the arc length of C, which terminates at M when one measures by starting from a fixed origin along C.

If α, β, γ are the direction cosines of a direction that moves with M then the following analogous formulas will be in force:

$$\frac{\delta\alpha}{ds} = \frac{d\alpha}{ds} - \frac{\gamma}{\rho} + 1, \qquad \frac{\delta\beta}{ds} = \frac{d\beta}{ds} - \frac{\gamma}{r} + 1, \qquad \frac{\delta\gamma}{ds} = \frac{d\gamma}{ds} + \frac{\alpha}{\rho} + \frac{\beta}{r}. \tag{2}$$

Those formulas are fundamental to the study of skew curves by the *instrinsic* method.

In particular, if one takes $\delta x = \delta y = \delta z = 0$ in (1) then one will have the immobility condition for the point (x, y, z), and if one takes $\delta\alpha = \delta\beta = \delta\gamma = 0$ in (2) then one will have the immobility condition for the direction (α, β, γ).

3. In order to perform an infinitesimal deformation, the point M must be subjected to a displacement MM' whose components with respect to the fundamental trihedron of C at M are denoted by εu, εv, εw, where u, v, w are three functions of s that are arbitrary, for now, and ε is an infinitesimal constant *for which one neglects powers higher than the first.*

Call the deformed curve C', which is the locus of points M', and s', ρ', r' are the arc lengths and radii of curvature at M', resp.

The necessary and sufficient condition for u, v, w to define an infinitesimal deformation of the inextensible curve C is that:

$$\frac{du}{ds} - \frac{w}{\rho} = 0. \tag{3}$$

The intrinsic equations of the deformed curve C' are:

$$\frac{1}{\rho'} = \frac{1}{\rho} - \varepsilon\left(\frac{dm}{ds} - \frac{n}{r}\right), \qquad \frac{1}{r'} = \frac{1}{r} - \varepsilon\left(\frac{n}{\rho} - \frac{dl}{ds}\right), \tag{4}$$

and the direction cosines of the tangent, binormal, and principal normal at M' are:

(*) E. CESÀRO, *Lezioni di Geometria intrinseca*, pp. 124.

$$\begin{array}{lll} a = 1, & b = \varepsilon\, n, & c = -\,\varepsilon\, m, \\ \alpha = -\,\varepsilon\, m, & \beta = 1, & \gamma = \varepsilon\, l, \\ \lambda = \varepsilon\, m, & \mu = -\,\varepsilon\, l, & \nu = 1, \end{array}$$

in which l, m, n are three functions of s that are defined by formulas:

(5) $$\frac{dv}{ds} - \frac{w}{r} = n, \qquad \frac{dw}{ds} + \frac{u}{\rho} + \frac{v}{r} = -\, m,$$

(6) $$\frac{dn}{ds} + \frac{l}{\rho} + \frac{m}{r} = 0.$$

Indeed, if one applies the fundamental formulas (1) to the point M whose coordinates are:

$$x = \varepsilon\, u, \qquad y = \varepsilon\, v, \qquad z = \varepsilon\, w$$

then one will have:

$$\frac{\delta x}{\delta s} = 1 + \left(\frac{du}{ds} - \frac{w}{\rho}\right), \qquad \frac{\delta y}{\delta s} = \varepsilon\left(\frac{dv}{ds} - \frac{w}{r}\right), \qquad \frac{\delta z}{\delta s} = \varepsilon\left(\frac{dw}{ds} + \frac{u}{\rho} + \frac{v}{r}\right),$$

from which:

$$\left(\frac{\delta s'}{\delta s}\right)^2 = \left(\frac{\delta x}{\delta s}\right)^2 + \left(\frac{\delta y}{\delta s}\right)^2 + \left(\frac{\delta z}{\delta s}\right)^2 = 1 + 2\varepsilon\left(\frac{du}{ds} - \frac{w}{\rho}\right);$$

it will then follow that (3) is necessary and sufficient for one to have $\delta s' = ds$. From (5), one will then have:

$$a = \frac{\delta x}{\delta s'} = 1, \qquad b = \frac{\delta y}{\delta s'} = \varepsilon\, n, \qquad c = \frac{\delta z}{\delta s'} = -\,\varepsilon\, n.$$

If one applies (2) to the direction (a, b, c) then one will have:

$$\frac{\delta a}{ds} = \varepsilon\frac{m}{\rho}, \qquad \frac{\delta b}{ds} = \varepsilon\left(\frac{dn}{ds} + \frac{m}{r}\right) = -\varepsilon\frac{l}{\rho}, \qquad \frac{\delta c}{ds} = \frac{1}{\rho} - \varepsilon\left(\frac{dm}{ds} - \frac{n}{r}\right),$$

so:

$$\frac{2}{\rho'^2} = \sum\left(\frac{\delta a}{\delta s'}\right)^2 = \sum\left(\frac{\delta a}{\delta s}\right)^2 = \frac{1}{\rho^2} - 2\frac{\varepsilon}{\rho}\left(\frac{dm}{ds} - \frac{n}{r}\right),$$

and therefore:

$$\frac{1}{\rho'} = \frac{1}{\rho} - \varepsilon\left(\frac{dm}{ds} - \frac{n}{r}\right).$$

It then follows that:

$$\lambda = \rho' \frac{\delta a}{\delta s'} = \rho' \frac{\delta a}{\delta s} = \varepsilon\, m, \qquad \text{etc.},$$

and then α, β, γ are easily calculated, because if one supposes that:

$$\begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \lambda & \mu & \nu \end{vmatrix} = 1$$

then one will have:

$$\alpha = \mu\, c - \nu\, b, \qquad \beta = \nu\, a - \lambda\, c, \qquad g = \lambda\, b - \mu\, a.$$

Finally:

$$\frac{1}{r'^2} = \sum \left(\frac{\delta\alpha}{\delta s'}\right)^2 = \sum \left(\frac{\delta\alpha}{ds}\right)^2 = \left(\frac{\delta\gamma}{ds}\right)^2,$$

because when one applies (2) to the direction (α, β, γ), $\left(\frac{\delta\alpha}{ds}\right)^2$ and $\left(\frac{\delta\beta}{ds}\right)^2$ will prove to be second-order infinitesimals; therefore:

$$\frac{1}{r'} = \frac{\delta\gamma}{ds} = \frac{1}{r} - \varepsilon\left(\frac{n}{\rho} - \frac{dl}{ds}\right).$$

OBSERVATION. – The kinematical significance of the three functions l, m, n is easily found. The direction cosines (1, 0, 0) of the tangent to C at M after the deformation will become (1, $\varepsilon\, n$, $-\varepsilon\, m$), so the components of the rotation of the tangent will be the minors:

$$\begin{vmatrix} 0 & 0 \\ \varepsilon\, n & -\varepsilon\, m \end{vmatrix} = 0, \qquad -\begin{vmatrix} 1 & 0 \\ 1 & -\varepsilon\, m \end{vmatrix} = \varepsilon\, m, \qquad \begin{vmatrix} 1 & 0 \\ 1 & \varepsilon\, n \end{vmatrix} = \varepsilon\, n$$

to the matrix:

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & \varepsilon\, n & -\varepsilon\, m \end{vmatrix},$$

when taken with convenient signs. Hence:

The rotation of the tangent has zero components with respect to that tangent and components $\varepsilon\, m$, $\varepsilon\, n$ with respect to the binormal and principal normal, resp.

One finds, analogously, that:

The components of the rotation of the binormal are $\varepsilon\, l$, 0, $\varepsilon\, n$, and those of the rotation of the principal normal are $\varepsilon\, l$, $\varepsilon\, m$, 0.

4. Let us consider some particular deformations.

a) First of all, it is obvious that *the deformations that do not alter the curvature are rigid motions*, since that already results directly from the preceding formulas. Indeed, it follows from (4) that in order to have $\rho = \rho'$, $r = r'$, it is necessary and sufficient that one must have:

$$\frac{dm}{ds} - \frac{n}{r} = 0, \qquad \frac{dl}{ds} - \frac{n}{\rho} = 0.$$

Now this, along with (3), (5), and (6) expresses the idea (*) that the line with coordinates (l, m, n, u, v, w) is a fixed line. Therefore, the infinitesimal deformation will be an infinitesimal rigid motion, which can be considered to be the resultant of an infinitesimal rotation around that line and an infinitesimal translation that is parallel to that line.

b) For $u = 0$, (3) will give only $w = 0$, while v will remain arbitrary, so:

In order for an infinitesimal deformation of a curve to subject each point to a displacement that is normal to the curve, it is necessary and sufficient that the displacement, which is otherwise arbitrary, should take place along the binormal to the curve.

c) For $v = 0$, one will have $w = \rho\,(du / ds)$, with u arbitrary, but not O, so:

There always exists an infinitesimal deformation for which any point is displaced in a given direction in the osculating plane, but not that of the principal normal.

d) For $w = 0$, one will have u = constant, and v will remain arbitrary, so:

There always exists an infinitesimal deformation for which any point is displaced in a given direction in the rectifying plane; the projection of the displacement onto the tangent is constant.

e) In particular, for $v = 0$, one has that:

Any curve will admit an infinitesimal deformation for which any point is displaced along the tangent and that displacement will be constant.

