INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

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THE MOTOR FIELDS OF THE THREE-DIMENSIONAL COSSERAT CONTINUUM IN THE CALCULUS OF DIFFERENTIAL FORMS

TRANSLATED BY
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COURSES AND LECTURES – No. 19

FOREWORD

Hermann Schaefer was one of the promoters of the "International Center for Mechanical Sciences." He took part in various meetings that were organized in Trieste, Aachen, Braunschweig, and other places with the purpose of drafting up the statutes and choosing the original department and activities of the center.

He was cordially connected with my Institute for Mechanics at the University of Trieste, at which he held seminars, courses, and lectures on various topics. In particular, his lectures on the Cosserat continuum were quite meaningful, which were lectures to which I also contributed to a modest extent, and whose publication by my institute was undertaken in Volume 7 of its "Lezioni e Conferenze."

The short lecture that I would like to present here following the death of Professor Schaefer was given by Hermann Schaefer at the Universities of Padua and Trieste in March of 1968. Professor Schaefer sent us the manuscript some days before his death.

I shall present this lecture as a CISM publication, instead of as part of the collection of the institute, since he was more closely linked with the former institution than the latter one.

As a colleague of the CISM and the University of Trieste and an unambiguous, consistent, lively, and amiable human being, Hermann Schaefer deserves the deepest gratitude.

We are all saddened by his passing.

Luigi Sobrero

Udine, April 1970

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CHAPTER I

SOME NOTIONS FROM THE CALCULUS OF DIFFERENTIAL FORMS

We shall consistently use Cartesian coordinates x_1, x_2, x_3 in Euclidian space E^3. We denote a differential form of degree p by $\overset{p}{\omega}$; e.g.:

$$\overset{p}{\omega} = a_1\, dx_1 + a_2\, dx_2 + a_3\, dx_3 = \sum_{k=1}^{3} a_k\, dx_k\ . \tag{1.1}$$

In what follows, the summation sign will always be omitted (viz., the Einstein convention). We then write:

$$\overset{1}{\omega} = a_k\, dx_k\ . \tag{1.2}$$

Let the a_k be functions of position, so:

$$a_k = a_k\,(x_1, x_2, x_3)\ ; \tag{1.3}$$

they can be regarded as components of a vector $\mathbf{a}$ (a_1, a_2, a_3).

The *exterior derivative* (*dérivation extérieure*), or more simply, the *differential* of the form $\overset{1}{\omega}$ will be defined as:

$$d\,\overset{1}{\omega} = (da_k)\, dx_k = \frac{\partial a_k}{\partial x_i}\, dx_i \wedge dx_k = \partial_i\, a_k\, dx_i \wedge dx_k\ . \tag{1.4}$$

The symbol ^ (i.e., wedge) signifies the *exterior product* (*multiplication extérieure*), which is antisymmetric in the indices i and k:

$$dx_i \wedge dx_k = -\, dx_k \wedge dx_i\ . \tag{1.5}$$

Naturally, one will have:

$$dx_k \wedge dx_k = 0. \tag{1.6}$$

Along with (1.5) and (1.6), (1.4) will read, in detail:

$$d\,\overset{1}{\omega} = (\partial_2\, a_3 - \partial_3\, a_2)\, dx_2 \wedge dx_3 + (\partial_3\, a_1 - \partial_3\, a_1)\, dx_3 \wedge dx_1 + (\partial_1\, a_2 - \partial_2\, a_1)\, dx_1 \wedge dx_2\ . \tag{1.7}$$

The *oriented volume element* for E^3 is:

$$dV = dx_1 \wedge dx_2 \wedge dx_3 \tag{1.8}$$

in Cartesian coordinates.

We introduce the oriented (vectorial) surface element:

$$d\mathbf{A} = (dA_1, dA_2, dA_3) = (dx_2 \wedge dx_3\,, dx_3 \wedge dx_1\,, dx_1 \wedge dx_2). \tag{1.9}$$

In more concise notation:

$$\varepsilon_{ikl}\, dA_l = dx_i \wedge dx_k\,. \tag{1.10}$$

(ε_{ikl} is the RICCI tensor, which is alternating in all three indices. Its components will be + 1 or – 1 whenever i, k, l defines an even or odd permutation of the numbers 1, 2, 3, respectively. The components with two or three equal indices will have the value 0.)

It follows from (1.8) and (1.9) that:

$$dx_i \wedge dA_k = dA_k \wedge dx_i = \delta_{ik}\, dV\,. \tag{1.11}$$

(δ_{ik} is the Kronecker symbol for the unit tensor.) With (1.10), we now write (1.4) as:

$$d\overset{1}{\omega} = \varepsilon_{ikl}\, \partial_i\, a_k\, dA_l\,. \tag{1.12}$$

This differential form of degree p = 2 represents the rotation of the vector **a**:

$$d\overset{1}{\omega} = (\text{rot}\,\mathbf{a})_l\, dA_l = \text{rot}\,\overset{1}{\omega}. \tag{1.13}$$

The differential of a form of degree p = 2:

$$\overset{2}{\omega} = a_k\, dA_k \tag{1.14}$$

will be defined as:

$$d\overset{2}{\omega} = (da_k)\, dA_k = \partial_i\, a_k\, dx_i \wedge dA_k = \partial_i\, a_k\, \delta_{ik}\, dV = \partial\, a_k\, dV. \tag{1.15}$$

The a_k in (1.14) can, in turn, be regarded as the components of a vector, and (1.15) will be nothing but:

$$d\overset{2}{\omega} = \text{div}\,\mathbf{a}\, dV = \text{div}\,\overset{2}{\omega}. \tag{1.16}$$

$d\overset{2}{\omega}$ is a form of degree p = 3. Forms of higher degree will not exist in E^3, since $dx_k \wedge dV = 0$ for k = 1, 2, 3. By contrast, a form degree p = 0 can be defined to be a scalar function:

$$\overset{0}{\omega} = a\,(x_1, x_2, x_3), \tag{1.17}$$

whose differential:

$$d\overset{0}{\omega} = \partial_i\, a\, dx_i = (\text{grad}\, a)_i\, dx_i = \text{grad}\,\overset{0}{\omega}, \tag{1.18}$$

is a form of degree $p = 1$.

One of the most important formulas of the calculus is:

$$dd\overset{p}{\omega} = 0, \tag{1.19}$$

which is also called *the Poincaré Lemma*.

The validity of formula (1.19) can be verified directly in our examples. One gets the identities:

$$\left.\begin{aligned} \text{rot grad}\,\overset{0}{\omega} = 0 \quad &\text{for } p = 0, \\ \text{div rot}\,\overset{1}{\omega} = 0 \quad &\text{for } p = 1. \end{aligned}\right\} \tag{1.20}$$

This brief extract from the calculus of differential forms in E^3 can suffice for our purposes to begin with.

Literature

One will find an elementary introduction to the calculus of differential forms in the book by R. C. Buck, *Advanced Calculus*, New York, 1956 on pages 309-321. The reader will enjoy perusing this stimulating book. An already challenging, but still easily-readable, presentation can be found in the chapter on "Alternierende Differentialformen" (written by Sommer, Reimann, and Rau) in the book by Behnke, Bachman, Fladt, and Süss, *Grundzüge der Mathematik* (for Gymnasium students, as well as mathematicians in industry and commerce), Band III, *Analysis*, Göttingen, 1962, pp. 133-200. I shall give further bibliographic references later at the end of Chapter 8.

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CHAPTER 2

DUAL NUMBERS AND DUAL VECTORS

In the book by **I. M. Yaglom**, *Complex Numbers in Geometry*, Academic Press, 1968, one reads on page 14:

> "*Dual numbers*, apparently, were first considered by the famous German geometer **E. Study** (1862-1930, University of Bonn) of the end of the last Century and the beginning of this one; double numbers were introduced by a contemporary of Study, the English geometer **W. Clifford** (1845-1879). Clifford, who was concerned with the use of these numbers in mechanics, called them *motors*."

That nice little book by Yaglom is generally too elementary for our purposes here. I recommend that the reader confer the book by **W. Blaschke** (a student of Study), *Vorlesungen über Differential-Geometrie,* Band I, Springer-Verlag Berlin, 1945, and in particular, page 261 of the chapter on line geometry in it.

A *dual number*:

$$A = a + \tau\,\hat{a}$$

consists of a pair $(a,\ \hat{a})$ of real numbers $a,\ \hat{a}$. The new unit τ shall satisfy the rule of calculation:

$$\tau^2 = 0.$$

The sums and products of two dual numbers $A_1 = a_1 + \tau\,\hat{a}_1$, $A_2 = a_2 + \tau\,\hat{a}_2$ will then be:

$$A_1 + A_2 = (a_1 + a_2) + \tau\,(\hat{a}_1 + \hat{a}_2)\,, \tag{2.1}$$

$$A_1\,A_2 = a_1\,a_2 + \tau\,(a_1\hat{a}_2 + a_2\hat{a}_1)\,. \tag{2.2}$$

We can regard τ as simply an auxiliary quantity that imparts a convenient overview to assigning:

$$(a_1 + a_2\,,\ \hat{a}_1 + \hat{a}_2)\,, \tag{2.3}$$

$$(a_1\,a_2\,,\ a_1\hat{a}_2 + a_2\hat{a}_1) \tag{2.4}$$

to the *sum* and *product* of the number-pairs $(a_1,\ \hat{a}_1)$, $(a_2,\ \hat{a}_2)$ on the basis of its property $\tau^2 = 0$.

