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Solving the Dirac equations with no specialization of the Dirac operators. II

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As a continuation of a previous investigation by the author ([1]) into the solution of the Dirac equations without the introduction of special assumptions on the γ_ν, the behavior of the quadratic expressions in the ψ that they yield will be discussed.

§ 1. – In a recently-appearing paper ([1]), the author gave a method that allowed one to solve the **Dirac** equations without requiring a special representation for the operators γ_ν that appear in those equations. Those operators are regarded as a higher type of complex unit that must satisfy the relations:

$$\gamma_\nu \, \gamma_\mu + \gamma_\mu \, \gamma_\nu = 2 \, \delta_{\mu\nu} \, ; \tag{1}$$

The question of their true nature remains completely untouched in that.

As was done in I, using the Ansatz that the desired function ψ is a linear combination of the sixteen linearly-independent operators that were given in I (2), any Dirac equation can be converted into a system of sixteen differential equations for the sixteen coefficients in the Ansatz whose solution will give one the eigenvalues and eigenfunctions of the problem. It was shown there that any solution of a Dirac equation will remain a solution when one multiplies it on the right (the transposed equation on the left, resp.) by an arbitrary constant **Dirac** operator ([2]) Γ that can be regarded as the integration constant of the homogeneous Dirac equation ([3]).

However, the arguments in I, which dealt mainly with finding the eigenvalues and eigenfunctions of a Dirac equation, can be considered to be incomplete as long as they

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([1]) **F. Sauter**, Zeit. Phys. **63** (1930), 803; cited as I in what follows.

([2]) Here, and in what follows, when we speak of an arbitrary operator, one should understand that to mean a linear combination of the sixteen linearly-independent operators of the system (2) that was given in I. The sixteen arbitrary coefficients that appear in the linear combination will be referred to as its coefficients.

([3]) The appearance of Γ is naturally connected with the fact that a certain arbitrariness exists in the integration constants that appear in the solution of the sixteen equations when they are free of operators. However, the multiply-infinite manifold of solutions that are obtained in that way can be derived from one basic solution for the free electron and two for the Kepler problem by multiplying by the arbitrary operator Γ. (The basic solutions were referred to as essential solutions in I.)

are not extended to an investigation of the behavior of the quantities that are quadratic in ψ, such as $\bar{\psi}\gamma_\nu\psi$, etc. Nonetheless, it is precisely those quantities that play an important role in most wave-mechanical problems (e.g., the calculation of perturbations, the presentation of selection rules and intensity rules).

The present efforts shall then deal with those quadratic quantities. It will be shown that solutions ψ of a Dirac equation that are found by the method was given in I must fulfill a certain condition in order for those quantities to be physically interpretable (§ **2**, **3**). As will be shown in § **4**, that condition can always be fulfilled by a suitable choice of integration constants Γ. A remarkable connection will be obtained from it between the solutions that are obtained with no specialization of the γ_ν and the solutions that one will be led to when one introduces four-rowed matrices for the γ_ν, as **Dirac** did (§ **5**, **6**).

§ 2. – In order to understand the difficulties that appear in connection with the method of solution that was given in I for the evaluation of the quantities that are quadratic in ψ, one might first show that those quantities never contain *c*-numbers in the **Dirac** sense, but always operators γ_ν.

In order to prove that, it is advantageous to introduce the concept of a "null operator." One will arrive at it when one seeks to find an operator γ^{-1} that is "inverse" to an arbitrarily-given Dirac operator γ, which will be defined by the equation:

$$\gamma\gamma^{-1} = \gamma^{-1}\gamma = 1. \tag{2}$$

If one lets γ^{-1} be general then one will get an inhomogeneous system of sixteen ordinary differential equations for its sixteen coefficients that will possess a solution only when its determinant, which is constructed from the coefficients of the γ, does not vanish.