For the deformed curve, one will have:

$$\frac{1}{\rho'} = \frac{1}{\rho} - \varepsilon u \frac{d\rho}{ds}, \qquad \frac{1}{r'} = \frac{1}{r} - \varepsilon u \frac{dr}{ds};$$

hence:

(*) CESÀRO, *loc. cit.*, pp. 124 and 125.

Under this deformation, the curves with constant flexion (skew circles) will keep their flexion unaltered, and the curves with constant torsion will keep their torsions unaltered.

The only curves with constant flexion and torsion – namely, the circular cylindrical helices – admit infinitesimal deformations into themselves.

f) It follows from the preceding that the displacement can take place along the tangent along the binormal, but not along the principal normal.

More generally, we wonder whether displacement can take place along a given direction (ξ, η, ζ) that is rigidly linked with the fundamental trihedron.

It is enough to determine u, v, w in such a way that:

$$\frac{u}{\xi} = \frac{v}{\eta} = \frac{w}{\zeta} = p, \qquad \text{in which} \qquad p = \sqrt{u^2 + v^2 + w^2}\,. \tag{7}$$

If ξ, η, ζ are constant then (3) will become:

$$\xi \frac{dp}{ds} - \zeta \frac{p}{\rho} = 0,$$

and give:

$$p = k \exp\left(\frac{\zeta}{\xi} \int \frac{ds}{\rho}\right) \tag{8}$$

if $\xi \neq 0$, with k and arbitrary constant. Therefore:

There exists an infinitesimal deformation for which any point is displaced along a direction (ξ, η, ζ) *that is arbitrarily rigidly linked with the fundamental trihedron, but which is not normal to the curve and is defined by* (7) *and* (8).

Theorem *b*) is true for a normal direction.

g) One wishes that C and C' should lie on the same cylinder whose generators have a given direction – viz., the curve C cannot escape from the cylinder upon which it is assumed to be traced.

If the fixed direction of the generators is given to be (ξ, η, ζ) then it will be necessary and sufficient that the generator, the displacement, and the tangent at each point M of the curve should be coplanar, and therefore it is necessary and sufficient that one should have:

$$\eta\, w - \zeta\, v = 0.$$

One then notes that in order to have an effective deformation, one must not have:

$$\frac{u}{\xi} = \frac{v}{\eta} = \frac{w}{\zeta},$$

because in that case, if p is the common value of those ratios then one will have $dp / ds = 0$ – namely, $p = 0$ – if one takes $\xi = u / p$, $\zeta = w / p$ in the first of the three immobility conditions for the direction (ξ, η, ζ), and one takes (3) into account.

h) The curve C is spherical, and one desires that it should deform without leaving the sphere upon which it is traced.

The normal planes to a spherical curve will coincide at a point – namely, the center of the sphere. If one lets (x_0, y_0, z_0) be the coordinates of that fixed point then one will have:

$$\frac{dx_0}{ds} - \frac{z_0}{\rho} + 1 = 0, \qquad \frac{dy_0}{ds} - \frac{z_0}{r} = 0, \qquad \frac{dz_0}{ds} + \frac{x_0}{\rho} + \frac{y_0}{r} = 0,$$

but obviously $x_0 = 0$, so it will result that:

$$z_0 = \rho, \qquad y_0 = r\frac{d\rho}{ds}, \qquad \frac{\rho}{r} + \frac{d}{ds}\left(r\frac{d\rho}{ds}\right) = 0,$$

and the last one is known to be the necessary and sufficient condition for a curve to be spherical.

If one wishes that the deformed curve C' should lie on that sphere then it will be necessary and sufficient that one should have:

$$(x_0 - \varepsilon u)^2 + (y_0 - \varepsilon v)^2 + (z_0 - \varepsilon w)^2 = x_0^2 + y_0^2 + z_0^2,$$

namely:

$$x_0 u + y_0 v + z_0 w = 0,$$

or

$$v r \frac{d\rho}{ds} = \rho w.$$

If u is given arbitrarily then (3) will determine w, and the preceding will determine v, namely:

$$w = \rho\frac{du}{ds}, \qquad V = \frac{\rho}{r}\frac{du}{d\rho}.$$

If the spherical curve C is (circular) planar then $1 / r$ will be zero, and therefore v, as well, so ρ will be constant. From (5), (6), $l = n = 0$, so from the second of (4), $1 / r' = 0$; hence, *that deformation will preserve circles.*

II. – Correspondence by equivalent arc lengths.

5. – The functions u, v, w define an infinitesimal deformation of C – namely, one that satisfies (3) – and one considers the curve that is the locus of the points (u, v, w).

From (1), the square of its elementary arc:

$$\delta\sigma^2 = \delta u^2 + \delta v^2 + \delta w^2$$

will be given by the formula:

$$\frac{\delta\sigma^2}{\delta s^2} = \left(\frac{du}{ds} - \frac{w}{\rho} + 1\right)^2 + \left(\frac{dv}{ds} - \frac{w}{r}\right)^2 + \left(\frac{dw}{ds} + \frac{u}{\rho} + \frac{v}{r}\right)^2 \tag{7}$$

namely, from (3):

$$\frac{\delta\sigma^2}{\delta s^2} = 1 + \left(\frac{dv}{ds} - \frac{w}{r}\right)^2 + \left(\frac{du}{ds} + \frac{u}{\rho} + \frac{v}{r}\right)^2 .$$

Now, the right-hand side will not change when one replaces u, v, w with $-u$, $-v$, $-w$. Therefore:

If u, v, w satisfies (3) *– i.e., it defines an infinitesimal deformation of the curve C – then the two curves that are the loci of the points* (u, v, w), ($-u$, $-v$, $-w$) *will correspond with equivalence of their arc lengths.*

Conversely:

If one establishes a correspondence with equivalence of arc lengths between the point of two curves then the curve C that is the locus of the midpoints of the conjugates to the corresponding points will admit an infinitesimal deformation for which any point will be displaced in the direction of that conjugate.

That is because if one is given (u, v, w), ($-u$, $-v$, $-w$) as the coordinates of the two corresponding points of the two curves with respect to the fundamental trihedron of the curve C at the midpoint of the line segment that links those points then (7) will give the elementary arc length of one of the curves, and by hypothesis, that of the other curve, as well, and therefore it must not change when one replaces u, v, w with $-u$, $-v$, $-w$. Now, that will happen only when (3) is satisfied, and therefore only when u, v, w define an infinitesimal deformation of C.

The study of infinitesimal deformations then takes on an initial well-defined aspect.

III. – Correspondence by orthogonality of elements.

6. – Draw a vector OM_1 with components u, v, w from a fixed point O (x_2, y_2, z_2) and consider the curve C_1 that is the locus of point M_1 that have the coordinates:

$$x_1 = x_2 + u, \qquad y_1 = y_2 + v, \qquad z_1 = z_2 + w.$$

From (1), one has:

$$\frac{\delta x_1}{ds} = \frac{dx_2}{ds} + \frac{du}{ds} - \frac{z_2 + w}{\rho} + 1 = \left(\frac{du}{ds} - \frac{w}{\rho}\right) + \frac{\delta x_2}{ds},$$

$$\frac{\delta y_1}{ds} = \frac{dy_2}{ds} + \frac{dv}{ds} - \frac{z_2 + w}{r} = \left(\frac{dv}{ds} - \frac{w}{r}\right) + \frac{\delta y_2}{ds},$$

$$\frac{\delta z_1}{ds} = \frac{dz_2}{ds} + \frac{dw}{ds} + \frac{x_2 + u}{\rho} + \frac{y_2 + v}{r} = \left(\frac{dw}{ds} + \frac{u}{\rho} + \frac{v}{r}\right) + \frac{\delta z_2}{ds},$$

namely, from (3), (5), and immobility conditions for the point O:

$$(8) \qquad \frac{\delta x_1}{ds} = 0, \qquad \frac{\delta y_1}{ds} = n, \qquad \frac{\delta z_1}{ds} = -m.$$

If one lets $\delta x_1 = 0$ then the tangent to C_1 at M_1 will be orthogonal to the tangent to C at M, so:

If one draws vectors from any point O that are equal to the displacements that the points M of a curve C experience under an infinitesimal deformation then the curve C_1 that is the locus of the extremes M_1 will correspond to C with orthogonality of the elements.

Conversely:

If C and C_1 are two curves that correspond with orthogonality of elements, and O is a fixed point then each of them will admit an infinitesimal deformations under which each of its points will displace in the direction of the ray that goes from the point O to the corresponding point of the other one.

