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The three-dimensional Cosserat continuum and the Cosserat shell in the calculus of differential forms (*)

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1. The parallel transport of a motor

It is known that a force-system on a rigid body can be reduced to a point P and represented by a single force vector $\mathbf{F}(P)$ whose line of action goes through P and a moment vector $\mathbf{M}(P)$. If one chooses another reduction point Q then one will have the *transport law:*

$$\begin{aligned} \mathbf{F}(Q) &= \mathbf{F}(P), \\ \mathbf{M}(Q) &= \mathbf{M}(P) + \mathbf{F} \times \overrightarrow{Q-P}. \end{aligned} \tag{1.1}$$

($\times$ is the symbol of vectorial multiplication.)

We call the two representations of the force system at P and Q *equivalent* or *equipollent.*

The same law of transport is true for the infinitesimal displacement state of a rigid body. Infinitesimal rotations and translations can be described completely when one is given a point P of the body, the vector $\boldsymbol{\varphi}(P)$ of infinitesimal rotation, and the vector $\mathbf{u}(P)$ of infinitesimal translation. For a different choice of point Q on the body, one will have the transport law:

$$\begin{aligned} \boldsymbol{\varphi}(Q) &= \boldsymbol{\varphi}(P), \\ \mathbf{u}(Q) &= \mathbf{u}(P) + \boldsymbol{\varphi} \times \overrightarrow{Q-P}. \end{aligned} \tag{1.2}$$

The pairs of vectors $\mathbf{F}$, $\mathbf{M}$ or $\boldsymbol{\varphi}$, $\mathbf{u}$ are called *screws*, and were originally called *Dynamen*. STUDY [**1**] and v. MISES [**2**] have introduced the concept of *motor* for vector-pairs that satisfy the transport law (1.1) or (1.2). A motor $\mathbf{V}$ is then composed of two vectors $\mathbf{v}$ and $\bar{\mathbf{v}}$:

$$\mathbf{V} = \begin{pmatrix} \mathbf{v} \\ \bar{\mathbf{v}} \end{pmatrix}, \tag{1.3}$$

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in which the second vector is the moment vector of **V**.

If the reduction points P and Q of a motor **V** are separated by the infinitesimal distance $d\mathbf{V}$ then the transport law will read:

(1.4)
$$\mathbf{v}\,(\mathbf{r} + d\mathbf{r}) = \mathbf{v}\,(\mathbf{r}),$$
$$\overline{\mathbf{v}}\,(\mathbf{r} + d\mathbf{r}) = \overline{\mathbf{v}}\,(\mathbf{r}) + \mathbf{v} \times d\mathbf{r},$$

or

(1.5)
$$d\mathbf{v} = 0,$$
$$d\,\overline{\mathbf{v}} + d\mathbf{r} \times \mathbf{v} = 0.$$

We define the *absolute differential* of the motor transport by:

(1.6)
$$d\mathbf{V} = \begin{pmatrix} d\mathbf{v} \\ d\overline{\mathbf{v}} + d\mathbf{r} \times \mathbf{v} \end{pmatrix}.$$

If (1.5) is true then $d\mathbf{V} = 0$ and we speak of *parallel transport*, in the sense of differential geometry. Hence, $d\mathbf{V}$ in (1.6) will be a measure for the deviation of a motor field from parallelism.

The Cosserat continuum is described by motor fields. Hence, to give an example, every point **r** of the continuum is associated with an infinitesimal rotation $\mathbf{v} = \boldsymbol{\varphi}\,(\mathbf{r})$ and an infinitesimal translation $\overline{\mathbf{v}} = \mathbf{u}\,(\mathbf{r})$, and here $d\mathbf{V}$ measures the deviation of the displacement state of the continuum in the neighborhood $\mathbf{r} + d\mathbf{r}$ of the displacement state of a rigid body. Hence, $d\mathbf{V}$ is a measure of the deformation state of the continuum at the field point **r**. To give a second example, if $\mathbf{V}\,(\mathbf{r})$ is a force-motor then $d\mathbf{V}$ will measure the deviation from equilibrium in the neighborhood $\mathbf{r} + d\mathbf{v}$ of the field point **r**.

2. The absolute differential of a motor in general coordinates.

We introduce the general coordinates with the parameters x^1, x^2, x^3 into three-dimensional Euclidian space E^3. With the help of the position vector:

(2.1)
$$\mathbf{r} = \mathbf{r}\,(x^1, x^2, x^3),$$

we define the natural basis:

(2.2)
$$\mathbf{g}_k = \frac{\partial \mathbf{r}}{\partial x^k} \qquad (k = 1, 2, 3),$$

and obtain the metric tensor g_{ik} from the scalar products of the three basis vectors:

(2.3)
$$< \mathbf{g}_i\,,\, \mathbf{g}_k > = g_{ik}\,.$$

(All Latin indices run from 1 to 3.)