The idea that one could use the ∞^4 lines in space as the building blocks for a spatial geometry goes back to **J. Plücker** (1801-1868, physicist and mathematician at the University of Bonn).

In his "line geometry," a line is determined by two vectors:

$$\mathbf{a},\ \hat{\mathbf{a}}\,, \tag{2.5}$$

such that one should have:

$$\mathbf{a}\cdot\mathbf{a} = 1, \qquad \mathbf{a}\cdot\hat{\mathbf{a}} = 0. \tag{2.6}$$

(· is the symbol of the scalar product of two vectors.) One calls the six coordinates of the vectors **a** and $\hat{\mathbf{a}}$ with the two auxiliary conditions (2.6) the *Plücker line coordinates.* $\hat{\mathbf{a}}$ is determined by the origin *O* of the Cartesian coordinate system: namely, $\hat{\mathbf{a}}$ is the static moment of the *direction vector* **a** with respect to *O*:

$$\mathbf{x}\times\mathbf{a} = \hat{\mathbf{a}}\,. \tag{2.7}$$

(× is the symbol for vectorial multiplication.) Refer to Fig. 1 for this.

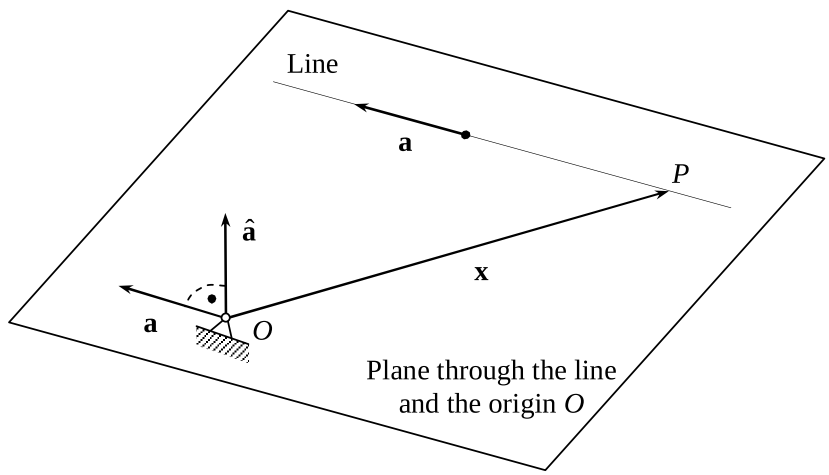

Figure 1.

The coordinates of the *position vector* **x** are the coordinates of the point *P* of the line.

Study combined the Plücker vectors ($\mathbf{a}$, $\hat{\mathbf{a}}$) into the *dual vector:*

$$\mathbf{A} = \mathbf{a} + \tau\,\hat{\mathbf{a}}\,. \tag{2.8}$$

The basis for that can be made clearer by the following argument: The scalar product of two dual vectors is:

$$\mathbf{A}_1\cdot\mathbf{A}_2 = (\mathbf{a}_1 + \tau\,\hat{\mathbf{a}}_1)\cdot(\mathbf{a}_2 + \tau\,\hat{\mathbf{a}}_2) = \mathbf{a}_1\cdot\mathbf{a}_2 + (\mathbf{a}_1\cdot\hat{\mathbf{a}}_2 + \mathbf{a}_2\cdot\hat{\mathbf{a}}_1). \tag{2.9}$$

Consider Fig. 2:

$\mathbf{A}_1$ and $\mathbf{A}_2$ represent two lines whose shortest distance is $p = |\,\mathbf{x}_2 - \mathbf{x}_1\,|$. Obviously, the "real part" of the scalar product (2.9) is:

$$\mathbf{a}_1\cdot\mathbf{a}_2 = \cos\Phi. \tag{2.10}$$

Furthermore, one has:

$$(\mathbf{x}_2 - \mathbf{x}_1)\cdot\mathbf{a}_1 = 0, \qquad (\mathbf{x}_2 - \mathbf{x}_1)\cdot\mathbf{a}_2 = 0. \tag{2.11}$$

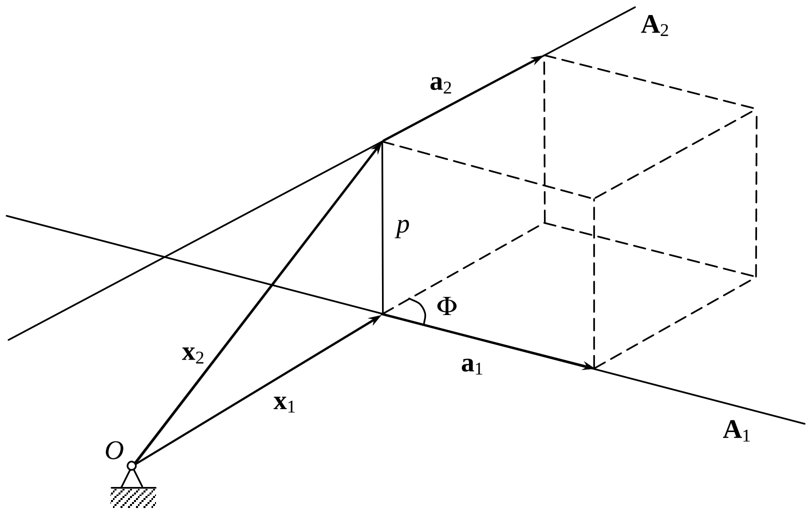

Figure 2.

We shall now consider the “dual part” of the scalar product (2.9), which one also calls the *mutual moment of the lines:*

$$\mathbf{a}_1 \cdot \hat{\mathbf{a}}_2 + \mathbf{a}_2 \cdot \hat{\mathbf{a}}_1 = \mathbf{a}_1 \cdot (\mathbf{x}_2 \times \mathbf{a}_2) + \mathbf{a}_2 \cdot (\mathbf{x}_1 \times \mathbf{a}_1) = (\mathbf{a}_1 \times \mathbf{a}_2) \cdot (\mathbf{x}_1 - \mathbf{x}_2) = - p \sin \Phi. \quad (2.12)$$

From (2.9), with (2.10) and (2.12), we have then obtained:

$$\mathbf{A}_1 \cdot \mathbf{A}_2 = \cos \Phi - \tau p \sin \Phi. \quad (2.13)$$

(By the way: By means of a formal series development, one has:

$$\cos (\Phi + \tau p) = \cos \Phi \cos \tau p - \sin \Phi \sin \tau p = \cos \Phi \cdot 1 - (\sin \Phi)\,(\tau p)$$

for the *dual angle* $\Phi + \tau p$.)

As long as $\mathbf{a}_1 \cdot \mathbf{a}_2 \neq 1$:

$$\mathbf{a}_1 \cdot \hat{\mathbf{a}}_2 + \mathbf{a}_2 \cdot \hat{\mathbf{a}}_1 = 0 \quad (2.14)$$

means that the lines that are represented by $\mathbf{A}_1$ and $\mathbf{A}_2$ intersect at the point $\mathbf{x} = \mathbf{x}_1 = \mathbf{x}_2$.

From now on, we shall employ Cartesian coordinates (x_1, x_2, x_3), which will be the case in all of this treatise.

At the point $\mathbf{x}$ (x_1, x_2, x_3), we construct the orthogonal dreibein $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ from $\mathbf{e}_i = \partial_i \mathbf{x}$. (∂_i is also the abbreviation for $\partial / \partial x_i$ that shall be employed from now on.) We shall now represent the three coordinate lines that go through the point $\mathbf{x}$ by the three dual vectors:

$$\mathbf{E}_i = \mathbf{e}_i + \tau\, \hat{\mathbf{e}}_i , \quad \hat{\mathbf{e}}_i = \mathbf{x} \times \mathbf{e}_i . \quad (2.15)$$

Obviously:

$$\mathbf{e}_i \cdot \mathbf{e}_k = \delta_{ik} , \qquad \mathbf{e}_i \cdot \hat{\mathbf{e}}_k + \mathbf{e}_k \cdot \hat{\mathbf{e}}_i = 0, \quad (2.16)$$

so the three coordinate lines do, in fact, intersect at the point **x**. With that, we have also "fixed" the three unit vectors $\mathbf{e}_i$; they are "line-bound" vectors, like forces on a rigid body. Since the $\mathbf{e}_i$ at each point of E^3 are unit vectors in the coordinate direction, one will have:

$$\partial_i \mathbf{e}_k = 0. \tag{2.17}$$

The *moment vectors* $\hat{\mathbf{e}}_k$ behave differently. Namely:

$$\partial_i \hat{\mathbf{e}}_k = \partial_i (\mathbf{x} \times \mathbf{e}_k) = \mathbf{e}_i \times \mathbf{e}_k = \varepsilon_{ikl}\, \mathbf{e}_l . \tag{2.18}$$

[ε_{ikl} is the Ricci symbol, which is alternating in all three indices. It has the value 1 (– 1, resp.) when *ikl* is an even (odd, resp.) permutation of the numbers 123. Its value will be zero when two or three indices are equal.]