One easily proves this by inverting the preceding argument. Indeed, if x_2, y_2, z_2 are the coordinates of O with respect to the fundamental trihedron to C at a point M, and x_1, y_1, z_1 are those of the corresponding point M_1 of C then one will have, by hypothesis:

$$\frac{\delta x_1}{ds} = \frac{dx_1}{ds} - \frac{x_1}{\rho} + 1 = 0,$$

namely:

$$\frac{du}{ds} + \frac{dx_2}{ds} - \frac{w + z_1}{\rho} + 1 = 0,$$

in which one sets:

$$x_1 - x_2 = u, \qquad y_1 - y_2 = v, \qquad z_1 - z_2 = w.$$

However, from the first of the immobility conditions for O:

$$\frac{dx_2}{ds} - \frac{z_2}{\rho} + 1 = 0,$$

so

$$\frac{du}{ds} - \frac{w}{\rho} = 0,$$

and therefore u, v, w will define an infinitesimal deformation for the curve C under which each point M will displace parallel to OM_1 .

7. – The problem of the infinitesimal deformations of a curve C will then take on a second well-defined aspect: *One studies the curves that correspond to C with orthogonality of elements,* or what amounts to the same thing, *one studies the curves that are loci of the point whose coordinates satisfy the first of the three immobility conditions.*

Set:

(9) $$n = f \sin \theta, \qquad - m = f \cos \theta,$$

so from the fundamental formulas (1), (8) can be written as:

(10) $$\frac{dx_1}{ds} - \frac{z_1}{\rho} + 1 = 0,$$

(11) $$f \sin \theta = \frac{dy_1}{ds} - \frac{z_1}{r}, \qquad f \cos \theta = \frac{dz_1}{ds} + \frac{x_1}{\rho} + \frac{y_1}{r}.$$

The first of these expresses the orthogonality condition between the elements of the two curves C and C_1, and the second one defines the functions $f(s)$ and $\theta(s)$ whose geometric significance follows immediately.

There is an infinitude of curves C_1 : One will get one when one assigns the two coordinates x_1, y_1 of the point M_1 arbitrarily and determines the third one z_1 by means of (10). (11) will then determine f and θ, and when one calculates the arc length, curvature, etc. of the curve, one can verify that:

(12) $$\delta s_1 = f\, ds;$$

the two curvatures are given by the formulas:

(13) $$\frac{f}{\rho_1} = \sqrt{\kappa^2 + \frac{\cos^2\theta}{\rho^2}},$$

(14) $$\frac{f}{r_1} = -\frac{\rho\kappa \sin\theta \dfrac{d\theta}{ds} + \cos\theta \dfrac{d}{ds}(\rho\kappa)}{\cos^2\theta + \rho^2\kappa^2} - \frac{\sin\theta}{\rho},$$

and the direction cosines of the tangent, the binormal, and the principal normal are:

$$(15)\qquad \begin{cases} a_1 = 0, & b_1 = \sin\theta, & c_1 = \sin\theta, \\ \alpha_1 = 0, & \beta_1 = \dfrac{\rho_1}{f}\dfrac{\cos^2\theta}{\rho}, & \gamma_1 = -\dfrac{\rho_1}{f}\dfrac{\sin\theta\cos\theta}{\rho}, \\ \lambda_1 = -\dfrac{\rho_1}{f}\dfrac{\cos\theta}{\rho}, & \mu_1 = \dfrac{\rho_1}{f}\kappa\cos\theta, & \nu_1 = -\dfrac{\rho_1}{f}\kappa\sin\theta, \end{cases}$$

in which:

$$(16)\qquad \kappa = \frac{d\theta}{ds} - \frac{1}{r}.$$

It then follows that:

f is the ratio of the two corresponding elementary arc lengths, and θ is the angle that the tangent to C_1 makes with the principal normal to C.

In order to prove these formulas, first observe that it follows from (8) and (9) that:

$$\left(\frac{\delta s_1}{ds}\right)^2 = \left(\frac{\delta x_1}{ds}\right)^2 + \left(\frac{\delta y_1}{ds}\right)^2 + \left(\frac{\delta z_1}{ds}\right)^2 = m^2 + n^2 = f^2,$$

and therefore (12). If one then divides (8) by (12) then one will have:

$$a_1 = 0, \qquad b_1 = \frac{n}{f} = \sin\theta, \qquad c_1 = -\frac{m}{f} = \cos\theta.$$

If one applies (2) to the direction (a_1, b_1, c_1) then one will have:

$$\frac{\delta a_1}{ds} = -\frac{\cos\theta}{\rho}, \qquad \frac{\delta b_1}{ds} = \kappa\cos\theta, \qquad \frac{\delta c_1}{ds} = -\kappa\sin\theta,$$

in which κ is defined by (16). It will then follow that:

$$\frac{1}{\rho_1^2} = \sum\left(\frac{\delta a_1}{\delta s_1}\right)^2 = \left(\frac{\delta s}{\delta s_1}\right)^2 \sum\left(\frac{\delta a_1}{ds_1}\right)^2 = \frac{1}{f^2}\left(\kappa^2 + \frac{\cos^2\theta}{\rho^2}\right),$$

and then (13). One will then have:

$$\lambda_1 = \rho_1\frac{\delta a_1}{\delta s_1} = \frac{\rho_1}{f}\cdot\frac{\delta a_1}{ds} = -\frac{\rho_1}{f}\frac{\cos^2\theta}{\rho},$$

and one calculates μ_1 and ν_1 analogously; one can then calculate α_1 , β_1, γ_1 from the formulas:

$$\alpha_1 = c_1\,\mu_1 - b_1\,\nu_1\,, \qquad \beta_1 = a_1\,\nu_1 - c_1\,\lambda_1\,, \qquad \gamma_1 = b_1\,\lambda_1 - a_1\,\mu_1\,.$$

The torsion $1 / r_1$ can be calculated directly by applying (2) to the direction $(\alpha_1, \beta_1, \gamma_1)$ and recalling that:

$$\frac{1}{r_1^2} = \sum\left(\frac{\delta\alpha_1}{\delta s_1}\right)^2 = \frac{1}{f^2}\sum\left(\frac{\delta\alpha_1}{ds_1}\right)^2 .$$

However, after some laborious calculation, one will have a formula of a complicated nature that does not allow one to recognize the great simplifications to which it is susceptible.

Nonetheless, the torsion can be calculated rapidly, given the reciprocity of the correspondence that intercedes between C and C_1, which permits one to switch the quantities that relate to C with the ones that relate to C_1 in the preceding formulas.

Therefore, if θ_1 is the angle that the tangent to C makes with the principal normal to C_1 then one will have, from (16), that:

$$\kappa_1 = \frac{\delta\theta_1}{\delta s_1} - \frac{1}{r_1},$$

and from the first of (15):

$$\frac{\rho}{f_1}\kappa_1 = \sin\,\theta, \qquad \text{in which} \qquad f_1 = \frac{1}{f},$$

because $\pi / 2 - \theta$ is the angle that the binormal to C makes with the tangent to C_1; it then follow that:

(17) $$\frac{1}{r_1} = \frac{\delta\theta_1}{\delta s_1} - \kappa_1 = \frac{\delta\theta_1}{\delta s_1} - \frac{f_1 \sin\theta}{\rho}.$$

Now:

(18) $$\cos\,\theta_1 = \lambda_1 = -\,\frac{\rho_1}{f}\frac{\cos\theta}{\rho},$$

so when one differentiates this with respect to s_1:

$$-\sin\,\theta_1\frac{\delta\theta_1}{\delta s_1} = -\,\frac{1}{f}\frac{d}{ds}\left(\frac{\rho_1}{f}\frac{\cos\theta}{\rho}\right).$$

However:

(19) $$\sin\,\theta_1 = \alpha_1 = \frac{\rho_1}{f}\,\kappa;$$

so

$$\frac{\delta\theta_1}{\delta s_1} = \frac{1}{\rho_1\kappa}\cdot\frac{d}{ds}\left(\frac{\rho_1}{f}\frac{\cos\theta}{\rho}\right),$$

and if one substitutes this in (19) then:

$$\frac{1}{r_1} = \frac{1}{\kappa\rho_1}\cdot\frac{d}{ds}\left(\frac{\rho_1}{f}\cdot\frac{\cos\theta}{\rho}\right) - \frac{\sin\theta}{\rho f};$$

finally, from (13):

$$\frac{1}{r_1} = \frac{1}{\kappa\rho_1}\cdot\frac{d}{ds}\left(\frac{\cos\theta}{\sqrt{\cos^2\theta+\rho^2\kappa^2}}\right) - \frac{\sin\theta}{\rho f},$$

from which (14) will follow easily.

8. – As we have observed already, in order to obtain a curve C_1, it is enough to assign the two functions x_1, y_1 arbitrarily and then determine z_1 from (10).

We shall now seek to put some semblance of order into this great multitude of curves; that will make the links that it has with the curve *C* stand out more distinctly.

Suppose that x_1 is fixed, and therefore z_1, but leave y_1 arbitrary. The infinitude of curves that corresponds to the infinitude of functions y_1 is all of the curves of a ruled surface that is generated by the parallels to the binormal to *C* at *M* that goes through the point *P* (x_1 or z_1) of the osculating plane.

In order to fix one or more particular curves in that ruled surface, one can assign one of the two auxiliary functions *f* and θ, instead of the function y_1 .