With $d\mathbf{r} = \mathbf{g}_i\, dx^i$, we write (1.6) as:

(2.4) $$d\mathbf{V} = \begin{pmatrix} d(v^k \mathbf{g}_k) \\ d(\bar{v}^k \mathbf{g}_k) + \mathbf{g}_i \, dx^i \times v^l \mathbf{g}_l \end{pmatrix}.$$

Now, one has:

(2.5) $$d\,(v^k\, \mathbf{g}_k) = \nabla_i\, v^k\, dx^i\, \mathbf{g}_k\,,$$

with

(2.6) $$\nabla_i\, v^k = \partial_i\, v^k + \begin{Bmatrix} k \\ i\,l \end{Bmatrix} v^l\,,$$

and the same equations are true for $\bar{v}^k$. (∇_i is the symbol of the covariant derivative with respect to x^i, $\partial_i = \partial / \partial x^i$, and $\begin{Bmatrix} k \\ i\,l \end{Bmatrix}$ is the CHRISTOFFEL symbol of the second kind for the metric g_{ik} .)

Furthermore, one has:

(2.7) $$\mathbf{g}_i \times \mathbf{g}_l = e_{ilr}\, g^{rk}\, \mathbf{g}_k\,,$$

with:

(2.8) $$g_{ri}\, g^{rk} = \delta_i^k = \begin{cases} 1 & \text{for} \quad i = k, \\ 0 & \text{for} \quad i \neq k \end{cases}$$

and the RICCI tensor:

(2.9) $$e_{ilr} = \sqrt{g}\; \varepsilon_{ilr}\,; \qquad g = \det\,(g_{ik}).$$

(ε_{ilr} is the symbol that is alternating in all indices and has the values + 1 or – 1 according to whether *ilr* is an even or odd permutation, resp., of the numbers 1, 2, 3. $\varepsilon_{ilr} = 0$ when two indices are equal to each other.)

In order to make the notation in (2.4) more intuitive, we introduce the transport symbols:

(2.10*a*, *b*) $$\Gamma_{il}^k = \begin{Bmatrix} k \\ i\,l \end{Bmatrix}; \qquad T_{il}^k = g^{rk}\, e_{ril}\,,$$

and ultimately get, with (2.5) to (2.10):

(2.11) $$d\mathbf{V} = \begin{pmatrix} \partial_i v^k dx^i\, \mathbf{g}_k + \Gamma_{il}^k\, dx^i v^l\, \mathbf{g}_k \\ \partial_i \bar{v}^k dx^i\, \mathbf{g}_k + \Gamma_{il}^k\, dx^i \bar{v}^l\, \mathbf{g}_k + T_{il}^k\, dx^i v^l\, \mathbf{g}_k \end{pmatrix}$$

or

(2.12) $$DV^k = \begin{pmatrix} \nabla_i v^k dx^i \\ \nabla_i \bar{v}^k dx^i + T_{il}^k\, dx^i v^l \end{pmatrix}.$$

An intuitive summary of the last result will be achieved by introducing *dual numbers:*

(2.13) $$V^k = v^k + \tau \bar{v}^k\,, \qquad L_{il}^k = \Gamma_{il}^k + \tau T_{il}^k\,,$$

which satisfy the rules of the elementary algebra that are required here, and in which τ is an undetermined quantity that obeys only the condition $\tau^2 = 0$. The summand that is endowed with the τ will be referred to as the *dual part* of the dual number, and the first summand as the *real part*. As one will convince oneself by a simple calculation:

(2.14) $$DV^k = \partial_i V^k dx^i + L^k_{il} \, dx^i V^l$$

or

(2.15) $$DV^k = \nabla_i V^k dx^i + \tau T^k_{il} \, dx^i V^l$$

will give the representation (2.12) after they are separated into real and dual parts.

(2.14) is the representation of a *linear connection* in the space D^3 of dual vectors. One must observe that this connection is not *symmetric*, since the dual part of the transport symbol L^k_{il} – namely, T^k_{il} – is skew-symmetric in its lower indices, from (2.13) and (2.10*b*).

3. The FRENET-CARTAN differential equations in the dual space D^3.

One can also represent a motor by its six coordinates v^k, $\bar{v}^k = v^{k+3}$ in a vector space V^6 and define the linear connection (2.12) in that V^6. That was done in [**3**]. Here, we would like to regard a motor as a dual vector $v^k + \tau \, \bar{v}^k$ in D^3, as in (2.13). We construct the basis of dual vectors that is required in D^3 as follows: We start from the natural basis in E^3 as in (2.2). We associate the basis vectors $\mathbf{g}_k$ with their moment vectors:

(3.1) $$\bar{\mathbf{g}}_k = \mathbf{r} \times \mathbf{g}_k$$

with respect to the origin of a Cartesian coordinate system in E^3. The dual vector:

(3.2) $$\mathbf{G}_k = \mathbf{g}_k + \tau \, \bar{\mathbf{g}}_k$$

represents a directed line in E^3 with the PLÜCKERian vectors $\mathbf{g}_k$ and $\bar{\mathbf{g}}_k$, in which, from (3.1), one will have:

(3.3) $$< \mathbf{g}_k , \, \bar{\mathbf{g}}_k > = 0.$$

The basis vector $\mathbf{g}_k$ in E^3 is fixed by $\bar{\mathbf{g}}_k$ in such a way that it can only be displaced along the tangent to its associated parameter curve.