With (2.15), (2.17), and (2.18), we can write:

$$\partial_i \mathbf{E}_k = \Gamma_{ikl}\, \mathbf{E}_l \qquad \text{or} \qquad d\mathbf{E}_k = \Gamma_{ikl}\, dx_i\, \mathbf{E}_l , \tag{2.19}$$

in which:

$$\Gamma_{ikl} = \tau\, \varepsilon_{ikl} . \tag{2.20}$$

(2.19) *defines a* ***linear connection*** *in the space D^3 of dual vectors.* One observes that the transport symbol Γ_{ikl} of this connection is skew-symmetric.

We define a dual vector $\mathbf{\Omega}$ in D^3 by way of:

$$\mathbf{\Omega} = \mathbf{E}_k\, \Omega_k , \qquad \Omega_k = \omega_k + \tau\, \hat{\omega}_k , \tag{2.21}$$

in which the ω_k and $\hat{\omega}_k$ are any three real numbers. With (2.19) and (2.20), the differential of the dual vector $\mathbf{\Omega}$ will become:

$$\begin{aligned} d\,\mathbf{\Omega} = d\,(\mathbf{E}_k\, \Omega_k) &= \mathbf{E}_k\, d\Omega_k + d\mathbf{E}_k\, \Omega_k \\ &= \mathbf{E}_k\, d\Omega_k + \Gamma_{ikl}\, dx_i\, \mathbf{E}_l\, \Omega_k \\ &= \mathbf{E}_k\, (d\Omega_k + \Gamma_{ikl}\, dx_i\, \Omega_k) \\ &= D\Omega_k\, \mathbf{E}_k , \end{aligned} \tag{2.22}$$

in which the *covariant differential* of the connection in D^3 is:

$$D\Omega_k = d\Omega_k + \tau\, \varepsilon_{ikl}\, dx_i\, \Omega_k . \tag{2.23}$$

The separation into real and dual parts yields:

$$\begin{aligned} D\omega_k &= d\omega_k , \\ D\hat{\omega}_k &= d\hat{\omega}_k + \varepsilon_{ikl}\, dx_i\, \omega_k . \end{aligned} \tag{2.24}$$

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CHAPTER 3

THE PARALLEL TRANSPORT OF A MOTOR

It is known that the individual forces that act upon a rigid body at a point P can be "reduced" and represented by a single force vector $\mathbf{F}$ (P) whose line of action goes through P, and a moment vector $\mathbf{M}$ (P). If one chooses another reduction point Q then one will have the *transport law:*

$$\mathbf{F}(Q) = \mathbf{F}(P), \qquad \mathbf{M}(Q) = \mathbf{M}(P) + \mathbf{F} \times \overrightarrow{\mathbf{Q}-\mathbf{P}}. \tag{3.1}$$

One calls these two representations of the reduced force systems at P and Q *equivalent* or *equipollent.*

The same transport law will be true for the infinitesimal displacement state of a rigid body. Infinitesimal rotations and translations can be described completely by giving a point P of the body, the infinitesimal rotation vector $\boldsymbol{\varphi}$ (P) and the infinitesimal translation $\mathbf{u}$ (P). For a different choice of point Q on the body, one will have the transport law:

$$\boldsymbol{\varphi}(Q) = \boldsymbol{\varphi}(P), \qquad \mathbf{u}(Q) = \mathbf{u}(P) + \boldsymbol{\varphi} \times \overrightarrow{\mathbf{Q}-\mathbf{P}}. \tag{3.2}$$

The vector-pair $\boldsymbol{\varphi}$, $\mathbf{u}$ will be called a *screw*, while the pair $\mathbf{F}$, $\mathbf{M}$ will be called a *force screw* or *dyname.* **E. Study** and **R. v. Mises** have adopted **Clifford**'s term *motor* for vector-pairs that satisfy the transport law (3.1) or (3.2), which is identical to it. A motor $\mathbf{V}$ is composed of the vectors $\mathbf{v}$ and $\hat{\mathbf{v}}$:

$$\mathbf{V} = \begin{pmatrix} \mathbf{v} \\ \hat{\mathbf{v}} \end{pmatrix}, \tag{3.3}$$

in which the second vector shall always be the moment vector of the motor.

If the reduction points P and Q of a motor have the infinitesimal distance $d\mathbf{x}$ then the transport law will read:

$$\mathbf{v}(\mathbf{x} + d\mathbf{x}) = \mathbf{v}(\mathbf{x}), \qquad \hat{\mathbf{v}}(\mathbf{x} + d\mathbf{x}) = \hat{\mathbf{v}}(\mathbf{x}) + \mathbf{v} \times d\mathbf{x}, \tag{3.4}$$

or

$$d\mathbf{v} = 0, \qquad d\hat{\mathbf{v}} + d\mathbf{x} \times \mathbf{v} = 0. \tag{3.5}$$

We shall define the *absolute differential* of the motor transport by:

$$d\mathbf{V} = \begin{pmatrix} d\mathbf{v} \\ d\hat{\mathbf{v}} + d\mathbf{x} \times \mathbf{v} \end{pmatrix}. \tag{3.6}$$

If (3.5) is true (so $d\mathbf{V} = 0$) then we can speak of a *parallel transport*, in the sense of differential geometry. Therefore, $d\mathbf{V}$ in (3.6) is a measure of the deviation of a motor field from parallelism.

With:

$$\mathbf{v} = v_k\,\mathbf{e}_k\,, \qquad \hat{\mathbf{v}} = \hat{v}_x\,\mathbf{e}_x\,, \qquad \text{and} \qquad d\mathbf{x} = \mathbf{e}_i\,dx_i\,,$$

we write:

$$d\mathbf{V} = \begin{pmatrix} d(v_k\mathbf{e}_k) \\ d(\hat{v}_k\mathbf{e}_k + \varepsilon_{kil}\,\mathbf{e}_i\,dx_i v_l)\,\mathbf{e}_k \end{pmatrix}. \tag{3.7}$$

A comparison of (3.8) with (2.24) shows that the parallel transport of a motor in E^3 corresponds to the existence of a linear connection in the space D^3 of dual vectors.

Literature and remarks about Chapters 2 and 3.

E. Study wrote an extensive book that was rich in content called *Geometrie der Dynamen* (Teubner, Leipzig, 1903), which is not easy to read. I know a woman who had studied that book from beginning to end, namely, the wife Elisabeth (*nee* Verständig) of my Braunschweiger colleague F. Rehbock, but she was killed in an air raid on Braunschweig in 1944. She had studied in Berlin under R. v. Mises, and he posed the problem to her as an examination paper that she should correct the all-too-numerous flawed figures in Study’s book. Unfortunately, her work no longer exists.

One has **R. v. Mises** to thank for the *motor calculus,* which is tailored completely to the needs of mechanics. Two major publications on that subject exist in ZAMM **4** (1924), which was issued by him at the time, and later reprinted in *Selected Papers of Richard v. Mises*, vol. 1, Amer. Math. Soc., 1964. A mechanically-sensible scalar and motorial product is defined in it and calculated with motor-dyadics. To my knowledge, that motor calculus has been used only a few times, and only in the German literature. One must refer to it, more precisely, as a linear motor algebra. The restricted multiplication of 6×6 matrices that is employed in it is a thorn in the reader’s side. V. Mises emphasized that he would make no use of Study’s dual vectors, in order to create a calculus that was free of dual numbers.