In particular, if one takes $\theta = \pi / 2$ in (11), *et seq.*, in § **7** then *one will have all of the elements:*

(20) $$y_1 = -\, r\left(\frac{x_1}{\rho}+\frac{dz_1}{ds}\right),$$

$$\delta s_1 = f\,ds, \qquad \text{in which} \qquad f = \frac{dy_1}{ds}-\frac{z_1}{r},$$

$$\rho_1 = -\,f\,r, \qquad r_1 = -\,f\,\rho,$$

$$a_1 = 0,\quad b_1 = 0,\quad c_1 = 0,\quad \alpha_1 = 0,\quad \beta_1 = 0,\quad \gamma_1 = 0,\quad \lambda_1 = 0,\quad \mu_1 = 0,\quad \nu_1 = -\,1$$

for a curve whose tangents are the generators of the ruled surface (the one parallel to the binormals to C) whose binormals are parallel to the tangents to C, and whose principal normals are parallel to the principal normals to C (but with opposite positive senses). In summation: *One has a B-transform of C.*

Therefore:

The infinitude of ruled surfaces that were defined just now and correspond to the infinitude of functions x_1 are developable and have the B-transforms of C for their edges of regression.

We shall call them *the B-developables of the curve C,* for brevity.

The normal to a *B*-developable along a generator, being parallel to the binormal to the edge of regression, is parallel to the tangents to the curve *C* at the corresponding points, so:

Any B-developable of a curve is the envelope of a simple infinitude of planes that are parallel to the normal planes to the curve (at a distance x_1).

It follows that it is the polar developable of the orthogonal trajectories to those planes, which are COMBESCURE transforms of *C*; therefore:

The B-developables of a curve are the polar developables of its COMBESCURE *transforms.*

(10) will not be altered if one adds an arbitrary constant to x_1, so:

If one gives an arbitrary translation to the rectilinear generator of a B-developable of a curve that is parallel to the tangents to the curve then that will once more general a B-developable.

9. – If θ_1 is the angle that the principal normal to C_1 makes with the tangent to *C* then it will also be the angle that the principal normal to C_1 makes with the normal to the *B*-developable that it lies on, and therefore (*):

$$N = \frac{\cos\theta_1}{\rho_1}, \qquad G = \frac{\sin\theta_1}{\rho_1}, \qquad T = \frac{\delta\theta_1}{\delta s_1} = \frac{1}{r_1},$$

namely, from (17), (18), (19):

$$N = -\frac{\cos\theta}{\rho f}, \qquad G = \frac{z}{f}, \qquad T = \frac{\sin\theta}{\rho f},$$

will be the normal curvature, geodetic curvature, and the geodetic torsion, resp., of the curve C_1 on the B-developable on which it lies.

Lines of curvature on a surface are the ones with zero geodetic torsion, so on a *B*-developable they will be the curves for which $\theta = 0$, and therefore they will have their tangents parallel to the principal normals to *C*. Hence:

The lines of curvature of a B-developable to a curve – i.e., the orthogonal trajectories to the generators – are N-transforms of the curve. For one of them:

(*) CESÀRO, *loc. cit.*, pp. 152.

(21) $$y_1 = \int \frac{z_1}{r} ds + k,$$

in which k is an arbitrary constant,

$$\delta s_1 = f\, ds \qquad \textit{in which} \qquad f = \frac{dz_1}{ds} + \frac{x_1}{\rho} + \frac{y_1}{r},$$

$$\frac{f}{\rho_1} = \sqrt{\frac{1}{\rho^2} + \frac{1}{r^2}}, \qquad \frac{f}{r_1} = \frac{d}{ds}\left(\arctan \frac{\rho}{r}\right),$$

$$a_1 = 0, \qquad b_1 = 0, \qquad c_1 = 0,$$

$$\alpha_1 = \frac{\rho}{\sqrt{\rho^2 + r^2}}, \qquad \beta_1 = \frac{r}{\sqrt{\rho^2 + r^2}}, \qquad \gamma_1 = 0,$$

$$\lambda_1 = -\frac{r}{\sqrt{\rho^2 + r^2}}, \qquad \mu_1 = -\frac{\rho}{\sqrt{\rho^2 + r^2}}, \qquad \nu_1 = 0.$$

The geodetics of a surface are the lines of zero geodetic curvature, so *the geodetics on a B-developable of a curve are the curves for which one has:*

$$\kappa = \frac{d\theta}{ds} - \frac{1}{r} = 0.$$

For an arbitrary geodetic, one will have:

$$\theta = \int \frac{ds}{r} + \tau,$$

(22) $$y_1 = \frac{1}{\cos\theta}\left(z_1 \sin\theta + \int \frac{x_1}{\rho} \sin\theta \, ds + A\right),$$

in which τ and A are arbitrary constants:

$$\delta s_1 = f\, ds, \qquad \textit{in which} \qquad f = \frac{d}{ds}(y_1 \sin\theta + z_1 \cos\theta) + \frac{x_1}{\rho}\cos\theta,$$

so

$$s_1 = y_1 \sin\theta + z_1 \cos\theta + \int \frac{x_1}{\rho} \cos\theta \, ds\,;$$

$$\frac{f}{\rho_1} = \frac{\cos\theta}{\rho}, \quad \frac{f}{r_1} = -\frac{\sin\theta}{\rho};$$

$$\begin{array}{lll} a_1 = 0, & b_1 = \sin\,\theta, & c_1 = \cos\,\theta, \\ \alpha_1 = 0, & \beta_1 = \cos\,\theta, & \gamma_1 = -\sin\,\theta, \\ \lambda_1 = -1, & \mu_1 = 0, & \nu_1 = 0. \end{array}$$

It then follows that:

θ is the angle that the tangent to C_1 makes with the principal normal to C and is just the angle between their binormals; the principal normal to C is parallel to the tangent to C.

Therefore:

Any curve is an N-transform of all the geodetics to all of its B-developables.

The preceding formulas are easily proved.

y_1 is obtained by integrating the linear differential equation:

$$\left(\frac{dy_1}{ds} - \frac{z_1}{r}\right)\cos\,\theta = \left(\frac{dz_1}{ds} + \frac{x_1}{\rho} + \frac{y_1}{r}\right)\sin\,\theta$$

that results from eliminating f from (11).

Set $y_1 = \xi\,\eta$ and determine η from the condition:

$$\cos\,\theta\,\frac{d\eta}{ds} - \frac{\sin\theta}{r}\eta = 0,$$

namely:

$$\frac{d\eta}{\eta} = \frac{\tan\theta}{r}ds = \tan\,\theta\,d\theta,$$

which gives:

$$\eta = \frac{1}{\cos\theta},$$

so the preceding equation will become:

$$\frac{d\xi}{ds} = \frac{\cos\theta}{r}z_1 + \sin\theta\left(\frac{dz_1}{ds} + \frac{x_1}{\rho}\right),$$

namely:

$$\frac{d\xi}{ds} = \frac{d}{ds}(z_1\sin\theta) + \frac{x_1}{\rho}\sin\theta,$$

so

$$\xi = z_1 \sin\theta + \int \frac{x_1}{\rho} \sin\theta\, ds + A\,;$$

the expression that was written for $y_1 = \xi\,\eta$ will then follow immediately.

It will then result from (11) that:

$$f = \left(\frac{dy_1}{ds} - \frac{z_1}{r}\right) \sin\theta + \left(\frac{dz_1}{ds} + \frac{x_1}{\rho} + \frac{y_1}{r}\right) \cos\theta,$$

namely:

$$f = \frac{d}{ds}(y_1 \sin\theta + z_1 \cos\theta) + \frac{x_1}{\rho}\cos\theta,$$

and therefore:

$$s_1 = \int f\, ds = y_1 \sin\theta + z_1 \cos\theta + \int \frac{x_1}{\rho}\cos\theta\, ds.$$

Ultimately, the remaining formulas are obtained immediately from (13), (14) when one then sets $\kappa = 0$.

Applications.

10. – The generality of the preceding results permits one to make numerous applications that are specialized by suitable choices of the arbitrary function x_1. I shall confine myself to the simplest ones.

First set $x_1 = 0$. It results from (10) that $z_1 = \rho$, so the corresponding B-developable is *the polar developable to the curve C that is the envelope of its normal planes, and is also the locus of the axes of the osculating circles.*

The theorems of §§ **8** and **9** specialize as follows:

The edge of regression of the polar developable to a curve is a B-transform of the curve. For it:

$$x_1 = 0, \qquad y_1 = -\,r\frac{d\rho}{ds}, \qquad z_1 = \rho,$$

$$s_1 = -\,r\frac{d\rho}{ds} - \int \frac{\rho}{r} ds\,,$$

$$\rho_1 = \rho + r\frac{d}{ds}\left(r\frac{d\rho}{ds}\right), \qquad r_1 = \frac{\rho^2}{r} + \rho\frac{d}{ds}\left(r\frac{d\rho}{ds}\right).$$

The lines of curvature, which are orthogonal trajectories of the generators, are N-transforms of the curve, and for them:

$$x_1 = 0, \qquad y_1 = \int \frac{\rho}{r} ds + k, \qquad z_1 = \rho,$$

$$\delta s_1 = f\, ds, \qquad \text{in which} \qquad f = \frac{d\rho}{ds} + \frac{1}{r}\left(\int \frac{\rho}{r} ds + k\right),$$

$$\frac{f}{\rho_1} = \sqrt{\frac{1}{\rho^2} + \frac{1}{r^2}}, \qquad \frac{f}{r_1} = \frac{d}{ds}\left(\arctan \frac{\rho}{r}\right),$$

in which k is an arbitrary constant.