We take the dual vectors $\mathbf{G}_k$ (k = 1, 2, 3) that were defined in (3.2) to be a basis for the dual space D^3, and their scalar products are:

(3.4) $$< \mathbf{G}_i , \mathbf{G}_k > = g_{ik} .$$

The dual part of that scalar product will then vanish. In detail:

$$\begin{aligned}\langle \mathbf{G}_i , \mathbf{G}_k \rangle &= \langle \mathbf{g}_i + \tau\, \overline{\mathbf{g}}_i , \mathbf{g}_k + \tau\, \overline{\mathbf{g}}_k \rangle \\ &= \langle \mathbf{g}_i , \mathbf{g}_k \rangle + \tau [\langle \mathbf{g}_i , \overline{\mathbf{g}}_k \rangle + \langle \overline{\mathbf{g}}_i , \mathbf{g}_k \rangle] \\ &= g_{ik} ,\end{aligned}$$

so

(3.5) $$\langle \mathbf{g}_i , \mathbf{g}_k \rangle = g_{ik} ,$$

(3.6) $$\langle \mathbf{g}_i , \overline{\mathbf{g}}_k \rangle + \langle \overline{\mathbf{g}}_i , \mathbf{g}_k \rangle = 0.$$

Obviously, (3.6) is fulfilled for any i and k, as a result of (3.1). Now, since the dual part of the scalar product is the "reciprocal moment" of the lines $\mathbf{G}_i$ and $\mathbf{G}_k$, the three directed lines $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{G}_3$ intersect the carriers of the basis vectors $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ in E^3 at the point $\mathbf{r}$ (x^1, x^2, x^3). ([**4**] and [**5**]) The vectors $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{G}_3$ of the dual space D^3 correspond to the basis vectors $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ in E^3 that are fixed at the point $\mathbf{r}$.

Since we are dealing with a line connection in D^3, the FRENET-CARTAN differential equations will take on the form:

(3.7) $$\partial_i \mathbf{G}_k = L^l_{ik} \mathbf{G}_l ,$$

in which

$$L^l_{ik} = \Gamma^l_{ik} + \tau T^l_{ik} ,$$

and we can show that the transport quantities L^l_{ik} are determined uniquely by the Euclidian metric. Upon separating the real and dual parts in (3.7), we will get:

(3.8) $$\partial_i \mathbf{g}_k = T^l_{ik} \mathbf{g}_l ,$$

(3.9) $$\partial_i \overline{\mathbf{g}}_k = \Gamma^l_{ik} \overline{\mathbf{g}}_l + T^l_{ik} \mathbf{g}_l .$$

It is known that:

(3.10) $$\Gamma^l_{ik} = \left\{ \begin{matrix} l \\ i\,k \end{matrix} \right\} .$$

With (3.1), (3.9) reads:

(3.11) $$\begin{aligned}\partial_i (\mathbf{r} \times \mathbf{g}_k) &= \mathbf{r} \times \partial_i \mathbf{g}_k + \mathbf{g}_i \times \mathbf{g}_k \\ &= \Gamma^l_{ik} \overline{\mathbf{g}}_l + e_{ikr}\, g^{rl} \mathbf{g}_l .\end{aligned}$$

A comparison of (3.11) and (3.9) will yield:

(3.12) $$T^l_{ik} = g^{lr} e_{rik} .$$

(3.10) and (3.12) agree with (2.10a, b).

4. The absolute differential of the linear connection in D^3.

In a three-dimensional Cosserat continuum, one deals with motors or dual vectors whose six coordinates are differential forms of degree p (p = 0, 1, 2, 3). For that reason, in what follows, we shall make use of the calculus of differential forms ([**3**], [**5**], [**6**]). Let us first review its notations.

The volume element is represented by:

(4.1) $$dV = \sqrt{g}\; dx^1 \wedge dx^2 \wedge dx^3$$

or

(4.2) $$dx^i \wedge dx^k \wedge dx^l = e^{ikl}\, dV.$$

The vector dF_i of the surface element is:

(4.3) $$dF_i = \tfrac{1}{2} e_{lik}\, dx^i \wedge dx^k$$

or

(4.4) $$dx^i \wedge dx^k = e^{ikl}\, dF_l\,.$$

In this, we have the upper-indexed RICCI tensor:

(4.5) $$e^{ikl} = \frac{1}{\sqrt{g}}\;\varepsilon_{lik}\,,$$

in which the right-hand side of (4.5) is explained by (2.9).