However, if that determinant is equal to zero then the inhomogeneous system cannot be solved, but the corresponding homogeneous system of equations probably can, and therefore the equation:

$$\gamma\bar{\gamma} = \bar{\gamma}\gamma = 0. \tag{3}$$

Such an operator γ – for which (3) can be solved, but not (2), and will thus possess no inverse operator – shall be referred to as a *null operator*. $\bar{\gamma}$, which is naturally a null operator, as well, shall be referred to as the null operator that is associated with γ. One easily convinces oneself that the product of a null operator with an arbitrary operator will once more represent a null operator.

It emerges from the foregoing that any solution of a Dirac equation must include at least one null operator as a factor. One considers, e.g., the case of the electron in field-free space that was treated in I. By way of the Ansatz:

$$\psi = u \exp \frac{2\pi i}{h}\sum_{\nu=1}^{4} p_\nu\, x_\nu ,$$

one will get the equation $\gamma u = 0$ for u, with:

$$\gamma = \sum_{\nu=1}^{4} p_\nu \, x_\nu + i\, mc,$$

from which it will follow that γ is a null operator, and u must be the null operator that is associated with it. The first condition leads to the known relativistic relationship between the impulses p_ν .

One can conclude analogously for an arbitrary Dirac equation that its solution must contain at least one null operator. Each such equation can then be regarded as a product of two expressions that contain γ_ν's. Should that product vanish then the latter would have to contain at least one null operator as a factor. From the definition of the null operator, a certain determinant that is constructed from its coefficients would have to vanish. If the null operator were not constant then, as a result of the criterion, a determinant that depends upon the coordinates would have to vanish, and therefore a relationship between the coordinates would exist, which is naturally impossible.

If one now forms a product of the form $\bar{\psi}\gamma_\nu\psi$ or $\bar{\psi}\gamma_\mu\gamma_\nu\psi$ then it will follow from the definition of a null operator that this product can never be a pure number (except possibly 0), since otherwise it would always have to contain a (null) operator, as was stated above.

§ 3. – The fact that the appearance of the γ_ν in the quadratic expressions will complicate the physical interpretation of those quantities [as components of the current (moment, resp.) vector, etc.] probably does not need to be stressed particularly.

In conjunction with that, however, it should be remarked expressly that one cannot normalize the wave functions to 1 now, but only to a constant operator Γ_0 :

$$\int \bar{\psi}\gamma_4\psi \, d\tau = \Gamma_0 \, . \tag{4}$$

Γ_0 shall be called the *normalization operator;* naturally, it contains the null operators that appear in ψ and $\bar{\psi}$.

If one would like to eliminate that difficulty with one stroke then I see no other way than to assert the postulate: "The integration constants Γ, which were previously left undetermined, are chosen in such a way that all of the quadratic expressions will exhibit the same form: namely, products of a constant operator (~ the normalization operator Γ_0) that is common to all of them with an expression that is free of operators and depends upon the coordinates." In the next paragraph, it will be shown that this postulate can always be fulfilled for an arbitrary problem.

Here, it might first be shown how fulfilling that postulate can remove all of the cited difficulties. Hence, e.g., the expressions $\bar{\psi}\gamma_\nu\psi$, which are ordinarily interpreted as components of the current vectors, are written in the form:

$$\bar{\psi}\gamma_\nu\psi = \Gamma_0 \cdot s_\nu \, , \tag{5}$$

in which the s_ν are pure numbers. Now, there is obviously nothing standing in the way of defining the s_ν to be the current components, in place of the $\bar{\psi}\gamma_\nu\psi$, since they indeed satisfy the divergence condition, just like to the $\bar{\psi}\gamma_\nu\psi$. Things are analogous when one interprets the moments of the electron by means of the equation:

$$\bar{\psi}\gamma_\mu\gamma_\nu\psi = \Gamma_0 \cdot m_{\mu\nu}\,. \tag{6}$$

Due to (5), the normalization on the basis of (4) will obviously yield:

$$\int \bar{\psi}\gamma_4\psi\,d\tau = \Gamma_0 \int s_4\,d\tau = \Gamma_0\,, \tag{4a}$$

or

$$\int s_4\,d\tau = 1, \tag{4b}$$

resp.