Any curve is the N-transform of all the geodetics of its polar developable, for which:

$$x_1 = 0, \qquad y_1 = \frac{1}{\cos\theta}(A + \rho \sin\theta), \qquad z_1 = \rho,$$

$$s_1 = \frac{1}{\cos\theta}(\rho + A \sin\theta),$$

in which:

$$\theta = \int \frac{ds}{r} + \tau, \qquad f = \frac{1}{r\cos^2\theta}\left(A + \rho \sin\theta + r\cos\theta \frac{d\rho}{ds}\right),$$

in which A and τ *are arbitrary constants.*

Suppose that ρ is constant, so the first theorem will give one a well-known "bouquet" of skew circles (*).

From the same theorem, one will have $\rho_1 / r_1 = r / \rho$, so *it is only for the cylindrical helix that the edge of regression of the polar developable will be a cylindrical helix.* In that case, the torsion of the lines of curvature will be zero, which would result from the second theorem, and one will then have the known theorem (**):

The only developables with lines of curvature that are all plane are the ones that have a cylindrical helix for their edge of regression.

From the presence of two arbitrary constants τ and A, the geodetics are doubly-infinite in number.

For $A = 0$, one has the ones whose tangents are supported by the curve C – namely, the *evolute* of the curve C; for it:

$$y_1 = \rho \tan\theta, \qquad s_1 = \frac{\rho}{\cos\theta}. \tag{23}$$

(*) BIANCHI, *loc. cit.*, v. I, pp. 34, or CESÀRO, *loc. cit.*, pp. 145.
(**) BIANCHI, *loc. cit.*, v. II, pp. 255.

From the presence of the constant τ, one has that:

All of the evolutes of a curve are obtained from one of them by rotating its tangents through the same angle around the curve.

When one passes from an evolute to another geodetic that corresponds to the same value of τ, y_1 will increase by $A / \cos \theta$, so A will be the distance between their tangents. If one takes into account that any geodetic is the evolute of either C or another trajectory that is orthogonal to the normal planes of C (which are, in fact, infinite) and that the polar developable to a generic curve is a generic developable then the last result can be stated in the form:

If one gives all of the tangents to a geodetic on a developable surface the same translation that lies in the tangent planes to the surface then its envelope will again be a geodetic.

Otherwise, let $\varphi = \pi / 2 - \theta$ be the inclination of a geodetic with respect to the rectilinear generator of the polar developable:

If one draws the segment $A / \sin \varphi$ from each point of a geodetic on a developable surfaces, where A is a constant, and φ is the inclination of the geodetic with respect to the generator, then the locus of the endpoints will be another geodetic that is inclined equally with respect to the generators.

The fusion of the third-to-last and the second-to-last theorems implies another:

All of the geodetics on a developable surface can be obtained from just one of them by giving the same arbitrary motion with respect to a trajectory that is orthogonal to its tangents to its tangents in the tangent plane to the surface.

From the second of (23), one will then have:

The orthogonal trajectory to a simple infinitude of planes can be imagined to be generated by the points of just as many flexible, inextensible filaments that were originally wrapped on the geodetic of the developable that is the envelope of those plane and are then unwrapped while always keeping them tense.

11. – More generally, suppose that x_1 is a constant. (10) will once again give $z_1 = \rho$, so:

Not only will the axis of the osculating circle to a curve generate developable surfaces, but also all of the lines g that are parallel to it, contained in the plane π, and drawn through the axis and parallel to the tangent.

For the edge of regression of any of those developables, y_1 is (§ **8**):

$$y_1 = -\rho\left(\frac{x_1}{\rho}+\frac{d\rho}{ds}\right).$$

Suppose that s is fixed and x_1 is variable, so that equation and the other one $z_1 = \rho$ are those of the line that goes through the center $\left(0, -r\dfrac{d\rho}{ds}, \rho\right)$ of the osculating sphere (first theorem in § **10**) and inclined with respect to the tangent (x-axis) through an angle ψ such that $\tan y = - r / \rho$; however, that is also the inclination of the generator of the rectifying developable of the curve C with respect to the tangent (*), so:

For each point of a curve, the locus of points for which the generators g of the above developable touch the corresponding edges of regression is the line p through the center of the osculating sphere and parallel to the generator of the rectifying developable.

It follows that the edges of regression of the developables that are generated by the line g are traced on the ruled surface that is generated by the line p. We then propose to look for the envelope of the planes π whose equation is $z_1 = \rho$. We must differentiate that equation with respect to s, while taking into account the fact that the points of the characteristic of the plane π satisfy the three immobility conditions, and in particular:

$$\frac{dz_1}{ds}+\frac{x_1}{\rho}+\frac{y_1}{r} = 0.$$

One then has:

$$\frac{x_1}{\rho}+\frac{y_1}{r}+\frac{d\rho}{ds} = 0,$$

which is the equation of a plane that cuts the plane π along the characteristic. However, that line is precisely the line p that was defined above, so the line p will generate a developable surface.

Hence:

The edges of regression of the developables that are generated by the line g are traced on the developable surface that is the envelope of planes p that are the loci of the line p. On that surface, they are geodetics and the meet the generators at the angle $\pi / 2 - \psi$, because their principal normals, being parallel to the principal normals to C (§ **8**), are normals to the developable.

12. – Set $x_1 = h - s$, with h constant, so one will have $z_1 = 0$ from (10), and if one takes into account the fact that the curve that is the locus of points Q ($h - s$, 0, 0) is a evolvent of C then one will have:

(*) CESÀRO, *loc. cit.*, pp. 137.

If one draws lines through the points of the evolvent of a curve that are parallel to the binormals to the corresponding points of the curve than one will have a developable surface.

If one makes the same assumptions as in (20), *et seq.*, in § **8** then one will have all of the elements of the edge of regression. In particular, one will have:

$$y_1 = -\frac{r}{\rho}x_1 = -\frac{r}{\rho}(h-s).$$

If one varies h, and then x_1, then the preceding equation will be that of the rectifying line of C in the rectifying plane, so:

The edges of regression of the developables that were just defined lie on the rectifying developable of the curve and are geodetics on it.

If one makes the same assumptions as in (21), *et seq.*, then one will have all of the elements of the lines of curvature. For $k = 0$, one will have the evolvents of the curve; hence:

The evolvents of a curve are N-transforms of it, and its intrinsic equations are obtained by eliminating s from:

$$s_1 = \int\frac{h-s}{\rho}ds, \qquad \frac{1}{\rho_1} = \frac{\sqrt{\rho^2+r^2}}{r(h-s)}, \qquad \frac{1}{r_1} = \frac{\rho}{h-s}\frac{d}{ds}\left(\arctan\frac{\rho}{r}\right).$$

IV. – Curves whose tangents are orthogonal to the binormal to another curve.

13. – The formulas of the theorem in § **7** are true for any curve that corresponds to the curve C by orthogonality of elements, namely, for every curve whose points have coordinates that satisfy the first (10) of the three immobility conditions.

By virtue of an elegant theorem of BIANCHI, those formulas will take on great importance, because suitable changes of the symbols in the theorem will permit one to write the analogous formulas for *the study of curves* C_1 *that are generated by the points* M_1 (x_1, y_1, z_1) *whose coordinates satisfy the second:*

$$\frac{dy_1}{ds} - \frac{z_1}{r} = 0 \tag{24}$$

of the three immobility conditions, namely, the curve C_1 *whose tangents are orthogonal to the binormals to another C.*

BIANCHI's theorem is the following one (*):

(*) BIANCHI, *loc. cit.*, v. I, pp. 53.

For any curve C, there exists another one C′ (which is defined up to a translation) that corresponds to C with equivalence arc lengths. The two curvatures and directions of the tangent and binormal are permuted by the transformation.

More precisely:

If one chooses the positive direction of the tangents to C to be that of the binormal to C′ then the positive direction of the binormal to C′ will be that of the tangent to C, while that of the principal normals will be parallel, but with the opposite sense; one will then have:

$$\rho' = -r, \qquad r' = -\rho.$$

The importance of the theorem comes from this: It establishes a law of duality under which, when one is given any relation that links the orientation of the fundamental trihedron and the curvatures of a curve C_1 to the fundamental trihedron and the curvatures of a curve C, one can immediately deduce another one by applying the same relation to the curves C_1 and C'.