It follows from (4.1) to (4.4) that:

(4.6) $$dx^i \wedge dx^k = \delta^i_k\, dV.$$

In the calculus of differential forms, one has:

(4.7) $$\begin{aligned} ddF_i &= \tfrac{1}{2}\partial_s (e_{lik})\, dx^s \wedge dx^i \wedge dx^k \\ &= \tfrac{1}{2}\;\varepsilon_{lik}\,\varepsilon_{sik}\,\frac{1}{\sqrt{g}}\partial_s\sqrt{g}\; dV \\ &= \frac{1}{\sqrt{g}}\partial_l\sqrt{g}\; dV. \end{aligned}$$

With (3.7), (3.10), and (3.12), the CARTAN torsion 2-form of our asymmetric connection reads:

(4.8) $$\begin{aligned} d\,(\mathbf{G}_k\, dx^k) &= \partial_i\, \mathbf{G}_k\, dx^i \wedge dx^k, \\ &= \tau\, T^l_{ik}\, dx^i \wedge dx^k\, \mathbf{g}_l \\ &= \tau\, g^{lr}\, e_{rik}\, e^{sik}\, dF_s\, \mathbf{g}_l \\ &= 2\tau\, \mathbf{g}^s\, dF_s\,. \end{aligned}$$

Furthermore, one has, with (4.7):

(4.9) $$0 = dd\,(\mathbf{G}_k\,dx^k) = 2\,\tau\,(\partial_l\,\mathbf{g}^s\,dx^l \wedge dF_s + \mathbf{g}^s\,ddF_i) = 2\,\tau\left(-\Gamma^l_{ls} + \frac{1}{\sqrt{g}}\partial_s\sqrt{g}\right)\mathbf{g}^s\,dV,$$

from which, it will follow that:

(4.10) $$ddF_s = \frac{1}{\sqrt{g}}\partial_s\sqrt{g} = \Gamma^l_{ls}\,dV.$$

We denote a dual differential form of degree p by:

(4.11) $$\overset{p}{\Omega}{}^k = \overset{p}{\omega}{}^k + \tau\,\overset{p}{\bar{\omega}}{}^k\,.$$

We obtain the absolute differential of such a form with the help of the differential equations (3.7):

(4.12) $$\begin{aligned} d(\mathbf{G}_k\,\overset{p}{\Omega}{}^k) &= \mathbf{G}_k\,d\overset{p}{\Omega}{}^k + d\mathbf{G}_k \wedge \overset{p}{\Omega}{}^k \\ &= (d\overset{p}{\Omega}{}^k + L^k_{il}\,dx^i \wedge \overset{p}{\Omega}{}^l)\,\mathbf{G}_k \\ &= D\overset{p}{\Omega}{}^k\,\mathbf{G}_k\,, \end{aligned}$$

with

(4.13) $$\boxed{D\overset{p}{\Omega}{}^k = d\overset{p}{\Omega}{}^k + L^k_{il}\,dx^i \wedge \overset{p}{\Omega}{}^l}$$

is the generalization of (2.14), in which V^k appears as the dual differential form of degree 0.

In the calculus of differential forms, one has:

(4.14) $$dd\,\overset{p}{\Omega}{}^k = 0$$

and likewise, since the basis vectors $\mathbf{G}_k$ of the connection are known as functions of position:

(4.15) $$dd\,\mathbf{G}_k = 0.$$

(The connection is integrable; teleparallelism prevails.)

With that, one will have:

(4.16) $$dd(\mathbf{G}_k\,\overset{p}{\Omega}{}^k) = d\mathbf{G}_k \wedge d\overset{p}{\Omega}{}^k + \mathbf{G}_k \wedge dd\overset{p}{\Omega}{}^k + dd\mathbf{G}_k \wedge d\overset{p}{\Omega}{}^k - d\mathbf{G}_k \wedge d\,\overset{p}{\Omega}{}^k\,,$$

such that from (4.14) and (4.15), one will have:

(4.17) $$dd(\mathbf{G}_k \overset{p}{\Omega}{}^k) = 0.$$

When (4.17) is applied to (4.12), that will yield:

(4.18)
$$\begin{aligned} 0 = d(D\overset{p}{\Omega}{}^k \mathbf{G}_k) &= (dD\overset{p}{\Omega}{}^k)\mathbf{G}_k + d\mathbf{G}_k \wedge D\overset{p}{\Omega}{}^k \\ &= (dD\overset{p}{\Omega}{}^k + L^k_{il}\, dx^i \wedge D\overset{p}{\Omega}{}^l)\,\mathbf{G}_k \\ &= DD\overset{p}{\Omega}{}^k\, \mathbf{G}_k, \end{aligned}$$

from which we get the important equation:

(4.19) $$\boxed{DD\overset{p}{\Omega}{}^k = 0.}$$

One will find examples of differential forms of degree p for the three-dimensional Cosserat continuum in my paper [**3**], and likewise the analogues of (4.13) and (4.19). For that reason, at this point, we need to clarify (4.13) and (4.19) with some examples. Moreover, we would like to treat the curved shell as a two-dimensional Cosserat continuum in what follows.

5. The shell as two-dimensional Cosserat continuum.

One has W. GÜNTHER [**7**] to thank for the first complete linear theory of the Cosserat shell. We will show how the basic kinematical and static equations in GÜNTHER's article follow almost immediately from our formulas (4.13) and (4.19). We shall preserve GÜNTHER's notations as much as possible in that demonstration.