§ 4. – In this paragraph, it shall be proved that for any problem, by a suitable choice of integration constants Γ, it will be possible to bring all quadratic expressions into the form: constant operator Γ_0 times an operator-free function of the coordinates.

A consideration of the solutions of the Kepler problem will show how that can be achieved. In that problem, the condition that was posed in § **3** will be fulfilled from the outset only when the Γ represent the same operator (up to a pure number that appears multiplicatively) for all solutions, which can be set to 1, with no loss of generality. The solutions of that problem were given in I (12) and (12a). It is characteristic of them that the null operators $(1 + i\,\gamma_1\,\gamma_2)$ and $(1 + \gamma_4)$ will appear, with which the solutions for ψ will inevitably be multiplied on the right and the solutions for $\bar{\psi}$, on the left ([1]).

By the construction of the expression $\bar{\psi}\gamma\psi$, in which γ might represent any arbitrary **Dirac** operator, one will then obtain a product of the form:

$$\bar{\Gamma}(1 + i\,\gamma_1\,\gamma_2)\,(1 + \gamma_4)\cdot\beta(1 + i\,\gamma_1\,\gamma_2)\,(1 + \gamma_4)\,\Gamma. \tag{7}$$

In this, β means the operator that one gets from γ upon multiplying it with the components of ψ and $\bar{\psi}$ that are independent of the coordinates. One easily convinces oneself that (7) can always be brought into the form:

$$\bar{\Gamma}(1 + i\,\gamma_1\,\gamma_2)\,(1 + \gamma_4)\,\Gamma\cdot b = \Gamma_0\,b, \tag{7a}$$

in which b is a pure number, and Γ_0 is the normalization operator. Namely, if one generally writes β as a linear combination of the sixteen operators of the system (2) in I then one will see that twelve of the sixteen terms will vanish when one performs the

([1]) The two null operators define an essential component of the solutions and cannot therefore be absorbed into the integration constants Γ.

multiplication, due to the null operators, while the rest of the terms with the operators 1, γ_4 , $\gamma_1\,\gamma_2$, $\gamma_1\,\gamma_2\,\gamma_4$ will lead to the expression (7a); one will then have:

$$\left.\begin{aligned}\gamma_4(1+\gamma_4) &= (1+\gamma_4),\\ \gamma_1\,\gamma_2(1+i\,\gamma_1\,\gamma_2) &= -i\,(1+i\,\gamma_1\,\gamma_2).\end{aligned}\right\} \tag{8}$$

The foregoing presentation shows the way to the fulfillment of the condition that was posed in § **3** for an arbitrary problem. If one writes the integration constant Γ in the form:

$$\Gamma = \delta(1 + i\,\gamma_1\,\gamma_2)\,(1 + \gamma_4), \tag{9}$$

with the arbitrary operator δ, then the construction of the quadratic expressions that correspond to (7) and (7a) will yield the product of a constant operator with a function of the coordinates that is free of operators, precisely as it did in the problem of the electron in a central field.

The Ansatz (9) can be simplified, insofar as the general sixteen-term Ansatz for δ in (9) will reduce to four terms, due to (8):

$$\Gamma = (a + b\,\gamma_1 + c\,\gamma_3 + d\,\gamma_1\,\gamma_3)(1 + i\,\gamma_1\,\gamma_2)(1 + \gamma_4)\,; \tag{9a}$$

in this, a, b, c, d are constant pure numbers.

If one introduces (9a) into the solution ψ of the problem then one will get four physically-possible solutions in connection with each basic solution of the problem, corresponding to the four arbitrary quantities in (9a); a solution might be referred to as physically-possible in that when it satisfies the postulate above. One can easily convince oneself that those four solutions can generally be derived from linear combinations of two linearly-independent solutions, and in special cases, one of them, when one considers that any basic solution that should be multiplied by (9a) must include at least one null operator that will effect that reduction.

If one applies the substitution (9a) to the Kepler problem then that will naturally show that the three terms with b, c, d will vanish, and only the one with a will remain. One will then get the two essentially-distinct basic solutions that were given in I (12) as physically-possible solutions.