Now, if a curve C_1 has its tangents orthogonal to the binormals to C then it will correspond by orthogonality to the elements of C', and therefore the formulas of the theorem in § **7** will be applicable to it as long as one changes:

$$\rho \quad \text{into} \quad -r, \qquad r \quad \text{into} \quad -\rho, \qquad \theta \quad \text{into} \quad \pi - \theta$$

and changes the signs of c_1, γ_1, ν_1 .

Therefore:

The elementary arc length and the curvatures of a curve C_1 whose tangents are orthogonal to the binormals of C are given by the formulas:

$$\delta s_1 = f\, ds, \qquad \frac{f}{\rho_1} = \sqrt{\kappa^2 + \frac{\cos^2\theta}{r^2}}, \qquad \frac{f}{r_1} = \frac{r\kappa \sin\theta \dfrac{d\theta}{ds} + \cos\theta \dfrac{d\theta}{ds}(r\kappa)}{\cos^2\theta + r^2\kappa^2} + \frac{\sin\theta}{r},$$

resp., and the direction cosines of the tangent, binormal, and principal normal are given by the formulas:

$$a_1 = \sin\theta, \qquad b_1 = 0, \qquad c_1 = \cos\theta,$$

$$\alpha_1 = -\frac{\rho_1}{f}\frac{\cos^2\theta}{r}, \qquad \beta_1 = \frac{\rho_1}{f}\kappa, \qquad \gamma_1 = \frac{\rho_1}{f}\cdot\sin\theta\cos\theta,$$

$$\lambda_1 = -\frac{\rho_1}{f}\kappa\cos\theta, \qquad \mu_1 = -\frac{\rho_1}{f}\frac{\cos\theta}{r}, \qquad \nu_1 = \frac{\rho_1}{f}\kappa\cdot\sin\theta,$$

in which:

$$\kappa = -\frac{d\theta}{ds} + \frac{1}{\rho}.$$

The ratio f of the elementary arc lengths of the two curves and the angle θ that the tangents to C_1 make with the principal normal of C are obtained from the formulas:

$$(25) \qquad f \sin\theta = \frac{dx_1}{ds} - \frac{z_1}{\rho} + 1, \qquad f \cos\theta = \frac{dz_1}{ds} + \frac{x_1}{\rho} + \frac{y_1}{r}.$$

Indeed, it is enough to observe that:

$$\sin\theta = a_1 = \frac{\delta x_1}{\delta s_1} = \frac{1}{f}\frac{\delta x_1}{ds_1}, \qquad \cos\theta = c_1 = \frac{1}{f}\frac{\delta z_1}{ds_1},$$

and to take into account the fundamental formulas (1) when they are applied to (x_1, y_1, z_1).

14. – There is an infinitude of curves C_1, because one can assign the functions x_1, y_1 arbitrarily and then determine z_1 from (24).

Suppose that y_1 has been assigned, and then z_1, but leave x_1 arbitrary.

The infinitude of curves that correspond to the infinitude of functions x_1 cover an entire ruled surface that is generated by the tangents to C at M that go through the point R $(0, y_1, z_1)$ of the normal plane.

In order to fix one or more curves of the ruled surface, one can assign one of the two functions f and θ, along with x_1 . In particular, if one takes $\theta = \pi / 2$ in the preceding formulas then *one will get all of the elements:*

$$(26) \qquad x_1 = -\rho\left(\frac{dz_1}{ds} + \frac{y_1}{r}\right),$$

$$\delta s_1 = f\,ds, \qquad \text{in which} \qquad f = \frac{dx_1}{ds} - \frac{z_1}{\rho} + 1,$$

$$a_1 = 1, \quad b_1 = 0, \quad c_1 = 0, \quad \alpha_1 = 0, \quad \beta_1 = 1, \quad \gamma_1 = 0, \quad \lambda_1 = 0, \quad \mu_1 = 1, \quad \nu_1 = 1,$$

of a curve whose tangents are the generators of the ruled surface and is a COMBESCURE *transform of C.*

Therefore: *Those ruled surfaces are developables and have* COMBESCURE *transforms of C for their edges of regressions.*

For brevity, we shall call them *the T-developables of C.*

As in § **8**, one proves that:

Any T-developable of a curve is the envelope of a simple infinitude of planes that are parallel to the osculating planes to the curve (at a distance y_1).

The T-developables of a curve are the polar developables of its B-transform.

If one gives an arbitrary translation to the rectilinear generator of a T-developable of a curve that is parallel to the binormal to the curve then it will once more generate a T-developable.

With a change of notation, one will have from § **9** that:

The normal curvature, geodetic curvature, and geodetic torsion of a curve C_1 of a T-developable are:

$$N = -\frac{\cos\theta}{f\,r}, \qquad G = -\frac{\kappa}{f}, \qquad T = -\frac{\sin\theta}{f\,r}.$$

resp.

The lines of curvature, which are orthogonal trajectories to the generators, are obtained for $\theta = 0$; *they are N-transforms of the curve C.*

The formulas (21), *et seq.*, that were established in § **9** are still valid.

It is then easy to show that:

For a geodetic, one has:

$$\theta = \int \frac{ds}{\rho} + \tau, \qquad x_1 = \frac{1}{\cos\theta}\left[z_1 \sin\theta + \int\left(\frac{y_1}{r}\sin\theta - \cos\theta\right)ds + A\right], \tag{27}$$

in which τ and A are arbitrary constants:

$$s_1 = x_1 \sin\theta + z_1 \cos\theta + \int\left(\frac{y_1}{r}\cos\theta + \sin\theta\right)ds, \qquad \frac{f}{\rho_1} = \frac{\cos\theta}{r}, \qquad \frac{f}{r_1} = \frac{\sin\theta}{r},$$

in which:

$$f = \frac{d}{ds}(x_1 \sin\theta + z_1 \cos\theta) + \frac{y_1}{r}\cos\theta + \sin\theta,$$

$$a_1 = 0, \qquad b_1 = \sin\theta, \qquad c_1 = \cos\theta,$$

$$\alpha_1 = -\cos\theta, \qquad \beta_1 = 0, \qquad \gamma_1 = \sin\theta,$$

$$\lambda_1 = 0, \qquad \mu_1 = -1, \qquad \nu_1 = 0.$$

It follows that: *The principal normal to the geodetic is parallel to the binormal to the curve C.*

Indeed, the geodetics are the lines with zero geodetic curvature, so they will be the curves for which $\kappa = 0$, and (27) will follow from that.

If one eliminates f from (25) then one will get the following linear differential equation for the determination of x_1 :

$$\left(\frac{dx_1}{ds}-\frac{z_1}{\rho}+1\right)\cos\theta=\left(\frac{dz_1}{ds}+\frac{x_1}{\rho}+\frac{y_1}{r}\right)\sin\theta.$$

If one sets $x_1 = \xi / \cos\theta$, as in § **9**, then this will become:

$$\frac{d\xi}{ds}-\frac{z_1\cos\theta}{\rho}+\cos\theta=\left(\frac{dz_1}{ds}+\frac{y_1}{r}\right)\sin\theta,$$

namely:

$$\frac{d\xi}{ds}=\frac{d}{ds}(z_1\sin\theta)+\frac{y_1}{r}\sin\theta-\cos\theta,$$

which will imply ξ, and then x_0 .

Also from (25), one will have:

$$\begin{aligned} f &= \left(\frac{dx_1}{ds}-\frac{z_1}{\rho}+1\right)\sin\theta+\left(\frac{dz_1}{ds}+\frac{x_1}{\rho}+\frac{y_1}{r}\right)\cos\theta \\ &= \frac{d}{ds}(x_1\sin\theta+z_1\cos\theta)+\frac{y_1}{r}\cos\theta+\sin\theta, \end{aligned}$$

Finally, s_1 is obtained by integrating $\delta s_1 = f\,ds$, and the remaining formulas will be deduced immediately from those of § **13** by setting $\kappa = 0$.

Applications.

15. – Let $y_1 = k$ (constant). From (24), one will have $z_1 = 0$, and the corresponding T-developable will be described by a line of the rectifying plane to C at M that is parallel to the tangent and at a distance of k. One will then have APPELL's theorem:

The lines of the rectifying plane of a curve that are parallel to the tangent and rigidly linked with it generate developable surfaces.

If one sets $y_1 = k$, $z_1 = 0$ in (26), *et seq.*, then one will have that *the edges of regression of those developables are* COMBESCURE *transforms of the curve, and for them:*

$$x_1 = -\frac{\rho}{r}k, \qquad s_1 = s - k\frac{\rho}{r}, \qquad \rho_1 = \rho\left[1-k\frac{d}{ds}\left(\frac{\rho}{r}\right)\right], \qquad r_1 = r\left[1-k\frac{d}{ds}\left(\frac{\rho}{r}\right)\right].$$

It is clear from the expression for s_1 that:

The difference between the arc lengths of the curve C and one of the aforementioned edges, or more generally, between any two of its edges of regression, at two corresponding points is equal to the distance between the normal planes at those points.