The position vector **r** in Euclidian space E^3 has the form:

(5.1) $$\mathbf{r}\,(x^1, x^2, x^3) = \mathbf{R}\,(x^1, x^2) + x^3\, \mathbf{E}\,(x^1, x^2),$$

such that the shell takes the form of $x^3 = 0$.

$\mathbf{r}\,(x^1, x^2, x^3) = \mathbf{R}\,(x^1, x^2)$ is position vector of a point on the shell, in which x^1, x^2 are the GAUSSian parameters of the shell-surface. On $x^3 = 0$, the basis vectors are:

(5.2) $$\frac{\partial \mathbf{R}}{\partial x^1} = \mathbf{a}_1, \qquad \frac{\partial \mathbf{R}}{\partial x^2} = \mathbf{a}_2, \qquad \frac{\partial \mathbf{r}}{\partial x^3} = \mathbf{E}.$$

The scalar product:

(5.3) $$< \mathbf{a}_\alpha , \mathbf{a}_\beta > = a_{\alpha\beta}$$

(Greek indices run from 1 to 3) defines the metric tensor of the shell-surface, which is also called the *first fundamental tensor*.

In order to define the transport quantities on $x^3 = 0$ by using (3.10) and (3.12), we shall require g_{ik} and its first derivatives for $x^3 = 0$.

One next has:

$$g_{\alpha\beta} = \left\langle \frac{\partial \mathbf{r}}{\partial x^\alpha}, \frac{\partial \mathbf{r}}{\partial x^\beta} \right\rangle = \left\langle \frac{\partial \mathbf{R}}{\partial x^\alpha} + x^3 \frac{\partial \mathbf{E}}{\partial x^\alpha}, \frac{\partial \mathbf{R}}{\partial x^\beta} + x^3 \frac{\partial \mathbf{E}}{\partial x^\beta} \right\rangle$$

(5.4)

$$= a_{\alpha\beta} + x^3 \left(\left\langle \frac{\partial \mathbf{E}}{\partial x^\alpha}, \mathbf{a}_\beta \right\rangle + \left\langle \frac{\partial \mathbf{E}}{\partial x^\beta}, \mathbf{a}_\alpha \right\rangle \right) + \ldots$$

(5.5) $$g_{\alpha 3} = \left\langle \frac{\partial \mathbf{R}}{\partial x^\alpha} + x^3 \frac{\partial \mathbf{E}}{\partial x^\alpha}, \mathbf{E} \right\rangle = \left\langle \frac{\partial \mathbf{E}}{\partial x^\alpha}, \mathbf{E} \right\rangle + x^3 \left\langle \frac{\partial \mathbf{E}}{\partial x^\alpha}, \mathbf{E} \right\rangle,$$

(5.6) $$g_{33} = \langle \mathbf{E}, \mathbf{E} \rangle.$$

From time to time, it might be appropriate to compute further with these g_{ik} . However, (5.4), (5.5), and (5.6) will become especially simple when one introduces **E** as the unit normal vector to the surface, which is customary in GAUSS's theory of surfaces. With:

(5.7) $$\mathbf{E} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} = \frac{1}{\sqrt{a}}\, \mathbf{a}_1 \times \mathbf{a}_2 ,$$

in which:

(5.8) $$a = a_{11}\, a_{22} - a_{12}^2 ,$$

one will get:

(5.9) $$g_{\alpha\beta} = a_{\alpha\beta} - 2x^3\, b_{\alpha\beta} + \ldots,$$

(5.10) $$g_{\alpha 3} = 0,$$

(5.11) $$g_{33} = 1.$$

$b_{\alpha\beta} = b_{\beta\alpha}$ is called the *second fundamental form*.

One calculates the contravariant spatial metric tensor from (2.8):

(5.12) $$g^{\alpha\beta} = a^{\alpha\beta} - 2x^3\, b^{\alpha\beta} + \ldots,$$

(5.13) $$g^{\alpha 3} = 0,$$

(5.14) $$g^{33} = 1.$$

in which:

(5.15) $$a_{\beta\mu}\, a^{\mu\alpha} = \delta_\mu^\alpha ; \qquad b_{\lambda\mu}\, a^{\lambda\alpha}\, a^{\mu\beta} = b^{\alpha\beta} .$$

For later purposes, we will need:

(5.16) $$a^{\alpha\mu}\, b_{\mu\rho} = b_\rho^\alpha .$$

On $x^3 = 0$, the RICCI tensor will become:

(5.17) $$e_{\alpha\beta 3} = e_{\alpha\beta} = \sqrt{a}\ \varepsilon_{\alpha\beta}\ ; \qquad e^{\alpha\beta 3} = e^{\alpha\beta} = \frac{1}{\sqrt{a}}\ \varepsilon_{\alpha\beta}\,,$$

with

(5.18) $$\varepsilon_{12} = -\,\varepsilon_{21} = 1; \qquad \varepsilon_{11} = \varepsilon_{22} = 0.$$