The introduction of (9a) into the solution for the free electron, which possesses only one basic solution, will yield two linearly-independent, physically-possible solutions. As a peculiarity of that, one might cite the fact that as long as one restricts oneself to the construction of the quadratic quantities that are the components of the current vector, the basic solution will already fulfill the condition that was posed in § **3** by itself. It is only when one makes the transition to the moments $m_{\mu\nu}$ that the introduction of the substitution (9a) will be necessary, and with it, the transition from one to two systems of solutions.

The question now arises of whether one has exhausted all possibilities for the fulfillment of the postulate by applying the Ansatz (9a). One can assume that one will obtain essentially different solutions by the introduction of another Ansatz, such as:

$$\Gamma = (a' + b' \gamma_4 + c' \gamma_3 + d' \gamma_4 \gamma_3)(1 + i \gamma_1 \gamma_3)(1 + \gamma_2), \tag{9b}$$

and therefore different current and moment vectors. However, the operator (9b) can always be derived from (9a) by right-multiplication by $(1 + i \gamma_1 \gamma_3)(1 + \gamma_2)$, such that the a' to d' can then be obtained as linear combinations of the a to d. No essential change in the system of possible solutions will be produced in that way.

§ 5. – On the basis of what was done in the last paragraph, in order to fulfill the requirement that was posed in § **3** on the structure of the forms that are quadratic in ψ from the outset, one does not generally need to make $\bar{\psi}$ the sum of sixteen terms, as was done in I, but it can take the special form:

$$\psi = (f_1 + f_2 \gamma_1 + f_3 \gamma_3 + f_4 \gamma_1 \gamma_3)(1 + i \gamma_1 \gamma_2)(1 + \gamma_4), \tag{10}$$

in which the f_i are operator-free functions that are determined from the differential equation. As was shown in the conclusion of the last paragraph, (10) can be regarded as the most general Ansatz for ψ.

A connection was suggested in I in conjunction with the solutions of the Kepler problem between those solutions and the ones to which one would be led by the use of four-rowed matrices in place of the γ_ν. For example, with the matrices that **Sommerfeld** employed ([1]):

$$\gamma_1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \quad \gamma_2 = \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \quad \gamma_3 = \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{vmatrix}, \quad \gamma_4 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \tag{11}$$

one will get the matrix:

$$(1 + i \gamma_1 \gamma_2)(1 + \gamma_4) = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \tag{12}$$

for the product of the two null operators in (10), and the expression:

$$\psi = 4 \begin{vmatrix} 0 & -i f_4 & 0 & 0 \\ 0 & f_1 & 0 & 0 \\ 0 & -i f_2 & 0 & 0 \\ 0 & -i f_3 & 0 & 0 \end{vmatrix} \tag{13}$$

([1]) **A. Sommerfeld**, *Wellenmechanischer Ergänzungsband*, Chap. II, § **10**.

for ψ. Just like in I, the four functions in the non-vanishing column of (13) correspond to precisely the four **Dirac** ψ-functions to which one will be led when one works out the problem on the basis of the matrix calculus.

Here, it shall be shown that a similar connection always exists between the two methods of solutions that is completely independent of the special choice of the type of representation for the γ_ν. It was shown in the course of § **4** that (10) can be regarded as the most general Ansatz for ψ. If one introduces that Ansatz into a Dirac equation:

$$\left.\begin{aligned} &\{p_1\,\gamma_1 + p_2\,\gamma_2 + p_3\,\gamma_3 + p_4\,\gamma_4 + i\,mc\}\psi = 0, \\ &\left(p_\nu = \frac{h}{2\pi i}\frac{\partial}{\partial x^\nu} + \frac{e}{c}A_\nu\right) \end{aligned}\right\} \tag{14}$$

and combines the terms with the same operators together then that will yield the four operator-free differential equations:

$$\left.\begin{aligned} (p_1 - i\,p_2)\,f_1 - p_3\,f_4 + (-p_4 + p_0)\,f_2 &= 0, \\ (p_1 + i\,p_2)\,f_2 + p_3\,f_3 + \quad (p_4 + p_0)\,f_1 &= 0, \\ (p_1 - i\,p_2)\,f_3 - p_3\,f_2 + \quad (p_4 + p_0)\,f_4 &= 0, \\ (p_1 + i\,p_2)\,f_4 + p_3\,f_1 + (-p_4 + p_0)\,f_3 &= 0, \end{aligned}\right\} \tag{15}$$

for the four functions f_i , and they will be independent of the meaning of γ_ν .