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CHAPTER 4

THE BASIC KINEMATIC AND STATIC EQUATIONS FOR A COSSERAT CONTINUUM

The Cosserat continuum can be described by motor fields. To give an example, any point $\mathbf{x}$ of the continuum is associated with an infinitesimal rotation $\mathbf{v} = \boldsymbol{\varphi}(\mathbf{x})$ and an infinitesimal translation $\hat{\mathbf{v}} = \mathbf{u}(\mathbf{x})$, and in it, $d\mathbf{V}$ in (3.6) measures the deviation of the displacement state of the continuum in the vicinity $\mathbf{x} + d\mathbf{x}$ from the displacement state of a rigid body. Hence, $d\mathbf{V}$ is a measure of the deformation state of the continuum at the field-point $\mathbf{x}$. To give a second example, if $\mathbf{V}(\mathbf{x})$ is a force-motor (or dyname) then $d\mathbf{V}$ will measure the deviation from equilibrium in the neighborhood $\mathbf{x} + d\mathbf{x}$ of the field point $\mathbf{x}$. We shall initially stick with our first example. From (2.94) or (3.8), the deformation of the continuum is defined by:

$$D\varphi_k = \partial_i\, \varphi_k\, dx_i\,, \qquad Du_k = (\partial_i\, u_k - \varepsilon_{ikl}\, \varphi_l)\, dx_i\,, \tag{4.1}$$

or

$$\kappa_{ik} = \partial_i\, \varphi_k\,, \qquad \varepsilon_{ik} = \partial_i\, u_k - \varepsilon_{ikl}\, \varphi_l\,. \tag{4.2}$$

κ_{ik} is the tensor of curvature and torsion, while ε_{ik} is the tensor of distortion and relative rotation. Both tensors are asymmetric, and for that reason, they have 18 components collectively. Naturally, they cannot be given arbitrarily, since, from (4.2), they must be expressible in terms of the six vector components of the φ_k and u_k. The compatibility condition for this reads:

$$\varepsilon_{sri}\, \partial_r\, \kappa_{ik} = 0, \qquad \varepsilon_{sri}\, (\partial_r\, \varepsilon_{ik} + \varepsilon_{krl}\, \partial_r\, \kappa_{il}) = 0. \tag{4.3}$$

One convinces oneself that (4.3) is fulfilled by φ_k and u_k identically by substituting (4.2) in (4.3).

We know that (4.1) or (4.2) can be combined into *a single* equation by way of the dual number τ. Even when (2.23) is unknown to us, after a brief attempt, we will come back to:

$$\begin{aligned}(\kappa_{ik} + \tau\, \varepsilon_{ik})\, dx_i &= \partial_i\, (\varphi_k + \tau\, u_k)\, dx_i - \tau\, \varepsilon_{ikl}\, (\varphi_l + \tau\, u_l)\, dx_i \\ &= d\,(\varphi_k + \tau\, u_k) - \tau\, \varepsilon_{ikl}\, dx_i\, (\varphi_l + \tau\, u_l) \\ &= D\,(\varphi_k + \tau\, u_k),\end{aligned} \tag{4.4}$$

and thus, to (2.23), when we set:

$$\Omega_k = \overset{\circ}{\Omega}_k = \varphi_k + \tau\, u_k\,. \tag{4.5}$$

We shall now proceed in an entirely analogous way with (4.3). However, if we are to do that then we must recall the calculus of differential forms, as they were set down in Chapter 1. Obviously, (4.3) has something to do with a rotor picture. We summarize (4.3) as:

$$\varepsilon_{sri}\, \partial_r\, (\kappa_{ik} + \tau\, \varepsilon_{ik})\, dA_s + \tau\, \varepsilon_{sri}\, \varepsilon_{krl}\, (\kappa_{ik} + \tau\, \varepsilon_{ik})\, dA_s = 0. \tag{4.6}$$

Now, since one has:

$$\varepsilon_{sri}\, dA_s = dx_r \wedge dx_i \tag{4.7}$$

from (1.10), (4.6) can then be written:

$$\partial_r\, (\kappa_{ik} + \tau\, \varepsilon_{ik})\, dx_r \wedge dx_i + \tau\, \varepsilon_{krl}\, (\kappa_{ik} + \tau\, \varepsilon_{ik})\, dx_r \wedge dx_i = 0 \tag{4.8}$$

or

$$d\, (\kappa_{ik} + \tau\, \varepsilon_{ik})\, dx_i + \tau\, \varepsilon_{krl}\, dx_r \wedge (\kappa_{ik} + \tau\, \varepsilon_{ik})\, dx_i = 0. \tag{4.9}$$

Now:

$$\Omega_k = \overset{1}{\Omega}_k = (\kappa_{ik} + \tau\, \varepsilon_{ik})\, dx_i \tag{4.10}$$

is a differential form of degree 1, while $\overset{o}{\Omega}_k$ in (4.5) was a form of degree 0.

With (4.10), (4.9) will become:

$$D\overset{1}{\Omega}_k = DD\overset{o}{\Omega}_k = d\,\overset{1}{\Omega}_k + \tau\, \varepsilon_{krl}\, dx_r \wedge \overset{1}{\Omega}_l = 0, \tag{4.11}$$

which one might compare with (2.23).

In a Cosserat continuum, along with the tensor σ_{ik} of force-stresses, there also exists the tensor μ_{ik} of moment-stresses. Both of them are asymmetric. One infers the orientation of their components from Figures 3 and 4.

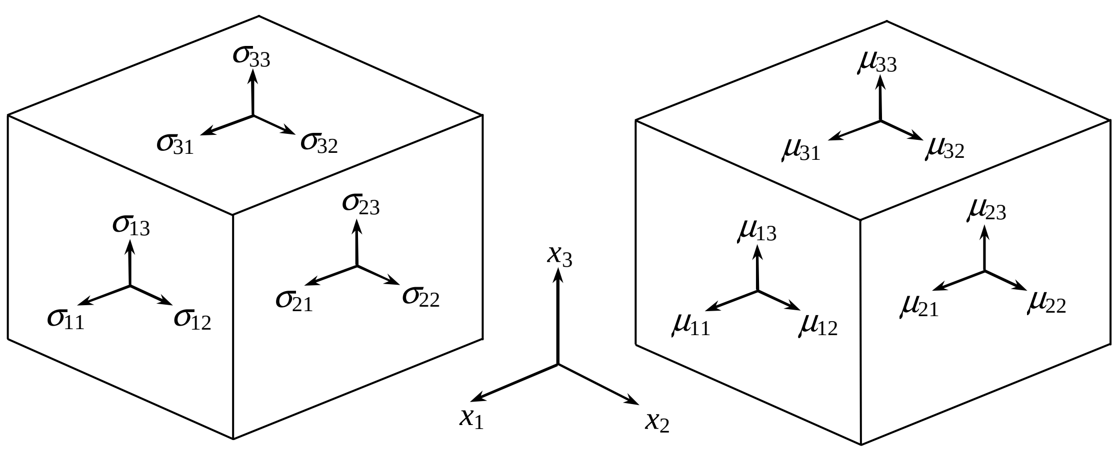

Figure 3. Figure 4.

One finds the equilibrium conditions:

$$\left.\begin{aligned}&\partial_i\sigma_{ik}+X_k=0,\\&\partial_i\mu_{ik}+\varepsilon_{krl}\,\sigma_{rl}+Y_k=0.\end{aligned}\right\}\tag{4.12}$$

In them, $X_k\,dV$ ($Y_k\,dV$, resp.) are the volume-forces (-moments, resp.).

We now set:

$$\overset{2}{\Omega}_k = (\sigma_{ik} + \tau\,\mu_{ik})\,dA_i \tag{4.13}$$

and calculate:

$$D\overset{2}{\Omega}_k = d\overset{2}{\Omega}_k + \tau\,\varepsilon_{rlk}\,dx_r \wedge \overset{2}{\Omega}_l\,. \tag{4.14}$$

We get:

$$D\overset{2}{\Omega}_k = \partial_s\,(\sigma_{ik} + \tau\,\mu_{ik})\,dx_s \wedge dA_i + \tau\,\varepsilon_{rlk}\,dx_r \wedge (\sigma_{il} + \tau\,\mu_{il})\,dA_l\,. \tag{4.15}$$

Now, from (1.11):

$$dx_s \wedge dA_i = \delta_{si}\,dV, \tag{4.16}$$

such that (4.15) can be written:

$$D\overset{2}{\Omega}_k = \partial_i\,(\sigma_{ik} + \tau\,\mu_{ik})\,dV + \tau\,\varepsilon_{rlk}\,(\sigma_{rl} + \tau\,\mu_{rl})\,dV\,. \tag{4.17}$$

After separating the real and dual parts, one will get:

$$\left.\begin{aligned}&D(\sigma_{ik}\,dA_i)=\partial_i\,\sigma_{ik}dV,\\&D(\mu_{ik}\,dA_i)=(\partial_i\,\mu_{ik}+\varepsilon_{rlk}\,\sigma_{rl})\,dV.\end{aligned}\right\}\tag{4.18}$$

(4.14) can then yield the equilibrium conditions (4.12) when we write it in the form:

$$\left.\begin{aligned}&D(\sigma_{ik}\,dA_i)+X_k\,dV=0,\\&D(\mu_{ik}\,dA_i)+Y_k\,dV=0.\end{aligned}\right\}\tag{4.18}$$

As an exercise, the reader might verify that:

$$DD\overset{1}{\Omega}_k = DD\,(\kappa_{ik} + \tau\,\varepsilon_{ik})\,dx_i$$

vanishes identically in κ_{ik} and ε_{ik} .