16. – In particular, for $k = 0$, one has the developable of the tangents to the curve C. Taking $y_1 = 0$, $z_1 = 0$ in the formulas of §§ **13** and **14**, one studies the noteworthy curves on a developable surface with a given edge of regression. Here, I shall confine myself to some curves that have not been considered up to now.

As for plane curves, we call the locus of feet of the perpendiculars that go from a point to the tangents of a skew curve the *pedal curve* or *pedal with respect to that curve.*

If one is given x, y, z as the coordinates of a fixed point P with respect to the fundamental trihedron of the curve C at a point M then the pedal curve of P with respect to C will be the curve C_1, which is the locus of points M_1 $(x, 0, 0)$.

As is obvious, and as would result from the observation that the coordinates of M_1 satisfy (24), moreover, the pedal curve of a point with respect to a curve has tangents that are orthogonal to the binormals of that curve, and it is also contained in its osculating plane. The fundamental formulas of § **13** are then applicable to it.

Set $x_1 = x$, $y_1 = 0$, $z_1 = 0$, so (25) will give:

$$f \sin \theta = \frac{dx}{ds} + 1, \qquad f \cos \theta = \frac{x}{\rho},$$

but from the immobility of the point P:

$$\frac{dx}{ds} - \frac{z}{\rho} + 1 = 0,$$

so

$$f \sin \theta = \frac{z}{\rho}, \qquad f \cos \theta = \frac{x}{\rho}.$$

Therefore:

The elements of the pedal curve of a point P (x, y, z) *with respect to a curve C are given by the formulas of theorem of* § **13**, *in which f and* θ *are the functions that are defined by the formulas:*

$$\tan \theta = \frac{z}{x}, \qquad f = \frac{\sqrt{x^2 + z^2}}{\rho}. \tag{28}$$

The tangent to C_1 is contained in the osculating plane to C at M and makes an angle θ with the principal normal to C. However, it follows from the first of (28) that θ is also the angle that the projection P of MP onto the osculating plane to C makes with the tangent, so:

The tangent to the pedal curve at a point M_1 and the projection of the ray MP onto the osculating plane to C at M are inclined equally with respect to the principal normal and the tangent to C, respectively.

It then follows that:

The normals to the pedal curve at M_1 and the projection of MP onto the osculating plane to C at M are anti-parallel with respect to the tangent to C at M.

Hence:

The normals to the pedal curve that are contained in the osculating plane intersect at the midpoint of the projection of a segment MP on the osculating plane, and therefore: *The normal plane to the pedal curve at a point M_1 intersects MP at its midpoint and the principal normal to C at the projection of P onto that line.*

A simple inspection of the figure will imply that:

The distance between two corresponding points M and M_1 of the curve and the pedal curve is the proportional mean of the segments that the tangents and the normal plane to the pedal curve cut out on the principal normal to the curve.

Recalling the direction cosines of the binormal and the principal normal to the pedal curve (§ **13**), one has that the equations of the osculating plane and the rectifying plane of the pedal curve are:

$$-(X-x)\frac{\cos^2\theta}{r}+Y\kappa+Z\,\frac{\sin\theta\cos\theta}{r}=0,$$

$$-(X-x)\,\kappa\cos\,\theta-Y\,\frac{\cos\theta}{r}+Z\kappa\sin\,\theta=0,$$

respectively. Hence, the pieces that this plane cuts out on the binormal to the curve *C* are $-x\cos^2\theta/r\,\kappa$; $x\,r\,\kappa$; whose product is $-x^2\cos^2\theta$, so:

In absolute value, the projection of MM_1 onto p is the proportional mean of the segments that the osculating plane and the rectifying plane at M_1 on the pedal curve cut out from the binormal to the curve at M.

17. – Some particular *T*-developables of the curve are the cones that are obtained when one draws the parallels to the tangents of that curve that go through a fixed point – viz., *the direction cone of the tangents to the curve.*

Indeed, if x_1, y_1, z_1 are the coordinates of a fixed point *P* then the immobility conditions will be true:

(28) $$\frac{dx}{ds} - \frac{z_1}{\rho} + 1 = 0, \qquad \frac{dy_1}{ds} - \frac{z_1}{r} = 0, \qquad \frac{dz_1}{ds} + \frac{x}{\rho} + \frac{y_1}{r} = 0,$$

the second of which is nothing but (24), and it is also satisfied by the coordinates:

$$x_1 = x + t, \qquad y_1, \qquad z_1$$

of any other point of the cone.

The study of the curves of one such cone is equivalent to the study of the curves of a cone that refers to a curve with tangents that are parallel to the rectilinear generators of the cone, i.e., to that thing that is intrinsic to only that cone, where one usually refers that study to a particular curve on that cone.

The general results of §§ **13** and **14** are valid for those curves.

The functions f and θ for any curve on the cone are obtained from the formulas:

(29) $$f \sin \theta = \frac{dt}{ds}, \qquad f \cos \theta = \frac{t}{\rho},$$

which are deduced from (25) by applying it to the point M_1 $(x + t, y_1, z_1)$ and taking (28) into account.

The lines of curvature, or the orthogonal trajectories to the generators of the cone, are obtained by drawing a constant segment t along the generators upon starting from the vertex, and one obtains their intrinsic equations by eliminating s from:

$$s_1 = t \int \frac{ds}{\rho}, \qquad \frac{1}{\rho_1} = \frac{\sqrt{\rho^2 + r^2}}{t\,r}, \qquad \frac{1}{r_1} = \frac{\rho}{t} \frac{d}{ds}\left(\arctan \frac{\rho}{\tau}\right).$$

Indeed, one will have $\theta = 0$ for the lines of curvature of a T-developable, so from (29):

$$t = \text{constant} \quad \text{and} \quad f = \frac{t}{\rho};$$

if one substitutes this in (21), *et seq.*, then one will get the formulas that were written down.

One will have that $1 / r_1 = 0$ only when ρ / r is constant, so ρ_1 will also be constant then – i.e., the lines of curvature will be circles: hence:

A characteristic property of a cylindrical helix is that the director cone of its tangents is circular.

The geodetics on a cone are obtained by starting at the vertex and measuring out the segment t = $A / \cos \theta$, *in which:*

$$\theta = \int \frac{ds}{\rho} + \tau,$$

with A and τ arbitrary constants; their intrinsic equations are obtained by eliminating s from:

$$(30) \qquad s_1 = A \tan \theta, \qquad \rho_1 = \frac{Ar}{\rho \cos^3 \theta}, \qquad r_1 = \frac{Ar}{\rho \sin\theta \cos^2 \theta}.$$

The inclination of a geodetic with respect to the generators is $\varphi = \pi / 2 - \theta$.

Indeed, for the geodetics of a T-developable of the curve C, one must set $\kappa = \frac{1}{\rho} - \frac{d\theta}{ds}$ = 0, from which, one gets θ by a quadrature. In addition to (29), one has:

$$\frac{dt}{t} = \frac{ds}{\rho} \tan \theta = \tan \theta \, d\theta,$$

from which, $t = A / \cos \theta$, with A an arbitrary constant. Hence:

$$f = \frac{dt}{ds} \sin \theta + \frac{t}{\rho} \cos \theta = \frac{d}{ds}(t \sin \theta) = A \frac{d}{ds} \tan \theta = \frac{A}{\rho \cos^2 \theta},$$

so

$$s_1 = \int f \, ds = A \tan \theta.$$

Finally, ρ_1 and r_1 are deduced from the general formulas of § **13** by setting $\kappa = 0$ and $f = A / \rho \cos^2 \theta$.

If one lets $t = A / \sin \varphi$ then it will follow that:

If one develops the cone onto the plane then its geodetics will rectify.

It also follows that A is the distance from the vertex of the cone to any tangent to a geodetic, so:

The vertex of a cone is equidistant from all of the tangents to one of its geodetics; i.e.: *The pedal curve of the vertex of a cone with respect to one of its geodetics is a spherical curve.*

If one sets $s_1 = A \tan \theta = A \cot \varphi$ then one will have that:

The length of the arc of a geodetic on a cone is the distance from the endpoint of the arc to the projection of the vertex of the cone onto the tangent at that extreme; i.e.: *The pedal curve to the vertex of a cone with respect to one of its geodetics is an evolvent of it.*

It follows easily from (30) that:

$$s_1 \, r_1 = A \, \rho_1 .$$

Conversely, it follows from that relation that the point $(-s_1, A, 0)$ satisfies the three immobility conditions with respect to the curve C_1, namely, that the rectifying planes of C_1 coincide at a point – i.e., they envelop a cone on which C_1 is a geodetic (because any curve is geodetic on its rectifying developable). Therefore:

A characteristic property of the geodetics of a cone is that their torsions vary like the product of the flexion with the arc length.