The system of equations (15) exhibits the same structure as the system that one will obtain from (14) when one employs the matrix calculus. It will coincide with it precisely only when one associates the f_i with the four Dirac functions in a suitable way. One immediately convinces oneself that one will likewise arrive at the system of equations (15) when one identifies the γ_ν that appear in (14) with, e.g., the four-matrices in (11) and sets:

$$\psi_1 = -\,i\,f_4\,, \qquad \psi_2 = f_1\,, \qquad \psi_3 = -\,f_2\,, \qquad \psi_4 = -\,i\,f_3\,. \tag{16}$$

Using other matrices than (11) would demand only a different association of the four ψ_i with the four f_i .

§ 6. – This result can then be summarized by saying that the four functions f_i that appear in the general Ansatz (10) for ψ would obey precisely the same rules of computation in the further calculations as if they represented the four Dirac functions (with a suitable ordering) and one worked out the problem with the help of four-rowed matrices for the γ_ν .

One can regard that result, to a certain degree, as the statement that the introduction of the postulate in § **3** is justified and compulsory, just like the argument in § **4** that led to the Ansatz (10). On the other hand, that shows that the **Dirac** method of employing four-rowed matrices can already be regarded as the most general method, since indeed the method of solution that leaves the γ_ν completely undetermined will always lead back to the **Dirac** method.

In conclusion, I would like to go briefly into the question of whether it is more preferable to work through a wave-mechanical problem while respecting the generality of the γ_ν or to employ the matrix calculus; in other words, whether it is more advantageous to introduce the substitution (9) [(9a), resp.] for the integration constants that was required by the postulate in § **3** only at the conclusion of the calculations or directly at the beginning, and therefore to start from the Ansatz (10).

As an advantage of the latter method, one can cite the fact that one works from the outset only with equations that are free of the γ_ν . On the other hand, one must consider that the consistent use of that method will increase only the scope of the work done in a problem, but in no way its clarity and intuitiveness. That is because, first of all, one must replace an operator equation with a system of four operator-free equations and replace a quadratic quantity $\bar{\psi}\gamma\psi$ with a sum of four terms in the four functions f_i in (10), which are not at all constructed symmetrically in the four f_i . Moreover, one does easily see whether the components of such a quadratic expression in the four f_i represent the current or moment vector, while that is not the case with the notations $\bar{\psi}\gamma_\nu\psi$ [$\bar{\psi}\gamma_\mu\gamma_\nu\psi$, resp.] with no further assumptions.

In particular, as long as general calculations such as perturbation calculations of conversions of quadratic expressions are carried out, it will certainly serve to elevate their clarity and simplicity when one proceeds while respecting the complete generality of the γ_ν and therefore the complete symmetry in the coordinates, as well. Even in the working out of special problems in which one must deal with the search for explicit expressions for the ψ, it will generally seem preferable to me to not proceed in connection with (10) and to first introduce the substitution (9) [(9a), resp.] at a later point ([1]).

I would like to thank the Österreichisch-Deutschen Wissenschaftshilfe for having granted me a stipend.

([1]) I would like to mention here, moreover, that it is possible to find the solution to the Kepler problem by a somewhat simpler method than the one in I, and which is most closely linked with the one that **Sommerfeld** employed (*loc. cit.*) with the use of the matrix calculus. It is largely analogous to that of **G. Temple** [Proc. Roy. Soc. London (A) **127** (1930), 349], but it will also lead to the explicit expressions for the dependency on the polar angles that were not given there.