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CHAPTER 5

COVARIANT DIFFERENTIAL AND DIFFERENTIAL OPERATORS Grad, Rot, Div IN THE COSSERAT CONTINUUM

In order to establish the calculations of the foregoing chapter, in which one deals with the covariant differentials of differential forms of degree p, we recall equations (2.19) and (2.20) for the linear connection in D^3:

$$d\mathbf{E}_k = \tau\,\varepsilon_{ikl}\,dx_i\,\mathbf{E}_l \tag{5.1}$$

are the **Frenet-Cartan** differential equations. We regard (5.1) as a differential form of degree 1 and define:

$$dd\mathbf{E}_k = \tau\,\varepsilon_{ikl}\,d\mathbf{E}_l \wedge dx_i\,. \tag{5.2}$$

Due to $\tau^2 = 0$, the substitution of $d\mathbf{E}_l$ from (5.1) in (5.2) will yield the result:

$$dd\mathbf{E}_k = 0. \tag{5.3}$$

(5.3) says that the connection possesses zero curvature; i.e., teleparallelism prevails. That result was to be expected, since a motor uniquely associates a moment vector at each point $\mathbf{x}$ of E^3, from the transport law.

The absolute differential of a dual-vectorial differential form of degree p will now be defined:

$$d(\mathbf{E}_k \overset{p}{\Omega}_k) = \mathbf{E}_k\, d\overset{p}{\Omega}_k + d\mathbf{E}_k \wedge \overset{p}{\Omega}_k = \mathbf{E}_k(d\,\overset{p}{\Omega}_k + \tau\,\varepsilon_{kil}\,dx_i \wedge \overset{p}{\Omega}_l) = \mathbf{E}_k D\overset{p}{\Omega}_k\,. \tag{5.4}$$

With that, we have obtained the covariant differential of a dual-differential form of degree p:

$$\boxed{D\overset{p}{\Omega}_k = d\,\overset{p}{\Omega}_k + \tau\,\varepsilon_{kil}\,dx_i \wedge \overset{p}{\Omega}_l\,.} \tag{5.5}$$

We define the next one by using (5.4):

$$dd(\mathbf{E}_k \overset{p}{\Omega}_k) = d\mathbf{E}_k \wedge d\,\overset{p}{\Omega}_k + \mathbf{E}_k dd\,\overset{p}{\Omega}_k + dd\mathbf{E}_k \wedge \overset{p}{\Omega}_k - d\mathbf{E}_k \wedge d\,\overset{p}{\Omega}_k\,. \tag{5.6}$$

Now, from the Poincaré Lemma (1.19) and the fact that $dd\mathbf{E}_k = 0$, due to (5.1), one will have $dd\mathbf{E}_k = 0$. One will then have:

$$0 = dd(\mathbf{E}_k \overset{p}{\Omega}_k) = d(\mathbf{E}_k D\overset{p}{\Omega}_k) = \mathbf{E}_k DD\overset{p}{\Omega}_k\,, \tag{5.7}$$

from (5.6) and (5.4). With that, we have obtained the important result that:

$$\boxed{DD\overset{p}{\Omega}_k = 0,} \tag{5.8}$$

which is an analogue of the Poincaré Lemma (1.19).

The reader might be confused about the negative sign on the fourth summand in (5.6). In the calculus of differential forms, one has the rule:

$$d(\overset{r}{\pi}\wedge\overset{p}{\omega}) = d\,\overset{r}{\pi}\wedge\overset{p}{\omega} + (-1)^r\,\overset{r}{\pi}\wedge d\,\overset{p}{\omega}. \tag{5.9}$$

However, due to (5.1), $d\mathbf{E}_k$ is a form of degree $r = 1$ in (5.6).

In analogy to (1.13), (1.16), (1.18), we now introduce the three differential operators Grad, Rot, and Div by way of:

$$D\overset{0}{\Omega}_k = \text{Grad }\overset{0}{\Omega}_k, \tag{5.10}$$

$$D\overset{1}{\Omega}_k = \text{Rot }\overset{1}{\Omega}_k, \tag{5.11}$$

$$D\overset{2}{\Omega}_k = \text{Div }\overset{2}{\Omega}_k. \tag{5.12}$$

(5.8) will then yield the identities:

$$\text{Rot Grad }\overset{0}{\Omega}_k = 0, \tag{5.13}$$

$$\text{Div Rot }\overset{1}{\Omega}_k = 0, \tag{5.14}$$

which are analogous to (1.20). If we now skip over dual numbers and differential forms for the moment then what will remain as a result will be the definition of the operators:

$$\text{Grad}\begin{pmatrix}\boldsymbol{\varphi}\\ \mathbf{u}\end{pmatrix} = \begin{pmatrix}\partial_i\,\varphi_k\\ \partial_i\,u_k - \varepsilon_{ikl}\,u_l\end{pmatrix}, \tag{5.15}$$

$$\text{Rot}\begin{pmatrix}\underline{\boldsymbol{\kappa}}\\ \underline{\boldsymbol{\varepsilon}}\end{pmatrix} = \begin{pmatrix}\varepsilon_{sri}\,\partial_r\,\kappa_{ik}\\ \varepsilon_{sri}(\partial_r\,\varepsilon_{ik} + \varepsilon_{krl}\,\kappa_{il})\end{pmatrix}, \tag{5.16}$$

$$\text{Div}\begin{pmatrix}\underline{\boldsymbol{\sigma}}\\ \underline{\boldsymbol{\mu}}\end{pmatrix} = \begin{pmatrix}\partial_i\,\sigma_{ik}\\ \partial_i\,\mu_{ik} + \varepsilon_{krl}\,\sigma_{rl}\end{pmatrix}. \tag{5.17}$$

In this, the bold quantities are vectors, while the underlined bold quantities are second-rank tensors.

With that, we now write the basic equations of the Cosserat continuum as follows:

$$\begin{pmatrix} \underline{\boldsymbol{\kappa}} \\ \underline{\boldsymbol{\varepsilon}} \end{pmatrix} = \text{Grad} \begin{pmatrix} \boldsymbol{\varphi} \\ \mathbf{u} \end{pmatrix} \qquad \text{[cf., 4.2)]} \tag{5.18}$$

$$\text{Rot} \begin{pmatrix} \underline{\boldsymbol{\kappa}} \\ \underline{\boldsymbol{\varepsilon}} \end{pmatrix} = 0 \qquad \text{[cf., (4.3)]} \tag{5.19}$$

$$\text{Div} \begin{pmatrix} \underline{\boldsymbol{\sigma}} \\ \underline{\boldsymbol{\mu}} \end{pmatrix} + \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} = 0; \qquad \text{[cf., (4.12)]} \tag{5.20}$$

(5.14) corresponds to:

$$\text{Div Rot} \begin{pmatrix} \underline{\boldsymbol{\kappa}} \\ \underline{\boldsymbol{\varepsilon}} \end{pmatrix} = 0. \tag{5.21}$$

Confirming this directly was posed as an exercise at the end of the foregoing chapter.

Let us consider the homogeneous equilibrium conditions (5.20):

$$\text{Div} \begin{pmatrix} \underline{\boldsymbol{\sigma}} \\ \underline{\boldsymbol{\mu}} \end{pmatrix} = 0. \tag{5.22}$$

They can be fulfilled identically in the tensors $\underline{\mathbf{F}}$, $\underline{\mathbf{G}}$ when:

$$\begin{pmatrix} \underline{\boldsymbol{\sigma}} \\ \underline{\boldsymbol{\mu}} \end{pmatrix} = \text{Rot} \begin{pmatrix} \underline{\mathbf{F}} \\ \underline{\mathbf{G}} \end{pmatrix}. \tag{5.23}$$

$\underline{\mathbf{F}}$ and $\underline{\mathbf{G}}$ are the tensors of the stress functions of the Cosserat continuum, which number eighteen in all.

Corresponding to (5.13), the compatibility conditions (5.19) will be fulfilled identically by (5.18).

Literature and remarks on Chapters 4 and 5

The basic equations of kinematics and statics were presented in 1958 by **W. Günther** (then at Braunschweig, now at Karlsruhe) in the Abhandlungen der Braunschweigerischen Wiss. Ges. **10** (1958). One will find the representation of force-stresses and moment-stresses by stress functions there. In the elastic Cosserat continuum, the stress functions must satisfy compatibility conditions. For quite some time, I sought to exhibit the differential equations for the two tensors F_{ik} and G_{ik} of the stress functions, which nonetheless remained mired in equations that were confusing for some time. From my calculations, I then recognized the possibility of introducing operators like Grad, Rot, Div, along with their identities. My young colleague **S. Kessel** (who works with **W.**

Günther at Karlsruhe) took up my incomplete work and has achieved the goal quite skillfully. His definitive paper appeared in ZAMM **47** (1967). One will find a brief summary of it in *Mechanics of Generalized Cosserat Continua*, IUTAM Symposium, Freudenstadt-Stuttgart, 1967, ed. E. Kröner, Springer-Verlag, 1968. Three more operators are required for the calculations with stress functions, namely, Grad^*, Rot^*, and Div^*. They will follow from the considerations of the next chapter.