The index 3 requires special attention when one constructs the absolute differential (4.13) on $x^3 = 0$. If we omit the degree notation p then we will have to write ($dx^3 = 0$):

(5.19) $$D\Omega^\alpha = d\Omega^\alpha + L^\alpha_{\rho\lambda}\, dx^\rho \wedge \Omega^\lambda + L^\alpha_{\rho 3}\, dx^\rho \wedge \Omega^3\,,$$

(5.20) $$D\Omega^3 = d\Omega^3 + L^3_{\rho\lambda}\, dx^\rho \wedge \Omega^\lambda + L^3_{\rho 3}\, dx^\rho \wedge \Omega^3\,,$$

(5.21) $$L^\alpha_{\rho\lambda} = \Gamma^\alpha_{\rho\lambda} + \tau T^\alpha_{\rho\lambda}\,.$$

A simple calculation yields:

(5.22) $$\Gamma^\alpha_{\rho\lambda} = \frac{1}{2} a^{\alpha\mu}\left(\frac{\partial a_{\mu\lambda}}{\partial x^\rho} + \frac{\partial a_{\mu\rho}}{\partial x^\lambda} - \frac{\partial a_{\rho\lambda}}{\partial x^\mu}\right) = \begin{Bmatrix} \alpha \\ \rho\ \lambda \end{Bmatrix},$$

(5.23) $$T^\alpha_{\rho\lambda} = 0.$$

Furthermore:

(5.24) $$\Gamma^3_{\rho 3} = 0; \qquad T^3_{\rho 3} = 0,$$

(5.25) $$L^\alpha_{\rho 3} = -\ b^\alpha_\rho + \tau\, a^{\alpha\mu}\, e_{\mu\rho}\,,$$

(5.26) $$L^\alpha_{\rho 3} = b_{\rho\lambda} + \tau\, e_{\rho\lambda}\,.$$

We now split (5.19) and (5.20) into real and dual parts and get, with (4.11):

(5.27) $$D\omega^\alpha = \nabla\omega^\alpha - b^\alpha_\rho\, dx^\rho \wedge \omega^3,$$

(5.28) $$D\omega^3 = d\omega^\alpha + b_{\rho\lambda}\, dx^\rho \wedge \omega^\lambda,$$

(5.29) $$D\varpi^\alpha = \nabla\varpi^\alpha - b^\alpha_\rho\, dx^\rho \wedge \varpi^3 + a^{\alpha\mu}\, e_{\mu\rho}\, dx^\rho \wedge \omega^3,$$

(5.30) $$D\varpi^3 = d\varpi^3 + b_{\rho\lambda}\, dx^\rho \wedge \varpi^\lambda + e_{\rho\lambda}\, dx^\rho \wedge \omega^\lambda.$$

In these expressions:

(5.31) $$\nabla\pi^\alpha = d\pi^\alpha + \Gamma^\alpha_{\rho\lambda}\ dx^\rho \wedge \pi^\lambda$$

means the absolute differential of the symmetric surface connection $\Gamma^{\alpha}_{\rho\lambda}$. Since $dx^3 = 0$ on the shell, the absolute differential operator can be applied to only differential forms of degree 0 (functions of position) and 1 (PFAFFiam forms). That corresponds to the fact that only differentials of degrees zero, one, and two will appear in the kinematics and statics of shells.

6. The basic equations of the kinematics and statics of shells.

We begin with differential forms of degree zero. Let ω^{α}, ω^3, **E** be the coordinates of a dual vector in the basis $\mathbf{a}_1$, $\mathbf{a}_2$, **E**.

As an example, we take:

$$\omega^{\alpha} = \varphi^{\alpha}, \qquad \bar{\omega}^{\alpha} = u^{\alpha}, \qquad \omega^3 = \varphi, \qquad \bar{\omega}^3 = u. \tag{6.1}$$

φ^{α}, φ are the coordinates of the rotation vector for a point on the shell, and u^{α}, u are the coordinate of its translation vector. From (3.31), we get:

$$\nabla \varphi^{\alpha} = \partial_{\rho}\, \varphi^{\alpha}\, dx^{\rho} + \Gamma^{\alpha}_{\rho\lambda}\, \varphi^{\lambda}\, dx^{\rho} \tag{6.2}$$

$$= \nabla_{\rho}\, \varphi^{\alpha}\, dx^{\rho}, \tag{6.3}$$

in which:

$$\nabla_{\rho}\, \varphi^{\alpha} = \partial_{\rho}\, \varphi^{\alpha} + \Gamma^{\alpha}_{\rho\lambda}\, \varphi^{\lambda} \tag{6.4}$$

means the covariant derivative of φ^{α} with the shell metric $a_{\alpha\beta}$. An equation that corresponds to (6.3) is true for ∇u^{α}. (5.27) to (5.30) yield the following first-degree differential forms:

$$D\varphi^{\alpha} = (\nabla_{\rho}\, \varphi^{\alpha} - b^{\alpha}_{\rho}\, \varphi)\, dx^{\rho}, \tag{6.5}$$