If one supposes that θ is constant then one will get a helix on the cone. One easily finds that:

The arc length and curvatures of a helix on a cone that meets the generators at an angle of θ are given by the formulas:

$$s_1 = \frac{t}{\sin\theta}, \qquad \frac{f}{\rho_1} = \sqrt{\frac{1}{r^2} + \frac{\sin^2\varphi}{\rho^2}}, \qquad \frac{f}{r_1} = -\frac{\rho^2 \sin\varphi \dfrac{d}{ds}\left(\dfrac{r}{\rho}\right)}{r^2 + \rho^2 \sin^2\varphi} - \frac{\cos\varphi}{r},$$

in which:

$$f = \frac{t}{\rho\sin\varphi}, \qquad t = h\exp\left(\cot\varphi\int\frac{ds}{\rho}\right),$$

and h is a constant.

Indeed, (29) gives:

$$\frac{dt}{t} = \frac{ds}{\rho}\tan\theta,$$

from which the value that was written down for t will follow when one lets the inclination of the helix with respect to its generators be $\varphi = \pi / 2 - \theta$. Therefore, from (29) itself and the preceding, one will have:

$$f = \frac{dt}{ds}\sin\theta + \frac{t}{\rho}\cos\theta = \frac{t}{\rho\sin\varphi'},$$

from which:

$$s_1 = \int f\,ds = \frac{1}{\sin\theta}\int\frac{t\,ds}{\rho} = \frac{1}{\sin\theta}\int dt = \frac{t}{\sin\varphi}.$$

Finally, ρ_1 and r_1 are obtained from the general formulas of § **13** when one sets:

$$\theta = \frac{\pi}{2} - \varphi, \qquad \frac{d\theta}{ds} = 0, \qquad \kappa = \frac{1}{\rho}.$$

One can write:

$$t = h\exp(\sigma\cot\varphi),$$

in which σ is the arc length of the spherical index of the generator of the cone, so:

If one develops a cone onto the plane then its helices will become logarithmic spirals.

It follows from the expression for s_1 that:

The arc length of a helix on a cone (as it is for a geodetic) is equal to the segment of the tangent that goes through the endpoint of the arc and is found between that endpoint and the projection of the vertex of the cone onto its tangent; i.e.: *The pedal curve of the vertex of a cone with respect to one of its helices is one of its evolvents.*

To conclude, consider the curve on the cone that one obtains for $t = -2x$ – i.e., the curve that is the locus of points $(-x_1, y_1, z_1)$ – *which is the curve that is described by the image of a light point P that is fixed with respect to a moving mirror* (viz., the normal plane to *C*).

The elements of that curve are given by the general formulas of § **13** if one sets:

$$\tan\theta = \frac{z_1 - \rho}{x}, \qquad f = \frac{2}{\rho}\sqrt{x^2 + (z_1 - \rho)^2}$$

in them.

Those values of f and θ will be deduced from (29) if one sets $t = -x$ and takes the first of (28) into account.

V. – Plane curves.

18. – None of the preceding results will cease to be valid in the case of a plane curve *C*; the formulas will also become somewhat simpler, since one will have $1 / r = 0$ then.

The *B*-developables are cylinders, and the *T*-developables are planes that are parallel to the plane of the curve.

19. – One will get much simpler and much more interesting results when one considers only those infinitesimal deformations for which the curve is not in its plane.

If we would like to apply all of the preceding results *without leaving the plane of the curve* then we would have to suppose that:

$$\frac{1}{r} = 0, \quad v = 0, \quad y = y_1 = y_2 = 0.$$

In order to perform an infinitesimal deformation, the generic point *M* will be subjected to a displacement whose components are $\varepsilon\, u$, $\varepsilon\, w$ along the tangent and the normal to the curve *C* at *M*.

The theorem in § **3** becomes:

The necessary and sufficient condition for the functions u, w to define an infinitesimal deformation of the curve C is that:

$$\frac{du}{ds} - \frac{w}{\rho} = 0. \tag{31}$$

The intrinsic equation of the deformed curve C′ is:

$$\rho' = \rho + \varepsilon \frac{dm}{ds}, \tag{32}$$

in which:

$$-m = \frac{dw}{ds} + \frac{u}{\rho}. \tag{33}$$

Each element of the curve will submit to a translation whose components are εu, εv, *and a rotation of moment* εm.

For the particular deformations that were considered in § **4**, one will have:

a) *The deformations that do not alter the curvature of the curve are rigid motions.*

That should be obvious, but it will also result directly from the preceding formulas. Indeed, $\rho' = \rho$ only when m is constant; with that hypothesis, if one sets:

$$x' = -\frac{w}{m}, \qquad y' = \frac{u}{m}$$

then (31) and (33) will become:

$$\frac{dx'}{ds} - \frac{y'}{\rho} + 1 = 0, \qquad \frac{dy'}{ds} + \frac{x'}{\rho} = 0,$$

resp.

Now, these express (*) the fact that the point (x', y') is immobile, so:

The deformation is not an infinitesimal rotation, and the point (x', y') *is the instantaneous center of rotation.*

b) *There exists no infinitesimal deformation for which any point is displaced normally to the curve.*

c) *There always exists an infinitesimal deformation for which any point is displaced in a given direction in the plane, but is not normal to the curve.*

(*) CESÀRO, *loc. cit.*, pp. 20.

d) *There exists an infinitesimal deformation for which any point is displaced along the tangent to the curve and the displacement is constant.*

The circles are preserved by that deformation; the circles are also the only plane curves that admit infinitesimal deformations into themselves.

20. – If one is given a fixed point O and one draws a vector OM_1 that is equal to the displacement MM' that M experiences under an infinitesimal deformation of the curve C then the curve C_1 that is the locus of points M_1 will correspond to C with orthogonality of elements, and conversely, as in § **6**. The problem of the infinitesimal deformations then takes on one initial aspect: One studies the curve whose points M_1 (x_1, z_1) satisfy the first of the two immobility conditions:

$$\frac{dx_1}{ds} - \frac{z_1}{\rho} + 1 = 0. \tag{34}$$

If one sets $1 / r = 0$, $y_1 = 0$ in (11) then one will have:

$$\theta = 0, \qquad f = \frac{dz_1}{ds} + \frac{x_1}{\rho}, \tag{35}$$

and the theorem of § **7** will become:

The elementary arc length and the curvature of a curve C_1 that corresponds to C by orthogonality of elements are given by the formulas:

$$\delta s_1 = f\, ds, \qquad \rho_1 = f\, \rho, \tag{36}$$

resp.

For $x_1 = 0$, (34) and (35) will give $z_1 = \rho$, $f = d\rho / ds$, so:

The arc length and the radius of curvature of the evolute of a curve are:

$$s_1 = \rho, \qquad \rho_1 = \rho \frac{d\rho}{ds},$$

resp.

One will also have $z_1 = \rho$ for $x = k$ (a constant), and the corresponding curve will be a curve that is *parallel* to the evolute of C.

For $x_1 = h - s$, with a constant, (34) will give $z_1 = 0$, and the curve C_1 will be an evolvent of C. From (35) and (36), one will have that:

The arc length and the curvature of an evolvent of a curve are:

$$s_1 = \int \frac{h-s}{\rho} ds, \qquad \rho_1 = h - s,$$

resp.

21. – Passing on to § **13**, *et seq.*, *all of the curves in the plane have tangents that are orthogonal to the binormals to C*, i.e., the coordinates (x_1, z_1) of the point M_1 are not constrained by any rule. The theorem of § **13** becomes:

If x_1 and z_1 are two functions that are given arbitrarily then the arc length and the curvature of the curve C_1 that is the locus of points M_1 (x_1, y_1) are given by the formulas:

$$s_1 = \int f\,ds, \quad \frac{f}{\rho_1} = \frac{1}{\rho} - \frac{d\theta}{ds}, \tag{36}$$

resp., in which f and θ are given by the formulas:

$$f \sin\theta = \frac{dx_1}{ds} - \frac{z_1}{\rho} + 1, \qquad f \cos\theta = \frac{dz_1}{ds} + \frac{x_1}{r};$$

θ is the inclination of the tangent to C with respect to the normal to C.

These are the formulas for the study of the geometric loci by the intrinsic method (*).

22. – From § **13**, one has that *the arc length and the curvature of the pedal curve to a point P* (x, z) *in a plane with respect to a curve are given by formulas* (36), *in which:*

$$\tan\theta = \frac{z}{x}, \qquad f = \frac{\sqrt{x^2 + z^2}}{\rho};$$

otherwise, they will be given by the formulas:

$$s_1 = \int \frac{t}{\rho}\,ds, \quad \rho_1 = \frac{t^2}{2t - \rho\cos\theta},$$

in which $t = \sqrt{x^2 + z^2} = MP$.

Indeed, $x = t \sin\theta$, $z = t \cos\theta$, so:

$$\frac{d\theta}{ds} = -\frac{1}{\rho} + \frac{\cos\theta}{t}, \qquad f = \frac{1}{\rho}, \text{ etc.}$$

The tangent to the pedal curve at M_1 and the ray MP are inclined equally with respect to the normal and the tangent to C at M, respectively. Therefore, the ray MP and the normal to the pedal curve are anti-parallel with respect to the tangent to C at M.

The normal to the pedal curve at M_1 bisects the segment MP.

(*) CESÀRO, *loc. cit.*, pp. 22.

Morcone, September 1905.

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