CHAPTER 6

FURTHER NOTIONS FROM THE CALCULUS OF DIFFERENTIAL FORMS

One requires two more identities in vector analysis that link the operators grad, rot, div with the Laplacian operator:

$$\Delta = \partial_i\,\partial_i = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_2^2}\,,$$

namely:

$$\text{div grad}\ \Phi = \Delta\Phi, \tag{6.1}$$

$$\text{rot rot}\ \mathbf{a} = -\,\Delta\mathbf{a} + \text{grad div}\ \mathbf{a}\,. \tag{6.2}$$

One calls Φ the *scalar potential*, while **a** is the *vector potential*. Both play an important role; e.g., in electrodynamics (viz., the theory of Maxwell's equations).

In the calculus of differential forms, (6.1) and (6.2) arise from the formula:

$$d\delta\overset{p}{\omega} + \delta d\overset{p}{\omega} = (d\delta + \delta d)\overset{p}{\omega} = -\,\Delta\overset{p}{\omega}. \tag{6.3}$$

δ is the so-called *codifferential*. In order to explain what it is, we must first become acquainted with the *star operator* *. In order to do that, I shall be content to define * for Euclidian space E^3 with Cartesian coordinates:

$$*dx_i = dA_i\,, \tag{6.4}$$

$$*dA_i = dx_i\,, \tag{6.5}$$

$$*dV = 1, \tag{6.6}$$

$$*1 = dV. \tag{6.7}$$

Obviously, one has:

$$** = 1. \tag{6.8}$$

For example, if:

$$\overset{1}{\omega} = a_i\,dx_i \tag{6.9}$$

then let:

$$*\overset{1}{\omega} = a_i\,dA_i\,. \tag{6.10}$$

[More generally: $*f\,(x_1, x_2, x_3)\,\overset{p}{\omega} = f *\overset{p}{\omega}$.]

The star operator then converts a form of degree p (p = 0, 1, 2, 3) into a form of degree 3 – p.

The codifferential of a form of degree p is now defined by:

$$\delta \overset{p}{\omega} = (-1)^p *d*\overset{p}{\omega}. \tag{6.11}$$

One has that:

$$\begin{array}{rl} *\overset{p}{\omega} & \text{has degree } 3 - p, \\ d*\overset{p}{\omega} & \text{has degree } 3 - p + 1 = 4 - p, \\ *d*\overset{p}{\omega} & \text{has degree } 3 - (4 - p) = p - 1. \end{array}$$

Whereas $d\overset{p}{\omega}$ is a form of degree $p + 1$, $\delta\overset{p}{\omega}$ will then be a form of degree $p - 1$. One will then have that $d\overset{3}{\omega} = 0$ and $\delta\overset{3}{\omega} = 0$.

We shall now apply (6.3) to the form $\overset{0}{\omega} = f(x_1, x_2, x_3)$. Since $\delta f = 0$, what will remain is:

$$\delta df = -\Delta f. \tag{6.12}$$

In detail:

$$df = \partial_i f\, dx_i , \tag{6.13}$$

$$*df = \partial_i f\, dA_i , \tag{6.14}$$

$$d*df = \partial_k \partial_i f\, dx_k \wedge dA_i = \partial_k \partial_i f\, \delta_{ki}\, dV = \Delta f\, dV,$$

$$*d*df = \Delta f. \tag{6.15}$$

Since df has degree 1, one will have:

$$\delta df = (-1)^1 *d*df. \tag{6.16}$$

(6.15), together with (6.16), confirm (6.12). By similar calculations, one confirms that $\delta\overset{1}{\omega}$ corresponds to the operator – div, and $\delta\overset{1}{\omega}$ corresponds to the operator rot.

When (6.3) is applied to $\overset{2}{\omega} = a_i\, dx_i$ that will yield:

$$d\delta\overset{1}{\omega} + \delta d\overset{1}{\omega} = -\Delta a_i\, dx_i , \tag{6.17}$$

or, in the same sequence:

$$-\text{grad div} + \text{rot rot} = -\Delta. \tag{6.18}$$

The cases $p = 2$, $p = 3$ imply nothing new in (6.3). However, it is noteworthy that:

$$\delta\delta\overset{p}{\omega} = 0. \tag{6.19}$$

In more detail, when one observes (6.8):

$$\delta\delta\overset{p}{\omega} = -(*d*)(*d*)\overset{p}{\omega} = -*dd*\overset{p}{\omega} = 0, \tag{6.20}$$

from the Poincaré Lemma. (6.19) once more implies the identities:

$$\begin{aligned} &\text{div rot} = 0 \qquad &&\text{for } p = 2, \\ &\text{rot grad} = 0 \qquad &&\text{for } p = 3. \end{aligned} \tag{6.21}$$

One should compare these with (1.19) and (1.20).

In order to prepare for a main result in Chapter 8, we shall give a proof of the theorem of Helmholtz that every vector field **v** (**x**) can be represented by:

$$\mathbf{v} = \text{rot}\,\mathbf{a} - \text{grad}\,\Phi. \tag{6.22}$$

In the calculus of differential forms, we must show that $\overset{2}{\omega} = v_i\, dA_i$ can be represented as:

$$\overset{2}{\omega} = d\overset{1}{\pi} + \delta\overset{3}{\sigma}, \tag{6.23}$$

in which one has $\overset{1}{\pi} = a_i\, dx_i$ and $\overset{3}{\sigma} = \Phi\, dV$.

In order to do that, we set:

$$\overset{1}{\pi} = \delta\overset{2}{\eta}, \qquad \overset{3}{\sigma} = d\overset{2}{\eta}, \tag{6.24}$$

in which $\overset{2}{\eta} = w_i\, dA_i$; when (6.24) is substituted in (6.23), and one observes (6.3), that will give:

$$\overset{2}{\omega} = (d\delta + \delta d)\,\overset{2}{\eta} = -\Delta\overset{2}{\eta}, \tag{6.25}$$

or

$$a_i = -\Delta w_i\,. \tag{6.26}$$

However, from the theorems of potential theory, (6.26) always possess a solution w_i . One substitutes w_i or $\overset{2}{\eta}$ in (6.24) and calculates $\overset{1}{\pi}$ and $\overset{3}{\sigma}$, with which, the proof will be complete.

We now further stipulate that the vector **v** should satisfy the equation:

$$\text{div}\,\mathbf{v} + \rho = 0. \tag{6.27}$$

$\overset{2}{\omega}$ in (6.23) should then be a solution of the equation:

$$d\overset{2}{\omega} + \rho\, dV = 0. \tag{6.28}$$

It follows from (6.23) that:

$$d\overset{2}{\omega} = d\delta\overset{3}{\sigma} = -\,\Delta\overset{3}{\sigma}, \tag{6.29}$$

such that, as a result of (6.28) and (6.29), one must have:

$$\Delta\overset{3}{\sigma} = -\,\rho\, dV. \tag{6.30}$$

In this, we replace $\overset{3}{\sigma}$ with $d\overset{2}{\eta}$ using (6.24) and obtain:

$$\Delta d\overset{2}{\eta} = d\Delta\overset{2}{\eta} = \rho\, dV\,. \tag{6.31}$$

(The operators Δ and d commute.) However, $\overset{2}{\eta}$ was determined by (6.25). Hence, (6.28) does, in fact, follow from (6.31). In summary, we have shown that any solution $\overset{2}{\omega}$ of equation (6.28) can be represented by (6.23), as long as (6.30) is fulfilled.

In the case of $\rho = 0$, (6.30) will say that $\overset{3}{\sigma}$ is harmonic. A deeper, more advanced examination will show that $\overset{3}{\sigma}$ can generally be set to zero only when the domain considered of the equation:

$$\text{div}\,\mathbf{v} = 0 \tag{6.32}$$

does not possess a cavity (i.e., a hole). Sources can indeed be present in such holes. We shall come back to this problem in Chapter 8.