$$D\varphi = (\partial_{\rho}\, \varphi + b_{\rho\lambda}\, \varphi^{\lambda})\, dx^{\rho}, \tag{6.6}$$

$$Du^{\alpha} = (\nabla_{\rho}\, u^{\alpha} - b^{\alpha}_{\rho}\, u + a^{\alpha\mu}\, e_{\mu\rho}\, \varphi)\, dx^{\rho}, \tag{6.7}$$

$$Du = (\partial_{\rho}\, u + b_{\rho\lambda}\, u^{\lambda} + e_{\rho\lambda}\, \varphi^{\lambda})\, dx^{\rho}. \tag{6.8}$$

As was mentioned in the beginning of Section 1, these absolute differentials are a measure of the deviation of the displacement state of the shell from that of a rigid body. The deformations of the shell are then defined by (6.5) to (6.8). We write:

$$D\varphi^{\alpha} = \chi_{\rho}^{\cdot\,\alpha}\, dx^{\rho}, \qquad Du^{\alpha} = \varepsilon_{\rho}^{\cdot\,\alpha}\, dx^{\rho}, \tag{6.9}$$

$$D\varphi = \chi_\rho \, dx^\rho , \qquad Du = \varepsilon_\rho \, dx^\rho .$$

The χ are the warping (*Verkrümmung*) deformations, while the ε are the membrane and shear deformations.

From (4.19), we have the compatibility conditions:

$$\text{(6.10)} \qquad \begin{aligned} DD\varphi^\alpha &= 0, & DDu^\alpha &= 0, \\ DD\varphi &= 0, & DDu &= 0, \end{aligned}$$

which one can verify by calculation with the use of the first-degree forms (6.5) to (6.8). However, the formulation of the compatibility conditions from the deformations from (6.10) is itself important. Thus, we have to construct the absolute differentials of the first-degree forms:

$$\text{(6.11)} \qquad \begin{aligned} \omega^\alpha &= \chi_\rho^{\cdot\alpha} \, dx^\rho, & \bar{\omega}^\alpha &= \varepsilon_\rho^{\cdot\alpha} \, dx^\rho, \\ \omega^3 &= \chi_\rho \, dx^\rho , & \bar{\omega}^3 &= \varepsilon_\rho \, dx^\rho . \end{aligned}$$

We next calculate (5.31) again, in which we have set π^α equal to ω^α in one case and to $\bar{\omega}^\alpha$ in the other, from (6.10):

$$\text{(6.12)} \qquad \begin{aligned} \nabla \omega^\alpha &= \nabla (\chi_\sigma^{\cdot\alpha} \, dx^\sigma) \\ &= \partial_\rho \chi_\sigma^{\cdot\alpha} dx^\rho \wedge dx^\sigma + \Gamma_{\rho\lambda}^\alpha dx^\rho \wedge \chi_\sigma^{\cdot\lambda} dx^\sigma \\ &= (\partial_\rho \chi_\sigma^{\cdot\alpha} + \Gamma_{\rho\lambda}^\alpha \chi_\sigma^{\cdot\lambda}) \, dx^\rho \wedge dx^\sigma . \end{aligned}$$

Now, the covariant derivative of the tensor $\chi_\sigma^{\cdot\alpha}$ in the metric $a_{\alpha\beta}$ is:

$$\text{(6.13)} \qquad \nabla_\rho \chi_\sigma^{\cdot\alpha} = \partial_\rho \chi_\sigma^{\cdot\alpha} + \Gamma_{\rho\lambda}^\alpha \, \chi_\sigma^{\cdot\lambda} - \Gamma_{\rho\sigma}^\beta \, \chi_\beta^{\cdot\alpha} ,$$

such that we can write:

$$\text{(6.14)} \qquad \nabla(\chi_\sigma^{\cdot\alpha} dx^\sigma) = \nabla_\rho \chi_\sigma^{\cdot\alpha} \, dx^\rho \wedge dx^\sigma,$$

instead of (6.12), since the last summand in (6.13) can be added into the bracket in (6.12), due to the symmetry in the indices ρ, σ. On the other hand, the result (6.12) should not be surprising, since we are dealing with the absolute differential, not of a tensor, but of the vectorial differential form $\chi_\sigma^{\cdot\alpha} \, dx^\sigma = \omega^\alpha$.