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CHAPTER 7

THE COVARIANT CODIFFERENTIAL AND THE ASSOCIATED DIFFERENTIAL OPERATORS

Grad*, Div*, Rot*

We recall the definition (5.5) of the covariant differential D and pose the problem of defining a codifferential ϑ such that the formula:

$$\boxed{(D\vartheta + \vartheta D)\overset{p}{\Omega}_k = -\Delta \overset{p}{\Omega}_k} \tag{7.1}$$

will be true, in analogy to (6.3). Since our linear connection is skew-symmetric in the indices i and l, a closely-related problem is to define ϑ in terms of the connection:

$$D^- \overset{p}{\Omega}_k = d\,\overset{p}{\Omega}_k - \tau\,\varepsilon_{ilk}\,dx_i \wedge \overset{p}{\Omega}_l\,. \tag{7.2}$$

We then get, in succession:

$$D^- {*}\overset{p}{\Omega}_k = d\,{*}\overset{p}{\Omega}_k - \tau\,\varepsilon_{ilk}\,dx_i \wedge {*}\overset{p}{\Omega}_l\,, \tag{7.3}$$

$$(-1)^p\,{*}D^-\,\overset{p}{\Omega}_k = (-1)^p\,{*}d\,{*}\overset{p}{\Omega}_k + \tau\,(-1)^{p+1}\,\varepsilon_{ilk}\,{*}(dx_i \wedge {*}\overset{p}{\Omega}_l\,), \tag{7.4}$$

$$\boxed{\vartheta\overset{p}{\Omega}_k = \delta\overset{p}{\Omega}_k + \tau(-1)^{p+1}\varepsilon_{ilk}\,{*}(dx_i \wedge {*}\overset{p}{\Omega}_k).} \tag{7.5}$$

Corresponding to (6.19), we shall show that:

$$\boxed{\vartheta\vartheta\overset{p}{\Omega}_k = 0.} \tag{7.6}$$

In analogy to (6.20), one will now have:

$$\vartheta\vartheta\overset{p}{\Omega}_k = -\,({*}D^-\,{*})\,({*}D^-\,{*})\overset{p}{\Omega}_k = -\,{*}D^-\,D^-\,{*}\overset{p}{\Omega}_k\,. \tag{7.7}$$

It will then suffice to show that:

$$D^-\,D^-\,\overset{p}{\Omega}_k = 0 \tag{7.8}$$

for the connection (7.2). It follows from (7.2) that:

$$\begin{aligned} D^-\,(D^-\overset{p}{\Omega}_k\,) &= d\,(D^-\overset{p}{\Omega}_k\,) - \tau\,\varepsilon_{ilk}\,dx_i \wedge D^-\overset{p}{\Omega}_l \\ &= dd\,\overset{p}{\Omega}_k - \tau\,\varepsilon_{ilk}\,d\,(dx_i \wedge \overset{p}{\Omega}_l\,) - \tau\,\varepsilon_{ilk}\,dx_i \wedge d\overset{p}{\Omega}_l\,. \end{aligned} \tag{7.9}$$

($\tau^2 = 0$!) Now, one has $dd\overset{p}{\Omega}_k = 0$, and from (5.9):

$$d\,(dx_i \wedge \overset{p}{\Omega}_l) = -\,dx_i \wedge d\overset{p}{\Omega}_l\,, \tag{7.10}$$

such that (7.8), and therefore (7.6), will actually be true.

Naturally, verifying the validity of (7.1) is essentially more complicated.

We next have ($\tau^2 = 0$!):

$$D\vartheta\overset{p}{\Omega}_l = d\vartheta\overset{p}{\Omega}_l + \tau\,\varepsilon_{ilk}\,dx_i \wedge \delta\overset{p}{\Omega}_l\,, \tag{7.11}$$

$$\vartheta D\overset{p}{\Omega}_l = \delta D\overset{p}{\Omega}_k + \tau\,(-1)^{p+2}\,\varepsilon_{ilk} *(dx_i \wedge d\overset{p}{\Omega}_l). \tag{7.12}$$

We will then have:

$$(D\vartheta + \vartheta D)\,\overset{p}{\Omega}_k = (d\delta + \delta d)\,\overset{p}{\Omega}_k + \tau\,\varepsilon_{ilk}\,[(-1)^{p+1}\,d*(dx_i \wedge \overset{p}{\Omega}_l) + \delta(dx_i \wedge \overset{p}{\Omega}_l) + dx_i \wedge \delta\overset{p}{\Omega}_l + (-1)^p *(dx_i \wedge d\overset{p}{\Omega}_l)]. \tag{7.13}$$

We must show that the square bracket in (7.13) vanishes. After some intermediate computations (going from δ to d, bring the sign out of the bracket, dropping the inessential index l), it will remain for us to show that the differential form of degree p:

$$\Sigma(\overset{p}{\Omega}) = -\,d*(dx_i \wedge \overset{p}{\Omega}) - *d*(dx_i \wedge \overset{p}{\Omega}) + dx_i \wedge *d*\overset{p}{\Omega} + *(dx_i \wedge d\overset{p}{\Omega}) \tag{7.14}$$

vanishes identically in $\overset{p}{\Omega}$. Since Σ is linear in $\overset{p}{\Omega} = \overset{p}{\omega} + \tau\overset{p}{\hat{\omega}}$, we might replace $\overset{p}{\Omega}$ with $\overset{p}{\omega}$ in (7.14). We immediately convince ourselves that:

$$\Sigma(*\overset{p}{\omega}) = *\Sigma(\overset{p}{\omega})\,, \tag{7.15}$$

since $** = 1$. If we can then verify that:

$$\Sigma(\overset{p}{\omega}) = 0 \tag{7.16}$$

for $p = 0, 1$ then the proof for $p = 2, 3$ will also follow from (7.15).

Showing that $\Sigma(\overset{0}{\omega}) = 0$ is simple.

The calculation for $\overset{1}{\omega} = a_k\,dx_k$ is somewhat complicated. One should note the relation:

$$\varepsilon_{ijk}\,\varepsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl} \tag{7.17}$$

in that regard.

The verification that $\Sigma(\overset{1}{\omega}) = 0$ might then be left to the reader.

The square bracket in (7.13) will then, in fact, vanish, and what will remain is:

$$(D\vartheta + \vartheta D)\overset{p}{\Omega} = (d\delta + \delta d)\overset{p}{\Omega} = -\Delta\overset{p}{\Omega}, \tag{7.18}$$

which was to be proved.

Taking the codifferential lowers the degree of a differential form by one step:

$$\vartheta\overset{p}{\Omega}_k = \overset{p-1}{\Omega}_k . \tag{7.19}$$

We once more introduce the operators:

$$\vartheta\overset{1}{\Omega}_k = -\operatorname{Div}^*\overset{1}{\Omega}_k , \tag{7.20}$$

$$\vartheta\overset{2}{\Omega}_k = \quad \operatorname{Rot}^*\overset{2}{\Omega}_k , \tag{7.21}$$

$$\vartheta\overset{3}{\Omega}_k = -\operatorname{Grad}^*\overset{3}{\Omega}_k . \tag{7.22}$$

(7.6) yields the identities:

$$\operatorname{Rot}^*\operatorname{Grad}^*\overset{3}{\Omega}_k = 0, \tag{7.23}$$

$$\operatorname{Div}^*\operatorname{Rot}^*\overset{2}{\Omega}_k = 0. \tag{7.24}$$

Ultimately, one will get from (7.1), with the operators of Chapter 5:

$$-\operatorname{Rot}^*\operatorname{Grad} \quad = -\Delta; \qquad (p = 0) \tag{7.25}$$

$$-\operatorname{Grad}^*\operatorname{Div}^* + \operatorname{Rot}^*\operatorname{Rot}^* = -\Delta; \qquad (p = 1) \tag{7.26}$$

$$-\operatorname{Grad}^*\operatorname{Div}^* + \operatorname{Rot}^*\operatorname{Rot}^* = -\Delta; \qquad (p = 1) \tag{7.27}$$

$$-\operatorname{Div}\operatorname{Grad}^* = -\Delta. \qquad (p = 1) \tag{7.28}$$

Explicitly, these new differential operators have the form:

$$\operatorname{Grad}^*\begin{pmatrix}\mathbf{S}\\ \mathbf{T}\end{pmatrix} = \begin{pmatrix}\partial_i S_k\\ \partial_i T_k + \varepsilon_{ikl}\, S_l\end{pmatrix} \tag{7.29}$$

$$\operatorname{Rot}^*\begin{pmatrix}\underline{\mathbf{Q}}\\ \underline{\mathbf{R}}\end{pmatrix} = \begin{pmatrix}\varepsilon_{sri}\,\partial_i Q_{ik}\\ \varepsilon_{sri}\,(\partial_i R_{ik} - \varepsilon_{krl}\, Q_{il})\end{pmatrix}, \tag{7.30}$$

$$\operatorname{Div}^*\begin{pmatrix}\underline{\mathbf{Q}}\\ \underline{\mathbf{R}}\end{pmatrix} = \begin{pmatrix}\partial_i Q_{ik}\\ \partial_i R_{ik} - \varepsilon_{ikl}\, Q_{rl}\end{pmatrix}. \tag{7.31}$$

A comparison of these with the operators Grad, Rot, Div in (5.15), (5.16), and (5.17) will produce only a sign difference in the second terms of the second row.

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CHAPTER 8

THE COMPLETE REPRESENTATION OF TENSOR FIELDS ON A COSSERAT CONTINUUM

I shall recall the considerations of Chapter 6, eq. (6.22) regarding the complete representation of a vector field, in particular, a vector field that satisfies the condition (6.28).