With these preparations, we need only to substitute (6.11) and (6.14) into (5.27) to (5.30):

$$\text{(6.15)} \qquad D(\chi_\sigma^{\cdot\alpha} dx^\sigma) = (\nabla_\rho \chi_{(\sigma)}^{\cdot\alpha} - b_\rho^\alpha \chi_\sigma) \, dx^\rho \wedge dx^\sigma,$$

$$\text{(6.16)} \qquad D\,(\chi_\sigma \, dx^\sigma) = (\partial_\rho \, \chi_\sigma + b_{\rho\lambda} \, \chi_\sigma^{\cdot\lambda}) \, dx^\rho \wedge dx^\sigma,$$

$$\text{(6.17)} \qquad D(\varepsilon_\sigma^{\cdot\alpha} dx^\sigma) = (\nabla_\rho \varepsilon_{(\sigma)}^{\cdot\alpha} - b_\rho^\alpha \varepsilon_\sigma + a^{\alpha\mu} e_{\mu\rho} \chi_\sigma) \, dx^\rho \wedge dx^\sigma,$$

(6.18) $$D\,(\varepsilon_\sigma\, dx^\sigma) = (\partial_\rho\, \varepsilon_\sigma + b_{\rho\lambda}\, \varepsilon_\sigma^{\cdot\,\lambda} + e_{\rho\lambda}\, \chi_\sigma^{\cdot\,\lambda})\, dx^\rho \wedge dx^\sigma.$$

The second-degree differential forms will become more intuitive when we introduce the surface element dF of the shell:

(6.19) $$dx^\rho \wedge dx^\sigma = e^{\rho\sigma}\, dF.$$

Ultimately, the compatibility conditions for the deformations read:

(6.20) $$e^{\rho\sigma}\,(\nabla_\rho \chi_{(\sigma)}^{\cdot\,\alpha} - b_\rho^\alpha \chi_\sigma) = 0,$$

(6.21) $$e^{\rho\sigma}\,(\partial_\rho\, \chi_\sigma + b_{\rho\lambda}\, \chi_\sigma^{\cdot\,\lambda}) = 0,$$

(6.22) $$e^{\rho\sigma}\,(\nabla_\rho \varepsilon_{(\sigma)}^{\cdot\,\alpha} - b_\rho^\alpha \varepsilon_\sigma + a^{\alpha\mu} e_{\mu\rho} \chi_\sigma) = 0,$$

(6.23) $$e^{\rho\sigma}\,(\partial_\rho\, \varepsilon_\sigma + b_{\rho\lambda}\, \varepsilon_\sigma^{\cdot\,\lambda} + e_{\rho\lambda}\, \chi_\sigma^{\cdot\,\lambda}) = 0.$$

In (6.15), (6.17), (6.20), and (6.22), the covariant index σ can be ignored in the covariant differentiation, as we did above. We have put that index in parentheses in order to recall that.

Our results, hence, the representation of the deformations by rotation and translation vectors as in (6.5) to (6.8), and furthermore, the compatibility conditions (6.20) to (6.23), coincide with the corresponding ones in GÜNTHER ([**7**], (3.10), (3.11), (3.38), (3.39)).

The basic static equations for the shell can be likewise obtained from equations (5.27) to (5.31) now. Instead of (6.11), we must now set the first-degree differential forms of the sectional forces and moments equal to:

(6.24) $$\omega^\alpha = K_\rho^{\cdot\,\alpha} dx^\rho, \qquad \bar{\omega}^\alpha = M_\rho^{\cdot\,\alpha} dx^\rho,$$
$$\omega^3 = K_\rho\, dx^\rho, \qquad \bar{\omega}^3 = M_\rho\, dx^\rho.$$

$K_\rho^{\cdot\,\alpha}$ is the tensor of membrane forces, $M_\rho^{\cdot\,\alpha}$ is the tensor of bending and torsional moments, K_ρ are the transverse forces, and M_ρ is the moment around the surface normal. The external loads on the shell consist of the surface forces p^α, p, and the surface moments q^α, q. The equilibrium equations will then read:

(6.25) $$D(K_\rho^{\cdot\,\alpha} dx^\rho) + p^\alpha\, dF = 0,$$

(6.26) $$D(K_\rho\, dx^\rho) + p\, dF = 0,$$

(6.28) $$D(M_\rho^{\cdot\,\alpha} dx^\rho) + q^\alpha\, dF = 0,$$

(6.29) $$D(M_\rho\, dx^\rho) + q\, dF = 0.$$

Their explicit forms can be read of immediately from (6.15) to (6.18).

When all of the surface forces and moments vanish, the first-degree differential forms (6.24) will be "closed," and under certain restricting conditions they will be "exact." The latter condition means that the first-degree forms (6.24) can be represented as absolute differentials of forms of degree zero, corresponding to (6.9):

(6.29) $$K_\rho^{\cdot\alpha} dx^\rho = D\Phi^\alpha, \qquad M_\rho^{\cdot\alpha} dx^\rho = D\Psi^\alpha,$$
$$K_\rho\, dx^\rho = D\Phi, \qquad M_\rho\, dx^\rho = D\Psi.$$

The vectors Φ^α, Φ, Ψ^α, Ψ are called *stress functions.* The explicit representation of (6.29) can also be read off immediately from equations (6.5) to (6.8).

The complete analogies between the basic kinematic and static equations for Cosserat shells are consistent with the tenor of GÜNTHER's study [**7**].

(4.13) and (4.19) can be regarded as the sources of those analogies in the calculus of differential forms that we employed. However, we would like to recall that our considerations began with the analogy between force and displacement screws that has been known since antiquity.

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