On the grounds of the definition of the codifferential ϑ, we are now in a position to give the complete representation of the motor fields on a Cosserat continuum. Of particular interest are the cases:

$$p = 1: \qquad \overset{1}{\Omega}_k = (\kappa_{ik} + \tau\,\varepsilon_{ik})\, dx_i\,, \tag{8.1}$$

$$p = 2: \qquad \overset{2}{\Omega}_k = (\sigma_{ik} + \tau\,\mu_{ik})\, dA_i\,. \tag{8.2}$$

We would initially like to treat them together and show that the representation:

$$\overset{p}{\Omega}_k = D\overset{p-1}{\pi}_k + \vartheta\overset{p+1}{\Sigma}_k \tag{8.3}$$

is always possible. In order to prove (8.3), we substitute:

$$\overset{p-1}{\pi}_k = \vartheta\overset{p}{H}_k\,, \qquad \overset{p+1}{\Sigma}_k = D\overset{p}{H}_k \tag{8.4}$$

in (8.3) and get the Poisson equation for $\overset{p}{H}_k$:

$$\overset{p}{\Omega}_k = (D\vartheta + \vartheta D)\overset{p}{H}_k = -\Delta\overset{p}{H}_k\,. \tag{8.5}$$

There always exists a solution $\overset{p}{H}_k$ for a given $\overset{p}{\Omega}_k$, which completes the proof.

Furthermore, $\overset{p}{\Omega}_k$ shall be a solution of the equation:

$$D\overset{p}{\Omega}_k + \overset{p+1}{\Psi}_k = 0, \tag{8.6}$$

in which $\overset{p+1}{\Psi}_k$ must satisfy the compatibility condition:

$$D\overset{p+1}{\Psi}_k = 0. \tag{8.7}$$

Substituting (8.3) in (8.6) will imply that:

$$D\vartheta \overset{p+1}{\Sigma}_k + \overset{p+1}{\Psi}_k = 0, \tag{8.8}$$

which we can write as:

$$-\Delta \overset{p+1}{\Sigma}_k - \vartheta D \overset{p+1}{\Sigma}_k + \overset{p+1}{\Psi}_k = 0, \tag{8.9}$$

since $D\vartheta + \vartheta D = -\Delta$. Now, since $\overset{p+1}{\Sigma}_k$ can be represented as:

$$\overset{p+1}{\Sigma}_k = D \overset{p}{H}_k, \tag{8.10}$$

from (8.4), the condition:

$$D \overset{p+1}{\Sigma}_k = 0 \tag{8.11}$$

will be superfluous. (8.9) then simplifies to:

$$\Delta \overset{p+1}{\Sigma}_k = \overset{p+1}{\Psi}_k . \tag{8.12}$$

Now, $\overset{p}{H}_k$ in (8.10) was calculated from the given $\overset{p}{\Omega}_k$ using (8.5), such that $\overset{p+1}{\Sigma}_k$ is established already by (8.10). It then remains to show that:

$$\overset{p+1}{\Sigma}_k = D \overset{p}{H}_k \qquad \text{with} \qquad \Delta \overset{p}{H}_k + \overset{p}{\Omega}_k = 0 \tag{8.13}$$

fulfills (8.12).

We get:

$$\Delta D \overset{p}{H}_k = D\Delta \overset{p}{H}_k = -D \overset{p}{\Omega}_k = \overset{p+1}{\Psi}_k . \tag{8.14}$$

The operators Δ and D commute, and thus $\overset{p}{\Omega}_k$ satisfies eq. (8.6), by assumption.

The case of $p = 1$ in eq. (8.2) is of especial interest, since one will then be dealing with the complete representation of the stresses by stress functions. One once more considers equations (4.12), (5.20), (5.22), and (5.23):

$$D \overset{2}{\Omega}_k + \overset{3}{\Psi}_k = 0, \tag{8.15}$$

with

$$\overset{3}{\Psi}_k = (X_k + \tau Y_k)\, dV, \tag{8.16}$$

to be the equilibrium conditions. In order to fulfill (8.15) identically, we make the Ansatz:

$$\overset{2}{\Omega}_k = D \overset{1}{\pi}_k + \vartheta \overset{3}{\Sigma}_k \tag{8.17}$$

according to (8.3), in which:

$$\overset{1}{\pi}_k = (F_{ik} + \tau G_{ik})\, dx_i , \tag{8.18}$$

$$\overset{3}{\Sigma}_k = (Q_k + \tau R_k)\, dV. \tag{8.19}$$

In symbolic notation, (8.17) reads:

$$\begin{pmatrix} \underline{\boldsymbol{\sigma}} \\ \underline{\boldsymbol{\mu}} \end{pmatrix} = \mathrm{Rot} \begin{pmatrix} \underline{\mathbf{F}} \\ \underline{\mathbf{G}} \end{pmatrix} - \mathrm{Grad}^* \begin{pmatrix} \underline{\mathbf{Q}} \\ \underline{\mathbf{R}} \end{pmatrix}. \tag{8.20}$$

The proof of the completeness of the Ansatz (8.17) [(8.20), resp.] will be simplified by the fact that (8.11) is fulfilled from the outset in the case of $p = 2$, since a form of degree 4 will have to vanish, due to the fact that $dV \wedge dx_i = 0$. Substituting (8.17) in (8.15) will yield:

$$D\vartheta \overset{3}{\Sigma}_k + \overset{3}{\Psi}_k = 0 \tag{8.21}$$

or

$$-\Delta \overset{3}{\Sigma}_k + \overset{3}{\Psi}_k = 0. \tag{8.22}$$

A variant of the method of proof above is the following one: We initially set:

$$\overset{1}{\pi}_k = \vartheta \overset{2}{H}_k , \tag{8.23}$$

or symbolically:

$$\begin{pmatrix} \underline{\mathbf{F}} \\ \underline{\mathbf{G}} \end{pmatrix} = \mathrm{Rot}^* \begin{pmatrix} \underline{\mathbf{A}} \\ \underline{\mathbf{B}} \end{pmatrix}, \tag{8.24}$$

with

$$\overset{2}{H}_k = (A_{ik} + \tau B_{ik})\, dA_i\, . \tag{8.25}$$

(8.17) will then become:

$$\overset{2}{\Omega}_k = D\vartheta \overset{2}{H}_k + \vartheta \overset{3}{\Sigma}_k \tag{8.26}$$

or

$$\overset{2}{\Omega}_k = -\, \Delta \overset{2}{H}_k - \vartheta (D \overset{2}{H}_k - \overset{3}{\Sigma}_k)\, . \tag{8.27}$$

Let $\overset{2}{\Omega}_k$ be any solution of (8.15), so it represents a stress state in equilibrium. We split (8.27) into:

$$\overset{2}{\Omega}_k = \Delta \overset{2}{H}_k , \tag{8.28}$$

$$D \overset{2}{H}_k = \overset{3}{\Sigma}_k , \tag{8.29}$$

and calculate $\overset{2}{H}_k$ from (8.28) (which is always possible), substitute it into (8.29), and get a $\overset{2}{H}_k$ that must fulfill (8.22). One convinces oneself that this is, in fact, the case. That achieves the proof of the completeness of the representation (8.17) [(8.20), resp.]. Naturally, (8.22) means nothing but:

$$-\Delta Q_k = X_k\,, \quad -\Delta R_k = Y_k\,. \tag{8.30}$$

(8.17) will then consist of the two stress states $D\overset{1}{\pi}_k$ and $\vartheta\overset{3}{\Sigma}_k$. One can now show that $D\overset{1}{\pi}_k$ represents a stress state for which the forces and moments preserve equilibrium on any closed surface inside the body. If the body has a cavity whose outer surface is loaded with forces and moments that are not themselves in equilibrium then the second term $\vartheta\overset{3}{\Sigma}_k$ in (8.17) must be added to them, and even when the volume forces and moments X_k and Y_k vanish. From (8.22) and (8.30), $\overset{3}{\Sigma}_k$ will then be harmonic in all of the body. However, $\overset{3}{\Sigma}_k$ is not harmonic in the domain of the hole, and the resultant dyname would then determine the loads on the hole. Nonetheless, pursuing that idea further would have to be a subject for a later lecture.

Literature and concluding remarks

As I have said before, I found the six differential operators Grad, Rot, Div, Grad*, Rot*, Div* by calculation using stress functions. However, I first arrived at the basis for that calculus after my Braunschweiger colleague Stickforth advised me that I should look into the calculus of differential forms. The result was my paper "Analysis der Motorfelder im Cosserat-Kontiuum," ZAMM **47** (1967). In it, I went into the connection with v. Mises's motor calculus and calculated in six-dimensional motor space. It was only later that I noticed that the use of dual numbers and dual vectors simplified the calculation essentially. I published an apparently-rigorous paper on the representation of the equilibrium states by stress functions in Bull. Acad. Pol. Sci. (1) **15** (1967), 63. The symbolic notation that was found in, e.g., (8.20) was employed in it.

In connection with the two Padua lectures that I documented here, on 4 April 1968, I lectured at a Symposium in Rome on "Das dreidimensionale Cosserat-Kontinuum und die Cosserat-Schale im Kalkül der Differentialformen." One will find the calculus in general coordinates there. Generally, I did not succeed in defining the codifferential on curved shells. In the meantime, the Rome lecture appeared in the Symposia Mathematica, Istituto Nazionale di Alta Matematica, **1** (1968). For those who would like to go deeper into the theory of differential forms, I recommend the book *Differential Forms with Applications to the Physical Sciences* by **H. Flanders**, Acad. Press, 1963.